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Developing a vertical quasi-two-dimensional surface-subsurface flow model using an approximation for hydraulic gradient

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Abstract:

Various models have been developed to predict rainfall-runoff. However, practical models often simplify the actual phenomena and do not always provide sufficient accuracy for field-observed parameter values, such as soil water retention and permeability. Other models based on Richards' equation can directly account for these factors but are not practical because of their huge computational cost. In this study, we developed a vertical quasi-two-dimensional surface-subsurface flow model (quasi-2D model) based on Richards' equation, in which the hydraulic gradient in the downward direction was approximated by the slope gradient. This method makes it possible to consider soil moisture distribution perpendicular to the slope and simplify the modeling of the runoff process. Rainfall-runoff simulations were conducted on a single slope using the quasi-2D model and compared with the results computed under the same conditions using a detailed model solving the two-dimensional Richards' equation (2D model). For both subsurface and surface flows, the quasi-2D model reproduced the results of the 2D model well (NSE > 0.99), and performed particularly well on steeper slopes. The computation time of the quasi-2D model was reduced to less than 1/10 of that of the 2D model, confirming the usefulness of the quasi-2D model.

KEYWORDS rainfall-runoff model; quasi-2D model; surface-subsurface flow; soil moisture profile; hydraulic gradient; Richards' equation

INTRODUCTION

Predicting runoff in response to rainfall is critical, and various rainfall-runoff models have been developed. Although runoff prediction models should represent physical processes, most practical models do not adequately capture complicated waterflows, such as the vertical flow component, which Tani *et al.* (2020) highlighted as important. In addition, it has been pointed out that the performance of such simplified models is influenced by simulation conditions (An *et al.*, 2010), and optimizing the model parameters to improve applicability fails to reflect watershed properties.

Richards' equation (Richards, 1931) is generally used to simulate the vertical unsaturated flow and two- or three-dimensional water movement. Such sophisticated models have the advantage that the observed parameter values can be used directly; thus, the actual soil properties can be reflected in the model. However, their large computational cost makes it difficult to apply them to basin-scale rainfall-runoff prediction, although various efforts have been made (Farthing and Ogden, 2017).

In our previous study (Fugami *et al.*, 2022), we developed a vertical quasi-two-dimensional subsurface flow model. The saturated-unsaturated subsurface flow was represented by the vertical two-dimensional Richards' equation, wherein the hydraulic gradient in the downward direction is approximated by the slope gradient, thus simplifying the modeling of the rainfall-runoff process. This model successfully reproduced the results of subsurface runoff simulated using a detailed model that assumed the true values. However, the previous model was inefficient, requiring iterative computation for both the main computation of subsurface flow and the estimation of infiltration capacity at the ground surface, and the estimated values were not always accurate. Moreover, the surface flow was not routed, but moved to the downstream end of the slope once it was generated.

In this paper, we propose a vertical quasi-two-dimensional surface-subsurface flow model (hereafter "quasi-2D model") that improves the previous model mentioned above. One of the notable advancements is the implementation of the surface flow routing scheme using the kinematic wave model, which is computed separately from subsurface flow computations. In addition, the presence or absence of surface flow is used to switch the boundary conditions at the ground surface. This improvement eliminates the need to estimate infiltration capacity and reduces the number of iterative computations.

The validity of the quasi-2D model is assessed by performing rainfall-runoff simulations on a single slope. The simulated results were compared with those of a vertical two-dimensional surface-subsurface flow model (hereafter "2D model") coupled with Richards' equation and the diffusion wave equation (An and Yu, 2014). Furthermore, we

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compared the computation times of the two models and confirmed the usefulness of the quasi-2D model; that is, rainfall-runoff simulations based on Richards' equation can be performed at a low computational cost with the quasi-2D model.

QUASI-2D MODEL

Model structure

Figure 1 shows the structure of the quasi-2D model, in which grid cells are applied to the slope, and the water movement is computed for each column sequentially from upstream to downstream. The subsurface and surface flows are described by Richards' equation and kinematic wave equation, respectively. Each equation is solved separately while maintaining the closure of the water balance between the subsurface and surface flows. This allows different Δt values to be set for each flow, with the faster surface flow having a smaller Δt value than the subsurface flow.

