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By

Shota TSUJIMURA

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract

Let p be a prime number. In the present paper, we discuss the relative/absolute version of the geometrically pro-p anabelian Grothendieck Conjecture (RpGC/ApGC). In the relative setting, we prove RpGC for hyperbolic curves of genus 0 over subfields of mixed characteristic valuation fields of rank 1 of residue characteristic p whose value groups have no nontrivial p-divisible element. In particular, one may take the completion of arbitrary tame extension of a mixed characteristic Henselian discrete valuation field of residue characteristic p as a base field. In light of the condition on base fields, this result may be regarded as a partial generalization of S. Mochizuki's classical anabelian result, i.e., RpGC for arbitrary hyperbolic curves over subfields of finitely generated fields of the completion of the maximal unramified extension of $\mathbb{Q}_p$. It appears to the author that this result suggests that much wider class of p-adic fields may be considered as base fields in anabelian geometry. In the absolute setting, under the preservation of decomposition subgroups, we prove ApGC for hyperbolic curves of genus 0 over mixed characteristic Henselian discrete valuation fields of residue characteristic p. This result may be regarded as the first absolute Grothendieck Conjecture-type result for hyperbolic curves in the pro-p setting. Moreover, by combining this ApGC-type result with combinatorial anabelian geometry, under certain assumptions on decomposition groups and dimensions, we prove ApGC for configuration spaces of arbitrary hyperbolic curves over unramified extensions of p-adic local fields or their completions. In light of the condition on the dimension of configuration spaces, this result may be regarded as a partial generalization of a K. Higashiyama's pro-p semi-absolute Grothendieck Conjecture-type result.

Contents

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Introduction

Let p be a prime number; K a field of characteristic 0. For any perfect field F, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$. For any algebraic variety X [i.e., a separated, of finite type, and geometrically connected scheme] over K, we shall write Π_X for the étale fundamental group of X, relative to a suitable choice of basepoint; $\Delta_X \stackrel{\text{def}}{=} \Pi_{X \times_K \overline{K}}$. For any nonempty set of prime numbers Σ; we shall write

$$\Pi_X^{(\Sigma)} \stackrel{\text{def}}{=} \Pi_X/\mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}),$$

where $\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}$ denotes the maximal pro-Σ quotient of Δ_X. In particular, we have an exact sequence

$$1 \longrightarrow \Delta_X^{\Sigma} \longrightarrow \Pi_X^{(\Sigma)} \longrightarrow G_K \longrightarrow 1$$

of profinite groups. We shall write $\Delta_X^p \stackrel{\text{def}}{=} \Delta_X^{\{p\}}$; $\Pi_X^{(p)} \stackrel{\text{def}}{=} \Pi_X^{(\{p\})}$. For any pair of K-schemes $S^\dagger$, $S^\ddagger$, we shall write

$$\mathrm{Isom}_K(S^\dagger, S^\ddagger)$$

for the set of K-isomorphisms between $S^\dagger$ and $S^\ddagger$. For any pair of schemes $S^\dagger$, $S^\ddagger$, we shall write

$$\mathrm{Isom}(S^\dagger, S^\ddagger)$$

for the set of isomorphisms $S^\dagger \xrightarrow{\sim} S^\ddagger$ of schemes. For any pair of continuous surjective homomorphisms $G^\dagger \twoheadrightarrow G$, $G^\ddagger \twoheadrightarrow G$ of profinite groups, if we write $H^\ddagger \subseteq G^\ddagger$ for the kernel of the surjection $G^\ddagger \twoheadrightarrow G$, then we shall write

$$\mathrm{Isom}_G(G^\dagger, G^\ddagger)/\mathrm{Inn}(H^\ddagger)$$

for the set of isomorphisms $G^\dagger \xrightarrow{\sim} G^\ddagger$ [in the category of profinite groups] over G, considered up to composition with inner automorphisms arising from elements $\in H^\ddagger$. For any pair of profinite groups $G^\dagger$, $G^\ddagger$, we shall write

$$\mathrm{Isom}(G^\dagger, G^\ddagger)/\mathrm{Inn}(G^\ddagger)$$

for the set of outer isomorphisms $G^\dagger \xrightarrow{\sim} G^\ddagger$ [in the category of profinite groups].

In anabelian geometry, we often consider an inexplicit problem

> *whether or not an algebraic variety X may be "reconstructed" from various variants of the étale fundamental group Π_X.*

In this research area, the Grothendieck Conjecture-type results may be regarded as central results [cf. for instance, [12], Theorem A; [23], Theorem 0.4]. Roughly speaking, the Grothendieck Conjecture tells us that "anabelian" varieties over "sufficiently arithmetic" fields [for example, hyperbolic curves over number fields, p-adic local fields or finite fields] may be reconstructed from their étale fundamental groups. Let us note that the known proofs for hyperbolic curves depend on the fact that the base fields [of algebraic varieties, in question] are not so "far" from the prime fields.

With regard to the above inexplicit problem, one of the explicit questions in anabelian geometry may be stated as follows:

> Question 1 (Relative version of the Grothendieck Conjecture — ($\Sigma\mathrm{RGC}_K$)): Let Σ be a nonempty set of prime numbers; $X^\dagger$, $X^\ddagger$ hyperbolic curves over K. Then is the natural map
>
> $$\mathrm{Isom}_K(X^\dagger, X^\ddagger) \longrightarrow \mathrm{Isom}_{G_K}(\Pi^{(\Sigma)}_{X^\dagger}, \Pi^{(\Sigma)}_{X^\ddagger})/\mathrm{Inn}(\Delta^{\Sigma}_{X^\ddagger})$$
>
> bijective? [Strictly speaking, Grothendieck conjectured that this natural map is bijective if Σ is the set of all prime numbers, and K is finitely generated over the field of rational numbers $\mathbb{Q}$ — cf. [2].]

Note that, if $K = \overline{K}$, then $G_K = \{1\}$, hence, in particular, ($\Sigma\mathrm{RGC}_K$) does not hold. As a corollary of an affirmative result of Question 1, one may reconstruct the isomorphism classes of hyperbolic curves over K from their fundamental groups. In particular, as a weaker question, one may ask the reconstructibility of isomorphism classes only. Usually, we shall refer to this type of question as the weak version of the Grothendieck Conjecture. With regard to Question 1, S. Mochizuki obtained the following remarkable result:

Theorem ([14], Theorem 4.12)**.** *Suppose that Σ contains p, and K is a generalized sub-p-adic field [i.e., a subfield of a finitely generated extension of the field of fractions of the Witt ring $W(\overline{\mathbb{F}}_p)$ — cf. [14], Definition 4.11]. Then ($\Sigma\mathrm{RGC}_K$) holds.*

On the other hand, in anabelian geometry, there exists an absolute version of Question 1:

> Question 2 (Absolute version of the Grothendieck Conjecture — ($\Sigma\mathrm{AGC}_{\mathcal{C}}$)): Let Σ be a nonempty set of prime numbers; $\mathcal{C}$ a class of fields; $X^\dagger$, $X^\ddagger$ hyperbolic curves over fields $\in \mathcal{C}$. Then is the natural map
>
> $$\mathrm{Isom}(X^\dagger, X^\ddagger) \longrightarrow \mathrm{Isom}(\Pi^{(\Sigma)}_{X^\dagger}, \Pi^{(\Sigma)}_{X^\ddagger})/\mathrm{Inn}(\Pi^{(\Sigma)}_{X^\ddagger})$$
>
> bijective?

With regard to Question 2, in a recent joint work with S. Mochizuki, by developing the theory of RNS [i.e., "Resolution of Nonsingularities"], we proved the following result:

Theorem ([20], Theorem D)**.** *Suppose that Σ is a set of cardinality ≥ 2 that contains p, and $\mathcal{C}$ is the class of p-adic local fields, i.e., finite extension fields of the field of p-adic numbers. Then ($\Sigma\mathrm{AGC}_{\mathcal{C}}$) holds.*

Here, we note that, in the proof of this theorem, the assumption on the cardinality of Σ may be applied both

- in establishing RNS for hyperbolic curves over p-adic local fields, and
- in reconstructing the underlying topological spaces of the Berkovich analytifications of hyperbolic curves over p-adic local fields

in essential ways. Therefore, to weaken the assumption on the cardinality of Σ, i.e., to consider the geometrically pro-p setting, we need an essentially new idea. On the other hand, we observe that one may not obtain an affirmative result for $(\Sigma\mathrm{AGC}_{\mathcal{C}})$ if we allow [not necessarily complete] Henselian valuation fields as base fields of hyperbolic curves. Indeed, this observation arises from the well-known fact that the operation forming the completions of Henselian valuation fields does not change the absolute Galois groups [cf. Remark 6.11.1, (iii)]. In particular, to extend the absolute Grothendieck Conjecture-type results to such a Henselian [or more general] setting, it is natural to consider the following question:

> Question 2' ($\mathcal{D}$-absolute version of the Grothendieck Conjecture — $(\Sigma\mathrm{DAGC}_{\mathcal{C}})$): Let Σ be a nonempty set of prime numbers; $\mathcal{C}$ a class of fields; $X^{\dagger}$, $X^{\ddagger}$ hyperbolic curves over fields $\in \mathcal{C}$. Write
> $$\mathrm{Isom}^{D}(\Pi^{(\Sigma)}_{X^{\dagger}}, \Pi^{(\Sigma)}_{X^{\ddagger}})/\mathrm{Inn}(\Pi^{(\Sigma)}_{X^{\ddagger}}) \subseteq \mathrm{Isom}(\Pi^{(\Sigma)}_{X^{\dagger}}, \Pi^{(\Sigma)}_{X^{\ddagger}})/\mathrm{Inn}(\Pi^{(\Sigma)}_{X^{\ddagger}})$$
> for the subset consisting of the outer isomorphisms that induce bijections between the respective sets of decomposition subgroups associated to the closed points. Then is the natural map
> $$\mathrm{Isom}(X^{\dagger}, X^{\ddagger}) \longrightarrow \mathrm{Isom}^{D}(\Pi^{(\Sigma)}_{X^{\dagger}}, \Pi^{(\Sigma)}_{X^{\ddagger}})/\mathrm{Inn}(\Pi^{(\Sigma)}_{X^{\ddagger}})$$
> bijective?

Historically, S. Mochizuki proved this type of results for hyperbolic curves over p-adic local fields [cf. [18], Corollary 2.9], which may be regarded as an intermediate step of the verification of the above absolute Grothendieck Conjecture-type result, i.e., [20], Theorem D.

In this context, in the present paper, we discuss the weak/relative/absolute version of the geometrically pro-p anabelian Grothendieck Conjecture for hyperbolic curves of genus 0 over certain p-adic fields that are not necessarily complete. In the relative setting, we treat situations that the base fields of hyperbolic curves can be non-discrete valuation fields. On the other hand, in the absolute setting, we treat the situation that the base fields of hyperbolic curves are mixed characteristic Henselian discrete valuation fields of residue characteristic p.

In order to state our main theorems, we introduce some notions and notations. For any field F and any positive integer n, we shall write

$$F^{\times p^{\infty}} \stackrel{\mathrm{def}}{=} \bigcap_{m\geq 1} (F^{\times})^{p^{m}}, \quad \mu_{n}(F) \stackrel{\mathrm{def}}{=} \{x \in F^{\times} \mid x^{n} = 1\},$$

$$\mu(F) \stackrel{\mathrm{def}}{=} \bigcup_{m\geq 1} \mu_{m}(F), \quad \mu_{p^{\infty}}(F) \stackrel{\mathrm{def}}{=} \bigcup_{m\geq 1} \mu_{p^{m}}(F),$$

where m ranges over the positive integers. Then:

- We shall say that M is a *stably $\mu_{p^{\infty}}$-finite* field if, for every finite field extension $M \subseteq M^{\dagger}$, it holds that $\mu_{p^{\infty}}(M^{\dagger})$ is finite.

- We shall say that M is *stably p-$\times\mu$-indivisible* if, for every finite field extension $M \subseteq M^{\dagger}$, it holds that
$$(M^{\dagger})^{\times p^{\infty}} \subseteq \mu(M^{\dagger}).$$

- We shall say that M is *p-divisibly nonreflexive* if the subset
$$\{x \in M^{\times p^{\infty}} \mid 1 - x \in M^{\times p^{\infty}}\} \subseteq M^{\times p^{\infty}}$$
is *finite*. Moreover, we shall say that M is *stably p-divisibly nonreflexive* if, for every finite field extension $M \subseteq M^{\dagger}$, it holds that $M^{\dagger}$ is p-divisibly nonreflexive.

- We shall say that M is *p-bounded* if $p^n \notin M^{\times p^\infty}$ for any positive integer n.

Note that, by considering norm maps, one may observe that every finite extension field of a p-bounded field is also p-bounded.

Now write

$$K^{p^\infty} \stackrel{\text{def}}{=} \bigcup_{K \subseteq L} \mathbb{Q}(L^{\times p^\infty}) \subseteq \overline{K},$$

where the union ranges over the finite field extensions $K \subseteq L$ $(\subseteq \overline{K})$. Then our first main result concerning the weak version of the Grothendieck Conjecture is the following [cf. Theorem 3.1]:

Theorem A. *Let Σ be a set of prime numbers that contains p; $X^\dagger$, $X^\ddagger$ hyperbolic curves of genus 0 over K;*

$$\phi : \Pi^{(\Sigma)}_{X^\dagger} \xrightarrow{\sim} \Pi^{(\Sigma)}_{X^\ddagger}$$

an isomorphism of profinite groups that lies over the identity automorphism on G_K. Suppose that the following conditions hold:

- *ϕ induces a bijection between the respective sets of cuspidal inertia subgroups.*
- *K is a stably p-divisibly nonreflexive p-bounded field.*
- *K^{p^∞} is a p-divisibly nonreflexive p-bounded field.*

Then there exists an isomorphism of K-schemes

$$X^\dagger \xrightarrow{\sim} X^\ddagger$$

that induces a bijection between the respective sets of cusps that is compatible with the bijection between the respective sets of cuspidal inertia groups induced by ϕ.

Here, we note that:

- Every stably p-$\times\mu$-indivisible field of characteristic 0 is a stably p-divisibly nonreflexive p-bounded field.
- The maximal abelian extension field $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ is a p-divisibly nonreflexive p-bounded field [cf. [24], Lemma D].

Thus, Theorem A may be regarded as a generalization of [24], Theorem 3.5.

Next, our second main result concerning the relative version of the Grothendieck Conjecture is the following [cf. Theorem 4.3]:

Theorem B. *Let M be a mixed characteristic valuation field of rank 1 of residue characteristic p whose value group has no nontrivial p-divisible element; L a stably μ_{p^∞}-finite Galois extension of M; $X^\dagger$, $X^\ddagger$ hyperbolic curves of genus 0 over K. Suppose that K is isomorphic to a subfield of L. Then the natural map*

$$\mathrm{Isom}_K(X^\dagger, X^\ddagger) \longrightarrow \mathrm{Isom}_{G_K}(\Pi^{(p)}_{X^\dagger}, \Pi^{(p)}_{X^\ddagger})/\mathrm{Inn}(\Delta^p_{X^\ddagger})$$

is bijective.

With regard to Theorem B, one may take any subfield of the p-adic completion of the maximal tamely ramified extension field of any mixed characteristic complete discrete valuation field of residue characteristic p as the base field, i.e., K. It appears to the author that

- this result suggests that much wider class of p-adic fields may be considered as base fields of anabelian geometry, and
- the analogous results in the more general geometrically pro-Σ [where $p \in \Sigma$] cases or in the case where U and V are hyperbolic curves of higher genus also hold.

Note that if one may prove the geometrically pro-p Grothendieck Conjecture for hyperbolic curves of arbitrary genus, then, by applying Galois descent, one may also complete the proof of general geometrically pro-Σ versions. In particular, it suffices to consider the geometrically pro-p Grothendieck Conjecture. In the case where the base field is a subfield of a mixed characteristic discrete valuation field, there exists an ongoing joint work with Y. Hoshi, S. Mochizuki, and G. Yamashita, to obtain results for higher genus hyperbolic curves.

Next, our third main result concerning the $\mathcal{D}$-absolute version of the Grothendieck Conjecture is the following [cf. Theorem 6.11, (i)]:

Theorem C. *Let $X^\dagger$, $X^\ddagger$ be hyperbolic curves of genus 0 over mixed characteristic Henselian discrete valuation fields of residue characteristic p. Then the natural map*

$$\mathrm{Isom}(X^\dagger, X^\ddagger) \longrightarrow \mathrm{Isom}^D(\Pi^{(p)}_{X^\dagger}, \Pi^{(p)}_{X^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{X^\ddagger})$$

is bijective.

It appears to the author that Theorem C may be regarded as the first absolute Grothendieck Conjecture-type result for hyperbolic curves in the pro-p setting [cf. Remark 6.11.1, (ii)]. Again, we note that if one could establish the analogous result for hyperbolic curves of arbitrary genus, then, by applying Galois descent, one may also obtain more general geometrically pro-Σ [where $p \in \Sigma$] version of Theorem C. One of the key points of the proof of Theorem C is an algebraicity criterion for certain *set-theoretic bijections* discussed in §5. In anabelian geometry, one often considers the reconstructibility of the additive structure from the multiplicative structure in respective situations. One interesting point of this algebraicity criterion is to discuss

> *the reconstructibility of the multiplicative structure from a partial multiplicative structure, together with a certain additive structure*

in some sense. In §6, we prove, in fact, a slightly stronger result with respect to the assumption on the preservation of decomposition subgroups [cf. Remark 6.11.1, (iv)]. Moreover, one may also obtain a certain arithmetically pro-p analogue of Theorem C [cf. Theorem 6.11, (ii)]. Here, we note that many portions of the proof of Theorem C also work in the case where the base fields of hyperbolic curves are mixed characteristic Henselian valuation field of rank 1 of residue characteristic p whose value group has no nontrivial p-divisible element. On the other hand, at the time of writing the present paper, the author could not prove several portions, especially,

- the elasticity [i.e., the property of a profinite group that every nontrivial topologically finitely generated normal closed subgroup of an open subgroup is open] of the absolute Galois groups of such valuation fields that is needed to verify the analogue of [11], Corollary 4.6, and
- the analogue of Proposition 6.6 for such valuation fields that is needed to apply the algebraicity criterion in §5.

Finally, in order to state our final main result, we introduce the notion of configuration spaces associated to hyperbolic curves. Let n be a positive integer; X a hyperbolic curve over K. Then we shall write

$$X_n \overset{\mathrm{def}}{=} X^{\times n} \setminus \bigcup_{1 \leq i \neq j \leq n} \Delta_{i,j},$$

where $X^{\times n}$ denotes the fiber product of n copies of X over K; $\Delta_{i,j}$ denotes the diagonal divisor on $X^{\times n}$ associated to the i-th and j-th components. Recall that K. Higashiyama proved a certain geometrically pro-p semi-absolute version of the Grothendieck Conjecture for higher dimensional configuration spaces associated to hyperbolic curves over generalized sub-p-adic fields [cf. [4], Theorem 0.1]. Our final main result is the following [cf. Theorem 7.5], which may be regarded as a partial generalization of this K. Higashiyama's result from the view point of dimension of configuration spaces:

Theorem D. *Let $n^\dagger$, $n^\ddagger$ be positive integers; Σ a set of prime numbers either $\{p\}$ or the set of all prime numbers; $K^\dagger$, $K^\ddagger$ mixed characteristic Henselian discrete valuation fields of residue characteristic p such that $(\Sigma\mathrm{RGC}_{K^\dagger})$ and $(\Sigma\mathrm{RGC}_{K^\ddagger})$ hold; $X^\dagger$, $X^\ddagger$ hyperbolic curves over $K^\dagger$, $K^\ddagger$, respectively. Write $(g^\dagger, r^\dagger)$ for the type of $X^\dagger$, i.e, the genus of $X^\dagger$ is $g^\dagger$, and the degree of the divisor of cusps of $X^\dagger$ is $r^\dagger$. Suppose that*

$$\begin{cases} n^\dagger \geq 2 & \textit{if } g^\dagger \geq 1, \textit{and } r^\dagger \geq 1, \\ n^\dagger \geq 3 & \textit{if } r^\dagger = 0. \end{cases}$$

Then the natural map

$$\mathrm{Isom}(X^\dagger_{n^\dagger}, X^\ddagger_{n^\ddagger}) \longrightarrow \mathrm{Isom}^D(\Pi^{(\Sigma)}_{X^\dagger_{n^\dagger}}, \Pi^{(\Sigma)}_{X^\ddagger_{n^\ddagger}})/\mathrm{Inn}(\Pi^{(\Sigma)}_{X^\ddagger_{n^\ddagger}})$$

— where $\mathrm{Isom}^D(-,-)$ denotes the set of the outer isomorphisms that induce bijections between the respective sets of decomposition groups associated to the closed points of suitable compactifications of configuration spaces [cf. Definition 7.2, (ii)] — is bijective.

