On Fujita's freeness conjecture in mixed characteristic

Hirotaka Onuki

Graduate School of Mathematical Sciences, The University of Tokyo

Background

Fujita's freeness conjecture

- Let X be an n-dimensional smooth projective variety over $\mathbb{C}$.
- Let $\mathcal{L}$ be an ample line bundle on X.

Then the adjoint bundle $\omega_X \otimes \mathcal{L}^{n+1}$ is globally generated.

If $\mathcal{L}$ is globally generated, Fujita's freeness conjecture is proved by Kodaira vanishing and Castelnuovo-Mumford regularity. Similar results are known in

- positive characteristic (Smith'97, Hara'03, Keeler'08)
- mixed characteristic [BM+23]

This globally generated case was generalized to the relative setting in characteristic zero:

Theorem (Popa-Schnell'14)

- Let $f\colon X \to Y$ be a surjection between projective varieties over $\mathbb{C}$ with X smooth and Y n-dimensional.
- Let $\mathcal{L}$ be an ample and globally generated line bundle on Y.

Then $f_*\omega_X^m \otimes \mathcal{L}^{m(n+1)}$ is globally generated for all $m \geq 1$.

- To see that this is indeed a generalization, set $f = \mathrm{id}_X$ and $m = 1$.

In positive characteristic, the analogous statement of this theorem is false (Moret-Bailly'81, Shentu-Zhang'20). However, an analog with some additional assumptions holds:

Theorem [Eji19] (cf. [Eji23])

Let k be a perfect field of characteristic $p > 0$.

- Let $f\colon X \to Y$ be a surjection between projective varieties over k with X regular and $n = \dim Y$.
- Let $\mathcal{L}$ be an ample and globally generated line bundle on Y.

If ω_X is f-ample, then $f_*\omega_X^m \otimes \mathcal{L}^{m(n+1)}$ is globally generated for all $m \gg 1$.

- If $f = \mathrm{id}_X$, the f-ampleness of ω_X is automatic.

The following table gives a very brief summary of the results mentioned above.

	char. zero	pos. char.	mixed char.
absolute	OK	OK	OK
relative	OK	OK	?

Figure 1: Results on the globally generated case of Fujita's conj. and its relative variant

We consider the relative case in mixed characteristic.

Main Theorem

Let $(R, \mathfrak{m})$ be a Noetherian complete local domain of residue characteristic $p > 0$.

Main Theorem

- Let $f\colon X \to Y$ be a surjection between integral projective R-schemes with X regular and let $n = \dim Y_{\mathfrak{m}}$ be the dimension of the closed fiber of Y.
- Let $\mathcal{L}$ be an ample and globally generated line bundle on Y.

If ω_X is f-ample, then $f_*\omega_X^m \otimes \mathcal{L}^{m(n+1)}$ is globally generated for all $m \gg 1$.

Sketch of the proof

Although Kodaira vanishing is only available in characteristic zero, Bhatt [BM+23] proved a Kodaira-type vanishing theorem "up to finite covers". Bhatt's vanishing theorem is employed to derive a global generation result for the sheaves defined below (Key Lemma). This result is used repeatedly to prove the Main Theorem.

Definition

Let X be a normal integral projective R-scheme.

- [BM+23] For a Weil divisor M on X, define
$$\mathbf{B}^0(X, \mathcal{O}_X(K_X + M)) \subset H^0(X, \mathcal{O}_X(K_X + M))$$
to be the submodule
$$\bigcap_{g\colon X' \to X} \mathrm{Im}(H^0(X', \mathcal{O}_{X'}(K_{X'} + g^*M)) \to H^0(X, \mathcal{O}_X(K_X + M)))$$
where $g\colon X' \to X$ runs over all finite surjection from a normal integral scheme X'.
- [HLS21] Define
$$\tau_+(\mathcal{O}_X) \subset \mathcal{O}_X$$
as the subsheaf such that $\tau_+(\mathcal{O}_X) \otimes \mathcal{A}$ is globally generated by $\mathbf{B}^0(X, \mathcal{A})$ for a sufficiently ample line bundle $\mathcal{A}$ on X.
- Let $f\colon X \to Y$ be a surjection to an integral R-scheme Y and N a Cartier divisor on X. Define
$$\mathbf{B}^0 f_*(\tau_+(\mathcal{O}_X) \otimes \mathcal{O}_X(N)) \subset f_*\mathcal{O}_X(N)$$
as the subsheaf such that $\mathbf{B}^0 f_*(\tau_+(\mathcal{O}_X) \otimes \mathcal{O}_X(N)) \otimes \mathcal{L}$ is globally generated by $\mathbf{B}^0(X, \mathcal{O}_X(N) \otimes f^*\mathcal{L}) \subset H^0(Y, f_*\mathcal{O}_Y(N) \otimes \mathcal{L})$ for a sufficiently ample line bundle $\mathcal{L}$ on Y.

We state some properties of these modules.

- If X is regular, $\tau_+(\mathcal{O}_X) = \mathcal{O}_X$ holds.
- If N is sufficiently f-ample, $\mathbf{B}^0 f_*(\tau_+(\mathcal{O}_X) \otimes \mathcal{O}_X(N)) = f_*(\tau_+(\mathcal{O}_X) \otimes \mathcal{O}_X(N))$ holds.

Key Lemma

- Let $f\colon X \to Y$ be a surjection between integral projective R-schemes with X normal and $n = \dim Y_{\mathfrak{m}}$.
- Let $\mathcal{L}$ be an ample and globally generated line bundle on Y.
- Let N be a Cartier divisor on X with $N - K_X$ big and semiample.

Then $\mathbf{B}^0 f_*(\tau_+(\mathcal{O}_X) \otimes \mathcal{O}_X(N)) \otimes \mathcal{L}^n$ is globally generated.

- If, furthermore, X is regular and N is sufficiently f-ample, then $f_*\mathcal{O}_X(N) \otimes \mathcal{L}^n$ is globally generated.

We prove the Key Lemma by using Bhatt's vanishing theorem and the Koszul/Skoda complex.

Now we prove the Main Theorem. We make two observations: for sufficiently large $m \gg 1$ and for any $s \geq 0$,

1. if $f_*\omega_X^m \otimes \mathcal{L}^s$ is globally generated, so is $\omega_X^m \otimes f^*\mathcal{L}^s$ (by the f-freeness of ω_X^m)
2. if $\omega_X^{m-1} \otimes f^*\mathcal{L}^{s-\delta}$ is semiample for some $\delta > 0$, then $f_*\omega_X^m \otimes \mathcal{L}^{s+n}$ is globally generated (by the Key Lemma)

We use these observations to show that $\omega_X \otimes f^*\mathcal{L}^{n+1}$ is semiample. Then we get the Main Theorem from 2.

References

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