Subsurface flow modeling

Subsurface flow is modeled by the vertical two-dimensional Richards' equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K\left(\frac{\partial \psi}{\partial z} + \cos\omega\right)\right] - \frac{\partial u}{\partial x} \tag{1}$$

where θ is the volumetric water content, K is the hydraulic conductivity, ψ is the pressure head, ω is the slope angle, t is time, z is the spatial coordinate perpendicular to the slope, x is the spatial coordinate in the downward direction, and u is the Darcy velocity in the x-direction.

Equation (1) is nonlinear, so it is discretized using a finite difference method as follows, and the numerical solution is obtained using the modified Picard iteration method (Celia *et al.*, 1990).

$$\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} = \frac{1}{\Delta z^2}\left[K_{i-1/2}^{n+1}\psi_{i-1}^{n+1} - \left(K_{i-1/2}^{n+1} + K_{i+1/2}^{n+1}\right)\psi_i^{n+1} + K_{i+1/2}^{n+1}\psi_{i+1}^{n+1}\right] + \frac{K_{i+1/2}^{n+1} - K_{i-1/2}^{n+1}}{\Delta z}\cos\omega + \frac{1}{\Delta x}\left(u_{\mathrm{UP},i}^{n+1} - u_{\mathrm{DW},i}^{n+1}\right) \tag{2}$$

where superscript n is the time level, Δt is the time step, and subscript i is the cell position in the z-direction. $i + 1/2$ and $i - 1/2$ denote the top and bottom interfaces of cell i, respectively. Δx and Δz are the spatial differences in each axis direction. $u_{\mathrm{DW},i}^{n+1}$ is the Darcy velocity in the downward direction of the water flowing out from the computational cells downstream, given by the following equation:

$$u_{\mathrm{DW},i}^{n+1} = K_i^{n+1}\sin\omega \tag{3}$$

Here, the value of the hydraulic gradient in the downward direction is definitively given by the slope gradient, which allows sequential saturated-unsaturated flow computation column by column. $u_{\mathrm{UP},i}^{n+1}$ is the Darcy velocity in the downward direction of the water flowing in from the upstream computational cells. At the time of computation for the target column, the computation for the upstream area has already been completed, and u_{DW} of the upstream column is used as the inflow boundary condition u_{UP} at the upstream face of the target column.

Surface flow modeling

Surface flow is modeled by the kinematic wave model stated as

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r - f_{\mathrm{in}} \tag{4}$$

$$q(h) = ah^m \tag{5}$$

where h is the surface water depth, q is the discharge per unit width, r is the rainfall intensity, f_{in} is the infiltration intensity into the soil, which is given by the results of subsurface flow computation, $a = \sqrt{\sin\omega}/n_{\mathrm{M}}$ where n_{M} is Manning's roughness coefficient, and $m = 5/3$. In the quasi-2D model, the equation transposing the second term on the left side of Equation (4) to its right side is solved by the fourth-order Runge-Kutta method, regarding it as an ordinary differential equation of surface water depth h with respect to time t.

Computation flow

Figure 2 shows the computational flow diagram for one time step for each column in the quasi-2D model. Here, the time step for the subsurface flow computation Δt is defined as one time step.

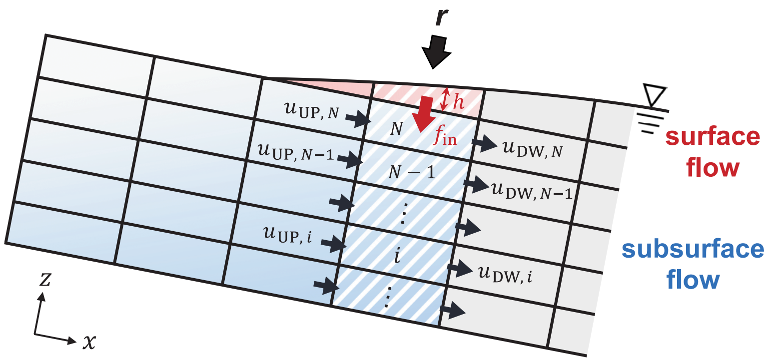

Figure 1. Schematic of the quasi-2D model. The hatched column is under computation for a computational time, and filled arrows indicate its boundary conditions for each face. At this moment, the colored area is already computed up to one time step ahead and grey area is not yet computed