Theorem D is proved by applying Theorem C, together with combinatorial anabelian geometry. Here, we note that if the base fields $K^\dagger$ and $K^\ddagger$ are unramified extensions of p-adic local fields or their p-adic completions, then $(\Sigma\mathrm{RGC}_{K^\dagger})$ and $(\Sigma\mathrm{RGC}_{K^\ddagger})$ hold [cf. [14], Theorem 4.12]. For these conditions, there exists an ongoing joint work with Y. Hoshi, S. Mochizuki, and G. Yamashita as mentioned above. With regard to further developments, it would be interesting to consider

to which extent the assumption on the dimensions of configuration spaces may be dropped.

However, at the time of writing of the present paper, the author does not have any clue on the further development of this direction.

The present paper is organized as follows. In §1, we introduce the notions of stably p-divisibly nonreflexive field and p-bounded field and prove that any subfield of a stably μ_{p^∞}-finite Galois extension of the mixed characteristic valuation field as in Theorem B is a stably p-divisibly nonreflexive p-bounded field [cf. Lemma 1.8]. In §2, we review some well-known facts concerning the étale fundamental groups of commutative algebraic groups. In §3, by establishing a pro-p analogue of the proof of [24], Theorem 3.5, together with a reviewed result in §2, we prove Theorem A. In §4, by applying Theorem A and a H. Nakamura's result concerning the computations of certain Galois centralizers, together with Lemma 1.8, we prove Theorem B. In §5, we establish a certain algebraicity criterion of set-theoretic bijections between suitable subsets of mixed characteristic Henselian valuation fields of rank 1 of residue characteristic p whose value groups have no nontrivial p-divisible element. In §6, by applying Theorem B, together with the algebraicity criterion established in §5, we prove Theorem C. In §7, as an application of Theorem C, together with combinatorial anabelian geometry, we prove Theorem D.

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Notations and Conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the pro-p completion of $\mathbb{Z}$. We shall refer to a finite extension field of the field of p-adic numbers $\mathbb{Q}_p$ as a *p-adic local field*. For each commutative ring A, the notation $A^{\times}$ will be used to denote the group of units of A.

Fields: Let p be a prime number; F a field. Then we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{sep} $(\subseteq \overline{F})$ for the separable closure of F; $F \subseteq F_p$ $(\subseteq F^{\mathrm{sep}})$ for the maximal pro-p extension; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$. If F is a mixed characteristic rank 1 valuation field of residue characteristic p [cf. Definition 1.1, (ii) below], then the notation v_p will be used to denote the normalized valuation on F whose normalization is determined by the condition that $v_p(p) = 1$.

Let n be a positive integer that is invertible in F. Then we shall fix a primitive n-th root of unity $\zeta_n \in F^{\mathrm{sep}}$.

Algebraic Varieties: Let F be a field; $F \subseteq E$ a field extension; X an algebraic variety [i.e., a separated, of finite type, and geometrically connected scheme] over F. Then we shall write $X_E \stackrel{\mathrm{def}}{=} X \times_F E$; $X(E)$ for the set of E-rational points of X. Suppose that X is an algebraic group over F. Then we shall write $0_X \in X(F)$ for the origin. Suppose, moreover, that X is commutative. Then we shall write $n_X : X \to X$ for the endomorphism induced by multiplication by n..

Profinite groups: Let Σ be a nonempty set of prime numbers; G a profinite group. Then we shall write G^{Σ} for the maximal pro-Σ quotient of G; $\mathrm{Aut}(G)$ for the group of automorphisms of G [in the category of profinite groups]; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the subgroup of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If p is a prime number, then we shall also write $G^p \stackrel{\mathrm{def}}{=} G^{\{p\}}$.

Fundamental groups: For a connected locally Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint [cf. [3]]. Let F be a field of characteristic 0; Σ a nonempty set of prime numbers; X an algebraic variety over F. Then we shall write

$$\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X_{\overline{F}}}; \quad \Pi_X^{(\Sigma)} \stackrel{\mathrm{def}}{=} \Pi_X/\mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\Sigma})$$

that fit into a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_X^{\Sigma} & \longrightarrow & \Pi_X^{(\Sigma)} & \longrightarrow & G_K & \longrightarrow & 1.
\end{array}$$

If $\Sigma = \{p\}$ for some prime number p, then we shall also write $\Pi_X^{(p)} \stackrel{\mathrm{def}}{=} \Pi_X^{(\{p\})}$.

1 Stably p-divisibly nonreflexive field

Let p be a prime number. In the present section, we first introduce some notions on fields including stably μ_{p^∞}-finite field, stably p-divisibly nonreflexive field, and p-bounded field [cf. Definitions 1.3, (i); 1.4]. Subsequent to this, we prove that any subfield of a stably μ_{p^∞}-finite Galois extension of a certain mixed characteristic valuation field is a stably p-divisibly nonreflexive p-bounded field [cf. Lemma 1.8]. For instance, the completion of any tamely ramified extension field of a mixed characteristic complete discrete valuation field of residue characteristic p may be taken to be such a mixed characteristic valuation field. In the later sections, we prove certain pro-p Grothendieck Conjecture-type results for hyperbolic curves of genus 0 over such fields.

Definition 1.1.

(i) Let Γ be an abelian group. Then we shall say that Γ is *p-reduced* if

$$\bigcap_{m\geq 1} p^m \cdot \Gamma \;=\; \{0\},$$

where m ranges over the positive integers.

(ii) Let Γ be an ordered abelian group. Then we shall say that Γ is of *rank* 1 if Γ is isomorphic to a nontrivial subgroup of $\mathbb{R}$ as ordered abelian groups. If the value group of a valuation ring is of rank 1, then we shall say that the valuation ring is of *rank* 1.

Definition 1.2. Let M be a field, n a positive integer. Then we shall write

$$M^{\times p^\infty} \stackrel{\text{def}}{=} \bigcap_{m\geq 1} (M^\times)^{p^m}, \quad \mu_n(M) \stackrel{\text{def}}{=} \{x \in M^\times \mid x^n = 1\},$$

$$\mu(M) \stackrel{\text{def}}{=} \bigcup_{m\geq 1} \mu_m(M), \quad \mu_{p^\infty}(M) \stackrel{\text{def}}{=} \bigcup_{m\geq 1} \mu_{p^m}(M),$$

where m ranges over the positive integers.

Definition 1.3. Let M be a field. Then we shall say that:

(i) M is a *stably μ_{p^∞}-finite* field if, for every finite field extension $M \subseteq M^\dagger$, it holds that $\mu_{p^\infty}(M^\dagger)$ is finite.

(ii) M is a *stably p-$\times\mu$-indivisible* field if, for every finite field extension $M \subseteq M^\dagger$, it holds that

$$(M^\dagger)^{\times p^\infty} \;\subseteq\; \mu(M^\dagger).$$

Definition 1.4. Let M be a field. Then we shall say that:

(i) M is a *p-divisibly nonreflexive* field if the subset

$$\{x \in M^{\times p^\infty} \mid 1 - x \in M^{\times p^\infty}\} \subseteq M^{\times p^\infty}$$

is *finite*.

(ii) M is a *stably p-divisibly nonreflexive* field if, for every finite field extension $M \subseteq M^{\dagger}$, it holds that $M^{\dagger}$ is p-divisibly nonreflexive.

(iii) M is a *p-bounded* field if, for each positive integer n, it holds that $p^n \notin M^{\times p^{\infty}}$.

Remark 1.4.1. Let M be a field of characteristic 0 such that $\zeta_6 \in M$. Then it holds that $1 - \zeta_6 = \zeta_6^{-1}$. This is a motivating example to allow the set $\{x \in M^{\times p^{\infty}} \mid 1 - x \in M^{\times p^{\infty}}\}$ finite [as opposed to impose a condition that the set $\{x \in M^{\times p^{\infty}} \mid 1 - x \in M^{\times p^{\infty}}\}$ is empty] in the definition of p-divisibly nonreflexive fields.

Remark 1.4.2.

(i) It follows immediately from the various definitions involved that any field of characteristic p is p-bounded.

(ii) In light of the norm maps associated to finite field extensions, one may observe that every finite extension field of a p-bounded field is also p-bounded.

(iii) Let M be a field of characteristic $\neq p$. Then, if $\mu_{p^{\infty}}(M)$ is infinite, then it follows immediately from the various definitions involved that M is p-bounded if and only if $p \notin M^{\times p^{\infty}}$.

Remark 1.4.3.

(i) It follows immediately from the various definitions involved that any subfield of a p-divisibly nonreflexive field (respectively, a stably p-divisibly nonreflexive field; a p-bounded field) is also a p-divisibly nonreflexive field (respectively, a stably p-divisibly nonreflexive field; a p-bounded field).

(ii) For any field M of characteristic 0, one may observe that $\{\zeta_6^{\pm 1}\} = \{x \in \mu(M) \mid 1 - x \in \mu(M)\}$. In particular, any stably p-$\times\mu$-indivisible field of characteristic 0 is a stably p-divisibly nonreflexive p-bounded field.

(iii) It follows immediately from the various definitions involved that any algebraically closed field of characteristic $\neq p$ is neither p-divisibly nonreflexive nor p-bounded.

Lemma 1.5. *Let K be a field of characteristic $\neq p$; $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ a(n) [possibly, infinite] Galois extension such that L is stably $\mu_{p^{\infty}}$-finite. Then, for each finite separable field extension $L \subseteq L^{\dagger}$ $(\subseteq K^{\mathrm{sep}})$, it holds that*

$$(L^{\dagger})^{\times p^{\infty}} \subseteq \bigcup_{K \subseteq K^{\dagger}} (K^{\dagger})^{\times p^{\infty}} \ (\subseteq K^{\mathrm{sep}}),$$

where $K \subseteq K^{\dagger}$ $(\subseteq K^{\mathrm{sep}})$ ranges over the finite separable field extensions.

Proof. Lemma 1.5 follows from a similar argument to the argument applied in the proof of [24], Lemma 3.4, (iv), (v). $\square$

Lemma 1.6. *Let A be a mixed characteristic valuation ring of residue characteristic p. Write $\mathfrak{m} \subseteq A$ for the maximal ideal of A; K for the field of fractions of A; $\Gamma \stackrel{\text{def}}{=} K^\times/A^\times$; $v : K^\times \twoheadrightarrow \Gamma$ for the natural surjection. Suppose that Γ is p-reduced and of rank 1. Then it holds that*

$$\bigcap_{i \geq 1} (1+\mathfrak{m})^{p^i} \ = \ \{1\},$$

where i ranges over the positive integers.

Proof. Let $x \in \cap_{i\geq 1} (1+\mathfrak{m})^{p^i}$ be an element. Write $a_0 \stackrel{\text{def}}{=} x-1$. Then, for each positive integer n, there exist elements $a_n, b_n \in \mathfrak{m}$ such that

$$a_0 \ = \ (1+a_n)^{p^n} - 1 \ = \ p \cdot b_n + a_n^{p^n}.$$

Suppose that $v(a_0) < v(p)$. Then it follows immediately from the above equality that $v(a_0) = v(a_n^{p^n}) = p^n \cdot v(a_n) \in p^n \cdot \Gamma$. Thus, since Γ is p-reduced, by varying n, we conclude that $v(a_0) = 0$, hence that $a_0 \in A^\times$. This is a contradiction. In particular, it holds that

$$v(x-1) \ = \ v(a_0) \ \geq \ v(p).$$

Next, since the inverse limit of an inverse system of nonempty finite sets is nonempty, we observe that there exists a family $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ consisting of elements $\in 1+\mathfrak{m}$ such that $x_0 = x$, and $x_{n+1}^p = x_n$ for each $n \in \mathbb{Z}_{\geq 0}$. Note that $x_n \in \cap_{i\geq 1} (1+\mathfrak{m})^{p^i}$ for each $n \in \mathbb{Z}_{\geq 0}$. Then, in light of the argument in the previous paragraph, it holds that $x_n - 1 \in p \cdot A$ for each $n \in \mathbb{Z}_{\geq 0}$. Thus, since Γ is of rank 1, we conclude that

$$x \ \in \ \bigcap_{i\geq 1} (1+p\cdot A)^{p^i} \ \subseteq \ \bigcap_{i \geq 1} 1 + p^{i+1} \cdot A \ = \ \{1\},$$

where i ranges over the positive integers. This completes the proof of Lemma 1.6. $\square$

Lemma 1.7. *In the notation of Lemma 1.6, let $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ be a finite field extension; $v^\dagger : (K^\dagger)^\times \twoheadrightarrow \Gamma^\dagger$ a valuation on $K^\dagger$ that extends v. Write $A^\dagger$ $(\subseteq K^\dagger)$ for the valuation ring associated to $v^\dagger$; $k^\dagger$ for the residue field of $A^\dagger$. Then the following hold:*

- *(i) The value group $\Gamma^\dagger$ is p-reduced and of rank 1.*
- *(ii) It holds that $(K^\dagger)^{\times p^\infty} \subseteq (A^\dagger)^\times$. Moreover, the natural composite $(K^\dagger)^{\times p^\infty} \subseteq A^\dagger \twoheadrightarrow k^\dagger$ is injective, and the image of this composite is contained in $(k^\dagger)^{\times p^\infty}$.*

Proof. Assertion (i) follows immediately from the facts that $\Gamma^\dagger$ is torsion-free, and the subgroup $\Gamma \subseteq \Gamma^\dagger$ is of finite index [cf. [1], Corollary 3.2.3].

Next, we verify assertion (ii). First, it follows immediately from the fact that $\Gamma^\dagger$ is p-reduced [cf. (i)] that $(K^\dagger)^{\times p^\infty} \subseteq (A^\dagger)^\times$. On the other hand, since $A^\dagger$ is a valuation ring, the image of the natural composite $(K^\dagger)^{\times p^\infty} \subseteq A^\dagger \twoheadrightarrow k^\dagger$ is contained in $(k^\dagger)^{\times p^\infty}$. Thus, it suffices to prove that this composite is injective.

Write $\mathfrak{m}^\dagger$ for the maximal ideal of $A^\dagger$. Let $x \in (K^\dagger)^{\times p^\infty} \cap (1+\mathfrak{m}^\dagger)$ be an element. For each positive integer m, let $y_m \in A^\dagger$ be such that $y_m^{p^m} = x$. Then since $k^\dagger$ is of characteristic p, and $x \in 1+\mathfrak{m}^\dagger$, it follows immediately that $y_m \in 1+\mathfrak{m}^\dagger$. In particular, we have

$$x \ \in \ \bigcap_{i\geq 1} (1+\mathfrak{m}^\dagger)^{p^i},$$

where i ranges over the positive integers. Thus, we conclude from Lemma 1.6 that $x = 1$. This completes the proof of assertion (ii), hence of Lemma 1.7. $\square$

Let K be a mixed characteristic valuation field of residue characteristic p whose value group is p-reduced and of rank 1. For instance,

> *the completion of [possibly, infinite] tame extension field of a mixed characteristic complete discrete valuation field of residue characteristic p*

may be such a valuation field. Write A for the valuation ring of K; k for the residue field of A; $\mathfrak{m}$ $(\subseteq A)$ for the maximal ideal of A;

$$v : K^{\times} \twoheadrightarrow \Gamma \stackrel{\text{def}}{=} K^{\times}/A^{\times}$$

for the natural surjection, i.e., the valuation on K. Let

$$K \subseteq L \ (\subseteq \overline{K})$$

be a Galois extension such that L is stably μ_{p^∞}-finite.

Lemma 1.8. *Let F be a subfield of L. Then F is a stably p-divisibly nonreflexive p-bounded field. Moreover, it holds that*

$$\{x \in F^{\times p^\infty} \mid 1 - x \in F^{\times p^\infty}\} \ \subseteq \ \{\zeta_6^{\pm 1}\}.$$

Proof. First, we may assume without loss of generality that $F = L$ [cf. Remark 1.4.3, (i)]. Next, we note that $p \in \mathfrak{m}$. Then it follows immediately from Lemma 1.5, together with Lemma 1.7, (ii), that, for each positive integer n, we have

$$p^n \ \notin \ L^{\times p^\infty}.$$

In particular, it holds that L is p-bounded. Thus, in light of Lemma 1.7, (i), by applying Lemma 1.5 again, we observe that it suffices to verify that

$$\{x \in K^{\times p^\infty} \mid 1 - x \in K^{\times p^\infty}\} \ \subseteq \ \{\zeta_6^{\pm 1}\}.$$

Next, let $x \in K^{\times p^\infty}$ be such that $1 - x \in K^{\times p^\infty}$. Then there exist elements $x_1, \ y_1 \in K^{\times p^\infty}$ such that

$$x_1^p \ = \ x, \quad y_1^p \ = \ 1 - x.$$

Fix such elements. Write

$$a_1 \ \stackrel{\text{def}}{=} \ x_1 + y_1 - 1.$$

Observe that the equality $x_1^p + y_1^p - 1 = 0$ implies that $a_1 \in \mathfrak{m}$.

Next, we verify the following assertion:

Claim 1.8.A : $v(a_1) \geq v(p)$.

Let t be a positive integer. Then since $x_1, \ y_1 \in K^{\times p^\infty}$, there exist $x_t, \ y_t \in A^{\times}$ such that $x_1 = x_t^{p^{t-1}}$, and $y_1 = y_t^{p^{t-1}}$. Note that there exists $c \in A$ such that

$$a_1 \ = \ x_1 + y_1 - 1 \ = \ (x_t + y_t - 1)^{p^{t-1}} - p \cdot c.$$

Suppose that $v(a_1) < v(p)$. Then it follows from the above equality that

$$p^{t-1} \cdot v(x_t + y_t - 1) \ = \ v(a_1) \ \in \ \Gamma.$$

Thus, by varying t, we conclude that

$$v(a_1) \ \in \ \bigcap_{t \geq 1} p^t \cdot \Gamma \ = \ \{0\},$$

where the equality follows from the fact that Γ is p-reduced. This contradicts the fact that $a_1 \in \mathfrak{m}$. This completes the proof of Claim 1.8.A.

Now we have

$$1 - x_1^p \ = \ y_1^p \ = \ (1 - x_1 + a_1)^p \ = \ (1 - x_1)^p + \sum_{1 \leq i \leq p} (1 - x_1)^{p-i} \cdot a_1^i \cdot \binom{p}{i}.$$

Then, by applying Claim 1.8.A, we have

$$\frac{1 - x_1^p - (1 - x_1)^p}{p} \ \in \ \mathfrak{m}.$$

Write $\overline{x}_1, \ \overline{x} \in k^\times$ for the images of $x_1, \ x \in A^\times$ via the natural surjection $A \twoheadrightarrow k$, respectively. Then it follows immediately that, if $p \geq 3$ (respectively, $p = 2$), then

$$\sum_{1 \leq i \leq p-1} \frac{1}{p} \cdot \binom{p}{i} \cdot (-\overline{x}_1)^i \ = \ 0 \quad \text{(respectively, } \overline{x}_1^2 - \overline{x}_1 \ = \ 0 \text{).}$$

Then since $\overline{x}_1 \neq 0$, it follows immediately that $\overline{x}_1 \in \mu(k)$, hence that $\overline{x} \in \mu(k)$. Thus, we conclude from Lemma 1.7, (ii), that $x \in \mu(K)$. On the other hand, by a similar argument, we also have $1 - x \in \mu(K)$. This implies that $x \in \{\zeta_6^{\pm 1}\}$. This completes the proof of Lemma 1.8. $\square$

2 Generalities on the étale fundamental groups of commutative algebraic groups

In the present section, we review some basic properties on the étale fundamental groups of commutative algebraic groups and a well-known argument surrounding the injectivity portion of the Section Conjecture [cf. Proposition 2.5; [27], §1, Example], which will be used only for the multiplicative group scheme $\mathbb{G}_\mathrm{m}$ in the next section.

Let K be a field of characteristic 0.

