# Moduli space of parabolic bundles and parabolic connections

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## Introduction

**Rank two case**

- $\circ$ $(C, \boldsymbol{t})$: an irreducible smooth projective curve over $\mathbb{C}$ with $n$ distinct points $\boldsymbol{t} = (t_1, \ldots, t_n)$
- $\circ$ $(L, \nabla_L)$: a $\mathrm{tr}(\boldsymbol{\nu})$-connection over $(C, \boldsymbol{t})$ of rank one and degree $2g-1$
- $\circ$ $M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) := \{(E, \nabla, l_*) \mid \boldsymbol{\alpha}\text{-stable}, (\det E, \mathrm{tr}\nabla) \cong (L, \nabla_L)\}/\sim$
- $\circ$ $P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L) := \{(E, l_*) \mid \boldsymbol{\alpha}\text{-stable}, \det E \cong L\}/\sim$
- $\circ$ $N := \frac{1}{2}\dim M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) = \dim P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L)$

**Theorem (Loray-Saito, Fassarella-Loray, Fassarella-Loray-Muniz, M)**

For a suitable $\boldsymbol{\alpha}$, the rational map

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) \dashrightarrow \mathbb{P}^N \times P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L)$$

is birational.

**Problem**

Is $\mathrm{App} \times \mathrm{Bun}$ also birational for higer rank case ?

In this poster, we investigate the moduli space of rank 3 parabolic bundles and parabolic logarithmic connections over $\mathbb{P}^1$ with 3 points, which provides a counterexample of the problem.

## Moduli space of parabolic connections

- $\circ$ $\boldsymbol{\nu} = (\nu_{i,j})^{1 \le i \le 3}_{0 \le j \le 2} \in \mathbb{C}^9$: $\sum_{i=1}^{3}\sum_{j=0}^{2}\nu_{i,j} = -d \in \mathbb{Z}$

**$\boldsymbol{\nu}$-parabolic connection**

$(E, \nabla, l_* = \{l_{i,*}\}_{1 \le i \le 3})$

- $E$ : a vector bundle of rank 3 and degree $d$,
- $\nabla \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$ : a logarithmic connection

$\triangleright$ $\nabla$ is locally written by

$$\nabla = d + \sum_{i=1}^{3}\frac{A_i}{z - t_i}dz + \text{holomorphic}, \ A_i \in M_3(\mathbb{C}).$$

- $l_{i,*}$ : a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$ such that $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j}\mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for $1 \le i \le 3, 0 \le j \le 2$.

- $\circ$ $\boldsymbol{\alpha} = \{\alpha_{i,j}\}^{1 \le i \le 3}_{1 \le j \le 3} \in (0,1)^9$: a set of rational numbers satisfying $0 < \alpha_{i,1} < \alpha_{i,2} < \alpha_{i,3} < 1$ for each $i = 1, 2, 3$, and $\alpha_{i,j} \neq \alpha_{i',j'}$ for $(i,j) \neq (i',j')$

**$\boldsymbol{\alpha}$-stability**

$(E, \nabla, l_*)$ is $\boldsymbol{\alpha}$-stable

$\overset{\text{def}}{\Longleftrightarrow}$

for any subbundle $0 \neq F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$, the inequality

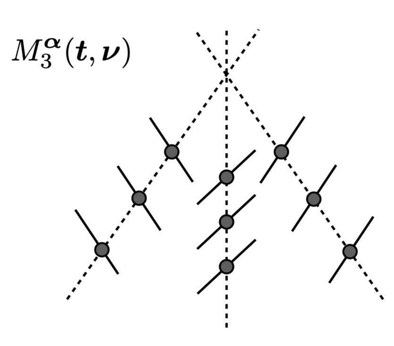
$$\frac{\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3}\alpha_{i,j}d_{i,j}(F)}{\mathrm{rank}\, F} < \frac{\deg E + \sum_{i=1}^{3}\sum_{j=1}^{3}\alpha_{i,j}}{\mathrm{rank}\, E}$$

holds, where $d_{i,j}(F) := \dim(F|_{t_i} \cap l_{i,j-1})/(F|_{t_i} \cap l_{i,j})$.