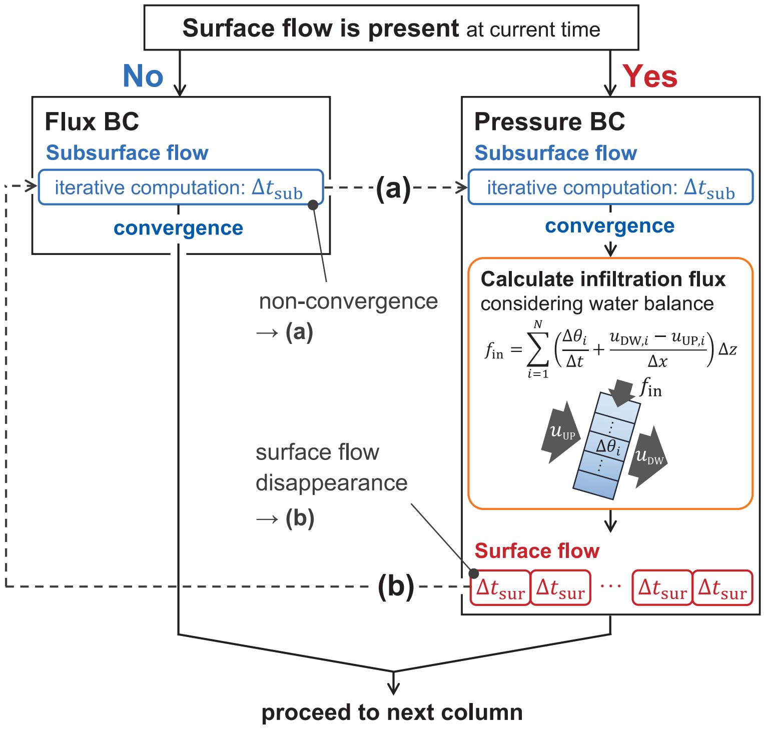

Figure 2. Computational flow diagram for one time step for each column in the quasi-2D model

First, based on the surface water depth in the column at the last time when the computation was completed (hereafter "current time"), the boundary condition (BC) of the ground surface is selected to conduct computations for the subsequent time step (hereafter "next time"). The flux BC given by rainfall intensity r is applied when surface flow is absent, and the pressure BC given by surface water depth h is applied when surface flow is present, as in Saito *et al.* (2006). These rules can be described by the following equations:

$$\begin{cases} -K(\psi)\left(\frac{\partial\psi}{\partial z} + \cos\omega\right)\Big|_{z=D} = r & (h = 0) \\ \psi_N = h & (h > 0) \end{cases} \quad (6)$$

where D is the soil depth and ψ_N is the pressure head of the top soil layer (N is the number of cells in the z-direction).

When the flux BC is selected, no surface flow occurs, and computation is performed only on subsurface flow. After completing the subsurface flow computation, it proceeds to compute one column downstream. Alternatively, when the pressure BC is selected, computations are performed for both subsurface and surface flows in that order. In the quasi-2D model, subsurface flow computation provides the inflow from the upstream column, outflow to the downstream column, and change in soil moisture content between the current time and the next time. The infiltration intensity into the soil is calculated by considering the water balance in the soil layer. Because Δt for surface flow is smaller than that for subsurface flow, as mentioned above, multiple surface flow computations are required per time step, using a constant value of infiltration intensity for all steps of surface flow computation.

Switching the boundary conditions at the ground surface

The boundary conditions at the ground surface can switch in the middle of a one-time step computation at the time of generation or disappearance of the surface flow.

The switch from the flux BC to the pressure BC is indicated by the dashed arrow (a) in the computational flow diagram (Figure 2). Under the flux BC, which attempts to infiltrate all rainwater into the soil, the subsurface flow computation will not converge when surface flow is generated. In this case, the flux BC is switched to the pressure BC and re-computed from the subsurface flow. Thus, the quasi-2D model assumes the saturation excess surface flow for the mechanism of surface flow generation, which occurs when the entire soil layer is saturated.

The switch from the pressure BC to the flux BC is indicated by the dashed arrow (b) in the computational flow diagram (Figure 2). When the surface flow disappears owing to the stopping or weakening of rain, there is no longer enough water on the surface to satisfy the infiltration intensity into the soil obtained from the results of the subsurface flow computation under the pressure BC. In this case, the pressure BC is switched to the flux BC, and the subsurface flow is re-computed. It should be noted that although the surface flow remains at the current time just before switching to the flux BC, the surface flow computation is omitted under the flux BC in accordance with the general framework of the computational procedure. However, in consideration of the water balance, the remaining surface flow water is allowed to flow from the upstream face of the uppermost cell in the downstream column.