Lemma 2.1. *Suppose that K is algebraically closed. Let Z, S be smooth varieties over K. Then the natural [outer] homomorphism $\Pi_{Z \times_K S} \to \Pi_Z \times \Pi_S$ is an isomorphism.*

Proof. Let Z^{cpt} be a smooth compactification of Z over K such that the complement $D \overset{\mathrm{def}}{=} Z^{\mathrm{cpt}} \setminus Z$ is a normal crossing divisor [cf. [5], [6]]. Write $Z^{\log}$ for the log regular log scheme whose underlying scheme is Z^{cpt} and whose log structure is determined by D. Note that $Z^{\log}$ is log smooth, proper over K, and S may be regarded as a log regular log scheme with respect to the trivial log structure. Then we have an isomorphism $\Pi_{Z^{\log} \times_K S} \overset{\sim}{\to} \Pi_{Z^{\log}} \times \Pi_S$ of log étale fundamental groups [cf. [7], §3, Proposition 3]. On the other hand, we observe that $Z^{\log} \times_K S$ is a log regular log scheme whose interior coincides with $Z \times_K S$. Therefore, since K is of characteristic 0, it follows immediately from [7], Proposition B.7, that the natural [outer] homomorphism $\Pi_{Z \times_K S} \to \Pi_Z \times \Pi_S$ is an isomorphism. This completes the proof of Lemma 2.1. $\square$

Let X be an algebraic group over K. In the remainder of the present section, whenever we consider the étale fundamental groups of algebraic groups over algebraically closed fields, we take the origins of them as basepoints.

Lemma 2.2. *Suppose that K is algebraically closed. Then Π_X is abelian.*

Proof. Note that since K is an algebraically closed field of characteristic 0, we have the composite of homomorphisms of profinite groups

$$f : \Pi_X \times \Pi_X \overset{\sim}{\to} \Pi_{X \times_K X} \longrightarrow \Pi_X,$$

where the first map is a natural isomorphism [cf. Lemma 2.1]; the second map is a natural homomorphism induced by the multiplication map $X \times_K X \to X$. Then since the respective restrictions of f on $\Pi_X \times \{1\}$ and $\{1\} \times \Pi_X$ are the identity automorphisms, it follows immediately from the fact that $\Pi_X \times \{1\}$ commutes with $\{1\} \times \Pi_X$ that Π_X is abelian. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Suppose that K is algebraically closed, and X is commutative. Let $f : Y \to X$ be a finite étale Galois covering over K. Then Y admits a structure of commutative algebraic group over K such that f is an isogeny.*

Proof. Write $M \overset{\text{def}}{=} f^{-1}(0_X) \subseteq Y(K)$. Note that since f is a finite étale Galois covering, Π_X acts on M regularly. Let $m \in M$ be a closed point. Recall that Π_X is abelian [cf. Lemma 2.2]. Then we have a natural map

$$\alpha_0 : M \times M \longrightarrow M, \quad (m^{\sigma_1}, m^{\sigma_2}) \mapsto m^{\sigma_1 \sigma_2}$$

[where $\sigma_1, \sigma_2 \in \Pi_X$] that is compatible with the multiplication map $\Pi_X \times \Pi_X \to \Pi_X$ and the natural actions of $\Pi_X \times \Pi_X$, Π_X on $M \times M$, M, respectively. Thus, in light of the theory of Galois categories, we obtain a commutative diagram of schemes

$$\begin{array}{ccc} Y \times_K Y & \xrightarrow[\alpha]{} & Y \\ {\scriptstyle f \times f}\downarrow & & {\scriptstyle f}\downarrow \\ X \times_K X & \longrightarrow & X, \end{array}$$

where the lower horizontal arrow is the multiplication map on X. Similarly, the bijection

$$M \overset{\sim}{\to} M, \quad m^{\sigma} \mapsto m^{\sigma^{-1}}$$

[where $\sigma \in \Pi_X$] induces a commutative diagram of schemes

$$\begin{array}{ccc} Y & \xrightarrow[i]{\sim} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{\sim} & X, \end{array}$$

where the lower horizontal arrow is the inversion map on X. Then it follows immediately that the morphisms α, i, and the closed point $m \in Y(K)$ determine a structure of algebraic group over K on Y such that f is an isogeny. On the other hand, we note that α_0 commutes with the bijection

$$M \times M \overset{\sim}{\to} M \times M, \quad (m_1, m_2) \mapsto (m_2, m_1)$$

that is compatible with the natural action of $\Pi_X \times \Pi_X$ on $M \times M$. Thus, we conclude that α commutes with the isomorphism

$$Y \times_K Y \overset{\sim}{\to} Y \times_K Y, \quad (y_1, y_2) \mapsto (y_2, y_1).$$

In particular, the structure of algebraic group on Y is commutative. This completes the proof of Lemma 2.3. □

Proposition 2.4. *Suppose that K is algebraically closed, and X is commutative. For each prime number l, write $T_l(X)$ for the l-adic Tate module associated to X. Then*

$$\Pi_X \xrightarrow{\sim} \prod_{l \in \mathfrak{Primes}} T_l(X).$$

Proof. Let $f : Y \to X$ be a finite étale Galois covering over K. Then it follows from Lemma 2.3 that Y admits a structure of commutative algebraic group over K such that f is an isogeny. Then since $\mathrm{Ker}(f)$ is a finite étale group scheme over K, there exists a positive integer n such that $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(n_Y)$. Thus, $n_Y : Y \to Y$ coincides with the composite of $f : Y \to X$ with a finite étale Galois covering $g : X \to Y$. Moreover, since f is surjective, it follows immediately that $f \circ g : X \to X$ coincides with $n_X : X \to X$. In summary, any finite étale Galois covering of X over K is dominated by n_X for some positive integer n. This completes the proof of Proposition 2.4. $\square$

Proposition 2.5. *Suppose that X is commutative. Write $Z \stackrel{\mathrm{def}}{=} X_{K^{\mathrm{sep}}}$ $(= X_{\overline{K}})$;*

$$\phi_1 : X(K) \longrightarrow H^1(G_K, \Pi_Z)$$

for the natural map obtained by taking the difference between the two sections of $\Pi_X \twoheadrightarrow G_K$ [each of which is well-defined up to composition with an inner automorphism induced by an element of Π_Z] induced by an element of $X(K)$ and $0_X \in X(K)$;

$$\phi_2 : X(K) \longrightarrow \varprojlim_n X(K)/n \cdot X(K) \hookrightarrow H^1(G_K, \Pi_Z)$$

for the composite of the natural homomorphism with an injection induced by the Kummer exact sequence associated to X [cf. Proposition 2.4], where n ranges over the positive integers. Then it holds that $\phi_1 = \phi_2$.

Proof. Let n be a positive integer; $\widetilde{X}$ a pro-universal étale covering of X; $x \in X(K) \subseteq Z(K^{\mathrm{sep}})$; $\tilde{x}$ a point of $\widetilde{X}$ that lies over x; $\sigma \in G_K$. Write $\tilde{0}_X$ for the point of $\widetilde{X}$ determined by the origins of finite étale Galois coverings [cf. Lemma 2.3] that appear in the projective system $\widetilde{X}$; $s_0, s_x : G_K \to \Pi_X$ for the section of the natural surjection $\Pi_X \twoheadrightarrow G_K$ associated to $\tilde{0}_X, (\tilde{x})^\sigma$, respectively; $x^{\frac{1}{n}} \in \mathrm{Ker}(n_Z)(K^{\mathrm{sep}})$ for the n-torsion point determined by $\tilde{x}$. Now observe that

$$s_x(\sigma)s_0(\sigma)^{-1}(x^{\frac{1}{n}}) \;=\; s_x(\sigma)((x^{\frac{1}{n}})^\sigma) \;=\; (x^{\frac{1}{n}})^\sigma.$$

Then the element $\in \mathrm{Ker}(n_Z)(K^{\mathrm{sep}})$ induced by $s_x(\sigma)s_0(\sigma)^{-1} \in \Pi_Z$ [cf. Proposition 2.4] coincides with $(x^{\frac{1}{n}})^{-1} \cdot (x^{\frac{1}{n}})^\sigma \in \mathrm{Ker}(n_Z)(K^{\mathrm{sep}})$. On the other hand, it follows immediately from the various definitions involved that the element $\in H^1(G_K, \mathrm{Ker}(n_Z)(K^{\mathrm{sep}}))$ induced by $\phi_2(x)$ is represented by a map

$$G_K \longrightarrow \mathrm{Ker}(n_Z)(K^{\mathrm{sep}}), \quad \sigma \mapsto (x^{\frac{1}{n}})^{-1} \cdot (x^{\frac{1}{n}})^\sigma.$$

Thus, by varying n, we conclude that $\phi_1 = \phi_2$. This completes the proof of Proposition 2.5. $\square$

3 Weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over stably p-divisibly nonreflexive p-bounded fields of characteristic 0

Let p be a prime number. In the present section, we generalize [24], Theorem F, and prove a weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over stably p-divisibly nonreflexive p-bounded field of characteristic 0. The proof proceeds in a similar way to [24], Theorem F. On the other hand, since we discuss in a pro-p setting, we need a certain technical discussion concerning cyclotomic rigidity [cf. Claim 3.1.A in the proof of Theorem 3.1 below].

Let K be a field of characteristic 0. Write

$$K^{p^\infty} \stackrel{\mathrm{def}}{=} \bigcup_{K \subseteq K^\dagger} \mathbb{Q}((K^\dagger)^{\times p^\infty}) \subseteq \overline{K},$$

where the union ranges over the finite field extensions $K \subseteq K^\dagger$ $(\subseteq \overline{K})$.

Theorem 3.1. *Let Σ be a set of prime numbers that contains p; U, V hyperbolic curves of genus 0 over K;*

$$\phi : \Pi_U^{(\Sigma)} \xrightarrow{\sim} \Pi_V^{(\Sigma)}$$

an isomorphism of profinite groups that lies over the identity automorphism on G_K. Suppose that the following conditions hold:

- *ϕ induces a bijection between the respective sets of cuspidal inertia subgroups.*
- *K is a stably p-divisibly nonreflexive p-bounded field.*
- *K^{p^∞} is a p-divisibly nonreflexive p-bounded field.*

Then there exists an isomorphism of K-schemes

$$U \xrightarrow{\sim} V$$

that induces a bijection between the respective sets of cusps that is compatible with the bijection between the respective sets of cuspidal inertia groups induced by ϕ.

Proof. First, by taking suitable quotients of $\Pi_U^{(\Sigma)}$ and $\Pi_V^{(\Sigma)}$, we may assume without loss of generality that $\Sigma = \{p\}$.

Next, it follows immediately from the theory of Galois descent [cf. the argument applied in the first paragraph of the proof of [24], Theorem 3.5] that we may assume without loss of generality that all cusps of U and V are K-rational. Thus, since ϕ preserves the cuspidal inertia subgroups, it follows immediately, by considering the quotients of $\Pi_U^{(p)}$ and $\Pi_V^{(p)}$ by the closed normal subgroups topologically generated by suitable collections of cuspidal inertia subgroups, that we may also assume without loss of generality that

- $U = \mathbb{P}^1_K \backslash \{0, 1, \lambda, \infty\}$, where $\lambda \in K \setminus \{0, 1\}$;
- $V = \mathbb{P}^1_K \backslash \{0, 1, \mu, \infty\}$, where $\mu \in K \setminus \{0, 1\}$;
- ϕ maps the cuspidal inertia subgroups of $\Pi_U^{(p)}$ associated to $* \in \{0, 1, \infty\}$ to the cuspidal inertia subgroups of $\Pi_V^{(p)}$ associated to $*$. [Note that this implies that ϕ maps the cuspidal inertia subgroups of $\Pi_U^{(p)}$ associated to λ to the cuspidal inertia subgroups of $\Pi_V^{(p)}$ associated to μ.]

Then our goal is to prove that

$$\lambda \; = \; \mu.$$

Write t for the standard coordinate on the projective line $\mathbb{P}^1_K$.

Next, we verify the following assertion:

> Claim 3.1.A: Let $* \in \{0, 1, \lambda, \infty\}$ be an element; $I_* \subseteq \Pi_U^{(p)}$ a cuspidal inertia subgroup associated to $*$. Consider the natural composite
>
> $$h_* : \mathbb{Z}_p(1) \; \xrightarrow{\sim} \; I_* \; \xrightarrow{\sim} \; \phi(I_*) \xleftarrow{\sim} \mathbb{Z}_p(1)$$
>
> — where "(1)" denotes the Tate twist; the first and final isomorphisms are the natural isomorphisms [obtained by considering the action of each cuspidal inertia subgroup on the roots of a uniformizer of the local ring of the compactified curve at the cusp under consideration]; the middle isomorphism is the isomorphism induced by ϕ. Then h_* is the identity automorphism.

Since ϕ induces a bijection between the respective sets of cuspidal inertia subgroups, it follows immediately, by considering suitable quotients of the abelianizations of Δ_U^p and Δ_V^p, that $h_0 = h_1 = h_\lambda = h_\infty$. Thus, it suffices to consider the case where $* = 1$. Write

- $(\mathbb{P}^1_K \supseteq)\; U_p \to U \; (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of U of degree p determined by $t \mapsto (1-t)^p$.
- $(\mathbb{P}^1_K \supseteq)\; V_p \to V \; (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of V of degree p determined by $t \mapsto (1-t)^p$.

Let ζ_p be a primitive p-th root of unity. Then, after possibly replacing K by a suitable finite extension of K and ϕ by the composite of ϕ with the inner automorphism of $\Pi_V^{(p)}$ determined by some element $\in \Delta_V^p$, we obtain an isomorphism of profinite groups

$$\phi_p : \Pi_{U_p}^{(p)} \; \xrightarrow{\sim} \; \Pi_{V_p}^{(p)}$$

such that

- ϕ_p induces the identity automorphism on G_K,
- there exists a bijection $b \in \mathrm{Aut}(\{1, \ldots, p-1\})$ of the set $\{1, \ldots, p-1\}$ such that ϕ_p maps the cuspidal inertia subgroups of $\Pi_{U_p}^{(p)}$ associated to 0, 1, ∞, $1 - \zeta_p^i$, where $i \in \{1, \ldots, p-1\}$, to the cuspidal inertia subgroups of $\Pi_{V_p}^{(p)}$ associated to 0, 1, ∞, $1 - \zeta_p^{b(i)}$, respectively.

By a slight abuse of notation, we shall also denote by ϕ_p the bijection between the cusps of U_p and V_p determined by the bijection between the respective sets of cuspidal inertia subgroups. Let I_p be a cuspidal inertia subgroup of $\Pi_{U_p}^{(p)}$ associated to $1 - \zeta_p$. Thus, since the cusp $1 - \zeta_p$ of U_p maps to the cusp 1 of U, we may assume without loss of generality that $I_p = I_1 \subseteq \Pi_U^{(p)}$. In particular, it suffices to prove that the natural composite

$$\mathbb{Z}_p(1) \; \xrightarrow{\sim} \; I_p \; \xrightarrow{\sim} \; \phi_p(I_p) \xleftarrow{\sim} \mathbb{Z}_p(1)$$

is the identity automorphism. Write

- $\epsilon_p \in \mathbb{Z}_p^\times$ for the element determined by this composite automorphism;
- $\kappa : K^\times \twoheadrightarrow K^\times / K^{\times p^\infty} \hookrightarrow H^1(G_K, \mathbb{Z}_p(1))$ for the Kummer map;
- $Y \overset{\mathrm{def}}{=} \mathbb{P}^1_K \backslash \{0, \infty\}$, $\Delta_Y \overset{\mathrm{def}}{=} \Pi_{Y \times_K \overline{K}}$.

Then it follows from Proposition 2.5 that κ coincides with the composite

$$K^\times \;=\; Y(K) \;\longrightarrow\; H^1(G_K, \Delta_Y) \;\longrightarrow\; H^1(G_K, \mathbb{Z}_p(1))$$

— where the first map is obtained by taking the difference between the two sections of $\Pi_Y \twoheadrightarrow G_K$ [each of which is well-defined up to composition with an inner automorphism induced by an element of Δ_Y] induced by an element of $Y(K)$ and $1 \in Y(K)$; the final map is induced by the natural surjection $\Delta_Y \twoheadrightarrow \Delta_Y^p \overset{\sim}{\to} \mathbb{Z}_p(1)$. Here, we recall that the image of such a section of $\Pi_Y \twoheadrightarrow G_K$ arising from an element of $Y(K)$ may also be thought of as the decomposition group in Π_Y of this element of $Y(K)$.

Next, let x be a cusp of U_p; I_x a cuspidal inertia subgroup of $\Delta^p_{U_p}$ associated to x. Recall that, since I_x is normally terminal in $\Delta^p_{U_p}$ [cf. [16], Proposition 1.2, (ii)], the normalizer $N_{\Pi^{(p)}_{U_p}}(I_x)$ is a decomposition subgroup $\subseteq \Pi^{(p)}_{U_p}$ associated to x. Similarly, since $\phi_p(I_x)$ is normally terminal in $\Delta^p_{V_p}$, the normalizer $N_{\Pi^{(p)}_{V_p}}(\phi_p(I_x))$ is a decomposition subgroup $\subseteq \Pi^{(p)}_{V_p}$ associated to $\phi_p(x)$.

Thus, since ϕ_p maps the cuspidal inertia subgroups of $\Pi^{(p)}_{U_p}$ associated to 1, $1-\zeta_p^i$ to the cuspidal inertia subgroups of $\Pi^{(p)}_{V_p}$ associated to 1, $1-\zeta_p^{b(i)}$, respectively, we conclude [by thinking of U_p and V_p as open subschemes of Y] that, for each $i \in \{1, \ldots, p-1\}$,

$$\epsilon_p \cdot \kappa(1-\zeta_p^i) \;=\; \kappa(1-\zeta_p^{b(i)}).$$

Let us note that

$$p \;=\; \prod_{1 \le i \le p-1} 1-\zeta_p^i.$$

Then we have

$$\epsilon_p \cdot \kappa(p) \;=\; \kappa(p).$$

On the other hand, since K is p-bounded, for each positive integer n, it holds that $p^n \notin K^{\times p^\infty}$. Then it follows that $\kappa(p)$ is not a torsion element, hence that $\mathbb{Z}_p \cdot \kappa(p) \overset{\sim}{\to} \mathbb{Z}_p$, which implies that $\epsilon_p = 1$. This completes the proof of Claim 3.1.A.

Next, we suppose that

$$\lambda \;\neq\; \mu.$$

In light of Claim 3.1.A, it follows by considering the Kummer classes of λ, μ, $1-\lambda$, and $1-\mu$ that there exist $a, b \in K^{\times p^\infty}$ such that

$$\mu \;=\; a \cdot \lambda, \quad 1-\mu \;=\; b \cdot (1-\lambda).$$

Since $\lambda \neq \mu$, it follows immediately that $a \neq 1$, $b \neq 1$, and $a \neq b$. In particular,

$$\lambda \;=\; \frac{1-b}{a-b} \;\in\; K^{p^\infty}.$$

Recall our assumption that K^{p^∞} ($\subseteq \overline{K}$) is a p-divisibly nonreflexive field. In particular, if $\lambda \in (K^{p^\infty})^{\times p^\infty}$, then there exist a positive integer m and a p^m-th root $\lambda^{\frac{1}{p^m}}$ of λ such that $1-\lambda^{\frac{1}{p^m}} \notin (K^{p^\infty})^{\times p^\infty}$. Note that even if we replace K by a finite extension field of K, the field K^{p^∞} does not change. Write ($\mathbb{P}^1_K \supseteq$) $U_{p^m} \to U$ ($\subseteq \mathbb{P}^1_K$) for the connected finite étale covering determined by the assignment $t \mapsto (1-t)^{p^m}$. Then, by replacing λ, K, U, by $1-\lambda^{\frac{1}{p^m}}$, a finite extension field of K, a suitable partial compactification of U_{p^m}, if necessary, we may assume without loss of generality that $\lambda \notin (K^{p^\infty})^{\times p^\infty}$. Let n be a positive integer such that some p^n-th root of λ is not an element of K^{p^∞}. Fix such a p^n-th root

$$\lambda^{\frac{1}{p^n}} \;\notin\; K^{p^\infty}.$$

Write

- $(\mathbb{P}^1_K \supseteq)\ U' \to U\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of U of degree p^n determined by $t \mapsto t^{p^n}$.
- $(\mathbb{P}^1_K \supseteq)\ V' \to V\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of V of degree p^n determined by $t \mapsto t^{p^n}$.

