Geometric transitions for Calabi–Yau hypersurfaces II

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October 2023

1 Introduction

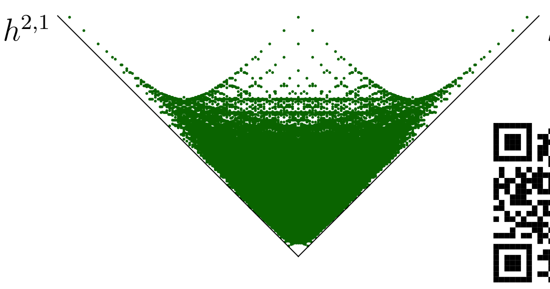
Geography of smooth Calabi–Yau 3-folds

A normal projective variety X over $\mathbb{C}$ is called *Calabi–Yau* if it has at worst Gorenstein canonical singularities and satisfies

- $K_X \sim 0$, and
- $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

In the case of a smooth Calabi–Yau 3-fold, the Hodge pair $(h^{1,1}, h^{2,1})$ is both a topological invariant and a derived invariant.

Example 1.1 (Calabi–Yau hypersurfaces in toric varieties)**.** In 2000, Kreuzer and Skarke provided the largest known dataset of Hodge pairs $(h^{1,1}, h^{2,1})$ for smooth Calabi–Yau 3-folds, derived from Calabi–Yau hypersurfaces in toric varieties:



Problems on the geography of smooth Calabi–Yau 3-folds

- Do they form a bounded family?
- Does topological mirror symmetry conjecture hold? It exchanges the Hodge numbers $h^{1,1}$ and $h^{2,1} \neq 0$.
- Are they all connected via geometric transitions?
- Is the ratio of Hodge pairs with odd $h^{1,1} - h^{2,1}$ equal to $1/e \fallingdotseq 0.3679$? (For Example 1.1, the ratio 0.3689 fails to reject this hypothesis statistically.)

Trinity: contractions, geometric transitions, and inclusions

A *geometric transition* is an operation connecting two (families of) smooth Calabi–Yau 3-folds via a birational contraction followed by a flat deformation. More precisely, a geometric transition is described as the composition $\iota \circ p : \mathcal{X} \to \mathcal{X}'$ of the following maps

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{p} & \overline{\mathcal{X}} & \xhookrightarrow{\iota} & \mathcal{X}' \\ \downarrow & & \downarrow & & \downarrow \\ B & = & B & \hookrightarrow & B' \end{array}$$

such that the squares in the diagram commute, where $\mathcal{X} \to B$ and $\mathcal{X}' \to B'$ are flat families of Calabi–Yau 3-folds over irreducible analytic spaces whose general fibers are smooth and p is a birational contraction over B shrunk if needed.

Example 1.2. A birational contraction of projective toric varieties $\mathbb{P}_\Sigma \to \mathbb{P}_{\Sigma'}$ often induces a geometric transition of Calabi–Yau hypersurfaces $\mathcal{X}_\Delta \to \mathcal{X}_{\Delta'}$. Here, $\mathbb{P}_\Sigma$ denotes a projective toric variety defined by a simplicial projective fan Σ, and Δ is the Newton polytope of anticanonical hypersurfaces in $\mathbb{P}_\Sigma$. These data defines a flat family $\mathcal{X}_\Delta \to B$. Note that it also induces the inclusions $G(\Sigma) \supset G(\Sigma')$ and $\operatorname{Conv} G(\Sigma) \supset \operatorname{Conv} G(\Sigma')$, where $G(\Sigma)$ is the set of primitive generators of one-dimensional cones in Σ and $\operatorname{Conv} G(\Sigma)$ is the convex hull of $G(\Sigma)$.

2 General results in toric geometry

From combinatorics to birational geometry

Let $N \simeq \mathbb{Z}^d$, and N^{prim} be the set of primitive lattice points of N. We call a finite set A a *primitive generating set* of $N_\mathbb{R}$ if A is a nonempty subset of N^{prim} and $\operatorname{Cone} A = N_\mathbb{R}$. For any complete fan Σ, the set of ray generators $G(\Sigma)$ is a primitive generating set.

