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Hom-versions of the Combinatorial Grothendieck Conjecture II: Outer Representations of PIPSC- and NN-type

By

Reiya TACHIHARA

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract

In the present paper, we continue our study, which was initiated in the previous paper of the present series of papers, of combinatorial anabelian geometry of (not necessarily bijective) continuous homomorphisms between PSC-fundamental groups of semi-graphs of anabelioids of PSC-type. In particular, we continue to study certain Hom-versions of the combinatorial versions of the Grothendieck conjecture established in some previous works, i.e., to study certain sufficient conditions of certain group-theoretic compatibility properties described in terms of outer representations.

The outer representations we mainly concern in the present paper are of PIPSC-type and of NN-type, both of which are of substantial importance in the study of algebro-geometric anabelian geometry of configuration spaces of hyperbolic curves. We also include, as a preparation for one of the main results, a presentation of a "reduction technique", namely, a technique of reduction to the "compactified quotients" of (various open subgroups of) the PSC-fundamental groups under consideration, in a similar vein to the previous paper where we included other two "reduction techniques".

Contents

Introduction

Semi-graphs of anabelioids of PSC-type and their PSC-fundamental groups are central objects in the study of combinatorial anabelian geometry. Among the various results obtained in the previous research in this realm, the combinatorial version of the Grothendieck conjecture (or "the combinatorial Grothendieck conjecture" for short), which is an analogue of (a certain version of) the original arithmetic Grothendieck conjecture, is of particular interest and importance. In the present paper, we continue our study, which was initiated in the previous paper [HmCbGCI] of the present series of papers, of combinatorial anabelian geometry of (not necessarily bijective) continuous homomorphisms between PSC-fundamental groups of semi-graphs of anabelioids of PSC-type. In particular, we continue our investigation of Hom-versions of the combinatorial Grothendieck conjecture in a similar vein to [HmCbGCI], §4, pivoting our attention to the outer representations of PIPSC-type and of NN-type (cf. Theorem A, Theorem B, and Theorem C below for our main results). In the remainder of the present Introduction section, the readers are assumed to have already read most of [HmCbGCI], Introduction, with the exception of the paragraph that starts with "Finally, in §4, we apply ...".

Now let us explain the flow of the present paper. We will adopt the notations and conventions introduced in [HmCbGCI], Introduction. Moreover, if $\mathcal{G}$ is a semi-graph of anabelioids of pro-Σ PSC-type and $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ its pro-Σ universal covering, then we shall write $\mathrm{Node}(\widetilde{\mathcal{G}})$ (resp. $\mathrm{Cusp}(\widetilde{\mathcal{G}})$) for the set of closed edges (resp. the set of open edges) of $\widetilde{\mathcal{G}}$. In addition, in the following discussion, the letter $\mathcal{G}$ (resp. $\mathcal{H}$) always denotes a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type (resp. of pro-$\Sigma_{\mathcal{H}}$ PSC-type). In particular, the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is fixed implicitly, the PSC-fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ is considered to be associated to that fixed pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ (i.e., we set $\Pi_{\mathcal{G}} \overset{\mathrm{def}}{=} \mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})^{\mathrm{op}}$), and similar conventions are adopted also for $\mathcal{H}$.

In §1, we review some generalities on *outer representations of PSC-type*. An outer representation of PSC-type is, by definition, a pair $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$, where I is a profinite group and ρ is a group homomorphism which is continuous with respect to the natural profinite topology of $\mathrm{Aut}(\mathcal{G})$ (cf. the discussion preceding [CbGC], Lemma 2.1). The class of l-graphically full outer representations, which we treated in the previous paper, is an intriguing class of outer representations of PSC-type, but the focus of the present paper lies on other kinds of outer representations of PSC-type. The following diagram illustrates the six types of outer representations which we deal with in the present paper, along with the logical implications between them:

$$\begin{array}{ccccc} \text{IPSC-type} & \Longrightarrow & \text{SNN-type} & \Longrightarrow & \text{SVA-type} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{PIPSC-type} & \Longrightarrow & \text{NN-type} & \Longrightarrow & \text{VA-type.} \end{array}$$

"IPSC" (resp. "PIPSC"; "NN"; "SNN"; "VA"; "SVA") stands for "inertial pointed stable curve" (resp. "potentially IPSC"; "nodally nondegenerate"; "strictly NN"; "veriticially admissible"; "strictly VA"). Outer representations of PSC-type which are of (P)IPSC-type and of (S)NN-type are particularly important. Roughly speaking, the former is defined to be outer representations of PSC-type which are (almost) isomorphic, in a certain sense, to an outer representation arising from a degenerating family of hyperbolic curves over a complete discrete valuation ring whose residue field is algebraically closed and of characteristic zero, while the latter, a generalization of the former, is defined entirely in group-theoretic language (i.e., without referring to any specific geometric situation) and encompasses (combinatorial Galois-category-theoretic counterparts of) other natural geometric situations additional to those mentioned in the definition of the former. The importance of them is evident from the fact that some versions of the combinatorial Grothendieck conjecture regarding these types of outer representations of PSC-type play crucial roles in the study of algebro-geometric anabelian geometry of configurations spaces of hyperbolic curves (cf., e.g., the proof of

[NodNon], Theorem 6.1, and the proof of [CbTpII], Theorem 2.3). The aim of the present paper is precisely to establish certain "Hom-versions" of such significant versions of the combinatorial Grothendieck conjecture.

Let $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ be an outer representation of PSC-type. Write $\Pi_\rho \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I$, where the "$\overset{\mathrm{out}}{\rtimes}$" is taken with respect to ρ (cf. the discussion entitled "Topological Groups" in [HmCbGCI], §0). In order to discuss the special types of the outer representations of PSC-type mentioned above, it is necessary to introduce beforehand the following special subgroups of Π_ρ. For $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ (resp. $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \amalg \mathrm{Node}(\widetilde{\mathcal{G}})$), we shall write $D_{\tilde{c}}$ (resp. $I_{\tilde{c}}$) for the normalizer subgroup (resp. the centralizer subgroup) of $\Pi_{\tilde{c}} \subset \Pi_{\mathcal{G}}$ in Π_ρ and refer to it as the *decomposition subgroup* (resp. the *inertia subgroup*) associated to $\tilde{c}$ (cf. the discussion entitled "Topological Groups" in §0 for the terminologies "normalizer subgroup" and "centralizer subgroup"). These subgroups $D_{\tilde{c}}$ and $I_{\tilde{c}}$ are also reviewed in §1 and will play crucial roles throughout the present paper.

Next, in §2, we give a brief exposition on a well-known "reduction technique", namely, a technique of reduction to the "compactified quotients" of (various open subgroups of) the PSC-fundamental groups under consideration. As the arguments are entirely similar to those of [HmCbGCI], §2, the details are largely omitted.

In §3 and §4, we formulate and prove certain "Hom-versions" of the combinatorial Grothendieck conjecture. In other words, we show that, for a continuous homomorphism α between PSC-fundamental groups, the compatibility of α with certain outer representations of PSC-type implies a certain group-theoretic compatibility property of α. Here, we recall that, for $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$, $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ is said to be (S,T)*-compatible* if, for every $\tilde{c} \in S$, either $\alpha(\Pi_{\tilde{c}}) = 1$ or there exists $\tilde{d} \in T$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$; we also recall that the properties of being (S,T)-compatible for some S, T are collectively referred to as the *group-theoretic compatibility properties*.

In §3, we focus on outer representations of PIPSC-type. The setting we consider in the statement of the Hom-versions of the combinatorial Grothendieck conjecture is the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Pi_{\mathcal{G}} \arrow[r] \arrow[d, "\alpha"] & \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I \arrow[r] \arrow[d, "\tilde{\alpha}"] & I \arrow[r] \arrow[d, "\beta"] & 1 \\ 1 \arrow[r] & \Pi_{\mathcal{H}} \arrow[r] & \Pi_{\mathcal{H}} \overset{\mathrm{out}}{\rtimes} J \arrow[r] & J \arrow[r] & 1, \end{tikzcd} \qquad (**)$$

where the two horizontal sequences are the respective exact sequences associated to outer representations of PSC-type $\big(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ and $\big(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ (cf. the discussion entitled "Topological Groups" in [HmCbGCI], §0). Then our first main result in the present paper is as follows.

Theorem A (Theorem 3.3)**.** *Let us consider the commutative diagram* $(**)$*. Suppose that* $\big(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ *is of VA-type, that* $\big(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ *is of PIPSC-type, and that* β *is an open homomorphism. Then* α *is* $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$*-compatible.*

Theorem A, along with its proof, can be considered as a "Hom-version" of [CbTpII], Theorem 1.9, (ii), at least if one disregards the point that, while [CbTpII], Theorem 1.9, (ii), includes the "group-theoretic nodality" (i.e., roughly, the Isom-version counterpart of the $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{H}})\big)$-compatibility) of the isomorphism under consideration, Theorem A says nothing about nodal subgroups.

The discrepancy between [CbTpII], Theorem 1.9, (ii), and Theorem A concerning nodal subgroups just pointed out above is inherent and cannot be remedied easily (cf. [HmCbGCI], Remark 1.7.6 for more detail). However, in light of a substantially different viewpoint presented in [GrphPIPSC], we find that under some additional assumptions we can say something nontrivial also on nodal subgroups. This is our second main result in the present paper, in the proof of which we apply the "reduction technique" reviewed in §2.

Theorem B (Theorem 3.7)**.** *Let us consider the commutative diagram* $(**)$*. Suppose that* $\big(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ *and* $\big(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ *are of PIPSC-type and that* α *is* $\big(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{H}})\big)$*-compatible. Then* α *is* $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$*-compatible (and thus, by assumption,* $\big(\mathrm{Edge}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$*-compatible).*

Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of PSC-type. Then the content of [GrphPIPSC] we quote in the proof of Theorem B is, essentially, a group-theoretic reconstruction algorithm of the nodal subgroups of $\Pi_{\mathcal{G}}$, regarded as subgroups of $\Pi_\rho \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$, from the abstract profinite group structure of Π_ρ under the assumptions that $\mathrm{Cusp}(\widetilde{\mathcal{G}}) = \emptyset$ and that $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is of PIPISC-type. This reconstruction algorithm should be considered as belonging to *absolute anabelian geometry* of outer representations of PSC-type (cf. the fact that the "input data" of the algorithm is a single abstract profinite group Π_ρ), in contrast to our situation that we are pursuing (certain Hom-versions of) *semi-absolute anabelian geometry* of outer representations of PSC-type (cf. the fact that our "input data" should be considered as including not only the profinite group Π_ρ but also the natural surjection $\Pi_\rho \twoheadrightarrow I$). We could have obtained a straightforward generalization of this "absolute" result of [GrphPIPSC] (cf. Remark 3.7.1), but we have chosen to pay the price of abandoning absoluteness and consider a semi-absolute situation in order to obtain the result that remains valid even for the cases where it does not necessarily hold that $\mathrm{Cusp}(\widetilde{\mathcal{G}}) = \mathrm{Cusp}(\widetilde{\mathcal{H}}) = \emptyset$ (cf. Remark 3.7.2).

Finally, in §4, we focus on outer representations of NN-type. The main result of §4 (cf. Theorem C below) states that, roughly, if a continuous homomorphism between PSC-fundamental groups is compatible with an outer representation of VA-type on the domain and an outer representation of NN-type on the codomain, and, moreover, it satisfies a certain group-theoretic compatibility property, then this group-theoretic compatibility "extends" to the neighbour vertices. Such "extension" phenomenon is typically observed in the context of existing Isom-versions of the combinatorial Grothendieck conjecture for outer representations of PSC-type which are of NN-type, and thus we may consider Theorem C as a nicely unified description of that phenomenon. On the other hand, since the proofs of those existing Isom-versions do not function in our Hom-version situation, we had to develop a substantially different technique (cf. Remark 4.7.2).

To state the result, we need another notation. If $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (resp. $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$), then we shall write $F_{\tilde{c}}$ for the subset of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ consisting of $\tilde{v}$ such that $\tilde{v}$ is equal to $\tilde{c}$ or there exists an edge $\tilde{e}$ which abuts both to $\tilde{c}$ and $\tilde{v}$ (resp. such that $\tilde{c}$ abuts to $\tilde{v}$). Moreover, if $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, then we shall write $F_S \stackrel{\mathrm{def}}{=} \bigcup_{\tilde{c} \in S} F_{\tilde{c}}$. Now our result is formulated as follows.

Theorem C (Theorem 4.7)**.** *Let us consider the commutative diagram* $(**)$*. Let* $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ *and* $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$*. Suppose that* $(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *is of VA-type, that* $(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H}))$ *is of NN-type, that* β *is non-trivial, that* α *is* (S,T)*-compatible, and that, for every* $\tilde{c} \in S$*, it holds that* $\alpha(\Pi_{\tilde{c}}) \neq 1$*. Then* α *is* (F_S, F_T)*-compatible.*

Theorem C may be considered as a unified generalized Hom-version of [NodNon], Theorem 4.1, [NodNon], Corollary 4.2, and [CbTpII], Theorem 1.9, (i), (2) $\implies$ (1) (cf. Remark 4.8.2 for the detail).

Finally, we note that we still do not have a Hom-version of [CbTpII], Theorem 1.9, (i), (3) $\implies$ (1), which is deeper than the implication (2) $\implies$ (1). We also note that, in contrast to §3 (cf. Theorem B), we regrettably cannot establish group-theoretic compatibility properties for edge-like subgroups in §4.