Note that the open subgroup $\Delta^p_{U'} \subseteq \Delta^p_U$ determined by the covering $U' \to U$ may be characterized as the unique normal open subgroup of index p^n such that

$$I_1 \subseteq \Delta^p_{U'}, \quad I_\lambda \subseteq \Delta^p_{U'}.$$

The open subgroup $\Delta^p_{V'} \subseteq \Delta^p_V$ determined by the covering $V' \to V$ admits a similar characterization. Thus, since ϕ is compatible with these characterizations, we conclude that, after possibly replacing K by a suitable finite extension of K and ϕ by the composite of ϕ with the inner automorphism of $\Pi^{(p)}_V$ determined by some element $\in \Delta^p_V$, we obtain an isomorphism of profinite groups

$$\phi_n : \Pi^{(p)}_{U'} \overset{\sim}{\to} \Pi^{(p)}_{V'}$$

such that

- ϕ_n induces the identity automorphism on G_K,
- ϕ_n maps the cuspidal inertia subgroups of $\Pi^{(p)}_{U'}$ associated to $*' \in \{0, 1, \infty\}$ to the cuspidal inertia subgroups of $\Pi^{(p)}_{V'}$ associated to $*'$,
- ϕ_n maps the cuspidal inertia subgroups of $\Pi^{(p)}_{U'}$ associated to $\lambda^{\frac{1}{p^n}}$ to the cuspidal inertia subgroups of $\Pi^{(p)}_{V'}$ associated to some p^n-th root $\mu^{\frac{1}{p^n}}$ of μ.

Let $M\ (\subseteq \overline{K})$ be a finite extension field of K such that $\lambda^{\frac{1}{p^n}}, \mu^{\frac{1}{p^n}} \in M$. Write

- $U'' \overset{\text{def}}{=} \mathbb{P}^1_M \backslash \{0, 1, \lambda^{\frac{1}{p^n}}, \infty\}$;
- $V'' \overset{\text{def}}{=} \mathbb{P}^1_M \backslash \{0, 1, \mu^{\frac{1}{p^n}}, \infty\}$.

Since $\lambda^{\frac{1}{p^n}} \neq \mu^{\frac{1}{p^n}}$ [by our assumption that $\lambda \neq \mu$], it follows, by considering the isomorphism

$$\Pi^{(p)}_{U''} \overset{\sim}{\to} \Pi^{(p)}_{V''}$$

induced by ϕ_n and applying a similar argument to the argument applied above to λ and μ, that

$$\lambda^{\frac{1}{p^n}} \in K^{p^\infty}.$$

This contradicts our choice of $\lambda^{\frac{1}{p^n}}$. Thus, we conclude that $\lambda = \mu$. This completes the proof of Theorem 3.1. $\square$

Remark 3.1.1. In the notation of Theorem 3.1, suppose that K is a stably p-$\times\mu$-indivisible field of characteristic 0 [cf. Remark 1.4.3, (ii)]. Then it follows immediately from [24], Lemma D, that K^{p^∞} $(\subseteq \mathbb{Q}^{\text{ab}})$ is also a stably p-$\times\mu$-indivisible field of characteristic 0, hence, in particular, a p-divisibly nonreflexive p-bounded field. Thus, Theorem 3.1 may be regarded as a generalization of [24], Theorem F.

Corollary 3.2. *Let M be a mixed characteristic valuation field of residue characteristic p whose value group is p-reduced and of rank 1; L a stably μ_{p^∞}-finite Galois extension of M; $\Sigma \subseteq \mathfrak{Primes}$ a subset that contains p; U, V hyperbolic curves of genus 0 over K;*

$$\phi : \Pi_U^{(\Sigma)} \stackrel{\sim}{\to} \Pi_V^{(\Sigma)}$$

an isomorphism of profinite groups such that ϕ lies over the identity automorphism on G_K. Suppose that K is isomorphic to a subfield of L. Then there exists an isomorphism of K-schemes

$$U \stackrel{\sim}{\to} V$$

that induces a bijection between the respective sets of cusps that is compatible with the bijection between the respective sets of cuspidal inertia groups induced by ϕ.

Proof. First, it follows from Lemma 1.8 that K is a stably p-divisibly nonreflexive p-bounded field. Next, since L is stably μ_{p^∞}-finite, the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open. Then it follows from [16], Corollary 2.7, (i), that ϕ induces a bijection between the respective sets of cuspidal inertia groups. On the other hand, it follows immediately from the definition of the field K^{p^∞} that

$$(K^{p^\infty})^{\times p^\infty} \subseteq \bigcup_{K \subseteq K^\dagger} (K^\dagger)^{\times p^\infty},$$

where $K \subseteq K^\dagger$ ranges over the finite field extensions. Therefore, by applying Lemmas 1.7, (ii); 1.8, we observe that K^{p^∞} is a p-divisibly nonreflexive p-bounded field. Thus, Corollary 3.2 follows immediately from Theorem 3.1. $\square$

Remark 3.2.1. Note that it follows immediately from the various definitions involved that, in the statement of Corollary 3.2, one may take K as any subfield of the p-adic completion of the maximal tamely ramified extension field of a mixed characteristic Henselian discrete valuation field of residue characteristic p. On the other hand, we recall that any Noetherian local domain is dominated by a discrete valuation ring that can be embedded into its Henselization. In particular, in the statement of Corollary 3.2, one may take K as the field of fractions of any mixed characteristic Noetherian local domain of residue characteristic p.

4 Geometrically pro-p Grothendieck Conjecture for hyperbolic curves of genus 0 over mixed characteristic valuation fields of rank 1 of residue characteristic p with p-reduced value groups

Let p be a prime number. In the present section, we apply the weak version of the Grothendieck Conjecture [cf. Theorem 3.1; Corollary 3.2] to prove the geometrically pro-p Grothendieck Conjecture for hyperbolic curves of genus 0 over subfields of certain mixed characteristic valuation fields [cf. for instance, the p-adic completion of [possibly, infinite] tamely ramified extension fields of mixed characteristic complete discrete valuation fields of residue characteristic p].

Let K be a field of characteristic 0.

Definition 4.1. Let U be a hyperbolic curve of genus 0 over K such that every cusp of U is K-rational. Write $\mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p) \subseteq \mathrm{Out}(\Delta_U^p)$ for the subgroup of outer automorphisms that induce the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Δ_U^p;

$$\rho : G_K \longrightarrow \mathrm{Out}(\Delta_U^p)$$

for the pro-p outer representation naturally determined by the natural homotopy exact sequence

$$1 \longrightarrow \Delta_U^p \longrightarrow \Pi_U^{(p)} \longrightarrow G_K \longrightarrow 1.$$

Lemma 4.2 ([21], Propositions 2.2.3, 2.2.4)**.** *In the notation of Definition 4.1, suppose that the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open. Write*

$$\chi_p : \mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p) \longrightarrow \mathbb{Z}_p^\times$$

for the p-adic cyclotomic character [obtained by considering the natural actions on cuspidal inertia subgroups ($\xrightarrow{\sim} \mathbb{Z}_p$) of Δ_U^p]. Then the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p)}(\mathrm{Im}(\rho)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p) \xrightarrow{\chi_p} \mathbb{Z}_p^\times$$

is injective.

Proof. Lemma 4.2 follows immediately from [21], Proposition 2.2.3, together with the latter half of the proof of [21], Proposition 2.2.4. □

Theorem 4.3. *Let U, V be hyperbolic curves of genus 0 over K. Write*

$$\mathrm{Isom}_K(U, V)$$

for the set of K-isomorphisms between U and V;

$$\mathrm{Isom}_{G_K}(\Pi_U^{(p)}, \Pi_V^{(p)})/\mathrm{Inn}(\Delta_V^p)$$

for the set of continuous group isomorphisms $\Pi_U^{(p)} \xrightarrow{\sim} \Pi_V^{(p)}$ that lies over the identity automorphism of G_K, considered up to composition with an inner automorphism arising from an element $\in \Delta_V^p$. Let M be a mixed characteristic valuation field of residue characteristic p whose value group is p-reduced and of rank 1; L a stably μ_{p^∞}-finite Galois extension of M. Suppose that K is isomorphic to a subfield of L. Then the natural map

$$\mathrm{Isom}_K(U, V) \longrightarrow \mathrm{Isom}_{G_K}(\Pi_U^{(p)}, \Pi_V^{(p)})/\mathrm{Inn}(\Delta_V^p)$$

is bijective.

Proof. The injectivity portion follows immediately from the structure of abelianization of Δ_V^p, together with the fact that an automorphism of the projective line over $\overline{K}$ that fixes three points is the identity automorphism.

Next, we consider the surjectivity portion. By applying Galois descent, we may assume without loss of generality that every cusp of U and V is K-rational. Let

$$\sigma \in \mathrm{Isom}_{G_K}(\Pi_U^{(p)}, \Pi_V^{(p)})/\mathrm{Inn}(\Delta_V^p)$$

be an element. Our goal is to prove that σ arises from an element $\in \mathrm{Isom}_K(U, V)$ via the natural map. Then, by applying Corollary 3.2, we may assume without loss of generality that $U = V$, and σ induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Δ_U^p. Here, we note that since Δ_U^p is center-free, there exists a natural isomorphism

$$\mathrm{Aut}_{G_K}(\Pi_U^{(p)})/\mathrm{Inn}(\Delta_U^p) \xrightarrow{\sim} Z_{\mathrm{Out}(\Delta_U^p)}(\mathrm{Im}(\rho))$$

[cf. Definition 4.1]. With respect to this isomorphism σ determines an element of $Z_{\mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p)}(\mathrm{Im}(\rho))$. Recall from Claim 3.1.A in the proof of Theorem 3.1 that $\chi_p(\sigma|_{\Delta_U^p}) = 1$, where

$$\chi_p : \mathrm{Out}^{|\mathrm{C}|}(\Delta_U^p) \longrightarrow \mathbb{Z}_p^\times$$

denotes the p-adic cyclotomic character. On the other hand, it follows from our assumption that the [usual] p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open. Thus, we conclude from Lemma 4.2 that $\sigma = 1$. This completes the proof of Theorem 4.3. $\square$

Remark 4.3.1. It appears to the author that Theorem 4.3 suggests that much wider class of p-adic fields may be considered as base fields of anabelian geometry. On the other hand, at the time of writing of the present paper, the author does not know

> *whether or not the analogous result of Theorem 4.3 holds in the case where U and V are hyperbolic curves of higher genus.*

The author hopes to be able to address such an issue in the future paper.

5 Algebraicity of certain set-theoretic bijections between the underlying sets of mixed characteristic valuation fields of rank 1 of residue characteristic p with p-reduced value groups

Let p be a prime number. In the present section, we prove an elementary lemma concerning the algebraicity of certain set-theoretic bijections between subsets of the underlying sets of mixed characteristic valuation fields of rank 1 of residue characteristic p whose value groups are p-reduced, which will be applied to prove geometrically/arithmetically pro-p absolute Grothendieck Conjecture-type results in the next section.

Definition 5.1. Let K be a valuation field. Write $\mathcal{O}_K$ for the ring of integers of K; $\mathfrak{m}_K$ for the maximal ideal of $\mathcal{O}_K$;

$$M_K \stackrel{\mathrm{def}}{=} \mathfrak{m}_K \cup 1 + \mathfrak{m}_K;$$
$$r_K : M_K \stackrel{\sim}{\to} M_K$$

for the *reflection map*, i.e., the bijection that maps $M_K \ni x \mapsto 1 - x \in M_K$.

Remark 5.1.1. In the notation of Definition 5.1, the subset $M_K \subseteq \mathcal{O}_K$ is multiplicatively closed, i.e., for each $x, y \in M_K$, it holds that $xy \in M_K$.

Remark 5.1.2. In the notation of Definition 5.1, suppose that the value group of K is p-reduced. Then it follows immediately from the various definitions involved that

$$K^{\times p^\infty} \subseteq \mathcal{O}_K^\times.$$

Suppose, moreover, that K is of rank 1, and the residue field of K is of characteristic p. Then it follows immediately from Lemma 1.7, (ii), that the natural composite map

$$K^{\times p^\infty} \subseteq \mathcal{O}_K^\times \twoheadrightarrow (\mathcal{O}_K/\mathfrak{m}_K)^\times$$

is *injective*.

Next, let $K^\dagger$, $K^\ddagger$ be mixed characteristic valuation fields of rank 1 of residue characteristic p whose value groups are p-reduced;

$$\alpha : M_{K^\dagger} \overset{\sim}{\to} M_{K^\ddagger}$$

a set-theoretic bijection satisfying the following conditions:

(1) $\alpha(\mathfrak{m}_{K^\dagger}) = \mathfrak{m}_{K^\ddagger}$.

(2) For each $x, y \in M_{K^\dagger}$, there exists an element $z \in (K^\ddagger)^{\times p^\infty}$ such that

$$\alpha(xy) \ = \ z \cdot \alpha(x) \cdot \alpha(y).$$

(3) α fits into a commutative diagram of sets

$$\begin{tikzcd} M_{K^\dagger} \arrow[r, "\sim", "\alpha"'] \arrow[d, "r_{K^\dagger}"', "\wr"] & M_{K^\ddagger} \arrow[d, "r_{K^\ddagger}"', "\wr"] \\ M_{K^\dagger} \arrow[r, "\sim", "\alpha"'] & M_{K^\ddagger}. \end{tikzcd}$$

In the remainder of the present section [except Corollary 5.10 below], we investigate an algebraicity of the set-theoretic bijection α.

Lemma 5.2. *The following hold:*

(i) Let $x, y \in \mathfrak{m}_{K^\dagger}$ $(\subseteq M_{K^\dagger})$ be elements. Then it holds that

$$\alpha(x + y - xy) \ = \ \alpha(x) + \alpha(y) - \alpha(x) \cdot \alpha(y).$$

(ii) $\alpha(0) = 0$, and $\alpha(1) = 1$.

Proof. First, we verify assertion (i). Note that it follows immediately from condition (1) that $\alpha(1+\mathfrak{m}_{K^\dagger}) = 1+\mathfrak{m}_{K^\ddagger}$. Then it follows immediately from condition (2), together with Remark 5.1.2, that the restriction of α to $1 + \mathfrak{m}_{K^\dagger}$ is a multiplicative bijection. In particular, since $1 - x, 1 - y \in 1 + \mathfrak{m}_{K^\dagger}$, it holds that

$$\alpha((1 - x)(1 - y)) \ = \ \alpha(1 - x) \cdot \alpha(1 - y).$$

On the other hand, it follows from condition (3) that

$$\alpha(1 - x) \cdot \alpha(1 - y) \ = \ (1 - \alpha(x))(1 - \alpha(y)) \ = \ 1 - \alpha(x) - \alpha(y) + \alpha(x) \cdot \alpha(y),$$

and

$$\alpha((1 - x)(1 - y)) \ = \ \alpha(1 - (x + y - xy)) \ = \ 1 - \alpha(x + y - xy).$$

Thus, by combining these equalities, we conclude that

$$\alpha(x + y - xy) \ = \ \alpha(x) + \alpha(y) - \alpha(x) \cdot \alpha(y),$$

as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). If we take x and y to be 0 in the equality of assertion (i), then $\alpha(0) = \alpha(0)^2$. Then it follows immediately from condition (1) that $\alpha(0) = 0$. Thus, we conclude from condition (3) that $\alpha(1) = 1$. This completes the proof of assertion (ii), hence of Lemma 5.2. $\square$

Lemma 5.3. *Let $x, y \in M_{K^\dagger}$ be such that $v_p(y) > v_p(x)$. Then it holds that*

$$v_p(\alpha(y)) \ > \ v_p(\alpha(x)).$$

Proof. Indeed, since $v_p(y) > v_p(x)$, it holds that $x^{-1}y \in \mathfrak{m}_{K^\dagger}$ ($\subseteq M_{K^\dagger}$). Then it follows immediately from condition (2) that there exists an element $z \in (K^\ddagger)^{\times p^\infty}$ such that

$$\alpha(y) \ = \ \alpha(x \cdot x^{-1}y) \ = \ z \cdot \alpha(x) \cdot \alpha(x^{-1}y).$$

Thus, we conclude from Remark 5.1.2 that $v_p(\alpha(y)) = v_p(\alpha(x)) + v_p(\alpha(x^{-1}y))$, hence that $v_p(\alpha(y)) > v_p(\alpha(x))$ [cf. condition (1)]. This completes the proof of Lemma 5.3. $\square$

Lemma 5.4. *Let $x \in M_{K^\dagger}$, $y \in 1 + \mathfrak{m}_{K^\dagger}$ be elements. Then it holds that*

$$\alpha(xy) \ = \ \alpha(x) \cdot \alpha(y).$$

Proof. First, suppose that $x \in 1 + \mathfrak{m}_{K^\dagger}$. Then it follows immediately from condition (2), together with Remark 5.1.2, that $\alpha(xy) = \alpha(x) \cdot \alpha(y)$. Thus, in light of Lemma 5.2, (ii), we may assume without loss of generality that

$$x \ \in \ \mathfrak{m}_{K^\dagger} \setminus \{0\}.$$

Next, let $w \in \mathfrak{m}_{K^\dagger}$ be such that $v_p(w) > v_p(x)$. Then it follows immediately from conditions (2), (3), that there exists an element $z \in (K^\ddagger)^{\times p^\infty}$ such that

$$\begin{aligned} \alpha(x + w - xw) \ &= \ \alpha(x \cdot (1 + x^{-1}w - w)) \\ &= \ z \cdot \alpha(x) \cdot \alpha(1 + x^{-1}w - w) \\ &= \ z \cdot \alpha(x) \cdot \alpha(1 - x^{-1}w \cdot (x - 1)) \\ &= \ z \cdot \alpha(x) \cdot (1 - \alpha(x^{-1}w \cdot (x - 1))). \end{aligned}$$

On the other hand, by applying Lemma 5.2, (i), we have

$$\alpha(x + w - xw) \ = \ \alpha(x) + \alpha(w) - \alpha(x) \cdot \alpha(w).$$

Thus, by combining the above equalities, we conclude that

$$(1 - z) \cdot \alpha(x) \ = \ \alpha(x) \cdot \alpha(w) - \alpha(w) - z \cdot \alpha(x) \cdot \alpha(x^{-1}w \cdot (x - 1)).$$

Next, in light of Lemma 5.3, since $v_p(w) > v_p(x)$, we observe that

$$v_p(\alpha(x) \cdot \alpha(x^{-1}w \cdot (x - 1))) \ > \ v_p(\alpha(x)), \quad v_p(\alpha(x) \cdot \alpha(w)) \ > \ v_p(\alpha(w)) \ > \ v_p(\alpha(x)).$$

In particular, it holds that $1 - z \in \mathfrak{m}_{K^\ddagger}$. However, this immediately implies that $z = 1$ [cf. Remark 5.1.2]. In summary, it follows from the second equality in the second display in the present proof that

$$\alpha(x \cdot (1 + x^{-1}w - w)) \ = \ \alpha(x) \cdot \alpha(1 + x^{-1}w - w).$$

Then, by replacing w by $\frac{x(1-y)}{x-1}$ [where we note that $v_p(\frac{x(1-y)}{x-1}) > v_p(x)$], we obtain the equality

$$\alpha(xy) \ = \ \alpha(x) \cdot \alpha(y),$$

as desired. This completes the proof of Lemma 5.4. $\square$

Lemma 5.5. *Let $x \in \mathfrak{m}_{K^\dagger}$, $y \in 1 + \mathfrak{m}_{K^\dagger}$ be elements. Then it holds that*

$$\alpha(x + y - xy) \ = \ \alpha(x) + \alpha(y) - \alpha(x)\alpha(y).$$

Proof. In light of Lemma 5.4, Lemma 5.5 follows from a similar argument to the argument applied in the proof of Lemma 5.2, (i). $\square$

Lemma 5.6. *Let $x \in \mathfrak{m}_{K^\dagger}$, $y \in 1 + \mathfrak{m}_{K^\dagger}$ be elements. Then it holds that*

$$\alpha(x + y) \ = \ \alpha(x) + \alpha(y).$$

Proof. First, since $x \in \mathfrak{m}_{K^\dagger}$, $y \in 1 + \mathfrak{m}_{K^\dagger}$, we observe that $x + y, 1 - \frac{xy}{x+y} \in 1 + \mathfrak{m}_{K^\dagger}$. Then it follows immediately from Lemma 5.4, together with condition (3), that

$$\begin{aligned}
\alpha(x + y - xy) \ &= \ \alpha\left((x + y) \cdot \left(1 - \frac{xy}{x + y}\right)\right) \\
&= \ \alpha(x + y) \cdot \alpha\left(1 - \frac{xy}{x + y}\right) \\
&= \ \alpha(x + y) \cdot \left(1 - \alpha\left(\frac{xy}{x + y}\right)\right) \\
&= \ \alpha(x + y) - \alpha(xy).
\end{aligned}$$

Thus, we conclude from Lemmas 5.4, 5.5, that

$$\alpha(x + y) \ = \ \alpha(x) + \alpha(y).$$

This completes the proof of Lemma 5.6. $\square$

Lemma 5.7. *Let $x \in \mathfrak{m}_{K^\dagger}$ be an element. Then it holds that*

$$\alpha(-x) \ = \ -\alpha(x).$$

Proof. Note that $-x \in \mathfrak{m}_{K^\dagger}$, $1 \in 1 + \mathfrak{m}_{K^\dagger}$. Then it follows immediately from Lemmas 5.2, (ii); 5.6, together with condition (3), that $1 - \alpha(x) = \alpha(-x + 1) = \alpha(-x) + \alpha(1) = \alpha(-x) + 1$. Thus, we conclude that $\alpha(-x) = -\alpha(x)$. This completes the proof of Lemma 5.7. $\square$

Proposition 5.8. *Let $x, y \in \mathfrak{m}_{K^\dagger}$ be elements. Then the following hold:*

(i) $\alpha(x + y) = \alpha(x) + \alpha(y)$.