- $\circ$ $M^{\boldsymbol{\alpha}}_3(\boldsymbol{t}, \boldsymbol{\nu}) := \{(E, \nabla, l_*) \mid \boldsymbol{\alpha}\text{-stable}, \deg E = -2\}/\sim$

**Description of $M^{\boldsymbol{\alpha}}_3(\boldsymbol{t}, \boldsymbol{\nu})$**

Take $\boldsymbol{\alpha} = (\alpha_{i,j})_{1 \le i,j \le 3}$ such that $0 < \alpha_{i,j} \ll 1$ for any $1 \le i, j \le 3$. Then $M^{\boldsymbol{\alpha}}_3(\boldsymbol{t}, \boldsymbol{\nu})$ is the surface obtained by removing the anti-canonical divisor from a blow-up of $\mathbb{P}^2$ at three points on each three lines meeting in a single point.




## Moduli space of parabolic bundles

**Parabolic bundle**

$(E, l_* = \{l_{i,*}\}_{1 \le i \le 3})$

- $E$ : a vector bundle of rank 3 and degree $d$, and
- $l_{i,*}$ : a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$.

**Assumption**

For any $1 \le i \le 3, j = 1, 2$,

$$\alpha_{i,j+1} - \alpha_{i,j} = \text{constant} =: w$$

- $\circ$ $P^w(-2) := \{(E, l_*) \mid w\text{-stable}, \deg E = -2\}/\sim$
- $\circ$ $(E, l_*)$ is $w$-stable $\Longrightarrow E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

**Description of $P^w(-2)$**

(1) If $0 < w < 2/9, 4/9 < w < 1/2$, then $P^w(-2) = \emptyset$.

(2) If $2/9 < w < 1/3$, then a $w$-stable parabolic bundle $(E, l_*)$ fits into a nonsplit exact sequence

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^1}, \emptyset) \longrightarrow (E, l_*) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*) \longrightarrow 0,$$

where $n(l'_*) := \max_{\mathcal{O}_{\mathbb{P}^1}(-1) \cong F \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}} \#\{i \mid F|_{t_i} = l'_i\} = 1$. In particular, $P^w(-2) \cong \mathbb{P}^1$.

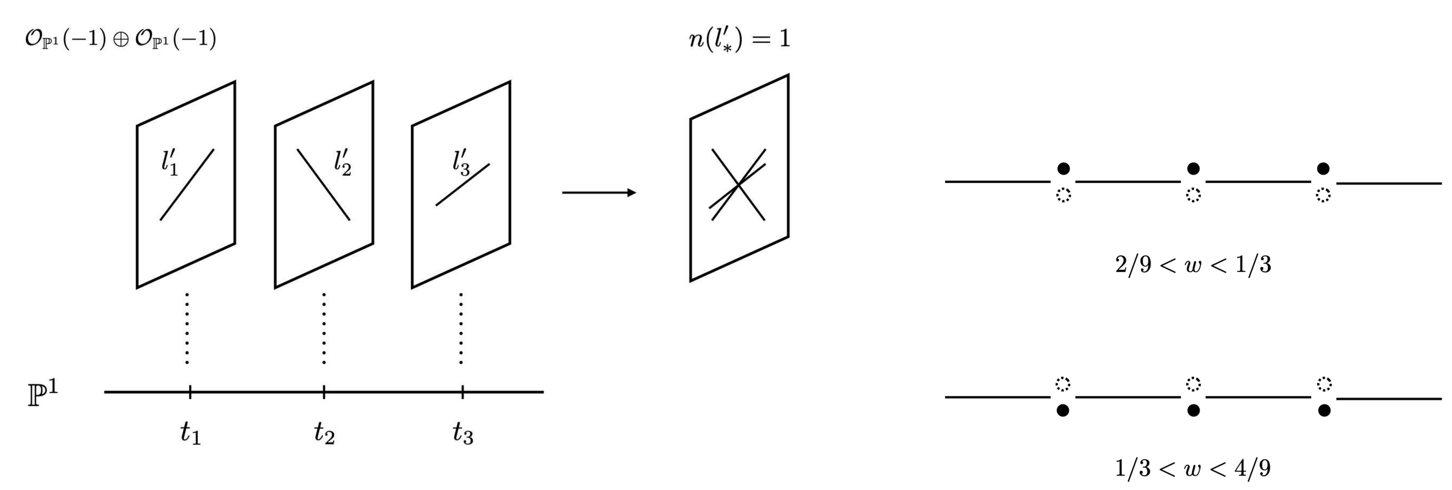
(3) If $1/3 < w < 4/9$, then a $w$-stable parabolic bundle $(E, l_*)$ is either type of the following:

(i) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ holds.

(ii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 1$, $n(l'_*) = 1$, and the condition $(*)$ holds.

In particular, $P^w(-2) \cong \mathbb{P}^1$. Here the condition $(*)$ is the following.

$(*)$ There is no subbundle $F \subset E$ such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l_{i,2} \subset F|_{t_i}$ and $F|_{t_j} = l_{j,1}$ for some $i$ and any $j \neq i$.




**A coordinate of $P^w(-2)$**

Assume that $2/9 < w < 1/3$. Then we can take a homogeneous coordinate $(a : b)$ of $P^w(-2) \cong \mathbb{P}^1$ by

$$l_{1,2} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix}, \ l_{1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}, l_{2,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}, \ l_{2,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$l_{3,2} = \mathbb{C}\begin{pmatrix}a+b\\1\\1\end{pmatrix}, \ l_{3,1} = \mathbb{C}\begin{pmatrix}a\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}b\\0\\1\end{pmatrix}.$$

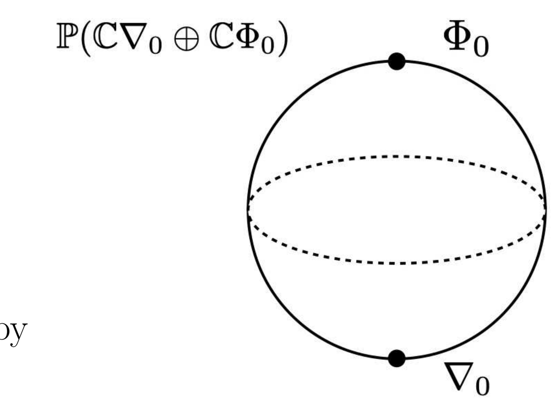
## Compactification by parabolic Higgs fields

**Parabolic Higgs field over $(E, l_*)$**

- $\Phi \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$ : a homomorphism satisfying $(\mathrm{res}_{t_i}(\Phi))(l_{i,j}) \subset l_{i,j+1}$ for $1 \le i \le 3, 0 \le j \le 2$.

- $\circ$ $\nabla_0$: a $\boldsymbol{\nu}$-parabolic connection over $(E, l_*)$
- $\circ$ $\Phi_0 \neq 0$: a parabolic Higgs field over $(E, l_*)$

$$\nabla_0 + \mathbb{C}\Phi_0 \underset{\text{open}}{\subset} \mathbb{P}(\mathbb{C}\nabla_0 \oplus \mathbb{C}\Phi_0)$$




- $\circ$ $M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0 := \{(E, \nabla, l_*) \mid (E, l_*) \in P^w(-2)\}/\sim$
- $\circ$ $\overline{M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0}$: the compactification of $M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0$ by parabolic Higgs fields

**Description of $\overline{M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0}$**

Assume that $2/9 < w < 1/3$. Then we have

$$\overline{M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0 \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0. \end{cases}$$

**Proof**

$$(\nabla_\infty(b), \Phi_\infty(b)) \cong (\nabla_0(a), \Phi_0(a))\begin{pmatrix} 1 & 0 \\ -(\nu_{1,0} + \nu_{2,0} + \nu_{3,0})a^{-1} & a^{-2} \end{pmatrix}$$

## App × Bun

Assume that $2/9 < w < 1/3$ and $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$.

