Rational curves on Fano threefolds with terminal factorial singularities

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This poster summarizes the results of my project in progress, in which we generalize the results of the paper [BLRT] regarding the Geometric Manin's Conjecture (GMC) for smooth Fano 3-folds to the case of terminal factorial Fano 3-folds. Our goal is to analyze the asymptotic behavior of irreducible components of the Hom scheme $\mathrm{Hom}(\mathbb{P}^1, X, \alpha)$, as $\alpha \in N_1(X)$ grows positively.

0 Settings

- X: terminal factorial Fano 3-fold/$\mathbb{C}$
- $\alpha \in N_1(X)$: effective curve class on X
- $M \subset \overline{M}_{0,0}(X, \alpha)$: irreducible component generically parametrizing stable maps $f\colon \mathbb{P}^1 \to X$ s.t. $f_*\mathbb{P}^1 = \alpha$
- $\mathcal{U} \xrightarrow{\mathrm{ev}} X$: evaluation morphism
  $\mathcal{U} \downarrow$ universal family
  M
- For a projective variety Y with a nef, big divisor L,

$$a := a(Y, L) = \inf\{t \in \mathbb{R} \mid K_{\tilde{Y}} + t\phi^* L \in \overline{\mathrm{Eff}}^1(\tilde{Y})\},$$

  which is indep. of the choice of a resolution $\phi\colon \tilde{Y} \to Y$
- $\kappa := \kappa(\tilde{Y}, K_{\tilde{Y}} + a\phi^* L)$: Iitaka dimension

1 Surfaces with higher a-invariants

a-invariants should detect "pathological" components.

Theorem 1 (cf. [BLRT], §.4)**.** $Y := \mathrm{ev}(\mathcal{U})$

- $Y = X \implies$ *general* $f \in M$ *is free,* $\dim M = -K_X \cdot \alpha$
- $Y \neq X \implies a(Y, -K_X) > 1$

Here is the list of surfaces with $a > 1$:

No.	$(Y, -K_X\vert_Y)$	κ	a	contractibility
1	swept out by $-K_X$-lines	1	2	F
2	$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$	0	2	T (E1 type)
3	$(\mathbb{P}^2, \mathcal{O}(2))$	0	3/2	T (E2 type)
4	$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$	0	2	T (E3 type)
5	(quadric cone, $\mathcal{O}(1)$)	0	2	T (E4 type)
6	$(\mathbb{P}^2, \mathcal{O}(1))$	0	3	T (E5 type)
7	its normalization $\cong (\mathbb{P}^2, \mathcal{O}(2))$	0	3/2	F

When X is smooth, the case 7 does not occur by [BLRT].

2 a-covers

Theorem 2 (work in progress, cf. [BLRT], §.5)**.**
Assume M is dominant and general fiber of ev *is not irreducible*
$\tilde{\mathcal{U}} \to \mathcal{U}$*: smooth resolution*
$\tilde{\mathcal{U}} \to Y \xrightarrow{f} X$*: Stein fact. of* $\tilde{\mathcal{U}} \to \mathcal{U} \to X$
$\implies f\colon Y \to X$ *: a-cover (i.e. $a(Y, -f^*K_X) = 1$)*

κ	Iitaka fib. $\pi\colon Y \dashrightarrow B$	X: smooth case ([BLRT])
2	general fiber maps birationally to a $-K_X$-conic	
1	(work in progress)	π: dP fib. if $-K_X$:very ample
0	$\nexists$ when $X \not\supseteq$ E5 divisor	$\nexists$ in any case

3 Movable Bend-and-Break

MBB reduces the irreducibility problem to low deg cases.

Theorem 3 (cf. [BLRT], §.6)**.**
M: dominant component s.t. $-K_X \cdot \alpha \geq 5$
$\implies \exists (f\colon C \to X) \in M$ *s.t. either*

(1) $C = C_1 + C_2$, $f|_{C_i}$: free, or
(2) $C = C_1 + \ell + C_2$, $f|_{C_i}$: free, $f(\ell)$: $-K_X$-line in E5 div.

Moreover, $-K_X \cdot \alpha \geq 10 \implies \exists f \in M$ *of the type (1)*

4 Geometric Manin's Conjecture

M: Manin component
$:\Longleftrightarrow \nexists f\colon Y \to X$: breaking map, $\nexists N \subset \overline{M}_{0,0}(Y)$: irred. comp.
s.t. f induces a dominant, gen. finite map $f_*\colon N \dashrightarrow M$

Conjecture 4 ([Tan])**.**
$\exists c \in \mathbb{Z}_{>0}$, $\exists \alpha \in \mathrm{Nef}_1(X)_{\mathbb{Z}}$ *s.t.* $\forall \beta \in \alpha + \mathrm{Nef}_1(X)_{\mathbb{Z}}$,
$\exists$ *exactly c Manin components in* $\overline{M}_{0,0}(X, \beta)$

5 Example: Cubic threefolds

$X \subset \mathbb{P}^4$: factorial terminal cubic 3-fold, $H := -K_X/2$

Theorem 5.
$\ell \in N_1(X)$: H-line class
$\implies \overline{M}_{0,0}(X, \ell)$ *is dominant, irreducible of dim* 2

(Strategy of the proof).
$q \in \mathrm{Sing}(X)$, C: 1-para. family of lines through q
Prove C is irreducible, $\overline{M}_{0,0}(X, \ell)$ is birational to $\mathrm{Sym}^2(C)$.

Theorem 6 (cf. [LT], §.7)**.** $\forall d \geq 2$,
$\overline{M}_{0,0}(X, d\ell) = \mathcal{R}_d \cup \mathcal{N}_d$*: union of two components of dim $2d$*

- *general $f \in \mathcal{R}_d$ is birational*
- *any $f \in \mathcal{N}_d$ is a multiple cover of an H-line of deg d*

In particular, $\forall d \geq 2$, $\exists! \mathcal{R}_d \subset \overline{M}_{0,0}(X, d\ell)$*: Manin component*

References

[BLRT] R. Beheshti et al. *Moduli spaces of rational curves on Fano threefolds.* Adv. Math., 408 Part A:Article 108557, 2022.

[LT] B. Lehmann and S. Tanimoto. *Geometric Manin's conjecture and rational curves.* Compos. Math., 155(5):833–862, 2019.

[Tan] S. Tanimoto. *An introduction to Geometric Manin's conjecture.* arXiv:2110.06660, 2021.