(ii) $\alpha(xy) = \alpha(x) \cdot \alpha(y)$.

Proof. First, we verify assertion (i). Observe that it follows immediately from Lemmas 5.2, (ii); 5.6, that

$$\begin{aligned}
1 + \alpha(x + y) \ &= \ \alpha(1 + x + y) \\
&= \ \alpha(x) + \alpha(1 + y) \\
&= \ \alpha(x) + 1 + \alpha(y).
\end{aligned}$$

Thus, we conclude that $\alpha(x + y) = \alpha(x) + \alpha(y)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that $x, y, -xy \in \mathfrak{m}_{K^\dagger}$. Then it follows immediately from Lemmas 5.2, (i); 5.7, together with assertion (i), that

$$\begin{aligned} \alpha(x) + \alpha(y) - \alpha(x) \cdot \alpha(y) \ &= \ \alpha(x + y - xy) \\ &= \ \alpha(x) + \alpha(y) + \alpha(-xy) \\ &= \ \alpha(x) + \alpha(y) - \alpha(xy). \end{aligned}$$

In particular, it holds that $\alpha(xy) = \alpha(x) \cdot \alpha(y)$. This completes the proof of assertion (ii), hence of Proposition 5.8. $\square$

Lemma 5.9. *It holds that*

$$\alpha(p) \ = \ p.$$

Proof. Note that $p \in \mathfrak{m}_{K^\dagger}$. Then, by applying Proposition 5.8, (i), repeatedly, we observe that $\alpha(p^2) = p \cdot \alpha(p)$. On the other hand, it follows from Proposition 5.8, (ii), that $\alpha(p^2) = \alpha(p)^2$. Thus, since $\alpha(p) \neq 0$ [cf. Lemma 5.2, (ii)], we conclude that $\alpha(p) = p$. This completes the proof of Lemma 5.9. $\square$

Corollary 5.10. *Let $K^\dagger$, $K^\ddagger$ be mixed characteristic Henselian rank 1 valuation fields of residue characteristic p whose value groups are p-reduced. For each $* \in \{\dagger, \ddagger\}$, write $K^* \subseteq (K^*)^p$ $(\subseteq \overline{K}^*)$ for the maximal pro-p extension. Let*

$$\sigma : G_{K^\dagger} \ \stackrel{\sim}{\to} \ G_{K^\ddagger} \quad (\text{respectively, } \sigma^p : G^p_{K^\dagger} \ \stackrel{\sim}{\to} \ G^p_{K^\ddagger} \ )$$

be an isomorphism of profinite groups;

$$\overline{\alpha} : M_{\overline{K}^\dagger} \ \stackrel{\sim}{\to} \ M_{\overline{K}^\ddagger} \quad (\text{respectively, } \overline{\alpha}^p : M_{(K^\dagger)^p} \ \stackrel{\sim}{\to} \ M_{(K^\ddagger)^p} \ )$$

a set-theoretic bijection. Suppose that the following conditions hold:

(1) $\overline{\alpha}$ and σ (respectively, $\overline{\alpha}^p$ and σ^p) are compatible with the respective natural actions of $G_{K^\dagger}$, $G_{K^\ddagger}$ (respectively, $G^p_{K^\dagger}$, $G^p_{K^\ddagger}$) on $M_{\overline{K}^\dagger}$, $M_{\overline{K}^\ddagger}$ (respectively, $M_{(K^\dagger)^p}$, $M_{(K^\ddagger)^p}$).

(2) $\overline{\alpha}(\mathfrak{m}_{\overline{K}^\dagger}) = \mathfrak{m}_{\overline{K}^\ddagger}$ (respectively, $\overline{\alpha}^p(\mathfrak{m}_{(K^\dagger)^p}) = \mathfrak{m}_{(K^\ddagger)^p}$).

(3) Let $K^\dagger \subseteq L^\dagger$ $(\subseteq \overline{K}^\dagger)$ (respectively, $K^\dagger \subseteq L^\dagger$ $(\subseteq (K^\dagger)^p)$) be a finite field extension. Write $K^\ddagger \subseteq L^\ddagger$ $(\subseteq \overline{K}^\ddagger)$ (respectively, $K^\ddagger \subseteq L^\ddagger$ $(\subseteq (K^\ddagger)^p)$) for the finite field extension determined by the open subgroup $\sigma(G_{L^\dagger}) \subseteq G_{K^\ddagger}$ (respectively, $\sigma^p(G^p_{L^\dagger}) \subseteq G^p_{K^\ddagger}$). Then, for each $x, y \in M_{L^\dagger}$, there exists an element $z \in (L^\ddagger)^{\times p^\infty}$ such that

$$\overline{\alpha}(xy) \ = \ z \cdot \overline{\alpha}(x) \cdot \overline{\alpha}(y) \quad (\text{respectively, } \overline{\alpha}^p(xy) \ = \ z \cdot \overline{\alpha}^p(x) \cdot \overline{\alpha}^p(y) \ ).$$

(4) $\overline{\alpha}$ (respectively, $\overline{\alpha}^p$) fits into a commutative diagram of sets

$$\begin{tikzcd} M_{\overline{K}^\dagger} \arrow[r, "\sim", "\overline{\alpha}"'] \arrow[d, "r_{\overline{K}^\dagger}"', "\wr"] & M_{\overline{K}^\ddagger} \arrow[d, "r_{\overline{K}^\ddagger}"', "\wr"] \\ M_{\overline{K}^\dagger} \arrow[r, "\sim", "\overline{\alpha}"'] & M_{\overline{K}^\ddagger} \end{tikzcd}$$

(respectively,

$$\begin{tikzcd} M_{(K^\dagger)^p} \arrow[r, "\sim", "\overline{\alpha}^p"'] \arrow[d, "r_{(K^\dagger)^p}"', "\wr"] & M_{(K^\dagger)^p} \arrow[d, "r_{(K^\dagger)^p}"', "\wr"] \\ M_{(K^\dagger)^p} \arrow[r, "\sim", "\overline{\alpha}^p"'] & M_{(K^\ddagger)^p}). \end{tikzcd}$$

Then σ (respectively, σ^p) arises from a [uniquely determined] commutative diagram of fields

$$\begin{array}{ccc} \overline{K}^{\dagger} & \xleftarrow{\sim} & \overline{K}^{\ddagger} \\ \uparrow & & \uparrow \\ K^{\dagger} & \xleftarrow{\sim} & K^{\ddagger} \end{array}$$

(respectively,

$$\begin{array}{ccc} (K^{\dagger})^p & \xleftarrow{\sim} & (K^{\ddagger})^p \\ \uparrow & & \uparrow \\ K^{\dagger} & \xleftarrow{\sim} & K^{\ddagger}), \end{array}$$

where the vertical arrows denote the natural injections.

Proof. Since the proof of the resp'd case is similar to the proof of the non-resp'd case, we verify the non-resp'd case only. Let $x \in \overline{K}^{\dagger}$ be an element; i a positive integer such that $p^i x \in \mathfrak{m}_{\overline{K}^{\dagger}}$. Write $\overline{\beta}(x) \overset{\text{def}}{=} \frac{\overline{\alpha}(p^i x)}{p^i} \in \overline{K}^{\ddagger}$. Observe that it follows immediately from Proposition 5.8, (ii); Lemma 5.9, that $\overline{\beta}(x)$ is independent of the choice of i. Then, by assigning x to $\overline{\beta}(x)$, we obtain a set-theoretic bijection

$$\overline{\beta} : \overline{K}^{\dagger} \xrightarrow{\sim} \overline{K}^{\ddagger}$$

that is compatible with $\sigma : G_{K^{\dagger}} \xrightarrow{\sim} G_{K^{\ddagger}}$ and the respective natural actions of $G_{K^{\dagger}}$, $G_{K^{\ddagger}}$ on $\overline{K}^{\dagger}$, $\overline{K}^{\ddagger}$ [cf. condition (1)]. Thus, in light of conditions (2), (3), (4), we conclude from Proposition 5.8; Lemma 5.9, together with the construction of $\overline{\beta}$, that $\overline{\beta}$ is an isomorphism of fields. Finally, since $\overline{\beta}$ is compatible with σ with respect to the natural actions, we observe that σ arises from $\overline{\beta}$, as desired. This completes the proof of Corollary 5.10. $\square$

Remark 5.10.1. In light of the automatic algebraicity of certain exotic set-theoretic objects, the algebraicity criterion proved in this section shares a similar spirit with the algebraicity criterion proved in [26], §3, via the notion of quasi-rational functions.

6 Pro-p absolute Grothendieck Conjecture for hyperbolic curves of genus 0 over mixed characteristic Henselian discrete valuation fields of residue characteristic p

Let p be a prime number. In the present section, by applying Corollary 5.10, we prove the geometrically pro-p version of the absolute Grothendieck Conjecture for hyperbolic curves of genus 0 over mixed characteristic Henselian discrete valuation fields of residue characteristic p under the assumption on the preservation of the decomposition groups associated to closed points. Note that since we consider arbitrary mixed characteristic [possibly, non-complete!] Henselian discrete valuation field, this assumption may not be dropped in general. Moreover, we also discuss an arithmetically pro-p analogue of these results under the assumption that the base fields contain a primitive p-th root of unity.

Definition 6.1. Let K be a field of characteristic 0. Then:

(i) We shall write

$$\widehat{K}^{\times} \stackrel{\text{def}}{=} \varprojlim_{n\geq 1} K^{\times}/(K^{\times})^{p^n}; \quad \Lambda_K \stackrel{\text{def}}{=} \varprojlim_{n\geq 1} \mu_{p^n}(\overline{K}); \quad X_K \stackrel{\text{def}}{=} \mathbb{P}^1_K\backslash\{0,1,\infty\},$$

where n ranges over the positive integers.

(ii) We shall write

$$r^{(p)}_{X_K} : \Pi^{(p)}_{X_K} \xrightarrow{\sim} \Pi^{(p)}_{X_K}$$

for the $\Delta^p_{X_K}$-outer automorphism that lies over the identity automorphism of G_K determined by the automorphism $X_K \xrightarrow{\sim} X_K$ over K that maps $(0,1,\infty) \mapsto (1,0,\infty)$.

(iii) Suppose that $\zeta_p \in K$. [Thus, we have a natural exact sequence $1 \to \Delta^p_{X_K} \to \Pi^p_{X_K} \to G^p_K \to 1$ of profinite groups.] Then we shall write

$$r^p_{X_K} : \Pi^p_{X_K} \xrightarrow{\sim} \Pi^p_{X_K}$$

for the $\Delta^p_{X_K}$-outer automorphism that lies over the identity automorphism of G^p_K determined by the automorphism $X_K \xrightarrow{\sim} X_K$ over K that maps $(0,1,\infty) \mapsto (1,0,\infty)$.

(iv) We shall write

$$\Pi^{(p\text{-ab})}_{X_K}$$

for the quotient of $\Pi^{(p)}_{X_K}$ by the kernel of the natural surjection $\Delta^p_{X_K} \twoheadrightarrow (\Delta^p_{X_K})^{\text{ab}}$.

(v) Suppose that $\zeta_p \in K$. Then we shall write

$$\Pi^{p\text{-ab}}_{X_K}$$

for the quotient of $\Pi^p_{X_K}$ by the kernel of the natural surjection $\Delta^p_{X_K} \twoheadrightarrow (\Delta^p_{X_K})^{\text{ab}}$.

Proposition 6.2. *Let $K^{\dagger}$, $K^{\ddagger}$ be mixed characteristic Henselian discrete valuation fields of residue characteristic p. Then the following hold:*

(i) Let

$$f : \Pi^{(p)}_{X_{K^{\dagger}}} \xrightarrow{\sim} \Pi^{(p)}_{X_{K^{\ddagger}}}$$

be an isomorphism of profinite groups. Then f induces a bijection between the respective sets of cuspidal inertia subgroups. In particular, this bijection induces a bijection

$$f_c : \{0,1,\infty\} \xrightarrow{\sim} \{0,1,\infty\}.$$

Suppose that f_c is the identity automorphism. Then it holds that

$$r^{(p)}_{X_{K^{\ddagger}}} \circ f = f \circ r^{(p)}_{X_{K^{\dagger}}},$$

where we regard the both sides of the equality as $\Delta^p_{X_{K^{\ddagger}}}$-outer isomorphisms.

(ii) Suppose that $\zeta_p \in K^\dagger$, and $\zeta_p \in K^\ddagger$. Let

$$f^p : \Pi^p_{X_{K^\dagger}} \xrightarrow{\sim} \Pi^p_{X_{K^\ddagger}}$$

be an isomorphism of profinite groups. Then f^p induces a bijection between the respective sets of cuspidal inertia subgroups. In particular, this bijection induces a bijection

$$f^p_c : \{0, 1, \infty\} \xrightarrow{\sim} \{0, 1, \infty\}.$$

Suppose that f^p_c is the identity automorphism. Then it holds that

$$r^p_{X_{K^\ddagger}} \circ f^p = f^p \circ r^p_{X_{K^\dagger}},$$

where we regard the both sides of the equality as $\Delta^p_{X_{K^\ddagger}}$-outer isomorphisms.

Proof. First, we observe that it follows from [11], Corollary 4.6 (respectively, [25], Theorem A, (ii)) that f (respectively, f^p) induces an isomorphism $G_{K^\dagger} \xrightarrow{\sim} G_{K^\ddagger}$ (respectively, $G^p_{K^\dagger} \xrightarrow{\sim} G^p_{K^\ddagger}$) via the natural surjections $\Pi^{(p)}_{X_{K^\dagger}} \twoheadrightarrow G_{K^\dagger}$ and $\Pi^{(p)}_{X_{K^\ddagger}} \twoheadrightarrow G_{K^\ddagger}$ (respectively, $\Pi^p_{X_{K^\dagger}} \twoheadrightarrow G^p_{K^\dagger}$ and $\Pi^p_{X_{K^\ddagger}} \twoheadrightarrow G^p_{K^\ddagger}$). Then since the p-adic cyclotomic character associated to a mixed characteristic discrete valuation field of residue characteristic p is open, the respective assertions concerning the preservations of cuspidal inertia subgroups follow immediately from [16], Corollary 2.7, (i). Next, write

$$g \stackrel{\text{def}}{=} (r^{(p)}_{X_{K^\dagger}})^{-1} \circ f^{-1} \circ r^{(p)}_{X_{K^\ddagger}} \circ f \quad \text{(respectively, } g^p \stackrel{\text{def}}{=} (r^p_{X_{K^\dagger}})^{-1} \circ (f^p)^{-1} \circ r^p_{X_{K^\ddagger}} \circ f^p \text{).}$$

Then it holds that g (respectively, g^p) is a $\Delta^p_{X_{K^\dagger}}$-outer automorphism of $\Pi^{(p)}_{X_{K^\dagger}}$ (respectively, $\Pi^p_{X_{K^\dagger}}$) that lies over the identity automorphism of $G_{K^\dagger}$ (respectively, $G^p_{K^\dagger}$). Moreover, since f_c (respectively, f^p_c) is the identity automorphism, it holds that g (respectively, g^p) induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi^{(p)}_{X_{K^\dagger}}$ (respectively, $\Pi^p_{X_{K^\dagger}}$). Thus, we conclude from Theorem 4.3 that g (respectively, g^p) is the identity $\Delta^p_{X_{K^\dagger}}$-outer automorphism, hence that $r^{(p)}_{X_{K^\ddagger}} \circ f = f \circ r^{(p)}_{X_{K^\dagger}}$ (respectively, $r^p_{X_{K^\ddagger}} \circ f^p = f^p \circ r^p_{X_{K^\dagger}}$) as $\Delta^p_{X_{K^\ddagger}}$-outer isomorphisms. This completes the proof of Proposition 6.2. $\square$

Definition 6.3. Let H, G be profinite groups; $\phi : H \twoheadrightarrow G$ a continuous surjective homomorphism. Then we shall write

$$\mathrm{Sect}(H \twoheadrightarrow G)$$

for the set of equivalence classes of sections of ϕ, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of $\mathrm{Ker}(\phi)$.

Proposition 6.4. *Let K be a mixed characteristic valuation field of rank 1 of residue characteristic p whose value group is p-reduced. Write $M_K \stackrel{\text{def}}{=} \mathfrak{m}_K \cup 1 + \mathfrak{m}_K$ [cf. Definition 5.1]. Then the following hold:*

(i) The natural composite map

$$s : M_K \setminus \{0, 1\} \subseteq K \setminus \{0, 1\} = X_K(K) \longrightarrow \mathrm{Sect}(\Pi^{(p\text{-ab})}_{X_K} \twoheadrightarrow G_K)$$

is injective.

(ii) Suppose that $\zeta_p \in K$. Then the natural composite map

$$s : M_K \setminus \{0,1\} \ \subseteq \ K \setminus \{0,1\} \ = \ X_K(K) \ \longrightarrow \ \mathrm{Sect}(\Pi_{X_K}^{p\text{-ab}} \ \twoheadrightarrow \ G_K^p)$$

is injective.

Proof. To verify assertion (i), by replacing K by a finite extension field of K [cf. Lemma 1.7, (i)], we may assume without loss of generality that $\zeta_p \in K$. Thus, to verify assertion (i), it suffices to verify assertion (ii).

Next, we verify assertion (ii). Write

$$h : K \setminus \{0,1\} \ = \ X_K(K) \ \longrightarrow \ \mathrm{Sect}(\Pi_{X_K}^{p\text{-ab}} \twoheadrightarrow G_K^p) \ \longrightarrow \ H^1(G_K^p, \Lambda_K) \ \stackrel{\sim}{\leftarrow} \ \widehat{K}^\times$$

for the natural composite map, where the second arrow denotes the natural map induced by the natural open immersion $X_K \hookrightarrow \mathbb{P}_K^1 \setminus \{0, \infty\}$ over K and the splitting of the surjection $\Pi_{\mathbb{P}_K^1 \setminus \{0,\infty\}}^p \twoheadrightarrow G_K^p$ determined by a cuspidal inertia subgroup of $\Delta_{X_K}^p$ associated to the cusp 1; the third arrow denotes the Kummer map. It follows from Proposition 2.5 that the map h coincides with the natural map $K \setminus \{0,1\} \to \widehat{K}^\times$. Let $x, y \in M_K$ be such that $s(x) = s(y)$. Then it holds that

$$h(x) \ = \ h(y), \quad h(1-x) \ = \ h(1-y).$$

In particular, there exist $s, t \in K^{\times p^\infty}$ such that

$$y \ = \ sx, \quad 1-y \ = \ t \cdot (1-x).$$

If $x \in 1 + \mathfrak{m}_K$, then since $M_K \ni y = sx$, it follows from Remark 5.1.2 that $s = 1$, hence that $x = y$. If $x \in \mathfrak{m}_K$, then since $M_K \ni 1 - y = t(1-x)$, it follows from Remark 5.1.2 that $t = 1$, hence that $x = y$. This completes the proof of assertion (ii), hence of Proposition 6.4. □

Lemma 6.5. *Let l be a prime number $\neq p$; K a Henselian discrete valuation field of residue characteristic p. Then the l-th power map on $1 + \mathfrak{m}_K$ is an isomorphism.*

Proof. In the case where K is complete, the assertion follows immediately from the fact that $1 + \mathfrak{m}_K$ admits the natural structure of a $\mathbb{Z}_p$-module. On the other hand, since K is Henselian, it holds that K is separably closed in the completion of K. Thus, the general Henselian case follows immediately from the complete case. This completes the proof of Lemma 6.5. □

Proposition 6.6. *Let $K^\dagger$, $K^\ddagger$ be Henselian discrete valuation fields of residue characteristic p;*

$$\sigma : (K^\dagger)^\times/(K^\dagger)^{\times p^\infty} \ \stackrel{\sim}{\to} \ (K^\ddagger)^\times/(K^\ddagger)^{\times p^\infty}$$

an isomorphism of groups. For each $* \in \{\dagger, \ddagger\}$, we regard $1 + \mathfrak{m}_{K^*}$ as a subgroup of $(K^*)^\times/(K^*)^{\times p^\infty}$ [cf. Remark 5.1.2]. Then the following hold:

(i) $\sigma(1 + \mathfrak{m}_{K^\dagger}) \subseteq \mathcal{O}_{K^\ddagger}^\times/(K^\ddagger)^{\times p^\infty}$.