0 Notations and Conventions

We shall continue to use the "Notations and Conventions" of [HmCbGCI], §0. In addition, we shall use the following notations and conventions:

Numbers

$\mathbb{C}$ denotes the field of complex numbers.

Topological Groups

Let K be a topological group and $G, H \subset K$ closed subgroups of K. Then we shall write $Z_G(H)$ (resp. $N_G(H)$; $C_G(H)$) for the *centralizer subgroup* (resp. *normalizer subgroup*; *commensurator subgroup*) of H in G, i.e.,

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g^{-1}hg = h \text{ for any } h \in H.\},$$

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g^{-1}Hg = H.\},$$

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap g^{-1}Hg \text{ is of finte index in } H \text{ and } g^{-1}Hg.\}.$$

It is immediate from the definitions that $Z_G(H) \subset N_G(H) \subset C_G(H) \subset G$; $G \cap H \subset N_G(H)$. Moreover, we shall write

$$R_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap g^{-1}Hg \neq 1.\}.$$

Note that the subset $R_G(H)$ of G is not necessarily a subgroup of G. If $H \neq 1$ (resp. H is infinite), then it is immediate from the definitions that $N_G(H) \subset R_G(H)$ (resp. $C_G(H) \subset R_G(H)$). Differing from the concepts of normalizer, centralizer, and commensurator, to the best of the author's knowledge, this construction "$R_G(H)$" does not have commonly shared symbols or names. The author decided to use the letter "R" for this construction in the present paper simply because "R" had not been utilized for any other purposes.

1 Review of Outer Representations of PSC-type

In this section, we review some basic definitions and results concerning outer representations of PSC-type, which is by definition a continuous homomorphism from a profinite group to the (profinite) automorphism group of a semi-graph of anabelioids of PSC-type. This section does not contain any new result.

A basic reference for the theory of semi-graphs of anabelioids of PSC-type is [CbGC]. We shall use the terminologies "*semi-graph of anabelioids of PSC-type*", "*PSC-fundamental group of a semi-graph of anabelioids of PSC-type*", "*finite étale covering of semi-graphs of anabelioids of PSC-type*", "*vertex*", "*edge*", "*node*", "*cusp*", "*verticial subgroup*", "*edge-like subgroup*", "*nodal subgroup*", and "*cuspidal subgroup*", as they are defined in [CbGC], Definition 1.1. Moreover, if $\mathcal{G}$ is a semi-graph of anabelioids of PSC-type, then we shall write $\Sigma_{\mathcal{G}}$ for the (necessarily unique — cf. [CbGC], Remark 1.1.2) set of prime numbers such that $\mathcal{G}$ is a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type. Also, we shall apply the various notational conventions established in [NodNon], Definition 1.1; in particular, if $\mathcal{G}$ is a semi-graph of anabelioids of PSC-type, then the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is fixed throughout the discussion, and the PSC-fundamental group $\Pi_{\mathcal{G}}$ is always considered to be associated to that fixed pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Thus there is a natural action $\widetilde{\mathcal{G}} \curvearrowleft \Pi_{\mathcal{G}}$, which induces a natural bijection

$$\begin{array}{ccc}
\Big(\text{the set of open subgroups of } \Pi_{\mathcal{G}}\Big) & \stackrel{\cong}{\longrightarrow} & \left(\begin{array}{c}\text{the set of the isomorphism classes}\\ \text{of the connected finite étale}\\ \text{subcoverings of } \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}\end{array}\right); \\
\big(U \subset \Pi_{\mathcal{G}}\big) & \longmapsto & \big(\widetilde{\mathcal{G}}/U \longrightarrow \mathcal{G}\big).
\end{array}$$

If $\mathcal{G}' \longrightarrow \mathcal{G}$ is a connected finite étale subcovering of the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, then we always choose the implicit structure morphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ as the pro-$\Sigma_{\mathcal{G}'}$ universal covering of $\mathcal{G}'$. (Here, we note that $\Sigma_{\mathcal{G}} =$

$\Sigma_{\mathcal{G}'}$.) Under this convention, the inverse of the natural bijection above is given by $(\mathcal{G}' \longrightarrow \mathcal{G}) \longmapsto (\Pi_{\mathcal{G}'} \subset \Pi_{\mathcal{G}})$. Finally, we shall refer to the "PSC-fundamental group of a semi-graph of anabelioids of PSC-type" simply as the "fundamental group" (of the semi-graph of anabelioids of PSC-type).

In the following, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type. Note that we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

Before proceeding, let us also recall the issue concerning the profinite topology on $\mathrm{Aut}(\mathcal{G})$. Since the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ is topologically finitely generated (cf. [CbGC], Remark 1.1.3), the profinite topology of $\Pi_{\mathcal{G}}$ induces (profinite) topologies on $\mathrm{Aut}(\Pi_{\mathcal{G}})$ and $\mathrm{Out}(\Pi_{\mathcal{G}})$ (cf. the discussion entitled "Topological Groups" in [HmCbGCI], §0). Moreover, if we write $\mathrm{Aut}(\mathcal{G})$ for the group of automorphisms of $\mathcal{G}$, then by the discussion preceding [CbGC], Lemma 2.1, the natural homomorphism $\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$ is an injection with closed image. (Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph — cf. [SemiAn], Definition 2.1.) Thus, by equipping $\mathrm{Aut}(\mathcal{G})$ with the topology induced via this homomorphism by the topology of $\mathrm{Out}(\Pi_{\mathcal{G}})$, we may regard $\mathrm{Aut}(\mathcal{G})$ as a profinite group.

Now we begin the review.

Definition 1.1 ([NodNon], Definition 2.1)**.** Let I be a profinite group and $\rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})$ a continuous homomorphism. Then we shall refer to the pair $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ as an *outer representation of PSC-type*. If moreover $\mathcal{G}$ is a semi-graph of anabelioids of pro-Σ PSC-type, i.e., $\Sigma_{\mathcal{G}} = \Sigma$, then we shall say that the pair $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is an *outer representation of pro-Σ PSC-type*. We have the evident notion of an isomorphism of outer representations of PSC-type.

Example 1.1.1. l-graphically full outer representations, which we focused on [HmCbGCI], §4, are outer representations of PSC-type by definition.

Example 1.1.2. Let Σ be a non-empty set of prime numbers, k a separably closed field whose characteristic is not contained in Σ, $S = \mathrm{Spec}\, k$, and $S^{\log}$ a log scheme obtained by equipping S with the log structure determined by the chart $\mathbb{N} \longrightarrow k$; $1 \longmapsto 0$. Then, as is discussed in [AbsTopII], Example 1.1, any stable log curve $X^{\log}$ over $S^{\log}$ (cf. the discussion entitled "Curves" in [CbGC], §0) naturally determines an outer representation of pro-Σ PSC-type

$$\left(\mathcal{G}^{\Sigma}_{X^{\log}}, \rho\colon I^{\Sigma}_{S^{\log}} \longrightarrow \mathrm{Aut}\big(\mathcal{G}^{\Sigma}_{X^{\log}}\big)\right).$$

Here, we write $\mathcal{G}^{\Sigma}_{X^{\log}}$ for the semi-graph of anabelioids of pro-Σ PSC-type associated to $X^{\log}$ and $I^{\Sigma}_{S^{\log}}$ for the maximal pro-Σ quotient of the log fundamental group $I_{S^{\log}}$ of $S^{\log}$. It is well-known that $I_{S^{\log}}$ is naturally isomorphic to $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, k^{\times})$ as a profinite group, hence also is (non-canonically) isomorphic to $\hat{\mathbb{Z}}^{\Sigma'}$ as a profinite group, where we write Σ' for the complement of the characteristic of k in the set of prime numbers. In particular, $I^{\Sigma}_{S^{\log}}$ is (non-canonically) isomorphic to $\hat{\mathbb{Z}}^{\Sigma}$ as a profinite group.

In the following, we are mainly concerned with the exact sequence of profinite groups and continuous homomorphisms

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\rho} \longrightarrow I \longrightarrow 1, \qquad (*)$$

where we write $\Pi_{\rho} \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$, associated to an outer representation of PSC-type $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$. In other words, we are interested in the exact sequence obtained by pulling back the exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \mathrm{Aut}(\Pi_{\mathcal{G}}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}) \longrightarrow 1$$

via the continuous homomorphism $I \stackrel{\rho}{\longrightarrow} \mathrm{Aut}(\mathcal{G}) \stackrel{\mathrm{natural}}{\hookrightarrow} \mathrm{Out}(\Pi_{\mathcal{G}})$ (cf. the discussion entitled "Topological Groups" in [HmCbGCI], §0). Here, we note that $\Pi_{\mathcal{G}}$ is topologically finitely generated and center-free (cf. [CbGC], Remark 1.1.3).

Definition 1.2. We shall write $\mathrm{VN}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\mathcal{G}) \amalg \mathrm{Node}(\mathcal{G})$ and $\mathrm{VN}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\widetilde{\mathcal{G}}) \amalg \mathrm{Node}(\widetilde{\mathcal{G}})$. We shall refer to a verticial or nodal subgroup of $\Pi_{\mathcal{G}}$ as a *VN-subgroup* of $\Pi_{\mathcal{G}}$.

Definition 1.3 ([NodNon], Definition 2.2)**.** Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of PSC-type. Let us consider the exact sequence $(*)$ associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$.

(1) For $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, we shall refer to $D_{\tilde{c}} \stackrel{\mathrm{def}}{=} N_{\Pi_I}(\Pi_{\tilde{c}})$ as the *decomposition subgroup* of Π_I associated to $\tilde{c}$.

(2) For $\tilde{c} \in \mathrm{VN}(\widetilde{\mathcal{G}})$, we shall refer to $I_{\tilde{c}} \stackrel{\mathrm{def}}{=} Z_{\Pi_I}(\Pi_{\tilde{c}})$ as the *inertia subgroup* of Π_I associated to $\tilde{c}$.

Remark 1.3.1. The following assertions follow immediately from the definitions and might be applied without a mention:

(1) For every $\tilde{c} \in \mathrm{VN}(\widetilde{\mathcal{G}})$, it holds that $I_{\tilde{c}} \subset D_{\tilde{c}}$.

(2) For every $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ and $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ to which $\tilde{e}$ abuts, it holds that $I_{\tilde{v}} \subset I_{\tilde{e}}$.

(3) For every $\tilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ to which $\tilde{e}$ abuts, it holds that $I_{\tilde{v}} \subset D_{\tilde{e}}$.

Lemma 1.4. *Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of PSC-type. Let us consider the exact sequence $(*)$ associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$. Then the following assertions hold.*

(1) For every $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, it holds that $D_{\tilde{c}} \cap \Pi_{\mathcal{G}} = \Pi_{\tilde{c}}$. In particular, the exact sequence $(*)$ induces the following exact sequence:

$$1 \longrightarrow \Pi_{\tilde{c}} \longrightarrow D_{\tilde{c}} \longrightarrow \mathrm{Im}(D_{\tilde{c}} \to I) \longrightarrow 1.$$

(2) For every $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, it holds that $I_{\tilde{v}} \cap \Pi_{\mathcal{G}} = 1$. In particular, the composite of the natural homomorphisms $I_{\tilde{v}} \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is injective.

(3) For every $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, it holds that $I_{\tilde{e}} \cap \Pi_{\mathcal{G}} = \Pi_{\tilde{e}}$. In particular, the exact sequence $(*)$ induces the following exact sequence:

$$1 \longrightarrow \Pi_{\tilde{e}} \longrightarrow I_{\tilde{e}} \longrightarrow \mathrm{Im}(I_{\tilde{e}} \to I) \longrightarrow 1.$$

(4) For every $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the natural inclusions $\Pi_{\tilde{v}}, I_{\tilde{v}} \hookrightarrow D_{\tilde{v}}$ determine an injective homomorphism $\Pi_{\tilde{v}} \times I_{\tilde{v}} \hookrightarrow D_{\tilde{v}}$.

(5) For every $\tilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ to which $\tilde{e}$ abuts, the natural inclusions $\Pi_{\tilde{e}}, I_{\tilde{v}} \hookrightarrow D_{\tilde{e}}$ determine an injective homomorphism $\Pi_{\tilde{e}} \times I_{\tilde{v}} \hookrightarrow D_{\tilde{e}}$. Moreover, if $\tilde{e} \in \mathrm{Node}(\mathcal{G})$, then this homomorphism factors through $I_{\tilde{e}} \subset D_{\tilde{e}}$.

Proof. Most part of the proof of Lemma 1.4 is essentially included in the proofs of [NodNon], Lemma 2.3, (i), [NodNon], Lemma 2.5, (iv), [NodNon], Lemma 2.7, and [GrphPIPSC], Lemma 1.5, (iv). However, we give the proof of them here for the sake of the reader.

The assertion (1) follows from the commensurable terminality of $\Pi_{\tilde{c}}$ in $\Pi_{\mathcal{G}}$ (cf. [CbGC], Proposition 1.2, (ii)). The assertion (2) follows from the assertion (1) and the center-freeness of $\Pi_{\tilde{v}}$ (cf. [CbGC], Remark 1.1.3). The assertion (3) follows from the assertion (1) and the abelianness of $\Pi_{\tilde{e}}$ (cf. [CbGC], Remark 1.1.3). The assertions (4) and (5) follow immediately from the assertions (2), (3) and the definitions. This completes the proof of Lemma 1.4. $\square$

Definition 1.5 ([NodNon], Definition 2.4, (ii))**.** An outer representation of PSC-type $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is said to be of *VA-type* (resp. of *SVA-type*) if, in the notation of Definition 1.3, the following two conditions hold:

- I is isomorphic to $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$ as a profinite group.
- For every $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the composite of the natural homomorphisms $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ (which is necessarily injective — cf. Lemma 1.4, (2)) is an open (resp. a bijective) homomorphism.