MODEL VALIDATION

Simulation conditions

Simulations to validate the quasi-2D model were conducted on a single slope, as shown in Figure 3. The slope length and soil layer thickness were 100 m and 1 m, respectively. Two slope angles, 30° and 5°, were chosen because it was expected that the error introduced by the assumption that the hydraulic gradient in the downward direction was approximated by the slope gradient used in the quasi-2D model would depend on the slope angle.

The water retention and permeability of the soil are expressed by the following Mualem-van Genuchten equations (Mualem, 1976; van Genuchten, 1980).

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \left\{ \frac{1}{1 + (\alpha|\psi|)^{n_v}} \right\}^{1 - \frac{1}{n_v}} \quad (7)$$

$$K = K_s S_e^{\frac{1}{2}} \left\{ 1 - \left(1 - S_e^{\frac{n_v}{n_v - 1}} \right)^{1 - \frac{1}{n_v}} \right\}^2 \quad (8)$$

where S_e is the effective saturation, θ_r and θ_s are the residual and saturated water contents, respectively, α and n_v are the model parameters, and K_s is the saturated hydraulic conductivity. These parameter values, such that both subsurface and surface flows occur, were given by Hopp and McDonnell (2009) as follows: $\theta_r = 0.325$, $\theta_s = 0.45$, $K_s = 1.81 \times 10^{-4}$ m/s, $\alpha = 4$/m, and $n_v = 2$. Manning's roughness coefficient, n_M, was 0.185 s/m$^{1/3}$.

The initial condition was given by a uniform effective saturation of $S_e = 0.5$ over the entire slope, and the runoff was simulated for 24 h. Impermeable boundary conditions were applied at the bottom, sides, and upper end of the slope and a constant hydraulic gradient (equal to the slope gradient) boundary condition was applied at the lower end. A constant rainfall of 20 mm/h perpendicular to the ground surface, that is, aligned with the z-axis, was provided for 6 h from the beginning of the simulations.

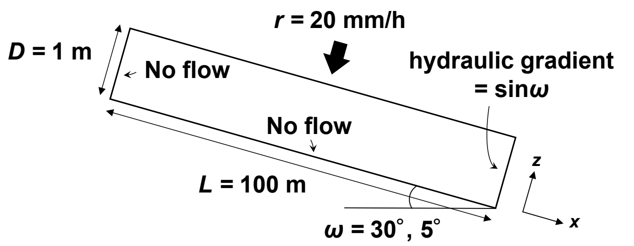

Figure 3. Hillslope settings (L: slope length, D: soil layer thickness, ω: slope gradient, and r: rainfall intensity)

Model settings

The size of the computational cells for both the 2D and quasi-2D models was 0.5 m in the x-direction and 0.05 m in the z-direction. In other words, 200 × 20 grids were used for the entire computation domain.

The computation time step Δt for the 2D model was maximum 30 s and varied depending on the number of iterations required for convergence. Because the subsurface and surface flows were computed simultaneously, the computation time step Δt for both flows were always the same in the 2D model. The computation time step Δt for the quasi-2D model were 30 s and 1 s for subsurface flow and surface flow, respectively, and these values were always constant throughout the simulation.

The value of the convergence tolerance of iterative computation was set to $\delta_\psi = 1.0 \times 10^{-6}$ m for both models.

RESULTS AND DISCUSSION

Hydrograph and soil moisture profile

Figure 4 shows a comparison of hydrographs at the lower end of the slope simulated by the 2D and quasi-2D models, where the left and right panels present the results for 30° and 5° slope angles, respectively. The solid and dashed lines indicate the 2D and quasi-2D models, respectively, and the blue and red lines represent the runoff heights of the subsurface and surface flows, respectively.

Overall, the quasi-2D model reproduced the results of the 2D model well. Evaluation using the Nash-Sutcliffe efficiency index (Nash and Sutcliffe, 1970) showed a very high accuracy of 0.9987 and 0.9995 for 30° and 5° slopes, respectively. The only difference in the surface flow was observed in the early stages of their occurrence on a gentle slope (5°).

Figure 5 shows the soil moisture profile computed by the 2D and quasi-2D models, color-coding by the effective saturation, and wet and dry areas shown in blue and red, respectively. The z-direction is enlarged by a factor of eight.