(ii) Suppose that the residue fields of $K^\dagger$ and $K^\ddagger$ are perfect. Then $\sigma(\mathcal{O}_{K^\dagger}^\times/(K^\dagger)^{\times p^\infty}) = \mathcal{O}_{K^\ddagger}^\times/(K^\ddagger)^{\times p^\infty}$.

Proof. Note that, for each prime number l, it holds that $\mathbb{Z}$ admits no nontrivial l-divisible elements. Then assertion (i) follows immediately from Lemma 6.5. Next, we verify assertion (ii). In light of assertion (i), it suffices to verify that, for each $* \in \{\dagger, \ddagger\}$, there exists no nontrivial homomorphism $\mathcal{O}^{\times}_{K^*}/(1+\mathfrak{m}_{K^*}) \to \mathbb{Z}$. On the other hand, since the residue field is perfect, it holds that $\mathcal{O}^{\times}_{K^*}/(1+\mathfrak{m}_{K^*})$ is p-divisible. This implies our conclusion, as desired. This completes the proof of assertion (ii), hence of Proposition 6.6. $\square$

Lemma 6.7. *Let K be a mixed characteristic Henselian valuation field of rank 1 of residue characteristic p; $\pi \in K$ an element such that $v_p(\pi) = \frac{1}{p-1}$; $\lambda \in K$. Write*

$$f(t) \stackrel{\text{def}}{=} 1 - (1 - \pi t)^p \in K[t],$$

where t denotes an indeterminate. Then the following hold:

(i) Suppose that $v_p(\lambda) > 2$. Then there exists a unique root $\lambda' \in \overline{K}$ of the equation $f(t) - \lambda = 0$ such that $v_p(\lambda') > 0$.

(ii) Suppose that $v_p(\lambda) = 0$. Then, for each root $\lambda' \in \overline{K}$ of the equation $f(t) - \lambda = 0$, it holds that $v_p(\lambda') < 0$.

Proof. Write $f(t) = \sum_{1 \leq i \leq p} a_i t^i$, where $a_i \in K$. Then, for each positive integer $i \leq p$, it holds that $a_i = (-1)^{i-1}\binom{p}{i}\pi^i$. In particular, since $v_p(\pi) = \frac{1}{p-1}$, and $v_p(\binom{p}{i}) = 1$ for each positive integer $i \leq p-1$, one may observe that:

- $v_p(a_1) = v_p(a_p) = \frac{p}{p-1}$.
- For each positive integer $i \leq p$, it holds that $\frac{p}{p-1} \leq v_p(a_i) \leq 2$.

Thus, in light of respective assumptions on λ, assertions (i), (ii) follow immediately from the well-known property of the Newton polygon associated to the polynomial $f(t) - \lambda$. This completes the proof of Lemma 6.7. $\square$

Proposition 6.8. *The following hold:*

(i) In the notation of Proposition 6.2, (i), write

$$\sigma : G_{K^\dagger} \xrightarrow{\sim} G_{K^\ddagger}$$

for the isomorphism of profinite groups induced by f [cf. the proof of Proposition 6.2, (i)];

$$f^{(p\text{-ab})} : \Pi^{(p\text{-ab})}_{X_{K^\dagger}} \xrightarrow{\sim} \Pi^{(p\text{-ab})}_{X_{K^\ddagger}}$$

for the isomorphism induced by f. Suppose that one of the following conditions holds:

(a) f induces a bijection between the respective sets of the decomposition subgroups associated to the closed points of hyperbolic curves $X_{K^\dagger}$ and $X_{K^\ddagger}$.

(b) The residue fields of $K^\dagger$ and $K^\ddagger$ are perfect, and $f^{(p\text{-ab})}$ induces a bijection between the respective sets of the decomposition subgroups associated to the closed points of hyperbolic curves $X_{K^\dagger}$ and $X_{K^\ddagger}$.

Then σ arises from a [uniquely determined] commutative diagram of fields

$$\begin{array}{ccc} \overline{K}^{\dagger} & \xleftarrow{\sim} & \overline{K}^{\ddagger} \\ \uparrow & & \uparrow \\ K^{\dagger} & \xleftarrow{\sim} & K^{\ddagger}, \end{array}$$

where the vertical arrows denote the natural injections.

(ii) In the notation and assumption of Proposition 6.2, (ii), write

$$\sigma^p : G^p_{K^\dagger} \xrightarrow{\sim} G^p_{K^\ddagger}$$

for the isomorphism of profinite groups induced by f^p [cf. the proof of Proposition 6.2, (ii)];

$$f^{p\text{-ab}} : \Pi^{p\text{-ab}}_{X_{K^\dagger}} \xrightarrow{\sim} \Pi^{p\text{-ab}}_{X_{K^\ddagger}}$$

for the isomorphism induced by f^p. Suppose that one of the following conditions holds:

(a) f^p induces a bijection between the respective sets of the decomposition subgroups associated to the closed points of hyperbolic curves $X_{K^\dagger}$ and $X_{K^\ddagger}$.

(b) The residue fields of $K^\dagger$ and $K^\ddagger$ are perfect, and $f^{p\text{-ab}}$ induces a bijection between the respective sets of the decomposition subgroups associated to the closed points of hyperbolic curves $X_{K^\dagger}$ and $X_{K^\ddagger}$.

Then σ^p arises from a [uniquely determined] commutative diagram of fields

$$\begin{array}{ccc} (K^{\dagger})^p & \xleftarrow{\sim} & (K^{\ddagger})^p \\ \uparrow & & \uparrow \\ K^{\dagger} & \xleftarrow{\sim} & K^{\ddagger}, \end{array}$$

where the vertical arrows denote the natural injections.

Proof. Since the proof of assertion (ii) is similar to the proof of assertion (i), we verify assertion (i) only. In light of Corollary 5.10, it suffices to verify that there exists a set-theoretic bijection

$$\overline{\alpha} : M_{\overline{K}^\dagger} \xrightarrow{\sim} M_{\overline{K}^\ddagger}$$

that satisfies conditions (1), (2), (3), (4), that appear in Corollary 5.10.

Let $K^\dagger \subseteq L^\dagger$ ($\subseteq \overline{K}^\dagger$) be a finite field extension; $x \in M_{L^\dagger} \setminus \{0,1\}$; $D_x \subseteq \Pi^{(p\text{-ab})}_{X_{K^\dagger}}$ a decomposition subgroup associated to $x \in M_{L^\dagger} \setminus \{0,1\} \subseteq L^\dagger \setminus \{0,1\} = X_{K^\dagger}(L^\dagger)$. Write $K^\ddagger \subseteq L^\ddagger$ ($\subseteq \overline{K}^\ddagger$) for the finite field extension determined by the open subgroup $\sigma(G_{L^\dagger}) \subseteq G_{K^\ddagger}$. For each $* \in \{\dagger, \ddagger\}$, write $F^* \subseteq \mathrm{Sect}(\Pi^{(p\text{-ab})}_{X_{L^*}} \twoheadrightarrow G_{L^*})$ for the subset of decomposition subgroups associated to the points $\in X_{K^*}(L^*)$. Then it follows immediately from our assumption [i.e., conditions (a), (b)] that f induces a commutative diagram

$$\begin{array}{ccccccc} F^{\dagger} & \longrightarrow & \mathrm{Sect}(\Pi^{(p\text{-ab})}_{X_{L^\dagger}} \twoheadrightarrow G_{L^\dagger}) & \longrightarrow & H^1(G_{L^\dagger}, \Lambda_{K^\dagger}) & \xleftarrow{\sim} & \widehat{L^\dagger}^{\times} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ F^{\ddagger} & \longrightarrow & \mathrm{Sect}(\Pi^{(p\text{-ab})}_{X_{L^\ddagger}} \twoheadrightarrow G_{L^\ddagger}) & \longrightarrow & H^1(G_{L^\ddagger}, \Lambda_{K^\ddagger}) & \xleftarrow{\sim} & \widehat{L^\ddagger}^{\times}, \end{array}$$

where the left-hand horizontal arrows denote the natural injections; the middle horizontal arrows denote the natural maps [cf. the first display in the proof of Proposition 6.4]; the right-hand horizontal arrows denote the Kummer maps. Note that the commutativity of the above diagram implies that the isomorphism $\widehat{L^\dagger}^\times \xrightarrow{\sim} \widehat{L^\ddagger}^\times$ [that appears in the above commutative diagram] induces an isomorphism

$$\phi : (L^\dagger)^\times/(L^\dagger)^{\times p^\infty} \xrightarrow{\sim} (L^\ddagger)^\times/(L^\ddagger)^{\times p^\infty}.$$

Recall that, in the case where condition (a) (respectively, condition (b)) holds, it follows from Proposition 6.6, (i) (respectively, Proposition 6.6, (ii)), that $\phi(1+\mathfrak{m}_{L^\dagger}) \subseteq \mathcal{O}^\times_{L^\ddagger}/(L^\ddagger)^{\times p^\infty}$ (respectively, $\phi(\mathcal{O}^\times_{L^\dagger}/(L^\dagger)^{\times p^\infty}) = \mathcal{O}^\times_{L^\ddagger}/(L^\ddagger)^{\times p^\infty}$). Next, we verify the following assertion:

> Claim 6.8.A: $f^{(p\text{-ab})}(D_x) \subseteq \Pi^{(p\text{-ab})}_{X_{K^\ddagger}}$ is a decomposition subgroup associated to a unique element $\in M_{L^\ddagger} \setminus \{0,1\} \subseteq L^\ddagger \setminus \{0,1\} = X_{K^\ddagger}(L^\ddagger)$. Moreover, if $x \in \mathfrak{m}_{L^\dagger}$, then $y \in \mathfrak{m}_{L^\ddagger}$.

Indeed, the uniqueness portion follows immediately from Proposition 6.4, (i). Next, let $y \in L^\ddagger \setminus \{0,1\}$ be such that $f^{(p\text{-ab})}(D_x) \subseteq \Pi^{(p\text{-ab})}_{X_{K^\ddagger}}$ is a decomposition subgroup associated to y. Then since $\phi(1+\mathfrak{m}_{L^\dagger}) \subseteq \mathcal{O}^\times_{L^\ddagger}/(L^\ddagger)^{\times p^\infty}$, it follows immediately from Proposition 6.2, (i), that

$$y \in \mathcal{O}_{L^\ddagger} \setminus \{0,1\}.$$

Moreover, in light of Proposition 6.2, (i), we may assume without loss of generality that

$$x \in \mathfrak{m}_{L^\dagger}.$$

In particular, since $y \in \mathcal{O}_{L^\ddagger} \setminus \{0,1\}$, it suffices to verify that $y \notin \mathcal{O}^\times_{L^\ddagger}$. In the case where condition (b) holds, since $\phi(\mathcal{O}^\times_{L^\dagger}/(L^\dagger)^{\times p^\infty}) = \mathcal{O}^\times_{L^\ddagger}/(L^\ddagger)^{\times p^\infty}$, it follows immediately from the various definitions involved that $y \notin \mathcal{O}^\times_{L^\ddagger}$.

Next, we consider the case where condition (a) holds. Suppose that $y \in \mathcal{O}^\times_{L^\ddagger}$. Then observe that f maps a decomposition subgroup associated to x to a decomposition subgroup associated to a unit. Then, by replacing x, y by a suitable power of x, a unit, respectively, we may assume without loss of generality that

- $v_p(x) > 2$, and
- f maps a decomposition subgroup associated to x to a decomposition subgroup associated to y.

Moreover, by replacing $L^\dagger$ by a finite extension field of $L^\dagger$, if necessary, we may assume without loss of generality that $\zeta_p \in L^\dagger$, and $\zeta_p \in L^\ddagger$. In the remainder, we shall use t for the standard coordinate of the projective line. For each $* \in \{\dagger, \ddagger\}$ and each positive integer $i \leq p-1$, write

$$\psi^i_{L^*} : \mathbb{P}^1_{L^*} \supseteq Y^i_{L^*} \longrightarrow X_{L^*} \subseteq \mathbb{P}^1_{L^*}$$

for the [connected] finite étale cyclic covering over L^* of hyperbolic curves of genus 0 determined by the assignment

$$t \mapsto 1-(1-(1-\zeta^i_p)t)^p.$$

Then since $v_p(x) > 2$, $v_p(y) = 0$, and $v_p(1-\zeta^i_p) = \frac{1}{p-1}$, after possibly replacing $L^\dagger$ by a finite extension field of $L^\dagger$, by applying Lemma 6.7, (i), (ii), we observe that:

- For each $* \in \{\dagger, \ddagger\}$, it holds that $\psi^i_{L^*}(0) = \psi^i_{L^*}(1) = 0$, and $\psi^i_{L^*}(\infty) = \infty$.
- There exists an element $x' \in (\psi^i_{L^\dagger})^{-1}(x) \subseteq Y^i_{L^\dagger}(L^\dagger) \subseteq L^\dagger$ such that $v_p(x') > 0$.

- Let $y' \in (\psi^i_{L^\ddagger})^{-1}(y) \subseteq Y^i_{L^\ddagger}(L^\ddagger) \subseteq L^\ddagger$ be an element. Then it holds that $v_p(y') < 0$.

On the other hand, we note that $\psi^i_{L^*}$ may be characterized [up to isomorphisms of coverings] as the finite étale cyclic covering over $L^\dagger$ of degree p such that

- $\psi^i_{L^*}$ is unramified over 0, and
- $\psi^i_{L^*}$ is totally ramified over $1, \infty$.

Then, by replacing f by the composite of f with the inner automorphism of $\Pi^{(p)}_{X_{K^\ddagger}}$ determined by an element $\in \Delta^p_{X_{K^\ddagger}}$, if necessary, we observe that there exists a positive integer $j \leq p-1$ such that f induces an isomorphism of profinite groups

$$f_Y : \Pi^{(p)}_{Y^1_{L^\dagger}} \xrightarrow{\sim} \Pi^{(p)}_{Y^j_{L^\ddagger}}$$

such that:

$(*_1)$ f_Y maps a cuspidal inertia subgroup associated to $c \in \{0, 1, \infty\}$ to a cuspidal inertia subgroup associated to c.

$(*_2)$ f_Y maps a decomposition subgroup associated to $\in \mathfrak{m}_{L^\dagger}$ to a decomposition subgroup associated to $\in L^\ddagger \setminus \mathcal{O}_{L^\ddagger}$.

Moreover, in light of $(*_1)$, it follows immediately from Theorem 4.3 that f_Y fits into a commutative diagram of profinite groups

$$\begin{array}{ccc}
\Pi^{(p)}_{Y^1_{L^\dagger}} & \xrightarrow[f_Y]{\sim} & \Pi^{(p)}_{Y^j_{L^\ddagger}} \\
\downarrow & & \downarrow \\
\Pi^{(p)}_{X_{L^\dagger}} & \xrightarrow[f]{\sim} & \Pi^{(p)}_{X_{L^\ddagger}},
\end{array}$$

where the vertical arrows denote the [geometrically outer] surjections induced by the natural open immersions $Y^1_{L^\dagger} \hookrightarrow X_{L^\dagger}$, $Y^j_{L^\ddagger} \hookrightarrow X_{L^\ddagger}$. Thus, we conclude from $(*_2)$ that f maps a decomposition subgroup associated to $\in \mathfrak{m}_{L^\dagger}$ to a decomposition subgroup associated to $\in L^\ddagger \setminus \mathcal{O}_{L^\ddagger}$. This contradicts the first display of the present proof of Claim 6.8.A. Thus, we conclude that $y \notin \mathcal{O}^\times_{L^\ddagger}$. This completes the proof of Claim 6.8.A.

Finally, by assigning x to the unique element in Claim 6.8.A, we obtain a bijection

$$\alpha_L : M_{L^\dagger} \xrightarrow{\sim} M_{L^\ddagger},$$

where $\alpha_L(0) \stackrel{\text{def}}{=} 0$; $\alpha_L(1) \stackrel{\text{def}}{=} 1$. Moreover, it follows immediately from the various definitions involved that:

(1) α_L and the isomorphism $\mathrm{Gal}(L^\dagger/K^\dagger) \xrightarrow{\sim} \mathrm{Gal}(L^\ddagger/K^\ddagger)$ induced by σ are compatible with the respective natural actions of $\mathrm{Gal}(L^\dagger/K^\dagger)$, $\mathrm{Gal}(L^\ddagger/K^\ddagger)$ on $M_{L^\dagger}$, $M_{L^\ddagger}$.

(2) $\alpha_L(\mathfrak{m}_{L^\dagger}) = \mathfrak{m}_{L^\ddagger}$ [cf. Claim 6.8.A].

(3) For each $x, y \in M_{L^\dagger}$, there exists an element $z \in (L^\ddagger)^{\times p^\infty}$ such that $\alpha_L(xy) = z \cdot \alpha_L(x) \cdot \alpha_L(y)$ [cf. the existence of ϕ].

(4) α_L fits into a commutative diagram

$$\begin{array}{ccc} M_{L^\dagger} & \xrightarrow[\alpha_L]{\sim} & M_{L^\ddagger} \\ {\scriptstyle r_{L^\dagger}} \downarrow \wr & & {\scriptstyle r_{L^\ddagger}} \downarrow \wr \\ M_{L^\dagger} & \xrightarrow[\alpha_L]{\sim} & M_{L^\ddagger} \end{array}$$

[cf. Proposition 6.4, (i)].

Thus, by varying L, we obtain a set-theoretic bijection $\overline{\alpha} : M_{\overline{K}^\dagger} \xrightarrow{\sim} M_{\overline{K}^\ddagger}$ that satisfies conditions (1), (2), (3), (4), that appear in Corollary 5.10, as desired. This completes the proof of Proposition 6.8. $\square$

Next, we recall a well-known fact concerning automorphisms of "anabelian" profinite groups.

Lemma 6.9. *Let G be a slim profinite group [i.e., a profinite group such that every open subgroup is center-free]; $\sigma \in \mathrm{Aut}(G)$. Suppose that σ induces the identity automorphism on a nontrivial normal open subgroup $N \subseteq G$. Then σ is the identity automorphism.*

Proof. Note that since G is slim, it follows immediately from the various definitions involved that the natural homomorphism $G \to \mathrm{Aut}(N)$ obtained by taking conjugates is injective. On the other hand, since σ induces the identity automorphism on N, we observe that σ induces the identity automorphism on $\mathrm{Aut}(N)$ compatible with the injection $G \hookrightarrow \mathrm{Aut}(N)$. Thus, we conclude that σ is the identity automorphism. This completes the proof of Lemma 6.9. $\square$

Definition 6.10. Let G_1, G_2 be profinite groups. Then we shall write

$$\mathrm{Isom}(G_1, G_2)/\mathrm{Inn}(G_2)$$

for the set of isomorphisms $G_1 \xrightarrow{\sim} G_2$ of profinite groups, considered up to composition with an inner automorphism arising from an element $\in G_2$.