We shall refer to an outer representation of PSC-type which is of VA-type (resp. of SVA-type) simply as an outer representation of VA-type (resp. of SVA-type).

Lemma 1.6 (essentially included in [NodNon], Lemma 2.5 and [NodNon], Lemma 2.7)**.** *Let $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ be an outer representation of VA-type (resp. of SVA-type). Let us consider the exact sequence $(*)$ associated to $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$. Then the following assertions hold.*

(1) The subgroups "$\mathrm{Im}(D_{\tilde{c}} \to I)$" and "$\mathrm{Im}(I_{\tilde{e}} \to I)$" of I which appear in Lemma 1.4, (1) and (3), are both open in I (resp. equal to I).

(2) The natural injective homomorphisms given in Lemma 1.4, (4) and (5), are both open homomorphisms (resp. isomorphisms).

(3) For any $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, $I_{\tilde{e}}$ is isomorphic to $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}} \times \hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$ as a profinite group, hence abelian.

Proof. The assertions (1) and (2) follow immediately from the definitions of "VA-type" and "SVA-type", in light of Remark 1.3.1, (1), (2), (3).

Let us verify the assertion (3). By the assertion (1), it holds that $\mathrm{Im}(I_{\tilde{e}} \to I) \cong \hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$. In particular, there exists a continuous section homomorphism $s\colon \mathrm{Im}(I_{\tilde{e}} \to I) \hookrightarrow I_{\tilde{e}}$ of the exact sequence in Lemma 1.4, (3). Since $\mathrm{Im}(s) \subset I_{\tilde{e}} = Z_{I_{\tilde{e}}}(\Pi_{\tilde{e}})$ by definition, the inclusion $\Pi_{\tilde{e}} \hookrightarrow I_{\tilde{e}}$ and the section s determine a continuous group isomorphism $\Pi_{\tilde{e}} \times \mathrm{Im}(I_{\tilde{e}} \to I) \xrightarrow{\cong} I_{\tilde{e}}$. Since both $\Pi_{\tilde{e}}$ and $\mathrm{Im}(I_{\tilde{e}} \to I)$ are isomorphic to $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$ (cf. [CbGC], Remark 1.1.3, and the assertion (1)), the assertion (3) follows. This completes the proof of Lemma 1.6. □

Lemma 1.7 ([NodNon], Remark 2.7.1)**.** *Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of SVA-type. Let us consider the exact sequence* $(*)$ *associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$. Then the following assertions hold.*

(1) For any $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, it holds that $I_{\tilde{e}} = D_{\tilde{e}}$.

(2) For any $\tilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ to which $\tilde{e}$ abuts, it holds that $D_{\tilde{e}} \subset D_{\tilde{v}}$.

Proof. The assertion (1) follows immediately from the last assertion of Lemma 1.4, (5), and the resp'd case of Lemma 1.6, (2). The assertion (2) follows immediately from the resp'd case of Lemma 1.6, (2), together with the fact that $\Pi_{\tilde{e}} \subset \Pi_{\tilde{v}}$. □

In §3 and §4, we are mainly interested in outer representations of PSC-type that satisfy the conditions defined below.

Definition 1.8 ([NodNon], Definition 2.4, (i), (iii), and [CbTpIII], Definition 1.3)**.** Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of PSC-type.

(1) $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is said to be of *IPSC-type* if it is, as an outer representation of PSC-type, isomorphic to the outer representation of PSC-type $\big(\mathcal{G}^{\Sigma}_{X^{\log}}, \rho\colon I^{\Sigma}_{S^{\log}} \longrightarrow \mathrm{Aut}\big(\mathcal{G}^{\Sigma}_{X^{\log}}\big)\big)$ associated to some pair "$(\Sigma, X^{\log} \longrightarrow S^{\log})$" as in Example 1.1.2. $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is said to be of *PIPSC-type* if the following two conditions hold:

- I is isomorphic to $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$ as a profinite group.
- There exists an open subgroup I' of I such that the restricted outer representation of PSC-type $\big(\mathcal{G}, \rho|_{I'}\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is of IPSC-type.

(2) $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is said to be of *NN-type* (resp. of *SNN-type*) if it is of VA-type (resp. of SVA-type), and, moreover, in the notation of Definition 1.3, the following condition holds:

> For every $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, if one writes $\tilde{v}_1$, $\tilde{v}_2$ for the two distinct vertices to which $\tilde{e}$ abuts, then the homomorphism $I_{\tilde{v}_1} \times I_{\tilde{v}_2} \longrightarrow I_{\tilde{e}}$ induced by the inclusions $I_{\tilde{v}_1}, I_{\tilde{v}_2} \subset I_{\tilde{e}}$ (which is well-defined since $I_{\tilde{e}}$ is abelian — cf. Lemma 1.6, (3)) is an injective open homomorphism.

We shall refer to an outer representation of PSC-type which is of PIPSC-type (resp. of IPSC-type; of NN-type; of SNN-type) simply as an outer representation of PIPSC-type (resp. of IPSC-type; of NN-type; of SNN-type).

Remark 1.8.1. The reader should be careful not to confuse the distinct terms "outer representation of PSC-type" and "outer representation of IPSC-type" due to their similar appearance. By definition, the latter constitutes a very specific instance of the former.

In passing, we recall the following.

Lemma 1.9 (essentially included in [NodNon], Lemma 2.6, (i), and [GrphPIPSC], Lemma 1.8, (iii)). *Let $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ be an outer representation of PSC-type. Let us consider the exact sequence $(*)$ associated to $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$. If $\Pi_{\rho'}$ is an open subgroup of Π_ρ, then write I' for the image of $\Pi_{\rho'}$ via the surjection $\Pi_\rho \twoheadrightarrow I$ and $\mathcal{G}' \longrightarrow \mathcal{G}$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to the open subgroup $\Pi_{\rho'} \cap \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$, i.e., $\Pi_{\mathcal{G}'} = \Pi_{\rho'} \cap \Pi_{\mathcal{G}}$. Then the following assertions hold.*

(1) For any open subgroup $\Pi_{\rho'} \subset \Pi_\rho$, the exact sequence $(*)$ naturally induces the following exact sequence:

$$1 \longrightarrow \Pi_{\mathcal{G}'} \longrightarrow \Pi_{\rho'} \longrightarrow I' \longrightarrow 1.$$

Moreover, this exact sequence arises from a unique outer representation of PSC-type $(\mathcal{G}', \rho'\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G}'))$.

(2) Let $\Pi_{\rho'} \subset \Pi_\rho$ be an open subgroup and $\square \in \{VA, NN, PIPSC\}$. Then, if $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is of $\square$-type, then $(\mathcal{G}', \rho'\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G}'))$ in (1) is also of $\square$-type.

(3) Suppose that $\bigl(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\bigr)$ is of VA-type (resp. of NN-type; of PIPSC-type). Then there exists an open subgroup $\Pi_{\rho'} \subset \Pi_\rho$ such that $\bigl(\mathcal{G}', \rho'\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G}')\bigr)$ in (1) is of SVA-type (resp. of SNN-type; of IPSC-type). Moreover, one can take such a $\Pi_{\rho'}$ as the inverse image of some open subgroup I' of I, or, equivalently, as an open subgroup of Π_ρ which includes $\Pi_{\mathcal{G}}$.

Proof. The assertion (1) is identical to [HmCbGCI], Lemma 4.2, (1), and the proof is given there.

The assertion (2) follows from [GrphPIPSC], Lemma 1.8, (iii).

Finally, let us verify the assertion (3). The assertion (3) in the case where $\bigl(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\bigr)$ is of PIPSC-type is immediate. The assertion (3) in the case where $\bigl(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\bigr)$ is of NN-type may be derived from the assertion (3) in the case where $\bigl(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\bigr)$ is of VA-type. Thus, to verify assertion (3), it suffices to verify the assertion (3) in the case where $\bigl(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\bigr)$ is of VA-type. To this end, observe that $\bigcap_{\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})} p(I_{\tilde{v}})$ is open in I, where we write p for the projection $\Pi_\rho \twoheadrightarrow I$, since this is essentially a finite intersection (of open subgroups of I — cf. Definition 1.5). The desired "$\Pi_{\rho'}$" is now obtained as the inverse image of this open subgroup of I. This completes the proof of the assertion (3), hence also the proof of Lemma 1.9. $\square$

2 Reduction to the Compactified Situations

In this section, we give a brief exposition on the argument of "reduction to the compactified situations", which is well-known to experts and will be of use in the proof of Theorem 3.7. The contents of §1 play no role in this section except that some remark concerning outer representation of PSC-type is given in Remark 2.2.2. As a consequence, we prove that, in order to show a certain group-theoretic compatibility property of a continuous homomorphism between PSC-fundamental groups, it suffices to verify certain group-theoretic compatibility properties of continuous homomorphisms between the PSC-fundamental groups of the "compactifications" of the various intermediate coverings (i.e., between appropriate quotients of various open subgroups). As the arguments are entirely similar to those of [HmCbGCI], §2, the details are largely omitted.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid in this section; in particular, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type; we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

To begin with, we recall the notion of sturdiness.

Definition 2.1 ([CbGC], Definition 1.1, (ii), and [CbGC], Remark 1.1.5)**.** $\mathcal{G}$ is said to be *sturdy* if every irreducible component of "the pointed stable curve that gives rise to $\mathcal{G}$" is of genus ≥ 2.

In general, one can "compactify" a semi-graph of anabelioids just by "removing the cusps of the underlying semi-graph" and "restricting the attention to the coverings unramified over the cusps". If the semi-graphs of anabelioids under consideration is of PSC-type and sturdy, then the compactification is again a semi-graph of anabelioids of PSC-type. We may concentrate on this case.

Definition 2.2. Suppose that $\mathcal{G}$ is sturdy.

(1) We shall write $\mathcal{G}^{\mathrm{cpt}}$ for the *compactification* of $\mathcal{G}$ (cf. [CbGC], Remark 1.1.6; also Remark 2.2.1 below), which is again a semi-graph of anabeliaids of pro-$\Sigma_{\mathcal{G}}$ PSC-type.

(2) We shall always consider the PSC-fundamental group $\Pi_{\mathcal{G}^{\mathrm{cpt}}}$ of $\mathcal{G}^{\mathrm{cpt}}$ as the quotient group of $\Pi_{\mathcal{G}}$ in the natural way (cf. [CbGC], Remark 1.1.6; also Remark 2.2.1 below).

Remark 2.2.1. Suppose that we are in the situation of Definition 2.2. Let us make the definition more explicit here.

The underlying semi-graph of $\mathcal{G}^{\mathrm{cpt}}$ is the graph obtained by removing all the cusps (i.e., the open edges) from the underlying semi-graph of $\mathcal{G}$. The constituent anabelioid $\mathcal{G}^{\mathrm{cpt}}_e$ of $\mathcal{G}^{\mathrm{cpt}}$ at each node $e \in \mathrm{Node}(\mathcal{G}^{\mathrm{cpt}}) = \mathrm{Node}(\mathcal{G})$ is equal to $\mathcal{G}_e$. The constituent anabelioid $\mathcal{G}^{\mathrm{cpt}}_v$ of $\mathcal{G}^{\mathrm{cpt}}$ at each vertex $v \in \mathrm{Vert}(\mathcal{G}^{\mathrm{cpt}}) = \mathrm{Vert}(\mathcal{G})$ is (the anabelioid determined by) the Galois category obtained as the full subcategory constituted by "the objects which is unramified over every $e \in \mathrm{Cusp}(\mathcal{G})$ abutting to v" of (the Galois category corresponding to) the constituent anabelioid $\mathcal{G}_v$ of $\mathcal{G}$. The constituent morphism of anabelioids at each branch b of $e \in \mathrm{Node}(\mathcal{G}^{\mathrm{cpt}})$ is (the morphism of anabelioids determined by) the composite exact functor $\mathcal{G}^{\mathrm{cpt}}_v \xrightarrow{\text{incl.}} \mathcal{G}_v \longrightarrow \mathcal{G}_e = \mathcal{G}^{\mathrm{cpt}}_e$, where the exact functor $\mathcal{G}_v \longrightarrow \mathcal{G}_e$ is (the exact functor corresponding to) the constituent morphism of anabelioids at b, a branch of $e \in \mathrm{Node}(\mathcal{G})$.

In particular, the Galois category $\mathcal{B}(\mathcal{G}^{\mathrm{cpt}})$ (cf. the discussion following [SemiAn], Definition 2.1, for the notation "$\mathcal{B}$") is naturally considered as a full subcategory of $\mathcal{B}(\mathcal{G})$. It is easily verified that the subsystem of (the projective system which gives rise to) the pro-Σ universal covering of $\mathcal{B}(\mathcal{G})$ constituted by the objects belonging to $\mathcal{B}(\mathcal{G}^{\mathrm{cpt}})$ (i.e., "the objects unramified over all the cusps of the base") determines the pro-Σ universal covering of $\mathcal{B}(\mathcal{G}^{\mathrm{cpt}})$. This determines the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}}$ by which we consider $\Pi_{\mathcal{G}^{\mathrm{cpt}}}$ as a quotient group of $\Pi_{\mathcal{G}}$, where the surjectivity of this natural homomorphism follows from the (easily verified) fact that an object of $\mathcal{B}(\mathcal{G}^{\mathrm{cpt}})$ is connected in $\mathcal{B}(\mathcal{G}^{\mathrm{cpt}})$ if and only if it is connected in $\mathcal{B}(\mathcal{G})$.