Definition 2.1. $A \gtrdot A'$ denotes an inclusion $A \supset A'$ of primitive generating sets of $N_\mathbb{R}$ such that $|A| = |A'| + 1$.

Definition 2.2. A *fiber structure* $A_\mathrm{f} \subset A$ is an inclusion of primitive generating sets $A \subset N$ and $A_\mathrm{f} \subset N_\mathrm{f} := \mathbb{R}A_\mathrm{f} \cap N$. The *base* can be naturally defined as a primitive generating set $A_\mathrm{b} \subset N_\mathrm{b} := N/N_\mathrm{f}$. $A_\mathrm{f} \Subset A$ denotes a fiber structure satisfying $|A| = |A_\mathrm{f}| + |A_\mathrm{b}|$ and $|A_\mathrm{f}| = \operatorname{rank} N_\mathrm{f} + 1$, which we call a *Mori fiber structure* on A.

Lemma 2.3 (a rough version)**.** *An inclusion $A \gtrdot A'$ can be lifted to an extremal divisorial contraction $\mathbb{P}_\Sigma \to \mathbb{P}_{\Sigma'}$ of projective toric varieties such that $G(\Sigma) = A$ and $G(\Sigma') = A'$. A Mori fiber structure $A_\mathrm{f} \Subset A$ can be lifted to a Mori fiber space $\mathbb{P}_\Sigma \to \mathbb{P}_{\Sigma_\mathrm{b}}$ such that $G(\Sigma) = A$ and $G(\Sigma_\mathrm{b}) = A_\mathrm{b}$, whose general fiber $\mathbb{P}_{\Sigma_\mathrm{f}}$ represents $G(\Sigma_\mathrm{f}) = A_\mathrm{f}$.*

The MMP and the Sarkisov program

A precise version of Lemma 2.3 guarantees that the toric MMP is coarse grained into a sequence of inclusions of primitive generating sets, which terminates at a Mori fiber structure:

$$A = A_0 \gtrdot A_1 \gtrdot \cdots \gtrdot A_m = A' \Supset A'_\mathrm{f}.$$

Similarly, the toric Sarkisov program is also coarse grained into a sequence of inclusions of primitive generating sets:

Theorem 2.4 (M. 2023–, arXiv:2207.01632)**.** *Primitive generating sets of $N_\mathbb{R}$ with Mori fiber structure are all connected via sequences of following diagrams and their inverses.*

type I_d	*type* I_m	*type* II_irr	*type* II_ni	*type* IV_m
$\begin{array}{ccc} A & \gtrdot & A' \\ \Cup & & \Cup \\ A_\mathrm{f} & = & A'_\mathrm{f} \end{array}$	$\begin{array}{ccccc} A & = & A'' & \gtrdot & A' \\ \Cup & & \cup & & \Cup \\ A_\mathrm{f} & \Subset & A''_\mathrm{f} & \gtrdot & A'_\mathrm{f} \end{array}$	$\begin{array}{ccccc} A & \lessdot & A'' & \gtrdot & A' \\ \Cup & & \cup & & \Cup \\ A_\mathrm{f} & \lessdot & A''_\mathrm{f} & \gtrdot & A'_\mathrm{f} \end{array}$	$\begin{array}{ccccc} A & \lessdot & A'' & \gtrdot & A' \\ \Cup & & \cup & & \Cup \\ A_\mathrm{f} & = & A''_\mathrm{f} & = & A'_\mathrm{f} \end{array}$	$\begin{array}{ccccc} A & = & A'' & = & A' \\ \Cup & & \cup & & \Cup \\ A_\mathrm{f} & \Subset & A''_\mathrm{f} & \Supset & A'_\mathrm{f} \end{array}$

3 Application in two dimensions

In two dimensions, the following statements of trinity is established by finding a "good" sequence of Sarkisov links:

Trinity in two dimensions

Theorem 3.1 (M. 2023–, arXiv:2207.01632)**.**

- *The classical Castelnuovo–Noether's theorem on the Cremona group* $\operatorname{Bir} \mathbb{P}^2$ *connects all smooth rational surfaces via "good" sequences of contractions.*
- *Elliptic curves described as anticanonical hypersurfaces in smooth proper surfaces are all connected via "geometric transitions" associated with the ambient contractions.*
- *Reflexive polygons are all connected via inclusions.*