The quasi-2D model reproduces the rainwater flow on

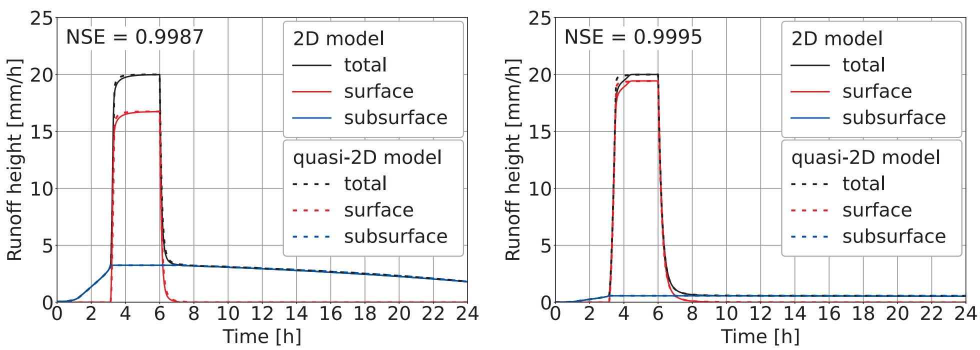

Figure 4. Simulated discharge hydrographs (left: 30° slope, right: 5° slope)

the entire slope simulated by the 2D model. At the slope angle of 30° shown in the upper panel, the results appear almost the same between the two models, but at an angle of 5° shown in the lower panel, there is a difference in the range and shape of the saturated zone in the upstream area of the slope.

Differences in the results between the 2D model and the quasi-2D model

The mechanisms that cause differences in the results between the 2D and quasi-2D models were investigated by dividing the runoff phase into three stages: the period of discharge increases, surface flow occurs, and discharge decreases.

As the discharge increased, from the beginning of the simulation to approximately 3 h later, there was only subsurface flow, and the quasi-2D model reproduced the runoff computed by the 2D model very well, as shown in Figure 4. Therefore, it is important to consider soil moisture distribution perpendicular to the slope. In addition, as shown in the soil moisture profile (Figure 5) at 3 h after the beginning, just before the generation of surface flow, the water table was formed roughly parallel to the slope on both 30° and 5° slopes, especially in the downstream area of the slopes. Therefore, the hydraulic gradient in the downward direction appeared to be almost equal to the slope gradient.

During surface flow occurring, subsurface runoff, shown by blue lines in Figure 4, is in almost perfect agreement between the 2D and quasi-2D models on both 30° and 5° slopes. This is because the entire soil layer was saturated at the lower cross-section of the slope where the hydrographs were created. However, looking at the entire slope, as can be seen from the soil moisture profile in Figure 5, the range of the saturated zone differs between the 2D and quasi-2D models, particularly on the 5° slope. Three hours after the beginning of the simulation, the saturated zone extended towards the upper area of the slope in the quasi-2D model compared to the 2D model. Conversely, the results obtained 4 h after the beginning indicate a reversal of this trend, with a greater expansion of the saturated zone towards the upper area in the 2D model than in the quasi-2D model. In areas where the height of the saturated zone reaches the ground surface, that is, the entire soil layer is saturated, the infiltration capacity of the ground surface is so small that more rainwater becomes surface flow than in unsaturated areas. It is possible that the disagreement in the extent of the saturated zone in the upstream area between the 2D and quasi-2D models may have contributed to the difference in surface runoff heights at the lower end of the slope.

The disagreement in the extent of the saturated zone between the two models can be attributed to the differences in their modeling. The 2D model considers the entire slope and computes the waterflow all at once, whereas the quasi-2D model computes it for each column commencing from upstream and does not consider the effect of the downstream moisture conditions at each location. The relative error was larger for a gentle slope than for a steep slope, and the differences in the simulated results between the 2D and quasi-2D models were larger for a 5° slope than for a 30° slope.

During the decrease in discharge, the slight difference in runoff height (Figure 4) between the two models appears to be due to the difference in the amount of water remaining on the slope, which was mainly caused by the different behaviors of the two models related to the occurrence of surface flow.