Finally, it follows from the tautological fact that a connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ is unramified over $e \in \mathrm{Cusp}(\mathcal{G})$ if and only if, for every $\tilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ lying over e, $\Pi_{\tilde{e}} \subset \Pi_{\mathcal{G}}$ is included in $\Pi_{\mathcal{G}'} \subset \Pi_{\mathcal{G}}$, that the kernel of the natural surjection $\Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}}$ is the closed normal subgroup of $\Pi_{\mathcal{G}}$ generated by all the cuspidal subgroups of $\Pi_{\mathcal{G}}$.

Remark 2.2.2. Let us consider the exact sequence $(*)$ associated to an outer representation of PSC-type $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ (cf. the discussion preceding Definition 1.2). Then it follows from the definitions (cf. also the final portion of Remark 2.2.1) that the kernel of the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}}$ is (regarded via the natural injection $\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\rho}$ as) a normal subgroup of Π_{ρ}, and thus the exact sequence $(*)$ naturally determines an exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}} \longrightarrow \Pi_{\rho}/\mathrm{Ker}(\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}}) \longrightarrow I \longrightarrow 1.$$

Write ϕ for the natural homomorphism $\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\mathcal{G}^{\mathrm{cpt}})$. Then it is easily verified that this new exact sequence is naturally identified with the exact sequence $(*)$ in the case where we take the "$(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$" of the discussion preceding Definition 1.2 to be $(\mathcal{G}^{\mathrm{cpt}}, \phi \circ \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{\mathrm{cpt}}))$. Also, it is easily verified that, if $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{\mathrm{cpt}}))$ is of PIPSC-type, then $(\mathcal{G}^{\mathrm{cpt}}, \phi \circ \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{\mathrm{cpt}}))$ is also of PIPSC-type.

Lemma 2.3. *The following assertions hold.*

(1) *There exists a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is sturdy.*

(2) *If $\mathcal{G}$ is sturdy, then, for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, $\mathcal{G}'$ is sturdy.*

Proof. Let us verify the assertion (1). First, let us show that, to verify the assertion (1), it suffices to show that, for every $v \in \mathrm{Vert}(\mathcal{G})$, there exists a connected finite étale Galois subcovering $\mathcal{G}'(v) \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that every $w \in \mathrm{Vert}(\mathcal{G}'(v))$ lying over v is "of genus ≥ 2", or, equivalently, such that there exists $w \in \mathrm{Vert}(\mathcal{G}'(v))$ lying over v that is "of genus ≥ 2" (cf. the condition that $\mathcal{G}' \longrightarrow \mathcal{G}$ is Galois). Indeed, if we have such $\mathcal{G}'(v) \longrightarrow \mathcal{G}$ for every $v \in \mathrm{Vert}(\mathcal{G})$, then a connected finite étale subcovering of $\widetilde{\mathcal{G}}$ dominating all of the "$\mathcal{G}'(v) \longrightarrow \mathcal{G}$" is manifestly sturdy (cf. the proof of the assertion (2)). On the other hand, the existence of such a $\mathcal{G}'(v) \longrightarrow \mathcal{G}$ follows immediately from the desired existence in the case where $\mathrm{Vert}(\mathcal{G})$ is of cardinality 1, which is easily verified, applied to (the semi-graph of anabelioids determined by) the constituent anabelioid $\mathcal{G}_v$, together with (the Galois-category-theoretic interpretation of) the well-known fact that the natural outer homomorphism $\pi_1(\mathcal{G}_v) = \Pi_v \longrightarrow \Pi_{\mathcal{G}}$ is injective. This completes the proof of the assertion (1).

In order to verify the assertion (2), it is immediate that it suffices to show the following statement:

> Claim 2.3.A: Let X, Y be smooth curves over $\mathbb{C}$ and $f\colon X \longrightarrow Y$ a finite morphism over $\mathbb{C}$. Suppose that Y is of genus ≥ 2. Then X is also of genus ≥ 2.

On the other hand, this follows immediately from a certain functoriality of the compactification of smooth curves over $\mathbb{C}$, together with the well-known "Riemann-Hurwitz formula" for finite separable morphisms between smooth proper curves. This completes the proof of the assertion (2), hence also the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let $\mathcal{G}' \longrightarrow \mathcal{G}$ be a connected finite étale Galois subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is sturdy. Write X for the set of $\Pi_{\mathcal{G}'}$-conjugacy classes of the VN-subgroups of $\Pi_{\mathcal{G}'}$; Y for the set of $\Pi_{(\mathcal{G}')^{\mathrm{cpt}}}$-conjugacy classes of the VN-subgroups of $\Pi_{(\mathcal{G}')^{\mathrm{cpt}}}$. Then we have a natural commutative diagram of right $\Pi_{\mathcal{G}}$-sets and $\Pi_{\mathcal{G}}$-equivariant bijections*

$$\begin{tikzcd} \mathrm{VN}(\mathcal{G}') \arrow[r, "\cong"] \arrow[d, "\cong"'] & X \arrow[d, "\cong"] \\ \mathrm{VN}\big((\mathcal{G}')^{\mathrm{cpt}}\big) \arrow[r, "\cong"] & Y. \end{tikzcd}$$

Proof. The right $\Pi_{\mathcal{G}}$-actions are defined in an evident way. The left-hand vertical arrow is the identity map. The right-hand vertical arrow maps each conjugacy class of a VN-subgroup of $\Pi_{\mathcal{G}'}$ to the conjugacy class of its image via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{(\mathcal{G}')^{\mathrm{cpt}}}$. The upper horizontal arrow is obtained by forming the quotient of the $\Pi_{\mathcal{G}}$-equivalent bijection from $\mathrm{VN}(\widetilde{\mathcal{G}})$ to the set of VN-subgroups of $\Pi_{\mathcal{G}'}$ given by $\tilde{c} \longmapsto \Pi_{\tilde{c}} \cap \Pi_{\mathcal{G}'}$ (cf. [HmCbGCI], Lemma 1.3, (2)). The lower horizontal arrow is defined similarly. The proof of the $\Pi_{\mathcal{G}}$-equivariance of each arrow and the commutativity of this diagram is left to the leader (cf. [HmCbGCI], Lemma 2.1, (5)). Then the bijectivity of the right-hand vertical arrow follows immediately. $\square$

Definition 2.5. Let $S \subset \mathrm{VN}(\widetilde{\mathcal{G}})$ and $\mathcal{G}' \longrightarrow \mathcal{G}$ a connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is sturdy. Then we shall write $\widetilde{S}(\mathcal{G}', \mathrm{cpt}) \subset \mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\mathrm{cpt}}}\big)$ for the inverse image of $S(\mathcal{G}') \subset \mathrm{VN}(\mathcal{G}') \cong \mathrm{VN}\big((\mathcal{G}')^{\mathrm{cpt}}\big)$ via the natural surjection $\mathrm{VN}\big(\widetilde{(\mathcal{G}')^{\mathrm{cpt}}}\big) \twoheadrightarrow \mathrm{VN}\big((\mathcal{G}')^{\mathrm{cpt}}\big)$.

Lemma 2.6. *Let $S \subset \mathrm{VN}(\widetilde{\mathcal{G}})$ and $H \subset \Pi_{\mathcal{G}}$ a closed subgroup. For every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is sturdy, write $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'})$ for the image of $H \cap \Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{(\mathcal{G}')^{\mathrm{cpt}}}$ (cf. Remark 2.2.1). Then the following conditions are equivalent.*

(i) $H = 1$ or H is included in an $\overline{S} \amalg \mathrm{Cusp}(\widetilde{\mathcal{G}})$-like subgroup of $\Pi_{\mathcal{G}}$.

(ii) There exists a cofinal subsystem $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda}$ of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ constituted by connected finite étale subcoverings of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that, for every $\lambda \in \Lambda$, $\mathcal{G}_\lambda$ is sturdy, and, moreover, $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) = 1$ or $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda})$ is included in an $\widetilde{S}(\mathcal{G}_\lambda, \mathrm{cpt})$-like subgroup of $\Pi_{\mathcal{G}_\lambda^{\mathrm{cpt}}}$.

Proof. The implication (i) $\Longrightarrow$ (ii) is easily verified in light of Lemma 2.3.

The verification of the implication (ii) $\Longrightarrow$ (i) in the case where there exists a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is sturdy and $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'}) \neq 1$ is entirely similar to the proof of [HmCbGCI], Lemma 2.4, (ii) $\Longrightarrow$ (i).

Finally, the implication (ii) $\Longrightarrow$ (i) in the case where the equality $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'}) = 1$ holds for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ with $\mathcal{G}'$ sturdy follows, in light of the final portion of Remark 2.2.1, easily from [HmCbGCI], Prop 3.1, (v) $\Longrightarrow$ (iv) and [HmCbGCI], Remark 3.1.2. This completes the proof of Lemma 2.6. $\square$

Lemma 2.7. *Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism, $S \subset \mathrm{VN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VN}(\widetilde{\mathcal{H}})$. Moreover, let $(\mathcal{H}_\lambda \longrightarrow \mathcal{H})_{\lambda\in\Lambda}$ be a cofinal subsystem of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{H}}$ universal covering $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ constituted by connected finite étale subcoverings. For every $\lambda \in \Lambda$, write $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ for the connected finte étale subcovering corresponding to the open subgroup $\alpha^{-1}(\Pi_{\mathcal{H}_\lambda}) \subset \Pi_{\mathcal{G}}$; write $\alpha_\lambda\colon \Pi_{\mathcal{G}_\lambda} \longrightarrow \Pi_{\mathcal{H}_\lambda}$ for the continuous homomorphism induced by α. Suppose that $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda$ are sturdy for every $\lambda \in \Lambda$ and that α is $\big(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{H}})\big)$-compatible. Then the following assertions hold.*

(1) There exists a unique homomorphism $\alpha_\lambda^{\mathrm{cpt}}\colon \Pi_{\mathcal{G}_\lambda^{\mathrm{cpt}}} \longrightarrow \Pi_{\mathcal{H}_\lambda^{\mathrm{cpt}}}$ compatible with α_λ relative to the natural surjections of Definition 2.2, (2).

(2) The following conditions are equivalent.

(i) α is $\big(\overline{S}, \overline{T} \amalg \mathrm{Cusp}(\widetilde{\mathcal{H}})\big)$-compatible.

(ii) For every $\lambda \in \Lambda$, $\alpha_\lambda^{\mathrm{cpt}}$ in the assertion (1) is $\big(\widetilde{S}(\mathcal{G}_\lambda, \mathrm{cpt}), \widetilde{T}(\mathcal{H}_\lambda, \mathrm{cpt})\big)$-compatible.

Proof. The assertion (1) follows, in light of the final portion of Remark 2.2.1, immediately from the $\big(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{H}})\big)$-compatibility of α (and hence also of α_λ — cf. [HmCbGCI], Corollary 1.8). The assertion (2) follows immediately from Lemma 2.6. This completes the proof of Lemma 2.7. □

3 Hom-version of the Combinatorial Grothendieck Conjecture for Outer Representations of PIPSC-type

The remainder of the present paper is devoted to formulating and proving certain "Hom-versions" of the combinatorial Grothendieck conjecture for outer representations of PIPSC- or NN-type. More specifically, we show that, for a continuous homomorphism α between PSC-fundamental groups, the compatibility of α with certain outer representations of PSC-type implies a certain group-theoretic compatibility property of α.

In this section, we will mainly focus on the outer representations of PIPSC-type. Roughly speaking, the first main result of this section, Theorem

3.3, states that, if a continuous homomorphism α between PSC-fundamental groups is compatible with an outer representation of VA-type on the domain and an outer representation of PIPSC-type on the codomain, then α maps verticial subgroups into verticial subgroups. This theorem, along with its proof, can be considered as a "Hom-version" of [CbTpII], Theorem 1.9, (ii), at least if one disregards the point that, while [CbTpII], Theorem 1.9, (ii), includes the group-theoretic nodality of the isomorphism under consideration, Theorem 3.3 says nothing about nodal subgroups.

The discrepancy between [CbTpII], Theorem 1.9, (ii), and Theorem 3.3 concerning nodal subgroups just pointed out above is due to the issue discussed in [HmCbGCI], Remark 1.7.6, and hence cannot be remedied easily. However, in light of a substantially different viewpoint presented in [GrphPIPSC], we find that under some additional assumptions we can say something non-trivial also on nodal subgroups. This is the second main result of this section, Theorem 3.7.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid in this section; in particular, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type; we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

In the following, we are mainly concerned with the following commutative diagram of profinite groups and continuous homomorphisms;

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{\mathcal{G}} \arrow[r] \arrow[d, "\alpha"'] & \Pi_{\rho_I} \arrow[r] \arrow[d, "\tilde{\alpha}"'] & I \arrow[r] \arrow[d, "\beta"'] & 1 \\
1 \arrow[r] & \Pi_{\mathcal{H}} \arrow[r] & \Pi_{\rho_J} \arrow[r] & J \arrow[r] & 1,
\end{tikzcd} \tag{$**$}$$

where the two horizontal sequences are the exact sequences $(*)$ associated to outer representations of PSC-type $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ and $\big(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ (cf. the discussion preceding Definition 1.2). Moreover, we are actually interested in the cases where each of the two outer representations of PSC-type $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ and $\big(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ is of VA-type, of NN-type, or of PIPSC-type.

