Stable rationality of hypersurfaces of mock toric varieties

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Theorem 1 (Y.)

A very general quartic hypersurface in $\mathrm{Gr}_{\mathbb{C}}(2,5)$ is not stably rational. In particular, it is not rational.

⟨Notation⟩
- Let R denote $\bigcup_{n\in\mathbb{Z}_{>0}} \mathbb{C}[[t^{\frac{1}{n}}]]$.
- Let K denote $\bigcup_{n\in\mathbb{Z}_{>0}} \mathbb{C}((t^{\frac{1}{n}}))$.
- Let F be a field, SB_F denote the set of stable birational equivalence classes $\{X\}_{\mathrm{sb}}$ of integral F-schemes X of finite type.
- Let $\mathbb{Z}[\mathrm{SB}_F]$ denote the free abelian group on SB_F.

Theorem 2 (Nicaise–Ottem 21)

There exist a unique ring morphism $\mathrm{Vol}_{\mathrm{sb}} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$ such that for every strictly toroidal proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}_{\mathrm{sb}}(\{X\}_{\mathrm{sb}}) = \sum_{E\in\mathcal{S}(\mathcal{X})} (-1)^{\mathrm{codim}(E)}\{E\}_{\mathrm{sb}}$$

, where $\mathcal{S}(\mathcal{X})$ denotes a set of stratum of $\mathcal{X}_{\mathbb{C}}$.
In particular, $\mathrm{Vol}_{\mathrm{sb}}(\{\mathrm{Spec}(K)\}_{\mathrm{sb}}) = \{\mathrm{Spec}(\mathbb{C})\}_{\mathrm{sb}}$

For applying this morphism to the rationality problem, there exist 3 obstructions.

(1) To construct such a model
(2) To enumerate all stratum of the closed fiber
(3) To check the rationality of all stratum

Theorem 3 (Nicaise–Ottem 22)

A very general quartic 4-fold is not stably rational.

⟨The sketch of the proof of Theorem 3⟩

(1), (2) We construct such a model as a hypersurfaces of a proper **toric variety** over R. (cf. tropical compactification) We can carry out the construction and computation **combinatorially**.

(3) There exists a strata E such that E is birational to a stably irrational quartic double solid (3-fold) cf.[Artin-Mumford 72]

Remark 4

*We doesn't need the **overall** toric structure, but need toric structure **locally**.*
Then we introduce mock toric varieties.

Definition 5 (Y.)

Let Z be a scheme over a field k.
*The **mock toric structure** of Z consists of the following 6 items which hold the following conditions (I) – (V):*

- *A lattice N of finite rank*
- *A strongly convex rational polyhedral fan Δ in $N_{\mathbb{R}}$*
- *A closed immersion $\iota : Z \hookrightarrow X(\Delta)$*
- *A finite set Φ*
- *A family of sublattices of N $\{N_\varphi\}_{\varphi\in\Phi}$*
- *A family of sub fans of Δ $\{\Delta_\varphi\}_{\varphi\in\Phi}$*

(I) *$N/((\langle\sigma\rangle \cap N) + N_\varphi)$ is torsion-free for any $\varphi \in \Phi$ and $\sigma \in \Delta_\varphi$.*
(II) *$\Delta = \bigcup_{\varphi\in\Phi} \Delta_\varphi$.*
(III) *For $\varphi \in \Phi$, let q_φ be a quotient map $N \to N/N_\varphi$. Then $(q_\varphi)_{\mathbb{R}}|_{\mathrm{Supp}(\Delta_\varphi)}$ is injective for any $\varphi \in \Phi$.*
(IV) *For any $\sigma \in \Delta$, $Z \cap O_\sigma \neq \emptyset$.*
(V) *Let $\Delta(\varphi) = \{(q_\varphi)_{\mathbb{R}}(\sigma) \mid \sigma \in \Delta_\varphi\}$ be a strongly convex rational polyhedral fan in $(N/N_\varphi) \otimes \mathbb{R}$. Then the following decomposition is an **open immersion** for any $\varphi \in \Phi$:*

$$Z \cap X(\Delta_\varphi) \xrightarrow{\iota} X(\Delta_\varphi) \xrightarrow{(q_\varphi)_*} X(\Delta(\varphi))$$

⟨Example⟩ Toric varieties and the **Del-Pezzo surface of degree 5** have mock toric structures.

Proposition 6 (Y.)

(a) *A mock toric variety is a rational toroidal variety over k.*
(b) *$Z \cap \overline{O_\sigma}$ has a mock toric structure for any $\sigma \in \Delta$.*
(c) *A toric resolution $X(\Delta') \to X(\Delta)$ induces a mock toric structure of $Z \times_{X(\Delta)} X(\Delta')$.*
(d) *The restriction of the action morphism $T_N \times Z \to X(\Delta)$ is faithfully flat.*

⟨The sketch of the proof of Theorem 1⟩

(1), (2) We construct such a model as a hypersurface of a proper mock toric variety over R. For the construction, we use a mock toric structure of the Del-Pezzo surface of degree 5. We can carry out the construction and computation combinatorially.

(3) There exists a strata E such that E is birational to a stably irrational quartic 4-fold (cf. thm 3).

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