**Apparent map**

There exists a unique filtration $E =: F_0 \supset F_1 \supset F_2 \supset 0$ such that $F_2 \cong \mathcal{O}_{\mathbb{P}^1}, F_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\nabla(F_2) \subset F_1 \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$. We define the apparent singularity $\mathrm{App}(E, \nabla, l_*)$ by the zero of the nonzero homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(-1) \cong F_1/F_2 \xrightarrow{\nabla} (E/F_1) \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \cong \mathcal{O}_{\mathbb{P}^1}.$$

- $\circ$ $V_0 := P^w(-2) \setminus \{(1:0), (1:1), (0:1)\}$
- $\circ$ $\mathrm{Bun} \colon \overline{M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0} \to P^w(-2), (E, \nabla, l_*) \mapsto (E, l_*)$

**App × Bun**

Then the morphism

$$\mathrm{App} \times \mathrm{Bun} \colon \mathrm{Bun}^{-1}(V_0) \longrightarrow \mathbb{P}^1 \times V_0$$

is finite and its generic fiber consists of three points.

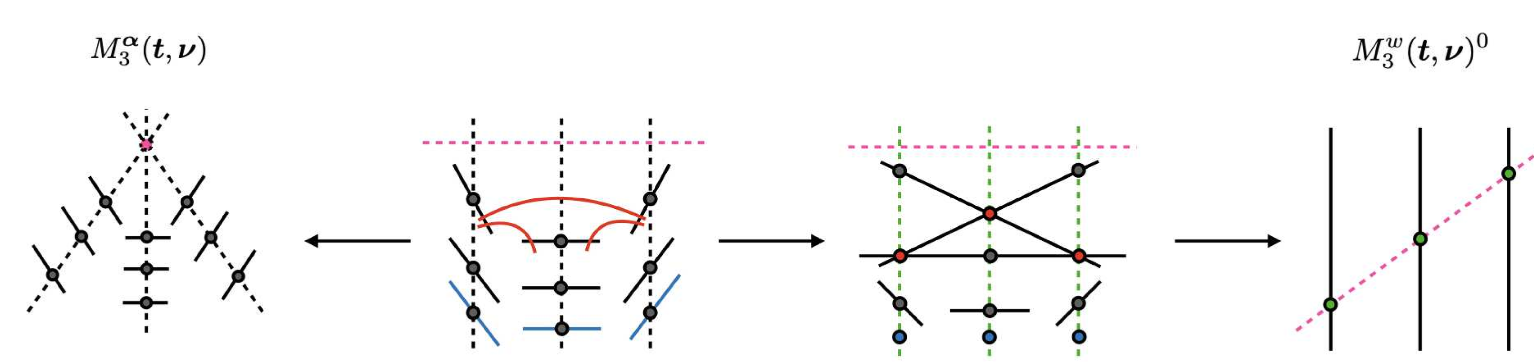
**Proof**

$\mathrm{Bun}^{-1}((E, l_{a,*})) = \mathbb{P}(\mathbb{C}\nabla_0(a) \oplus \mathbb{C}\Phi_0(a))$

$$\mathrm{App}(\lambda\nabla_0(a) + \mu\Phi_0(a)) = (f_1(a;\lambda,\mu) + f_2(a;\lambda,\mu) : t_1 f_1(a;\lambda,\mu) + t_2 f_2(a;\lambda,\mu)),$$

where $f_1(a;\lambda,\mu), f_2(a;\lambda,\mu)$ are homogeneous polynomials of degree 3 in variable $\lambda, \mu$.

## Relation between $M^{\boldsymbol{\alpha}}_3(\boldsymbol{t}, \boldsymbol{\nu})$ and $M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0$




## References

[1] T. Matsumoto, Moduli space of rank three logarithmic connections on the projective line with three poles, in preparation.