Lemma 3.1. *Let us consider the commutative diagram $(**)$. Let $I' \subset \Pi_{\rho_I}$ be (the image of) a section of $\Pi_{\rho_I} \twoheadrightarrow I$. Then the following assertions hold.*

(1) Suppose further that β is non-trivial. Then it holds that $\tilde{\alpha}(I') \neq 1$.

(2) *Suppose further that* $I \cong \hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$, *that* $J \cong \hat{\mathbb{Z}}^{\Sigma_{\mathcal{H}}}$, *and that* β *is surjective. Then* $\tilde{\alpha}(I') \subset \Pi_{\rho_J}$ *is (the image of) a section of* $\Pi_{\rho_J} \twoheadrightarrow J$.

Proof. This is immediate. □

We recall the following crucial result from [CbTpII].

Lemma 3.2. *Let us consider the exact sequence* $(*)$ *associated to an outer representation of IPSC-type* $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *(cf. the discussion preceding Definition 1.2). Let* $I_0 \subset \Pi_\rho$ *be (the image of) a section of the natural homomorphism* $\Pi_\rho \twoheadrightarrow I$. *Then there exists* $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *such that* $Z_{\Pi_{\mathcal{G}}}(I_0) \subset \Pi_{\tilde{v}}$.

Proof. This is [CbTpII], Theorem 1.6, (iii). □

The following is the first main result in this section, which may be thought of as a Hom-version of [CbTpII], Theorem 1.9, (ii).

Theorem 3.3. *Let us consider the commutative diagram* $(**)$. *Suppose further that the following three conditions hold.*

(i) The outer representation of PSC-type $\big(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ *is of VA-type.*

(ii) The outer representation of PSC-type $\big(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ *is of PIPSC-type.*

(iii) β *is an open homomorphism.*

Then α *is* $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$*-compatible.*

Proof. By replacing Π_{ρ_I} and Π_{ρ_J} by their appropriate open subgroups, we may assume without loss of generality that $\big(\mathcal{G}, \rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is of SVA-type, that $\big(\mathcal{H}, \rho_J\colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ is of IPSC-type, and that β is surjective (cf. Lemma 1.9, (3)). Let $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. It follows from Definition 1.5 that the centralizer $I_{\tilde{v}} \subset \Pi_{\rho_I}$ of $\Pi_{\tilde{v}}$ is (the image of) a section of $\Pi_{\rho_I} \twoheadrightarrow I$. Then it follows from Lemma 3.1, (2), that $\tilde{\alpha}(I_{\tilde{v}})$ is (the image of) a section of $\Pi_{\rho_J} \twoheadrightarrow J$. Thus, by Lemma 3.2, there exists $\tilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $Z_{\Pi_{\mathcal{H}}}(\tilde{\alpha}(I_{\tilde{v}})) \subset \Pi_{\tilde{w}}$. On the other hand, it follows immediately from the obvious inclusion $\Pi_{\tilde{v}} \subset Z_{\Pi_{\mathcal{G}}}(I_{\tilde{v}})$ that $\alpha(\Pi_{\tilde{v}}) \subset Z_{\Pi_{\mathcal{H}}}(\tilde{\alpha}(I_{\tilde{v}}))$. By combining these two inclusions, we obtain the desired inclusion $\alpha(\Pi_{\tilde{v}}) \subset \Pi_{\tilde{w}}$. This completes the proof of Theorem 3.3. □

Remark 3.3.1. Theorem 3.3, along with its proof, can be considered as a Hom-version of (the "group-theoretic verticiality" portion of) [CbTpII], Theorem 1.9, (ii). Indeed, these two theorems share, not only the appearance of the statement, but also the technical core of the proof, say, the equivalence between the property of being a closed subgroup of a verticial subgroup and the property of being a closed subgroup of a section centralizer (cf. Lemma 3.2; [CbTpII], Theorem 1.6, (iii)). However, (the author believes that) [CbTpII], Theorem 1.9, (ii), cannot be obtained as a formal consequence of Theorem 3.3, because of the fact that an outer representation of NN-type is not necessarily of PIPSC-type (cf. [CbTpI], Remark 5.9.2, (iii)).

As is discussed in the beginning of this section, the conclusion of Theorem 3.3 does not lead to the $(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{H}}))$-compatibility of α (cf. also [HmCbGCI], Remark 1.7.6). In relation to this, one can derive a non-trivial result from the following observation given in [GrphPIPSC].

Definition 3.4. Let G be a profinite group. Then we shall write $G^{\text{ab-free}}$ for the maximal torsion-free quotient of the maximal abelian qutient G^{ab} of G.

Lemma 3.5. *Let $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ be an outer representation of IPSC-type and C a non-trivial pro-cyclic closed subgroup of $\Pi_\rho = \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I$, where the "$\overset{\text{out}}{\rtimes}$" is taken with respect to ρ. Suppose further that $\mathrm{Cusp}(\mathcal{G}) = \emptyset$. Then the following conditions are equivalent.*

(i) C is included in a nodal subgroup of $\Pi_{\mathcal{G}} \subset \Pi_\rho$.

(ii) For every open subgroup $H \subset \Pi_\rho$, the image of the composition of the natural homomorphisms

$$C \cap H \hookrightarrow H \twoheadrightarrow H^{\text{ab-free}}$$

is trivial.

Proof. This is [GrphPIPSC], Lemma 2.5. □

Remark 3.5.1. Suppose that we are almost in the situation of Lemma 3.5 but we assume that $C = 1$. Then we observe that, if $\mathrm{Node}(\mathcal{G}) = \emptyset$, then the condition (i) of Lemma 3.5 does not hold while the condition (ii) of Lemma 3.5 holds.

Remark 3.5.2. It follows immediately from [HmCbGCI], Proposition 3.1, (i) $\iff$ (iii), that the assumption that C is pro-cyclic in Lemma 3.5 is superfluous.

Lemma 3.6. *Let* $(\mathcal{G}, \rho_I : I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *and* $(\mathcal{H}, \rho_J : J \longrightarrow \mathrm{Aut}(\mathcal{H}))$ *be outer representation of PIPSC-type. Let us consider the exact sequence* $(*)$ *associated to each of them (cf. the discussion preceding Definition 1.2). We consider* $\Pi_{\mathcal{G}}$ *(resp.* $\Pi_{\mathcal{H}}$*) as a subgroup of* Π_{ρ_I} *(resp.* Π_{ρ_J}*) via the natural injection. Let* $\tilde{\alpha} : \Pi_{\rho_I} \longrightarrow \Pi_{\rho_J}$ *be a continuous homomorphism.*

Suppose that $\mathrm{Cusp}(\mathcal{G}) = \emptyset$ *and that* $\mathrm{Cusp}(\mathcal{H}) = \emptyset$*. Then, for every* $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$*,* $\tilde{\alpha}(\Pi_{\tilde{e}}) = 1$ *or there exists* $\tilde{f} \in \mathrm{Node}(\widetilde{\mathcal{H}})$ *such that* $\tilde{\alpha}(\Pi_{\tilde{e}}) \subset \Pi_{\tilde{f}}$*. In particular, if we suppose moreover that* $\tilde{\alpha}(\Pi_{\mathcal{G}}) \subset \Pi_{\mathcal{H}}$*, then the continuous homomorphism* $\alpha : \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ *obtained by restricting the domain and the codomain of* $\tilde{\alpha}$ *is* $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{H}})\big)$*-compatible.*

Proof. Lemma 3.6 follows immediately from Lemma 3.5 and Lemma 1.9, (2), in light of the fact that the images of the nodal subgroups of $\Pi_{\mathcal{G}}$ via a continuous homomorphism are pro-cyclic (cf. [CbGC], Remark 1.1.3). $\square$

Now we are ready to show the following theorem.

Theorem 3.7. *Let us consider the commutative diagram* $(**)$*. Suppose that* $\big(\mathcal{G}, I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ *and* $\big(\mathcal{H}, J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ *are of PIPSC-type and that* α *is* $\big(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{H}})\big)$*-compatible. Then* α *is* $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$*-compatible (and thus, by assumption,* $\big(\mathrm{Edge}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$*-compatible).*

Proof. In light of [HmCbGCI], Corollary 1.8, it suffices to show the $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$-compatibility of the restriction of α to some open subgroups of domain and codomain. Thus, by replacing Π_{ρ_I} and Π_{ρ_J} by appropriate open subgroups (cf. Lemma 1.9, (2), and Lemma 2.3, (1)), we may assume that $\mathcal{G}$ and $\mathcal{H}$ are sturdy. Note that, for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, $\mathcal{G}'$ is also sturdy by Lemma 2.3, (2). The same assertion holds also for $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$.

By an argument entirely similar to the argument in the second paragraph of the proof of [HmCbGCI], Theorem 4.4, where we replace the use of [HmCbGCI], Lemma 2.5, by Lemma 2.7 of the present paper, we are reduced to showing that $\alpha^{\mathrm{cpt}} : \Pi_{\mathcal{G}^{\mathrm{cpt}}} \longrightarrow \Pi_{\mathcal{H}^{\mathrm{cpt}}}$ induced by α is $\big(\mathrm{Node}(\widetilde{\mathcal{G}^{\mathrm{cpt}}}), \mathrm{Node}(\widetilde{\mathcal{H}^{\mathrm{cpt}}})\big)$-compatible. On the other hand, this follows immediately from Remark 2.2.2 and Lemma 3.6. This completes the proof of Theorem 3.7. $\square$

Remark 3.7.1. At first glance, the reader may think that Theorem 3.7 should have been formulated in an "absolute anabelian" fashion similarly to Lemma 3.6, i.e., in a fashion where we do not assume that the homomorphism $\Pi_{\rho_I} \longrightarrow \Pi_{\rho_J}$ under consideration is compatible with the natural surjections $\Pi_{\rho_I} \twoheadrightarrow I$ and $\Pi_{\rho_J} \twoheadrightarrow J$. Indeed, if we were to focus only on the case where $\mathrm{Cusp}(\mathcal{G}) = \mathrm{Cusp}(\mathcal{H}) = \emptyset$, then we could obtain an "absolute" result straightforwardly, which we leave the detail to the interested readers. We prioritized dealing with the case where it does not necessarily hold that $\mathrm{Cusp}(\mathcal{G}) = \mathrm{Cusp}(\mathcal{H}) = \emptyset$, and for this reason, we only obtained the "semi-absolute" result. For further detail on this point, we refer to Remark 3.7.2 below.

Remark 3.7.2. Here, we briefly explain why our argument unfortunately leads to a substantially weaker conclusion if we modify the assumption of Theorem 3.7 so that the homomorphism $\Pi_{\rho_I} \longrightarrow \Pi_{\rho_J}$ under consideration is not necessarily compatible with the natural surjections $\Pi_{\rho_I} \twoheadrightarrow I$ and $\Pi_{\rho_J} \twoheadrightarrow J$. Recall that our argument is ultimately summarized as follows (cf. the proof of Lemma 2.6).

> The image in $\Pi_{\mathcal{H}}$ of the nodal subgroup of $\Pi_{\mathcal{G}}$ under consideration stabilizes something with respect to an appropriate action, hence is included in the stabilizer subgroup corresponding to it. On the other hand, one observes that the stabilizer subgroup of $\Pi_{\mathcal{H}}$ under consideration is a nodal subgroup of $\Pi_{\mathcal{H}}$.

If we do not assume the compatibility of the homomorphism $\Pi_{\rho_I} \longrightarrow \Pi_{\rho_J}$ with the natural surjections $\Pi_{\rho_I} \twoheadrightarrow I$ and $\Pi_{\rho_J} \twoheadrightarrow J$, then we have to treat "the image in Π_{ρ_J} of the nodal subgroup of $\Pi_{\mathcal{G}}$" rather than "the image in $\Pi_{\mathcal{H}}$ of the nodal subgroup of $\Pi_{\mathcal{G}}$". Then the image is not necessarily included in "the stabilizer subgroup of $\Pi_{\mathcal{H}}$" but is included in "the stabilizer subgroup of Π_{ρ_J}". In particular, the image is not necessarily included in a nodal subgroup of $\Pi_{\mathcal{H}}$, but is included in a nodal decomposition subgroup of Π_{ρ_J}.

4 Hom-version of the Combinatorial Grothendieck Conjecture for Outer Representations of NN-type

In this section, we continue our study of "Hom-versions" of the combinatorial Grothendieck conjecture, pivoting our attention to the outer representations of NN-type. Theorem 4.7, the main result of this section, states that, roughly, if a continuous homomorphism between PSC-fundamental groups is compatible with an outer representation of VA-type on the domain and an outer representation of NN-type on the codomain, and, moreover, it satisfies a certain group-theoretic compatibility property, then this group-theoretic compatibility "extends" to the neighbour vertices. This theorem may be considered as a unified Hom-version of (the "group-theoretic verticiality" portion of) [NodNon], Theorem 4.1, (the "group-theoretic verticiality" portion of) [NodNon], Corollary 4.2, and [CbTpII], Theorem 1.9, (i), (2) $\implies$ (1) (cf. Remark 4.8.2), while the proofs of these Isom-versions do not function in our Hom-version situation, and thus we developed a substantially different technique. We also note that, in contrast to the previous section, we regrettably cannot establish group-theoretic compatibility properties for edge-like subgroups in this section.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid in this section; in particular, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type; we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

To begin with, we prepare some terminologies and notations.