Theorem 6.11. *Let $K^\dagger$, $K^\ddagger$ be mixed characteristic Henselian discrete valuation fields of residue characteristic p; $U^\dagger$, $U^\ddagger$ hyperbolic curves of genus 0 over $K^\dagger$, $K^\ddagger$, respectively. Write*

$$\mathrm{Isom}(U^\dagger, U^\ddagger)$$

for the set of isomorphisms of schemes between $U^\dagger$ and $U^\ddagger$. Then the following hold:

(i) Write

$$\mathrm{Isom}^D(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger}) \;\subseteq\; \mathrm{Isom}(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger})$$

for the subset of outer isomorphisms that induce bijections between the respective sets of conjugacy classes of decomposition subgroups associated to the closed points of the hyperbolic curves $U^\dagger$ and $U^\ddagger$. Then the natural map

$$\mathrm{Isom}(U^\dagger, U^\ddagger) \;\longrightarrow\; \mathrm{Isom}^D(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger})$$

is bijective.

(ii) Suppose that, for each $* \in \{\dagger, \ddagger\}$, it holds that:

- *$\zeta_p \in K^*$.*
- *The cusps of U^* are $(K^*)^p$-rational.*

Write

$$\mathrm{Isom}^D(\Pi^p_{U^\dagger}, \Pi^p_{U^\ddagger})/\mathrm{Inn}(\Pi^p_{U^\ddagger}) \;\subseteq\; \mathrm{Isom}(\Pi^p_{U^\dagger}, \Pi^p_{U^\ddagger})/\mathrm{Inn}(\Pi^p_{U^\ddagger})$$

for the subset of outer isomorphisms that induce bijections between the respective sets of conjugacy classes of decomposition subgroups associated to the closed points of the hyperbolic curves $U^\dagger$ and $U^\ddagger$. Then the natural map

$$\mathrm{Isom}(U^\dagger, U^\ddagger) \;\longrightarrow\; \mathrm{Isom}^D(\Pi^p_{U^\dagger}, \Pi^p_{U^\ddagger})/\mathrm{Inn}(\Pi^p_{U^\ddagger})$$

is bijective.

Proof. First, we verify assertion (i). Note that any isomorphism of schemes between $U^\dagger$ and $U^\ddagger$ necessarily lies over an isomorphism of fields between $K^\dagger$ and $K^\ddagger$. [Indeed, this follows immediately by considering subgroups of the groups of units $\Gamma(U^\dagger, \mathcal{O}^\times_{U^\dagger})$, $\Gamma(U^\ddagger, \mathcal{O}^\times_{U^\ddagger})$ whose unions with $\{0\}$ are closed under addition.] Then, in light of [10], Theorem, (4), the injectivity follows immediately from Theorem 4.3. Next, we verify the surjectivity. It follows immediately from the various definitions involved that we may assume without loss of generality that

$$\mathrm{Isom}^D(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger}) \;\neq\; \emptyset.$$

Let $f \in \mathrm{Isom}^D(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger})$; $\tilde{f} : \Pi^{(p)}_{U^\dagger} \xrightarrow{\sim} \Pi^{(p)}_{U^\ddagger}$ a lifting of f. Then $\tilde{f}$ induces an isomorphism

$$\sigma : G_{K^\dagger} \;\xrightarrow{\sim}\; G_{K^\ddagger}$$

of profinite groups [cf. [11], Corollary 4.6]. Therefore, since the p-adic cyclotomic character associated to a mixed characteristic discrete valuation field of residue characteristic p is open, it follows from [16], Corollary 2.7, (i), that $\tilde{f}$ induces a bijection between the respective sets of cuspidal inertia subgroups of $\Pi^{(p)}_{U^\dagger}$, $\Pi^{(p)}_{U^\ddagger}$. On the other hand, recall that $G_{K^\dagger}$ is slim [cf. [11], Theorem C]. Then, to verify that $\tilde{f}$ arises from an isomorphism $U^\dagger \xrightarrow{\sim} U^\ddagger$ of schemes, in light of Lemma 6.9, by replacing $K^\dagger$, $K^\ddagger$ by suitable finite extension fields of $K^\dagger$, $K^\ddagger$, respectively, we may assume without loss of generality that, for each $* \in \{\dagger, \ddagger\}$, the cusps of U^* are K^*-rational. Next, observe that, in light of Theorem 4.3, to verify that $\tilde{f}$ arises from an isomorphism $U^\dagger \xrightarrow{\sim} U^\ddagger$ of schemes, it suffices to verify that σ arises from a commutative diagram of fields

$$\begin{array}{ccc} \overline{K}^\dagger & \xleftarrow{\sim} & \overline{K}^\ddagger \\ \uparrow & & \uparrow \\ K^\dagger & \xleftarrow{\sim} & K^\ddagger, \end{array}$$

where the vertical arrows denote the natural injections. In particular, since $\tilde{f}$ preserves cuspidal inertia subgroups, we may assume without loss of generality that $U^\dagger = X_{K^\dagger}$, and $U^\ddagger = X_{K^\ddagger}$. However, under this assumption, since $f \in \mathrm{Isom}^D(\Pi^{(p)}_{U^\dagger}, \Pi^{(p)}_{U^\ddagger})/\mathrm{Inn}(\Pi^{(p)}_{U^\ddagger})$, the field-theoreticity of σ follows immediately from Proposition 6.8, (i), as desired. This completes the proof of assertion (i).

Finally, in light of our assumptions in assertion (ii), together with [10], Lemma 1.4; [25], Theorem A, (ii); Proposition 6.8, (ii), one may observe that assertion (ii) follows from a similar argument to the argument applied in the proof assertion (i). This completes the proof of Theorem 6.11. □

Remark 6.11.1.

(i) It is natural to ask whether or not the geometrically pro-Σ [where Σ is a set of prime numbers that contains p] or the higher genus analogue of Theorem 6.11, (i), holds. On the other hand, by applying Galois descent, one may observe that the geometrically pro-Σ analogue of Theorem 6.11, (i) follows from the higher genus analogue of Theorem 6.11, (i). In particular, we can restrict our attention to pro-p settings for the future development.

(ii) With regard to the geometrically pro-Σ [where $p \in \Sigma$] analogue of Theorem 6.11, (i), suppose that the cardinality of Σ is of ≥ 2, and the base fields are p-adic local fields. Then one may prove a much stronger result, i.e., the geometrically pro-Σ absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves [cf. [20], Theorem D]. On the other hand, in the proof of [20], Theorem D, the assumption on the cardinality of Σ may be applied both

- in establishing RNS [i.e., "Resolution of Nonsingularities"] for hyperbolic curves over p-adic local fields, and
- in reconstructing the underlying topological spaces of the Berkovich analytifications of hyperbolic curves over p-adic local fields.

At the time of writing of the present paper, the author has no clue to establish geometrically pro-p analogues of the statements in the above bullets. Moreover, we note that the reconstruction of topological Berkovich spaces requires combinatorial anabelian geometry that does not work well for geometrically pro-p fundamental groups over fields of characteristic p. Therefore, it appears to the author that we need a drastically new idea to verify the preservation of decomposition subgroups in the geometrically pro-p setting even if the base fields are p-adic local fields.

(iii) Note that the operation of forming the completion of Henselian valuation fields does not change its absolute Galois group. In particular, for each algebraic variety Z over a Henselian valuation field M, the natural projection morphism $Z_{\widehat{M}} \to Z$ induces an isomorphism $\Pi_{Z_{\widehat{M}}} \xrightarrow{\sim} \Pi_Z$, where $\widehat{M}$ denotes the completion of M. On the other hand, the cardinalities of M and $\widehat{M}$ are different in general. Therefore, in Theorem 6.11, (i), one may not drop the assumption on the preservation of the decomposition subgroups associated to closed points.

(iv) In the notation of Theorem 6.11, (i), suppose that the residue fields of $K^\dagger$ and $K^\ddagger$ are perfect. Then, in light of Proposition 6.8, (i), one may weaken the assumption on the preservation of the decomposition subgroups as the preservation of the decomposition subgroups in the level of the geometrically abelianized setting.

Finally, let $K^\dagger$, $K^\ddagger$ be p-adic local fields. Then we observe that any isomorphism $\Pi^{(p)}_{X_{K^\dagger}} \xrightarrow{\sim} \Pi^{(p)}_{X_{K^\ddagger}}$ induces a bijection between the respective sets of decomposition subgroups in the level of the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$.

Proposition 6.12. *In the notation of Definition 6.1, the following hold:*

(i) Let $\sigma : G_{K^\dagger} \xrightarrow{\sim} G_{K^\ddagger}$ be an isomorphism; $g : \Lambda_{K^\dagger} \xrightarrow{\sim} \Lambda_{K^\ddagger}$ an isomorphism compatible with σ and the respective natural actions of $G_{K^\dagger}$, $G_{K^\ddagger}$ on $\Lambda_{K^\dagger}$, $\Lambda_{K^\ddagger}$. Write

$$\sigma_\kappa : \widehat{K^\dagger}^\times \xrightarrow{\sim} H^1(G_{K^\dagger}, \Lambda_{K^\dagger}) \xrightarrow{\sim} H^1(G_{K^\ddagger}, \Lambda_{K^\ddagger}) \xleftarrow{\sim} \widehat{K^\ddagger}^\times$$

for the natural composite, where the first and final arrows denote the Kummer maps; the second arrow denotes the natural isomorphism determined by σ and g;

$$\sigma_c : \widehat{K^\dagger}^\times \xrightarrow{\sim} (G^{\mathrm{ab}}_{K^\dagger})^p \xrightarrow{\sim} (G^{\mathrm{ab}}_{K^\ddagger})^p \xleftarrow{\sim} \widehat{K^\ddagger}^\times$$

for the natural composite, where the first and final arrows denote the reciprocity maps; the second arrow denotes the natural isomorphism determined by σ. Then there exists a unique element $l_{\sigma,g} \in \mathbb{Z}_p^\times$ such that

$$\sigma_c = \sigma_\kappa^{l_{\sigma,g}}.$$

(ii) Suppose that $\zeta_p \in K^\dagger$. Let $\sigma^p : G^p_{K^\dagger} \xrightarrow{\sim} G^p_{K^\ddagger}$ be an isomorphism; $g^p : \Lambda_{K^\dagger} \xrightarrow{\sim} \Lambda_{K^\ddagger}$ an isomorphism compatible with σ^p and the respective natural actions of $G^p_{K^\dagger}$, $G^p_{K^\ddagger}$ on $\Lambda_{K^\dagger}$, $\Lambda_{K^\ddagger}$. Write

$$\sigma^p_\kappa : \widehat{K^\dagger}^\times \xrightarrow{\sim} H^1(G^p_{K^\dagger}, \Lambda_{K^\dagger}) \xrightarrow{\sim} H^1(G^p_{K^\ddagger}, \Lambda_{K^\ddagger}) \xleftarrow{\sim} \widehat{K^\ddagger}^\times$$

for the natural composite, where the first and final arrows denote the Kummer maps; the second arrow denotes the natural isomorphism determined by σ^p and g^p;

$$\sigma^p_c : \widehat{K^\dagger}^\times \xrightarrow{\sim} (G^p_{K^\dagger})^{\mathrm{ab}} \xrightarrow{\sim} (G^p_{K^\ddagger})^{\mathrm{ab}} \xleftarrow{\sim} \widehat{K^\ddagger}^\times$$

for the natural composite, where the first and final arrows denote the reciprocity maps; the second arrow denotes the natural isomorphism determined by σ^p. Then there exists a unique element $l_{\sigma^p,g^p} \in \mathbb{Z}_p^\times$ such that

$$\sigma^p_c = (\sigma^p_\kappa)^{l_{\sigma^p,g^p}}.$$

Proof. Assertions (i), (ii) follow immediately from local Tate duality [cf. [22], Theorem 7.2.6; [22], Proposition 7.5.8]. This completes the proof of Proposition 6.12. □

Proposition 6.13. *The following hold:*

(i) In the notation of Propositions 6.2, (i); 6.12, (i), suppose that

$$\sigma : G_{K^\dagger} \xrightarrow{\sim} G_{K^\ddagger}$$

is the isomorphism induced by f via the natural surjections $\Pi^{(p)}_{X_{K^\dagger}} \twoheadrightarrow G_{K^\dagger}$ and $\Pi^{(p)}_{X_{K^\ddagger}} \twoheadrightarrow G_{K^\ddagger}$ [cf. [17], Theorem 2.6, (v)], and

$$g : \Lambda_{K^\dagger} \xrightarrow{\sim} \Lambda_{K^\ddagger}$$

is the isomorphism determined by the isomorphisms of cuspidal inertia subgroups induced by f. [Note that g is compatible with σ and the respective natural actions of $G_{K^\dagger}$, $G_{K^\ddagger}$ on $\Lambda_{K^\dagger}$, $\Lambda_{K^\ddagger}$.] Then it holds that

$$l_{\sigma,g} = 1.$$

In particular, since σ_c induces a bijection between the images of the multiplicative groups of p-adic local fields $K^\dagger$, $K^\ddagger$ [cf. e.g., [15], Proposition 1.2.1, (iii)], one concludes that σ_κ also induces a bijection between the images of the multiplicative groups of p-adic local fields $K^\dagger$, $K^\ddagger$.

(ii) In the notation of Propositions 6.2, (ii); 6.12, (ii), suppose that

$$\sigma^p : G^p_{K^\dagger} \xrightarrow{\sim} G^p_{K^\ddagger}$$

is the isomorphism induced by f^p via the natural surjections $\Pi^p_{X_{K^\dagger}} \twoheadrightarrow G^p_{K^\dagger}$ and $\Pi^p_{X_{K^\ddagger}} \twoheadrightarrow G^p_{K^\ddagger}$ [cf. [25], Theorem A, (ii)], and

$$g^p : \Lambda_{K^\dagger} \xrightarrow{\sim} \Lambda_{K^\ddagger}$$

is the isomorphism determined by the isomorphisms of cuspidal inertia subgroups induced by f^p. [Note that g^p is compatible with σ^p and the respective natural actions of $G^p_{K^\dagger}$, $G^p_{K^\ddagger}$ on $\Lambda_{K^\dagger}$, $\Lambda_{K^\ddagger}$.] Suppose, moreover, that σ^p preserves inertia subgroups and Frobenius liftings. Then it holds that

$$l_{\sigma^p,g^p} = 1.$$

Proof. First, we verify assertion (i). By replacing $K^\dagger$, $K^\ddagger$ by respective finite extension fields of $K^\dagger$, $K^\ddagger$, we may assume without loss of generality that $\zeta_p \in K^\dagger$, and $\zeta_p \in K^\ddagger$. Write

- $(\mathbb{P}^1_{K^\dagger} \supseteq)\ U^\dagger \to X_{K^\dagger}\ (\subseteq \mathbb{P}^1_{K^\dagger})$ for the connected finite étale covering of $X_{K^\dagger}$ of degree p determined by $t \mapsto (1-t)^p$.
- $(\mathbb{P}^1_{K^\ddagger} \supseteq)\ U^\ddagger \to X_{K^\ddagger}\ (\subseteq \mathbb{P}^1_{K^\ddagger})$ for the connected finite étale covering of $X_{K^\ddagger}$ of degree p determined by $t \mapsto (1-t)^p$.

Let $I_1 \subseteq \Delta^p_{X_{K^\dagger}}$ be a cuspidal inertia subgroup associated to the cusp 1. Note that the open subgroup $\Delta^p_{U^\dagger} \subseteq \Delta^p_{X_{K^\dagger}}$ determined by the covering $U^\dagger \to X_{K^\dagger}$ may be characterized as the unique open subgroup of index p such that $I_1 \subseteq \Delta^p_{U^\dagger}$. The open subgroup $\Delta^p_{U^\ddagger} \subseteq \Delta^p_{X_{K^\ddagger}}$ determined by the covering $U^\ddagger \to X_{K^\ddagger}$ admits a similar characterization. Thus, since f is compatible with these characterizations, we conclude that, after possibly replacing f by the composite of f with the inner automorphism of $\Pi^{(p)}_{X_{K^\ddagger}}$ determined by some element $\in \Delta^p_{X_{K^\ddagger}}$, we obtain an isomorphism of profinite groups

$$f_U : \Pi^{(p)}_{U^\dagger} \xrightarrow{\sim} \Pi^{(p)}_{U^\ddagger}$$

such that

- f_U induces the isomorphism $\sigma : G_{K^\dagger} \xrightarrow{\sim} G_{K^\ddagger}$ via the natural surjections $\Pi^{(p)}_{U^\dagger} \twoheadrightarrow G_{K^\dagger}$ and $\Pi^{(p)}_{U^\ddagger} \twoheadrightarrow G_{K^\ddagger}$,
- f_U maps the cuspidal inertia subgroups of $\Pi^{(p)}_{U^\dagger}$ associated to $* \in \{0, 1, \infty\}$ to the cuspidal inertia subgroups of $\Pi^{(p)}_{U^\ddagger}$ associated to $*$.

Write $q^\dagger : \Pi^{(p)}_{U^\dagger} \twoheadrightarrow \Pi^{(p)}_{X_{K^\dagger}}$ (respectively, $q^\ddagger : \Pi^{(p)}_{U^\dagger} \twoheadrightarrow \Pi^{(p)}_{X_{K^\ddagger}}$) for the $\Delta^p_{X_{K^\dagger}}$-outer (respectively, $\Delta^p_{X_{K^\ddagger}}$-outer) surjection that lies over the identity automorphism of $G_{K^\dagger}$ (respectively, $G_{K^\ddagger}$) determined by the natural open immersion $U^\dagger \hookrightarrow X_{K^\dagger}$ over $K^\dagger$ (respectively, $U^\ddagger \hookrightarrow X_{K^\ddagger}$ over $K^\ddagger$). Then it follows immediately from [12], Theorem A, that the following diagram

$$\begin{array}{ccc} \Pi^{(p)}_{U^\dagger} & \xrightarrow[f_U]{\sim} & \Pi^{(p)}_{U^\ddagger} \\ {\scriptstyle q^\dagger}\downarrow & & {\scriptstyle q^\ddagger}\downarrow \\ \Pi^{(p)}_{X_{K^\dagger}} & \xrightarrow[f]{\sim} & \Pi^{(p)}_{X_{K^\ddagger}} \end{array}$$

commutes. Let $I_{1-\zeta_p} \subseteq \Delta^p_{U^\dagger}$ be a cuspidal inertia subgroup associated to $1-\zeta_p$. Then there exists a positive integer $i \leq p-1$ such that $f(I_{1-\zeta_p}) \subseteq \Delta^p_{U^\ddagger}$ is a cuspidal inertia subgroup associated to $1-\zeta_p^i$. Observe that the image of the normalizer $N_{\Pi^{(p)}_{U^\dagger}}(I_{1-\zeta_p})$ (respectively, $N_{\Pi^{(p)}_{U^\ddagger}}(f(I_{1-\zeta_p}))$) via $q^\dagger$ (respectively, $q^\ddagger$) is a decomposition subgroup of $\Pi^{(p)}_{X_{K^\dagger}}$ (respectively, $\Pi^{(p)}_{X_{K^\ddagger}}$) associated to $1-\zeta_p$ (respectively, $1-\zeta_p^i$). Thus, f maps the decomposition subgroups associated to $1-\zeta_p$ to decomposition subgroups associated to $1-\zeta_p^i$. In particular, it follows immediately from the various definitions involved that

$$\sigma_\kappa(1-\zeta_p) = 1-\zeta_p^i.$$

On the other hand, it follows immediately from [15], Proposition 1.2.1, (iii), (iv), that there exists an element $u \in \mathcal{O}^\times_{K^\ddagger}$ such that

$$\sigma_c(1-\zeta_p) = u \cdot (1-\zeta_p).$$

Then since $\sigma_c = \sigma_\kappa^{l_{\sigma,g}}$ [cf. Proposition 6.12, (i)], it holds that

$$u \cdot (1-\zeta_p) = (1-\zeta_p^i)^{l_{\sigma,g}} \in \widehat{K^\ddagger}^\times.$$

Thus, since $v_p(u \cdot (1-\zeta_p)) = v_p(1-\zeta_p^i)$, we conclude that $l_{\sigma,g} = 1$. This completes the proof of assertion (i).