Computation time

Figure 6 compares the computation time between the 2D and quasi-2D models required for the 24-hour rainfall-

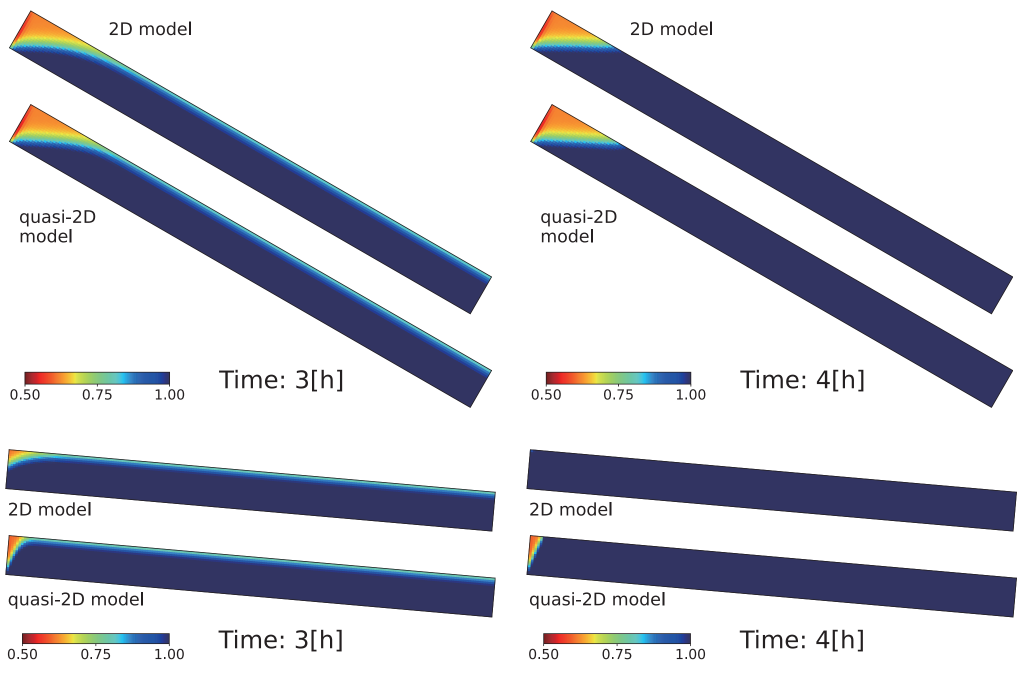

Figure 5. Simulated soil moisture profile 3 and 4 hours after the beginning of the simulations, color-coding by effective saturation (top: 30° slope, bottom: 5° slope)

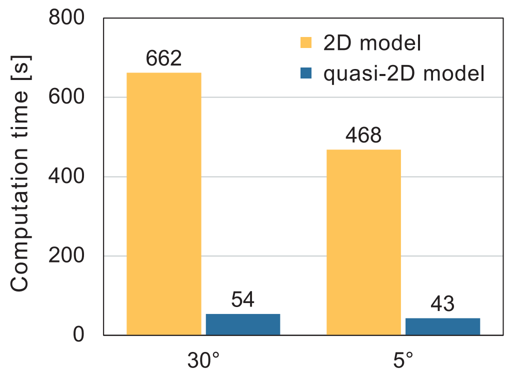

Figure 6. Comparison of computation time required to simulate 24 h of rainfall-runoff between the 2D and quasi-2D models

runoff simulations on the 30° and 5° single slope using the following computer. Note that parallel computing was not employed in either model.

CPU: Intel Core i7-10700
Memory: 16GB DDR4-2933
OS: Ubuntu 20.04 LTS

The computation time for the quasi-2D model was less than 1/10 of that of the 2D model. We can conclude that this quasi-two-dimensional modeling method can reduce the computational costs of saturated-unsaturated flow simulations. However, for practical use in basin-scale simulations, further computation time reduction is necessary, and consideration such as implementing parallel computing and modifying numerical solution methods will be our future issues.

CONCLUSIONS

In this study, we developed a vertical quasi-2D surface-subsurface flow model aiming to simulate rainfall-runoff while properly reflecting both the rainfall-runoff process and soil properties with a smaller computation cost. This model is characterized by using Richards' equation for saturated-unsaturated subsurface flow to account for soil moisture distribution perpendicular to the slope and simplifying the modeling of the runoff process in the downward direction to reduce computational cost. The quasi-2D model was applied to rainfall-runoff simulations on a single slope, and its validity was assessed through comparison with a detailed model based on the two-dimensional Richards' equation (2D model). The quasi-2D model reproduced the results simulated by the 2D model very well (NSE > 0.99), in approximately 1/10 of the computation time of the 2D model. These findings could lead to the development of detailed rainfall-runoff models that can be applied to basin-scale predictions.

In this study, the quasi-2D model was validated by comparison with a more sophisticated model for a single slope. In the future, we plan to apply our quasi-2D model to an actual river basin and validate it by comparing the simulated results with the observational data.

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