Definition 4.1. Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism, $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$.

(1) We shall say that α is S-*visible* if, for every $\tilde{c} \in S$, it holds that $\alpha(\Pi_{\tilde{c}}) \neq 1$.

(2) We shall say that α is strictly (S,T)-*compatible* if α is S-visible and (S,T)-compatible.

Remark 4.1.1. Suppose that we are in the situation of Definition 4.1. Then a result similar to [HmCbGCI], Corollary 1.8, holds also for S-visibility and strict (S,T)-compatibility. The proof is also entirely similar and left to the reader.

Definition 4.2. (i) Let $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then we shall say that $\tilde{v}$ is *neighbour to* $\tilde{c}$ if one of the following conditions holds:

- $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{c}$ abuts to $\tilde{v}$.
- $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and $\tilde{v} = \tilde{c}$.
- $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and there exists a (necessarily unique — cf. [HmCbGCI], Lemma 1.2, (3)) node $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ which abuts both to $\tilde{v}$ and to $\tilde{c}$.

(ii) Let $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then we shall write $F_{\tilde{c}}$ for the following subset of $\mathrm{Vert}(\widetilde{\mathcal{G}})$:

$$\left\{\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \;\middle|\; \tilde{v} \text{ is neighbour to } \tilde{c}.\right\} \subset \mathrm{Vert}(\widetilde{\mathcal{G}}).$$

(iii) Let $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then we shall write F_S for the following subset of $\mathrm{Vert}(\widetilde{\mathcal{G}})$:

$$\bigcup_{\tilde{c} \in S} F_{\tilde{c}} = \left\{\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \;\middle|\; \tilde{v} \text{ is neighbour to } \tilde{c} \text{ for some } \tilde{c} \in S.\right\} \subset \mathrm{Vert}(\widetilde{\mathcal{G}}).$$

Remark 4.2.1. If $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$, then, by definition, $F_{\tilde{c}}$ coincides with the set of vertices to which $\tilde{c}$ abuts. In particular, if $\tilde{c}$ is a cusp (resp. node), then it follows immediately (resp. from [HmCbGCI], Remark 1.2.1, or Remark 4.5.1 below) that the cardinality of $F_{\tilde{c}}$ is equal to one (resp. two).

Remark 4.2.2. It follows from [HmCbGCI], Lemma 1.2, (2), (3), that, for $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, $\tilde{v}$ is neighbour to $\tilde{c}$ if and only if $\Pi_{\tilde{v}} \cap \Pi_{\tilde{c}} \neq 1$.

Before proceeding, we note the following result (essentially) given in [NodNon].

Lemma 4.3. *Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of SNN-type. Let us consider the exact sequence $(*)$ associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ (cf. the discussion preceding Definition 1.2). Let $\tilde{c}, \tilde{d} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then the following equivalence holds:*

$$D_{\tilde{c}} \cap D_{\tilde{d}} \neq 1 \iff F_{\tilde{c}} \cap F_{\tilde{d}} \neq \emptyset.$$

Proof. If $\tilde{c}, \tilde{d} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ (resp. $\tilde{c}, \tilde{d} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$), then the assertion follows immediately from [NodNon], Proposition 3.8, (i) (resp. [NodNon], Proposition 3.9, (i)). Thus we may assume that $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and $\tilde{d} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. Though this case is, in fact, not of use in the present paper, the proof of this case is short enough that the author decided to include it.

The implication "$\Longleftarrow$" follows from the observation that, if $\tilde{v} \in F_{\tilde{c}} \cap F_{\tilde{d}}$, then it holds that $1 \neq I_{\tilde{v}} \subset D_{\tilde{c}} \cap D_{\tilde{d}}$ (cf. Remark 1.3.1, (3), and Lemma 1.7, (2)). Let us verify (the contrapositive of) the converse. Suppose that $F_{\tilde{c}} \cap F_{\tilde{d}} = \emptyset$. Take $\tilde{v} \in F_{\tilde{d}}$. Then it holds that $D_{\tilde{d}} \subset D_{\tilde{v}}$ (cf. Lemma 1.7, (2)), hence $D_{\tilde{c}} \cap D_{\tilde{d}} \subset D_{\tilde{c}} \cap D_{\tilde{v}}$. Moreover, it follows from [NodNon], Proposition 3.9, (i), (iii), that either $D_{\tilde{c}} \cap D_{\tilde{v}} = 1$ or $D_{\tilde{c}} \cap D_{\tilde{v}} = I_{\tilde{u}}$ holds, where in the latter case it holds that $\tilde{u} \in F_{\tilde{c}}$, which thus (cf. our assumption that $F_{\tilde{c}} \cap F_{\tilde{d}} = \emptyset$) implies that $\tilde{u} \notin F_{\tilde{d}}$. In the former case, the desired triviality $D_{\tilde{c}} \cap D_{\tilde{d}} = 1$ follows formally from the inclusion $D_{\tilde{c}} \cap D_{\tilde{d}} \subset D_{\tilde{c}} \cap D_{\tilde{v}}$. On the other hand, in the latter case, we compute as follows:

$$D_{\tilde{c}} \cap D_{\tilde{d}} = (D_{\tilde{c}} \cap D_{\tilde{v}}) \cap D_{\tilde{d}} = I_{\tilde{u}} \cap D_{\tilde{d}} = 1.$$

Note that the third "=" follows, in light of [NodNon], Proposition 3.5, from $\tilde{u} \notin F_{\tilde{d}}$. This completes the proof of Lemma 4.3. $\square$

To state Lemma 4.6, which plays a crucial role in the proof of Theorem 4.7, we have to recall the notion of untangledness.

Definition 4.4 ([NodNon], Definition 1.2)**.** $\mathcal{G}$ is said to be *untangled* if every node of $\mathcal{G}$ abuts to two distinct vertices of $\mathcal{G}$.

Lemma 4.5 ([NodNon], Remark 1.2.1, (i), (ii))**.** *The following assertions hold.*

(1) There exists a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}'$ is untangled.

(2) If $\mathcal{G}$ is untangled, then any connected finite étale covering of $\mathcal{G}$ is also untangled.

Proof. Let us verify the assertion (1). Let $l \in \Sigma$. For each $e \in \mathrm{Node}(\mathcal{G})$, we construct a finite étale covering $\mathcal{G}'(e) \longrightarrow \mathcal{G}$ of degree l as follows. First we prepare a trivial covering $\mathcal{G}_1 \amalg \cdots \amalg \mathcal{G}_l \longrightarrow \mathcal{G}$ of degree l, where each $\mathcal{G}_i$ is a copy of $\mathcal{G}$. Write b, b' for the two branches of e and, moreover, e_i

(resp. b_i; b'_i) for the "e" (resp. "b"; "b'") in $\mathcal{G}_i$. Then by "cutting" each e_i and "connecting" b_i with b'_{i+1} (where we set $b'_{l+1} \stackrel{\mathrm{def}}{=} b'_1$), we obtain a finite étale covering $\mathcal{G}'(e) \longrightarrow \mathcal{G}$ of degree l. Write $L \stackrel{\mathrm{def}}{=} \{e \in \mathrm{Node}(\mathcal{G}) \mid$ both of two branches of e abut to the same vertex of $\mathcal{G}.\}$. Then, for any $e \in L$, it follows from the definition of L, together with the construction of $\mathcal{G}'(e)$, that $\mathcal{G}'(e)$ is connected and, more precisely, that $\mathcal{G}'(e) \longrightarrow \mathcal{G}$ is a connected finite étale Galois covering of degree l. Since any node of $\mathcal{G}'(e)$ lying over e abuts to two distinct vertices, any connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ which dominates all of $\big(\mathcal{G}'(e) \longrightarrow \mathcal{G}\big)_{e \in L}$ satisfies the desired condition. This completes the proof of the assertion (1).

The assertion (2) is obvious. This completes the proof of Lemma 4.5. $\square$

Remark 4.5.1. The fact that any $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ abuts to precisely two vertices follows immediately from Lemma 4.5. In fact, this deduction aligns precisely the verification presented in [NodNon], Remark 1.2.1, (iii).

Remark 4.5.2. Utilizing the construction of "$\mathcal{G}'(e) \longrightarrow \mathcal{G}$" given in the proof of Lemma 4.5, one can also verify the well-known fact that the underlying (profinite) semi-graph of the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ is a "tree", i.e., does not have any closed loop. Indeed, suppose to the contrary that there exists a closed loop in $\widetilde{\mathcal{G}}$, i.e., a sequence $\tilde{e}_0, \ldots, \tilde{e}_n \in \mathrm{Node}(\widetilde{\mathcal{G}})$ ($n \geq 0$) and a sequence $\tilde{v}_0, \ldots, \tilde{v}_n \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfying the following two conditions:

- The nodes $\tilde{e}_0, \ldots, \tilde{e}_n \in \mathrm{Node}(\widetilde{\mathcal{G}})$ are all distinct.

- For $0 \leq i \leq n$, $\tilde{e}_i$ abuts both to $\tilde{v}_i$ and to $\tilde{v}_{i+1}$, where we set $\tilde{v}_{n+1} \stackrel{\mathrm{def}}{=} \tilde{v}_0$.

Replacing $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ by $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}^\dagger$ for some suitable connected finite étale subcovering $\mathcal{G}^\dagger \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, we may assume without loss of generality that $e_0, \ldots, e_n \in \mathrm{Node}(\mathcal{G})$ are all distinct, where we write $e_i \stackrel{\mathrm{def}}{=} \tilde{e}_i(\mathcal{G})$. We consider the finite étale covering $\mathcal{G}'(e_n) \longrightarrow \mathcal{G}$ in order to derive a contradiction. It follows from the fact that $e_0, \ldots, e_n$ form a closed loop of $\mathcal{G}$, together with the construction of $\mathcal{G}'(e_n)$, that $\mathcal{G}'(e_n)$ is connected and, more precisely, that $\mathcal{G}'(e_n) \longrightarrow \mathcal{G}$ is a connected finite étale Galois covering of degree l. Let us fix a morphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'(e_n)$ over $\mathcal{G}$, with respect to which we consider $\mathcal{G}'(e_n) \longrightarrow \mathcal{G}$ as a subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Then it follows from the construction of $\mathcal{G}'(e_n)$ that $\tilde{v}_0\big(\mathcal{G}'(e_n)\big), \ldots, \tilde{v}_n\big(\mathcal{G}'(e_n)\big)$ lie in "the same sheet", i.e., the same copy of $\mathcal{G}$ among the l copies of $\mathcal{G}$ that appear in the construction of $\mathcal{G}'(e_n)$. Moreover, again by the construction of $\mathcal{G}'(e_n)$, $\tilde{v}_n\big(\mathcal{G}'(e_n)\big)$

and $\tilde{v}_{n+1}\big(\mathcal{G}'(e_n)\big) = \tilde{v}_0\big(\mathcal{G}'(e_n)\big)$ do not lie in "the same sheet". These two observations obviously lead to a contradiction. This completes the proof of the fact that the underlying (profinite) semi-graph of the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ is a tree.

Lemma 4.6. *Let $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ be an outer representation of SNN-type. Let us consider the exact sequence $(*)$ associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ (cf. the discussion preceding Definition 1.2). Let $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Suppose that $\mathcal{G}$ is untangled. Then the following assertions hold.*

(1) $R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}}) = \Pi_{\tilde{c}}$.

(2) $R_{\Pi_I}(\Pi_{\tilde{c}}) = D_{\tilde{c}}$.

(3) For $\tilde{v}, \tilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, write $M_{\tilde{v},\tilde{w}} \overset{\mathrm{def}}{=} \{\gamma \in \Pi_{\mathcal{G}} \mid \tilde{v}^{\gamma} = \tilde{w}.\} \subset \Pi_{\mathcal{G}}$. (In particular, it holds that $M_{\tilde{v},\tilde{v}} = \Pi_{\tilde{v}}$ — cf. [HmCbGCI], Lemma 1.3, (1).) Then it holds that

$$R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \bigcup_{\tilde{v},\tilde{w} \in F_{\tilde{c}}} M_{\tilde{v},\tilde{w}}.$$

(4) If a closed subgroup H of $\Pi_{\mathcal{G}}$ satisfies $H \subset R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}})$, then $H \subset \Pi_{\tilde{v}}$ for some $\tilde{v} \in F_{\tilde{c}}$.

Proof. In the following discussion, we will use [HmCbGCI], Lemma 1.3, (1), without further mention.

Let us verify the assertion (1). We have only to verify the inclusion $R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{c}}$, since the converse is a formal consequence of the fact that $\Pi_{\tilde{c}} \neq 1$ (cf. [CbGC], Remark 1.1.3). Suppose that $\gamma \in R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}})$, or, equivalently, $\Pi_{\tilde{c}} \cap \Pi_{\tilde{c}^{\gamma}} \neq 1$. If $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$, then the assertion (1) follows immediately from [HmCbGCI], Lemma 1.2, (1). On the other hand, if $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, then it follows from Remark 4.2.2 that $\tilde{c}^{\gamma}$ is neighbour to $\tilde{c}$. However, if $\tilde{c} \neq \tilde{c}^{\gamma}$, then, since $\tilde{c}^{\gamma}$ is neighbour to $\tilde{c}$, there exists $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ which abuts both to $\tilde{c}$ and to $\tilde{c}^{\gamma}$, in contradiction to the untangledness assumption on $\mathcal{G}$ (cf. the fact that $\tilde{c}(\mathcal{G})$ must coincide with $\tilde{c}^{\gamma}(\mathcal{G})$). Thus, for any $\gamma \in R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}})$, it must hold that $\tilde{c} = \tilde{c}^{\gamma}$, or, equivalently, $\gamma \in \Pi_{\tilde{c}}$. This completes the proof of the assertion (1).

Let us verify the assertion (2). If $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (resp. $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$), then let $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be equal to $\tilde{c}$ (resp. a vertex to which $\tilde{c}$ abuts). Then, since $I_{\tilde{v}}$ is (the image of) a section of $\Pi_I \twoheadrightarrow I$ (cf. Definition 1.8, (2)), it holds that $\Pi_I = I_{\tilde{v}} \cdot \Pi_{\mathcal{G}}$, i.e., any element $\gamma \in \Pi_I$ can be written uniquely as $\gamma = ab$ with $a \in I_{\tilde{v}}$ and $b \in \Pi_{\mathcal{G}}$. Moreover, if we write $\gamma = ab$ as above, then the equivalence $\gamma \in R_{\Pi_I}(\Pi_{\tilde{c}}) \iff b \in R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}})$ holds, since $a \in I_{\tilde{v}}$ commutes with any element of $\Pi_{\tilde{c}} \subset \Pi_{\tilde{v}}$. Put another way, it holds that $R_{\Pi_I}(\Pi_{\tilde{c}}) = I_{\tilde{v}} \cdot R_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}})$. Thus the assertion (2) follows formally from the assertion (1) and the resp'd case of Lemma 1.6, (2). This completes the proof of the assertion (2).

The assertion (3) follows from the following computation (where the untangledness assumption on $\mathcal{G}$ is unnecessary);

$$\begin{aligned} R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) &= \{\gamma \in \Pi_{\mathcal{G}} \mid D_{\tilde{c}} \cap D_{\tilde{c}^\gamma} \neq 1.\} \\ &= \{\gamma \in \Pi_{\mathcal{G}} \mid F_{\tilde{c}} \cap F_{\tilde{c}^\gamma} \neq \emptyset.\} \\ &= \{\gamma \in \Pi_{\mathcal{G}} \mid \text{There exist } \tilde{v}, \tilde{w} \in F_{\tilde{c}} \text{ such that } \tilde{v}^\gamma = \tilde{w}.\} \\ &= \bigcup_{\tilde{v}, \tilde{w} \in F_{\tilde{c}}} M_{\tilde{v}, \tilde{w}}, \end{aligned}$$

where we note that the second "=" follows from Lemma 4.3.

Let us verify the assertion (4). We divide our argument to three cases: the case where $\tilde{c} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$, the case where $\tilde{c} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, and the case where $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$.

If $\tilde{c} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$, then it follows immediately from the assertion (3) that $R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \Pi_{\tilde{v}}$, where we write $\tilde{v}$ for the unique element of $F_{\tilde{c}}$. Thus the desired assertion follows.

Next, suppose that $\tilde{c} \in \mathrm{Node}(\widetilde{\mathcal{G}})$. Write $\tilde{v}$, $\tilde{w}$ for the two distinct elements of the set $F_{\tilde{c}}$. Then it follows immediately from the assertion (3) that $R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \Pi_{\tilde{v}} \cup \Pi_{\tilde{w}} \cup M_{\tilde{v},\tilde{w}} \cup M_{\tilde{w},\tilde{v}}$. If $M_{\tilde{v},\tilde{w}} \cup M_{\tilde{w},\tilde{v}} \neq \emptyset$, then the equality $\tilde{v}(\mathcal{G}) = \tilde{w}(\mathcal{G})$ holds, which thus implies that both of the two branches of $\tilde{c}(\mathcal{G})$ abut to the single vertex $\tilde{v}(\mathcal{G}) = \tilde{w}(\mathcal{G})$, in contradiction to the assumption that $\mathcal{G}$ is untangled. Thus it follows that $M_{\tilde{v},\tilde{w}} \cup M_{\tilde{w},\tilde{v}} = \emptyset$, hence $R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \Pi_{\tilde{v}} \cup \Pi_{\tilde{w}}$. Now suppose that a closed subgroup H of $\Pi_{\mathcal{G}}$ satisfies $H \subset R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \Pi_{\tilde{v}} \cup \Pi_{\tilde{w}}$. In this situation, the following elementary argument shows that $H \subset \Pi_{\tilde{v}}$ or $H \subset \Pi_{\tilde{w}}$, as desired: if there exist $h \in H \setminus \Pi_{\tilde{v}} \subset \Pi_{\tilde{w}}$ and $k \in H \setminus \Pi_{\tilde{w}} \subset \Pi_{\tilde{v}}$, then it follows that $hk \in H \setminus (\Pi_{\tilde{v}} \cup \Pi_{\tilde{w}})$, in contradiction to $H \subset \Pi_{\tilde{v}} \cup \Pi_{\tilde{w}}$. (Instead of this elementary argument, we could also apply [HmCbGCI], Proposition 3.1, (iii) $\implies$ (i), to "S" $= F_{\tilde{c}} = \{\tilde{v}, \tilde{w}\}$.) This completes the proof of the assertion (4) in the case where $\tilde{c} \in \mathrm{Node}(\widetilde{\mathcal{G}})$.

Finally, we consider the case where $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. By the assertion (3), we have

$$R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \left(\bigcup_{\tilde{v}\in F_{\tilde{c}}} \Pi_{\tilde{v}}\right) \cup \left(\bigcup_{\tilde{v}\in F_{\tilde{c}}\setminus\{\tilde{c}\}} M_{\tilde{v},\tilde{c}}\right) \cup \left(\bigcup_{\tilde{v}\in F_{\tilde{c}}\setminus\{\tilde{c}\}} M_{\tilde{c},\tilde{v}}\right) \cup \left(\bigcup_{\substack{\tilde{v},\tilde{w}\in F_{\tilde{c}}\setminus\{\tilde{c}\},\\ \tilde{v}\neq\tilde{w}}} M_{\tilde{v},\tilde{w}}\right).$$

Further, by the same argument as the argument applied in the case where $\tilde{c} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, it follows from the untangledness assumption on $\mathcal{G}$ that $M_{\tilde{v},\tilde{c}} = M_{\tilde{c},\tilde{v}} = \emptyset$ for any $\tilde{v} \in F_{\tilde{c}} \setminus \{\tilde{c}\}$. Thus we have

$$R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \left(\bigcup_{\tilde{v}\in F_{\tilde{c}}} \Pi_{\tilde{v}}\right) \cup \left(\bigcup_{\substack{\tilde{v},\tilde{w}\in F_{\tilde{c}}\setminus\{\tilde{c}\},\\ \tilde{v}\neq\tilde{w}}} M_{\tilde{v},\tilde{w}}\right) = \left(\bigcup_{\tilde{v}\in F_{\tilde{c}}} \Pi_{\tilde{v}}\right) \cup \left(\bigcup_{\substack{\tilde{v},\tilde{w}\in F_{\tilde{c}}\setminus\{\tilde{c}\},\\ \tilde{v}\neq\tilde{w}}} (M_{\tilde{v},\tilde{w}}\setminus\Pi_{\tilde{c}})\right).$$

Now we claim as follows:

> Claim 4.6.A: If $\gamma \in \bigcup_{\substack{\tilde{v},\tilde{w}\in F_{\tilde{c}}\setminus\{\tilde{c}\},\\ \tilde{v}\neq\tilde{w}}} (M_{\tilde{v},\tilde{w}} \setminus \Pi_{\tilde{c}})$, then $\gamma^2 \notin R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}})$.

Let us verify Claim 4.6.A. Let $\gamma \in \Pi_{\mathcal{G}}$ and $\tilde{v}, \tilde{w} \in F_{\tilde{c}} \setminus \{\tilde{c}\}$. Suppose that $\tilde{v} \neq \tilde{w}$, $\tilde{v}^{\gamma} = \tilde{w}$ and $\tilde{c}^{\gamma} \neq \tilde{c}$. In light of Lemma 4.3, it suffices to show that $F_{\tilde{c}} \cap F_{\tilde{c}^{\gamma^2}} = \emptyset$. Since the underlying (profinite) semi-graph of $\widetilde{\mathcal{G}}$ is a "tree" (cf. Remark 4.5.2), it suffices to give a sequence of distinct nodes $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in \mathrm{Node}(\widetilde{\mathcal{G}})$ and a sequence of vertices $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfying the following condition:

> For each $i \in \{1, 2, 3, 4\}$, $\tilde{e}_i$ abuts both to $\tilde{u}_{i-1}$ and to $\tilde{u}_i$, where we set $\tilde{u}_0 \overset{\mathrm{def}}{=} \tilde{c}$ and $\tilde{u}_4 \overset{\mathrm{def}}{=} \tilde{c}^{\gamma^2}$.

To this end, we set $\tilde{u}_1 = \tilde{w}$, $\tilde{u}_2 = \tilde{c}^{\gamma}$, and $\tilde{u}_3 = \tilde{w}^{\gamma}$. It follows from the various assumptions involved that, for each $i \in \{1, 2, 3, 4\}$, there exists a (necessarily unique — cf. Remark 4.5.2 or [HmCbGCI], Lemma 1.2, (3)) node $\tilde{e}_i \in \mathrm{Node}(\widetilde{\mathcal{G}})$ which abuts both to $\tilde{u}_{i-1}$ and to $\tilde{u}_i$. Thus, to verify Claim 4.6, it suffices to show that the nodes $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ are all distinct. Since two nodes $\tilde{e}$ and $\tilde{f}$ coincide if and only if $F_{\tilde{e}} = F_{\tilde{f}}$ (cf. [HmCbGCI], Lemma 1.2, (1), (3), and [HmCbGCI], Remark1.2.1), it suffices to verify the following three assertions:

(a) $\{\tilde{w}, \tilde{w}^{\gamma}\} \cap \{\tilde{c}, \tilde{c}^{\gamma}, \tilde{c}^{\gamma^2}\} = \emptyset$.

(b) $\tilde{w} \neq \tilde{w}^{\gamma}$.

(c) $\tilde{c} \neq \tilde{c}^{\gamma} \neq \tilde{c}^{\gamma^2}$.

The assertion (a) follows from the observation that, if the intersection under consideration is nonempty, then it is immediate that $\tilde{c}(\mathcal{G}) = \tilde{w}(\mathcal{G})$, which thus implies (cf. the fact that $\tilde{w} \in F_{\tilde{c}} \setminus \{\tilde{c}\}$) a contradiction to our assumption that $\mathcal{G}$ is untangled. The assertion (b) follows from the assumption that $\tilde{v} \neq \tilde{w}$ and $\tilde{v}^{\gamma} = \tilde{w}$. The assertion (c) follows from the assumption that $\tilde{c}^{\gamma} \neq \tilde{c}$. This completes the proof of Claim 4.6.A.

Now we proceed to complete the proof of the assertion (4) in the case where $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Let H be a closed subgroup of $\Pi_{\mathcal{G}}$ such that $H \subset R_{\Pi_{\mathcal{G}}}(D_{\tilde{c}}) = \left(\bigcup_{\tilde{v} \in F_{\tilde{c}}} \Pi_{\tilde{v}}\right) \cup \left(\bigcup_{\substack{\tilde{v}, \tilde{w} \in F_{\tilde{c}} \setminus \{\tilde{c}\}, \\ \tilde{v} \neq \tilde{w}}} (M_{\tilde{v},\tilde{w}} \setminus \Pi_{\tilde{c}})\right)$. Then, by Claim 4.6.A, in fact, it holds that $H \subset \bigcup_{\tilde{v} \in F_{\tilde{c}}} \Pi_{\tilde{v}}$. This readily implies, in light of [HmCbGCI], Proposition 3.1, (iii) $\Longrightarrow$ (i), applied to "S" $= F_{\tilde{c}}$, that $H \subset \Pi_{\tilde{v}}$ for some $\tilde{v} \in F_{\tilde{c}}$. This completes the proof of the assertion (4) in the case where $\tilde{c} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, hence also the proof of Lemma 4.6. □

Theorem 4.7. *Let us consider the commutative diagram* $(**)$ *(cf. the discussion preceding Lemma 3.1). Let* $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ *and* $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$*. Suppose further that the following four conditions hold.*

(i) $\left(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\right)$ *is of VA-type.*

(ii) $\left(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\right)$ *is of NN-type.*

(iii) α *is strictly* (S,T)*-compatible.*

(iv) β *is non-trivial.*

Then α *is* (F_S, F_T)*-compatible.*

Proof. To begin with, we note, in light of [HmCbGCI], Corollary 1.8, its analogue for strict compatibility (cf. Remark 4.1.1), Lemma 1.9, (2), (3), and Lemma 4.5, (1), (2), that, by replacing Π_{ρ_I} and Π_{ρ_J} by their appropriate open subgroups, we may assume that $\left(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\right)$ is of SVA-type, that $\left(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\right)$ is of SNN-type, and that $\mathcal{H}$ is untangled.

Let $\tilde{v} \in F_S$. Fix $\tilde{c} \in S$ such that $\tilde{v}$ is neighbour to $\tilde{c}$. By assumption, there exists $\tilde{d} \in T$ such that $1 \neq \alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$. We would like to show that $\alpha(\Pi_{\tilde{v}}) \subset \Pi_{\tilde{w}}$ for some $\tilde{w} \in F_{\tilde{d}}$.

To this end, first we claim as follows:

Claim 4.7.A: It holds that

$$1 \neq \tilde{\alpha}(I_{\tilde{v}}) \subset R_{\Pi_J}\big(\alpha(\Pi_{\tilde{c}})\big) \subset R_{\Pi_J}(\Pi_{\tilde{d}}) = D_{\tilde{d}}.$$

We verify Claim 4.7.A as follows. The "$\neq$" follows from the non-triviality of β, in light of Definition 1.5 and Lemma 3.1, (1). The first inclusion follows immediately from the assumption $1 \neq \alpha(\Pi_{\tilde{c}})$, together with the fact that $I_{\tilde{v}} \subset D_{\tilde{c}}$ (cf. Remark 1.3.1, (1), (3), and Lemma 1.7, (2)). The second inclusion follows formally from the assumption $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$. The "$=$" follows immediately from Lemma 4.6, (2), where we note that we have assumed that $\mathcal{H}$ is untangled here. This completes the proof of Claim 4.7.A.

It follows from Claim 4.7.A that

$$\alpha(\Pi_{\tilde{v}}) \subset R_{\Pi_{\mathcal{H}}}\big(\tilde{\alpha}(I_{\tilde{v}})\big) \subset R_{\Pi_{\mathcal{H}}}(D_{\tilde{d}}).$$

Indeed, the first inclusion follows immediately from $1 \neq \tilde{\alpha}(I_{\tilde{v}})$, together with the fact that $\Pi_{\tilde{v}} \subset Z_{\Pi_{\mathcal{G}}}(I_{\tilde{v}})$, while the second inclusion follows formally from the inclusion $\tilde{\alpha}(I_{\tilde{v}}) \subset D_{\tilde{d}}$. Then, by Lemma 4.6, (4), there exists $\tilde{w} \in F_{\tilde{d}}$ such that $\alpha(\Pi_{\tilde{v}}) \subset \Pi_{\tilde{w}}$. This completes the proof of Theorem 4.7. $\square$

Remark 4.7.1. We make some preparations for the following discussion here. In the following, we shall refer to the following statement as "Theorem 4.7.1.A":

> Let us consider the commutative diagram $(**)$ (cf. the discussion preceding Lemma 3.1). Suppose further that the following conditions hold.
>
> (i) The vertical arrows α, $\tilde{\alpha}$, and β of the diagram $(**)$ are all isomorphisms.
>
> (ii) $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ and $\big(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ are both of NN-type.
>
> (iii) $\mathrm{Cusp}(\widetilde{\mathcal{G}}) \neq \emptyset$ and α is group-theoretically cuspidal (cf. [CbGC], Definition 1.4, (iv)).
>
> Then there exist $\tilde{v}_{\mathcal{G}} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ and $\tilde{v}_{\mathcal{H}} \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $\alpha(\Pi_{\tilde{v}_{\mathcal{G}}}) = \Pi_{\tilde{v}_{\mathcal{H}}}$.

Moreover, we shall refer to as "Theorem 4.7.1.B" the statement obtained by replacing the condition (iii) of Theorem 4.7.1.A by the following weaker condition:

(iii)′ There exist $\tilde{e}_{\mathcal{G}} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{e}_{\mathcal{H}} \in \mathrm{Edge}(\widetilde{\mathcal{H}})$ such that $\alpha(\Pi_{\tilde{e}_{\mathcal{G}}}) = \Pi_{\tilde{e}_{\mathcal{H}}}$.

With regard to these two statements, we make the following four observations.

(a) Theorem 4.7.1.B is true. Indeed, Theorem 4.7.1.B follows formally from [CbTpII], Theorem 1.9, (i), (3) $\implies$ (1). (For a more direct proof, see Remark 4.7.3.)

(b) Theorem 4.7.1.A follows formally from Theorem 4.7.1.B.

(c) [NodNon], Corollary 4.2, follows formally from Theorem 4.7.1.A and [NodNon], Theorem 4.1.

(d) The basic strategy of the proof of [NodNon], Corollary 4.2, is, roughly, to prove Theorem 4.7.1.A and apply the observation (c), while the logical structure of the proof actually carried out in [NodNon] is, in fact, more complicated.

Remark 4.7.2. As is discussed in detail in Remark 4.8.2 below, Theorem 4.7 can be thought of as a unified generalized Hom-version of Theorem 4.7.1.B (cf. Remark 4.7.1), [NodNon], Theorem 4.1 (hence also [NodNon], Corollary 4.2 — cf. the observations (b) and (c) in Remark 4.7.1), and [CbTpII], Theorem 1.9, (i), (2) $\implies$ (1). However, the strategy of the proof of Theorem 4.7 is definitely different from the proofs of the three theorems quoted above (where we mean the observation (a) in Remark 4.7.1 by "the proof of Theorem 4.7.1.B").

Let us consider [NodNon], Theorem 4.1, here. We leave the consideration of other theorems to the interested reader. At a rough level, we must admit that the proof of Theorem 4.7 is similar to that of [NodNon], Theorem 4.1, in that they both depend deeply on the "graph-theoretic geometry via the decomposition and inertia subgroups" (cf. the lemmas in [NodNon] quoted in the proof of Lemma 4.3 of the present paper). However, it should not be ignored that the "sandwiching technique" (cf. [NodNon], Lemma 1.15) plays no role in the proof of Theorem 4.7, though it is crucial in the proof of [NodNon], Theorem 4.1. Not only that, in fact, (the author believes that) such a technique cannot be applied to the "Hom-version" situation of (a certain special case, corresponding to [NodNon], Theorem 4.1 — cf. Corollary 4.8, (4) or (5), below — of) Theorem 4.7. The problem which occurs when one tries to apply such a technique to the "Hom-version" situation of (a certain special case of) Theorem 4.7 is similar to the problem discussed in [HmCbGCI], Remark 1.7.6. That is to say, even if the image via $\tilde{\alpha}$ of the inertia subgroup of a vertex of $\widetilde{\mathcal{G}}$ is included in the intersection of the decomposition subgroups of two components of $\widetilde{\mathcal{H}}$, it might be still possible that those two components coincide.

Remark 4.7.3. In this context, it might be of interest to note that Theorem 4.7.1.B (cf. Remark 4.7.1) (hence also [NodNon], Corollary 4.2 — cf. the observations (b) and (c) in Remark 4.7.1) may be proved more directly in the context of [NodNon], §4, by utilizing a certain variant of the "sandwiching technique" (cf. [NodNon], Lemma 1.15) in a similar vein to [NodNon], Theorem 4.1. In particular, the complexity pointed out in the observation (d) in Remark 4.7.1 may be avoided.

Let us explain the outline here. To prove Theorem 4.7.1.B, we may assume (cf. the argument in the first paragraph of the proof of Theorem 4.7) that $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ and $\big(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ are both of SNN-type. By assumption, there exist $\tilde{e}_{\mathcal{G}} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{e}_{\mathcal{H}} \in \mathrm{Edge}(\widetilde{\mathcal{H}})$ such that $\alpha(\Pi_{\tilde{e}_{\mathcal{G}}}) = \Pi_{\tilde{e}_{\mathcal{H}}}$. Let $\tilde{v}_{\mathcal{G}} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be a vertex to which $\tilde{e}_{\mathcal{G}}$ abuts. Then it is easily verified (by constructing a suitable finite étale covering of $\mathcal{G}$ — cf. the proof of [NodNon], Lemma 1.15) that there exists $\gamma \in \Pi_{\mathcal{G}}$ such that $\tilde{e}_{\mathcal{G}}^{\gamma} \neq \tilde{e}_{\mathcal{G}}$ and $\tilde{e}_{\mathcal{G}}^{\gamma}$ also abuts to $\tilde{v}_{\mathcal{G}}$. It follows from [NodNon], Proposition 3.8, (ii), that $D_{\tilde{e}_{\mathcal{G}}} \cap D_{\tilde{e}_{\mathcal{G}}^{\gamma}} = I_{\tilde{v}_{\mathcal{G}}}$, hence $D_{\tilde{e}_{\mathcal{H}}} \cap D_{\tilde{e}_{\mathcal{H}}^{\alpha(\gamma)}} = \tilde{\alpha}(I_{\tilde{v}_{\mathcal{G}}})$. Thus, by [NodNon], Proposition 3.8, (ii), there exists $\tilde{v}_{\mathcal{H}} \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $\tilde{\alpha}(I_{\tilde{v}_{\mathcal{G}}}) = I_{\tilde{v}_{\mathcal{H}}}$, hence (by taking the "$Z_{\Pi_{\mathcal{H}}}(-)$" of each side of the equality — cf. [NodNon], Lemma 3.6, (i)) $\alpha(\Pi_{\tilde{v}_{\mathcal{G}}}) = \Pi_{\tilde{v}_{\mathcal{H}}}$. This completes the direct proof of Theorem 4.7.1.B in the context of [NodNon], §4.

Corollary 4.8. *Let us consider the commutative diagram* $(**)$ *(cf. the discussion preceding Lemma 3.1). Suppose further that the following two conditions hold.*

(i) The outer representations of PSC-type $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ *and* $\big(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H})\big)$ *are both of NN-type.*

(ii) β is non-trivial.

Then the following assertions hold.

(1) If α is strictly $\big(\mathrm{Edge}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$-compatible, and $\mathrm{Edge}(\widetilde{\mathcal{G}}) \neq \emptyset$, then α is strictly $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible.

(2) If α is strictly $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$-compatible, and $\mathrm{Node}(\widetilde{\mathcal{G}}) \neq \emptyset$, then α is strictly $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible.

(3) If α is strictly $\big(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}})\big)$-compatible, then α is strictly $\big(F_{\mathrm{Cusp}(\widetilde{\mathcal{G}})}, \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible.

(4) Let $S \subset \mathrm{Vert}(\widetilde{\mathcal{G}})$. If α is strictly $\big(S, \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible, then α is $\big(F_S, \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible.

(5) Let $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. If α is $\mathrm{Vert}(\widetilde{\mathcal{G}})$-visible, and $\big(\{\tilde{v}\}, \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible, then α is (necessarily strictly) $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible.

Proof. The assertions (1), (2), (3), and (4) follow immediately from Theorem 4.7.

Let us verify the assertion (5). By [HmCbGCI], Lemma 1.10, together with the visibility assumption, it follows that α is strict $(\tilde{v}^{\Pi_{\mathcal{G}}}, \mathrm{Vert}(\widetilde{\mathcal{H}}))$-compatible (where we use the notation "$(-)^{\Pi_{\mathcal{G}}}$" as in [HmCbGCI], Lemma 1.10). We start from this compatibility and apply Theorem 4.7 repeatedly (cf. the visibility assumption). Then, because the underlying semi-graph of $\mathcal{G}$ is connected and finite, we finally reach the desired $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatibility of α. This completes the proof of the assertion (5), hence also the proof of Corollary 4.8. □

Remark 4.8.1. It is immediatethat the assumption in the statement of Corollary 4.8 that $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is of NN-type is superfluous and may be weakened to the assumption that $\big(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is of VA-type.

Remark 4.8.2. Let us explain how Theorem 4.7 can be thought of as a unified generalized Hom-version of Theorem 4.7.1.B (cf. Remark 4.7.1), [NodNon], Theorem 4.1 (hence also [NodNon], Corollary 4.2 — cf. the observations (b) and (c) in Remark 4.7.1), and [CbTpII], Theorem 1.9, (i),

(2) $\Longrightarrow$ (1) (cf. Remark 4.7.2). Since Corollary 4.8 is essentially a formal consequence of Theorem 4.7, we may also apply Corollary 4.8.

First let us consider [NodNon], Theorem 4.1. Suppose that we are in the situation of [NodNon], Theorem 4.1. In light of [HmCbGCI], Remark 1.7.5 (or [HmCbGCI], Lemma 1.9, (ii) $\Longrightarrow$ (i)), it suffices to show that α is $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible and that α^{-1} is $\big(\mathrm{Vert}(\widetilde{\mathcal{H}}), \mathrm{Vert}(\widetilde{\mathcal{G}})\big)$-compatible. On the other hand, these follow immediately from Corollary 4.8, (5).

Theorem 4.7.1.B follows, in light of [HmCbGCI], Lemma 1.9, (ii) $\Longrightarrow$ (i), applied to "(S,T)" $= (\{\tilde{e}_{\mathcal{G}}\}, \{\tilde{e}_{\mathcal{H}}\})$, formally from Theorem 4.7, applied to "$(\mathcal{G},\mathcal{H},S,T)$" $= (\mathcal{G},\mathcal{H},\{\tilde{e}_{\mathcal{G}}\},\{\tilde{e}_{\mathcal{H}}\})$ and "$(\mathcal{G},\mathcal{H},S,T)$" $= (\mathcal{H},\mathcal{G},\{\tilde{e}_{\mathcal{H}}\},\{\tilde{e}_{\mathcal{G}}\})$.

Finally, [CbTpII], Theorem 1.9, (i), (2) $\Longrightarrow$ (1) follows, by an argument similar to the arguments given above, from Corollary 4.8, (2).

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