Finally, in light of our assumptions on σ^p, assertion (ii) follows from a similar argument to the argument applied in the proof of assertion (i). This completes the proof of Proposition 6.13. □

Remark 6.13.1. In light of Remark 6.11.1, (iv), together with Proposition 6.13, (i), if the base fields are p-adic local fields, then it would be interesting to pursue the difference between the assumption on the preservation of the decomposition subgroups in the level of the geometrically abelianized setting and the automatic preservation of the decomposition subgroups in the level of $\mathbb{G}_\mathrm{m}$.

7 Pro-p absolute Grothendieck Conjecture for configuration spaces over mixed characteristic Henselian discrete valuation fields of residue characteristic p

Let p be a prime number. In this section, by applying Theorem 6.11, together with combinatorial anabelian geometry, we prove a geometrically pro-p absolute Grothendieck Conjecture-type result for higher dimensional configuration spaces associated to hyperbolic curves over Henselian discrete valuation fields of residue characteristic p. In the case where the base fields are generalized sub-p-adic fields that admit Henselian discrete valuations, from the viewpoint of the dimensions of configuration spaces, this result generalizes Higashiyama's geometrically pro-p semi-absolute Grothendieck Conjecture-type result [cf. [4], Theorem 0.1] that depends on the use of the $\mathfrak{S}_5$-symmetry of the second configuration space associated to the projective line minus three points.

Definition 7.1. Let n, g, r be positive integers; k a field; X a hyperbolic curve over k of type (g, r), i.e, the genus of X is g, and the degree of the divisor of cusps is r.

(i) Let m be a nonnegative integer. Observe that, by considering the permutations on the first r marked points of pointed stable curves of type $(g, r+m)$, we obtain natural actions of the symmetric group $\mathfrak{S}_r$ on the [log] algebraic stacks $\mathcal{M}_{g,r+m}$, $\overline{\mathcal{M}}^{\log}_{g,r+m}$. [The log structure on $\overline{\mathcal{M}}_{g,r+m}$ is determined by the normal crossing divisor $\overline{\mathcal{M}}_{g,r+m} \setminus \mathcal{M}_{g,r+m} \subseteq \overline{\mathcal{M}}_{g,r+m}$.] Then we shall write $\mathcal{M}_{g,[r]+m}$ (respectively, $\overline{\mathcal{M}}^{\log}_{g,[r]+m}$) for the algebraic stack (respectively, log algebraic stack) obtained by forming the quotient of $\mathcal{M}_{g,r+m}$ (respectively, $\overline{\mathcal{M}}^{\log}_{g,r+m}$) via the natural action of $\mathfrak{S}_r$.

(ii) Note that the hyperbolic curve X determines a classifying morphism $\mathrm{Spec}\ k \to \mathcal{M}_{g,[r]}$ $(\subseteq \overline{\mathcal{M}}^{\log}_{g,[r]})$. Then we shall write

$$X_n$$

for the pull-back of the natural morphism $\mathcal{M}_{g,[r]+n} \to \mathcal{M}_{g,[r]}$ [obtained by forgetting the final n marked points] via the classifying morphism $\mathrm{Spec}\ k \to \mathcal{M}_{g,[r]}$. Equivalently, X_n denotes the open subscheme of the product of n copies of X whose complement consists of various diagonals. We shall refer to X_n as the *n-th configuration space* associated to X.

(iii) We shall write
$$\overline{X}_n^{\log}$$
for the pull-back of the natural morphism $\overline{\mathcal{M}}_{g,[r]+n}^{\log} \to \overline{\mathcal{M}}_{g,[r]}^{\log}$ [obtained by forgetting the final n marked points] via the classifying morphism Spec $k \to \overline{\mathcal{M}}_{g,[r]}^{\log}$;
$$\overline{X}_n$$
for the underlying scheme of $\overline{X}_n^{\log}$. We shall refer to $\overline{X}_n^{\log}$ as the *n-th log configuration space* associated to X. Here, we note that the interior of the log scheme $\overline{X}_n^{\log}$ coincides with X_n. Moreover, in the case where the characteristic of k is 0, it follows from log purity theorem [cf. [13], Theorem B] that the natural morphism $X_n \hookrightarrow \overline{X}_n^{\log}$ induces an isomorphism $\Pi_{X_n} \xrightarrow{\sim} \Pi_{\overline{X}_n^{\log}}$ of [log étale] fundamental groups.

Definition 7.2. Let Σ be a nonempty set of prime numbers.

(i) Let n be a positive integer; X a hyperbolic curve over a field of characteristic 0. Then we shall write
$$D_{X_n}$$
for the set of decomposition subgroups of $\Pi_{X_n}^{(\Sigma)}$ ($\xrightarrow{\sim} \Pi_{\overline{X}_n^{\log}}^{(\Sigma)}$) associated to the closed points of $\overline{X}_n$.

(ii) Let $n^\dagger$, $n^\ddagger$ be positive integers; $X^\dagger$, $X^\ddagger$ hyperbolic curves over fields of characteristic 0. Then we shall write
$$\mathrm{Isom}^D(\Pi_{X_{n^\dagger}^\dagger}^{(\Sigma)}, \Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)})/\mathrm{Inn}(\Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)}) \ (\subseteq \mathrm{Isom}(\Pi_{X_{n^\dagger}^\dagger}^{(\Sigma)}, \Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)})/\mathrm{Inn}(\Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)}))$$
for the set of outer isomorphisms $\Pi_{X_{n^\dagger}^\dagger}^{(\Sigma)} \xrightarrow{\sim} \Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)}$ of profinite groups that induce bijections $D_{X_{n^\dagger}^\dagger} \xrightarrow{\sim} D_{X_{n^\ddagger}^\ddagger}$ of sets.

Definition 7.3. Let k be a field of characteristic 0; Σ a nonempty set of prime numbers. Then we shall say that (ΣRGC$_k$) [i.e., "pro-Σ Relative Grothendieck Conjecture" over k] holds if for each pair of hyperbolic curves $X^\dagger$, $X^\ddagger$ over k, the natural map
$$\mathrm{Isom}_k(X^\dagger, X^\ddagger) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_{X^\dagger}^{(\Sigma)}, \Pi_{X^\ddagger}^{(\Sigma)})/\mathrm{Inn}(\Delta_{X^\ddagger}^{\Sigma})$$
is bijective.

Proposition 7.4. *Let $n^\dagger$, $n^\ddagger$ be positive integers; Σ a set of prime numbers whose cardinality is equal to* 1 *if $\Sigma \subsetneq \mathfrak{Primes}$; k a field of characteristic* 0 *such that* (ΣRGC$_k$) *holds; $X^\dagger$, $X^\ddagger$ hyperbolic curves over k. Suppose that the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ is open for some prime number $l \in \Sigma$. Then the natural map*
$$\mathrm{Isom}_k(X_{n^\dagger}^\dagger, X_{n^\ddagger}^\ddagger) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_{X_{n^\dagger}^\dagger}^{(\Sigma)}, \Pi_{X_{n^\ddagger}^\ddagger}^{(\Sigma)})/\mathrm{Inn}(\Delta_{X_{n^\ddagger}^\ddagger}^{\Sigma})$$
is bijective.

Proof. First, it follows immediately from [8], Theorem A, (i), (ii), together with our assumption that (ΣRGC$_k$) holds, that:

- We may assume without loss of generality that
$$n \overset{\text{def}}{=} n^{\dagger} = n^{\ddagger} \geq 2, \quad X \overset{\text{def}}{=} X^{\dagger} = X^{\ddagger}.$$
- Let $\sigma : \Pi_{X_n}^{(\Sigma)} \overset{\sim}{\to} \Pi_{X_n}^{(\Sigma)}$ be an automorphism that induces the identity automorphism on the set of generalized fiber subgroups and on $\Pi_X^{(\Sigma)}$ via the natural surjections $\Pi_{X_n}^{(\Sigma)} \twoheadrightarrow \Pi_X^{(\Sigma)}$ that arise from the projection morphisms. Then it suffices to verify that σ is an inner automorphism determined by an element $\in \Delta_{X_n}^{\Sigma}$.

Moreover, to verify the statement in the second bullet, by induction on n, we may assume without loss of generality that σ induces the identity automorphism on $\Pi_{X_{n-1}}^{(\Sigma)}$ via the natural surjections $\Pi_{X_n}^{(\Sigma)} \twoheadrightarrow \Pi_{X_{n-1}}^{(\Sigma)}$ that arise from the projection morphisms $X_n \to X_{n-1}$. Let $x \in X_{n-1}$ be a closed point. Write k_x for the residue field of X_{n-1} at x; $U \subseteq X_{k_x}$ for the open subscheme that arises as the fiber of the projection morphism $X_n \to X_{n-1}$ at x; $\rho_x : G_{k_x} \to \mathrm{Out}(\Delta_U^{\Sigma})$ for the natural outer representation. Then σ determines a commutative diagram of profinite groups

$$\begin{array}{ccc} G_{k_x} & \xrightarrow{\rho_x} & \mathrm{Out}(\Delta_U^{\Sigma}) \\ \| & & \downarrow \wr \\ G_{k_x} & \xrightarrow{\rho_x} & \mathrm{Out}(\Delta_U^{\Sigma}), \end{array}$$

where the right-hand vertical arrow denotes the automorphism induced by the restriction $\sigma|_{\Delta_U^{\Sigma}}$ of σ to Δ_U^{Σ}. Note that the field extension $k \subseteq k_x$ is finite. Therefore, it follows from [16], Corollary 2.7, (i), together with our assumption on the cyclotomic character associated to k, that $\sigma|_{\Delta_U^{\Sigma}}$ induces a bijection on the set of cuspidal inertia subgroups of Δ_U^{Σ}. Then, by applying [9], Theorem B, we observe that σ induces an inner automorphism on $\Delta_{X_n}^{\Sigma}$. Thus, we conclude from the center-freeness of $\Delta_{X_n}^{\Sigma}$ [cf. [19], Proposition 2.2, (ii)] that σ is an inner automorphism determined by an element $\in \Delta_{X_n}^{\Sigma}$. This completes the proof of Proposition 7.4. $\square$

Remark 7.4.1. Let n be a positive integer; Σ a set of prime numbers whose cardinality is equal to 1 if $\Sigma \subsetneq \mathfrak{Primes}$; k a field of characteristic 0 such that ($\Sigma\mathrm{RGC}_k$) holds; X a hyperbolic curve over k. Then the natural map
$$X_n(k) \longrightarrow \mathrm{Sect}(\Pi_{X_n}^{(\Sigma)} \twoheadrightarrow G_k)$$
[cf. Definition 6.3] is injective. Indeed, by considering the projections, we may assume without loss of generality that $n = 1$. In this case, the desired injectivity follows from a similar argument to the argument applied in the proof of [12], Theorem C.

Theorem 7.5. *Let p be a prime number; $n^{\dagger}$, $n^{\ddagger}$ positive integers; Σ a set of prime numbers either $\{p\}$ or $\mathfrak{Primes}$; $K^{\dagger}$, $K^{\ddagger}$ mixed characteristic Henselian discrete valuation fields of residue characteristic p such that ($\Sigma\mathrm{RGC}_{K^{\dagger}}$) and ($\Sigma\mathrm{RGC}_{K^{\ddagger}}$) hold; $X^{\dagger}$, $X^{\ddagger}$ hyperbolic curves over $K^{\dagger}$, $K^{\ddagger}$, respectively. Write $(g^{\dagger}, r^{\dagger})$ for the type of $X^{\dagger}$. Suppose that*
$$\begin{cases} n^{\dagger} \geq 2 & \text{if } g^{\dagger} \geq 1, \text{ and } r^{\dagger} \geq 1, \\ n^{\dagger} \geq 3 & \text{if } r^{\dagger} = 0. \end{cases}$$
Then the natural map
$$\mathrm{Isom}(X_{n^{\dagger}}^{\dagger}, X_{n^{\ddagger}}^{\ddagger}) \longrightarrow \mathrm{Isom}^D(\Pi_{X_{n^{\dagger}}^{\dagger}}^{(\Sigma)}, \Pi_{X_{n^{\ddagger}}^{\ddagger}}^{(\Sigma)})/\mathrm{Inn}(\Pi_{X_{n^{\ddagger}}^{\ddagger}}^{(\Sigma)})$$
is bijective.

Proof. First, in light of [23], Lemma 4.2, together with [10], Theorem, (4); the injectivity portion of Proposition 7.4, the injectivity of the natural map follows from a similar argument to the argument applied in the proof of Theorem 6.11, (i).

Next, we verify the surjectivity of the natural map. Let

$$\sigma : \Pi^{(\Sigma)}_{X^{\dagger}_{n^{\dagger}}} \xrightarrow{\sim} \Pi^{(\Sigma)}_{X^{\ddagger}_{n^{\ddagger}}}$$

be an isomorphism of profinite groups that induces a bijection $D_{X^{\dagger}_{n^{\dagger}}} \xrightarrow{\sim} D_{X^{\ddagger}_{n^{\ddagger}}}$. Then it follows immediately from [11], Corollary 4.6, that σ induces isomorphisms

$$\sigma_{\Delta} : \Delta^{\Sigma}_{X^{\dagger}_{n^{\dagger}}} \xrightarrow{\sim} \Delta^{\Sigma}_{X^{\ddagger}_{n^{\ddagger}}}, \quad \sigma_{\mathrm{Gal}} : G_{K^{\dagger}} \xrightarrow{\sim} G_{K^{\ddagger}}$$

of profinite groups. To verify the surjectivity, it follows from the surjectivity portion of Proposition 7.4 that it suffices to verify that σ_{Gal} arises from an isomorphism $K^{\ddagger} \xrightarrow{\sim} K^{\dagger}$ of fields. In particular, we may assume without loss of generality that

$$\Sigma = \{p\}.$$

Observe that $n \overset{\mathrm{def}}{=} n^{\dagger} = n^{\ddagger}$, and σ_{Δ} induces a bijection between the respective sets of generalized fiber subgroups [cf. [8], Theorem A, (i), (ii)]. In particular, it follows immediately from Theorem 6.11, (i), together with the assumption on $n^{\dagger}$ in the statement, we may assume without loss of generality that:

- It holds that $g^{\dagger} \geq 1$, and $n = 2$ (respectively, $n = 3$) in the case where $X^{\dagger}$ is affine (respectively, proper).
- The isomorphism σ and the respective first projection morphisms determine an isomorphism

  $$\sigma_1 : \Pi^{(\Sigma)}_{X^{\dagger}} \xrightarrow{\sim} \Pi^{(\Sigma)}_{X^{\ddagger}}$$

  that lies over σ_{Gal}. Moreover, the isomorphism σ_1 induces a bijection $D_{X^{\dagger}} \xrightarrow{\sim} D_{X^{\ddagger}}$.

On the other hand, since $G_{K^{\dagger}}$ is slim [cf. [11], Theorem C], in light of Lemma 6.9, by replacing $K^{\dagger}$ by a finite extension field of $K^{\dagger}$, we may assume without loss of generality that

$$X^{\dagger}(K^{\dagger}) \neq \emptyset, \text{ and the cusps of } X^{\dagger} \text{ are } K^{\dagger}\text{-rational points.}$$

Let $x^{\dagger} \in X^{\dagger}(K^{\dagger})$ be a point; $x^{\ddagger} \in X^{\ddagger}(K^{\ddagger})$ a [uniquely determined — cf. Remark 7.4.1] point such that $\sigma_1(D_{x^{\dagger}}) = D_{x^{\ddagger}}$, where $D_{(-)}$ denotes a decomposition group associated to a point $(-)$. In particular, in the case where $X^{\dagger}$ is proper, the isomorphism σ induces an isomorphism

$$\Pi^{(\Sigma)}_{(X^{\dagger}\setminus\{x^{\dagger}\})_2} \xrightarrow{\sim} \Pi^{(\Sigma)}_{X^{\dagger}_3} \times_{\Pi^{(\Sigma)}_{X^{\dagger}}} D_{x^{\dagger}} \xrightarrow{\sim} \Pi^{(\Sigma)}_{X^{\ddagger}_3} \times_{\Pi^{(\Sigma)}_{X^{\ddagger}}} D_{x^{\ddagger}} \xleftarrow{\sim} \Pi^{(\Sigma)}_{(X^{\ddagger}\setminus\{x^{\ddagger}\})_2}$$

that lies over σ_{Gal}. Therefore, by replacing $X^{\dagger}$, $X^{\ddagger}$ by $X^{\dagger} \setminus \{x^{\dagger}\}$, $X^{\ddagger} \setminus \{x^{\ddagger}\}$, respectively, we may assume without loss of generality that $X^{\dagger}$ and $X^{\ddagger}$ are affine curves [so $n = 2$] whose genus is of ≥ 1 [cf. [8], Theorem A, (i)]. Next, let $c^{\dagger} \in \overline{X^{\dagger}} \setminus X^{\dagger}(K^{\dagger})$ be a cusp. Write $c^{\ddagger} \in \overline{X^{\ddagger}} \setminus X^{\ddagger}(K^{\ddagger})$ for the cusp such that $\sigma_1(D_{c^{\dagger}}) = D_{c^{\ddagger}}$. For each $* \in \{\dagger, \ddagger\}$, write $\mathcal{G}^{*}$ for the semi-graph of anabelioids of PSC-type associated to the geometric fiber of the first projection morphism $(X^{*}_2)^{\log} \to (X^{*})^{\log}$ at c^{*}: $\Pi_{\mathcal{G}^{*}}$ for the pro-Σ fundamental group of $\mathcal{G}^{*}$. In particular, the isomorphism σ determines an isomorphism

$$\sigma_f : \Pi_{\mathcal{G}^{\dagger}} \overset{\mathrm{out}}{\rtimes} D_{c^{\dagger}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\ddagger}} \overset{\mathrm{out}}{\rtimes} D_{c^{\ddagger}}$$

that lies over the isomorphism $D_{c^{\dagger}} \xrightarrow{\sim} D_{c^{\ddagger}}$. Here, we note that the p-adic cyclotomic characters associated to $K^{\dagger}$ and $K^{\ddagger}$ are open. Then it follows from [16], Corollary 2.7, (i), that the isomorphism $\Pi_{\mathcal{G}^{\dagger}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\ddagger}}$

[determined by σ_f] induces a bijection between the respective sets of cuspidal inertia subgroups. Thus, we conclude from [9], Theorem A, (iii), that the isomorphism $\Pi_{\mathcal{G}^\dagger} \xrightarrow{\sim} \Pi_{\mathcal{G}^\ddagger}$ is graphic. Therefore, since the genera of $X^\dagger$ and $X^\ddagger$ are of ≥ 1, we observe that σ_f induces an isomorphism

$$\Pi^{(\Sigma)}_{\mathbb{P}^1_{K^\dagger}\setminus\{0,1,\infty\}} \xrightarrow{\sim} \Pi^{(\Sigma)}_{\mathbb{P}^1_{K^\ddagger}\setminus\{0,1,\infty\}}$$

that lies over σ_{Gal}. Moreover, since σ induces a bijection $D_{X_2^\dagger} \xrightarrow{\sim} D_{X_2^\ddagger}$, we also observe that this isomorphism induces a bijection between the respective sets of decomposition subgroups associated to closed points. Thus, we conclude from Theorem 6.11, (i), that σ_{Gal} arises from an isomorphism $K^\ddagger \xrightarrow{\sim} K^\dagger$ of fields. This completes the proof of the surjectivity, hence of Theorem 7.5. $\square$

Remark 7.5.1. Note that the key point of the proof of Theorem 7.5 is to construct the pro-p fundamental groups of *tripods* in the pro-p fundamental group of configuration spaces. On the other hand, in the proof of [4], one needs to construct the pro-p fundamental groups of the *second configuration spaces associated to tripods*. In particular, in light of the combinatorial complexity, our situation is much easier to handle. This is the reason why our proof is shorter.

Remark 7.5.2. Recall that if K is a generalized sub-p-adic field, then ($\Sigma\mathrm{RGC}_K$) holds [cf. [14], Theorem 4.12]. In particular, one may apply Theorem 7.5 in the case where $K^\dagger$ and $K^\ddagger$ are unramified [algebraic] extension fields of p-adic local fields or their completions. With regard to a further development on the relative Grothendieck Conjecture-type results [that is related to the assumption in the statement of Theorem 7.5], there is an ongoing joint work with Y. Hoshi, S. Mochizuki, and Go Yamashita.

Remark 7.5.3. By applying a similar argument to the argument applied in the proof of Theorem 7.5, together with [25], Theorem A, (ii), one may also obtain an arithmetically pro-p Grothendieck Conjecture-type result as in [4], Theorem 0.1. We leave the routine details to the interested reader.

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(Shota Tsujimura) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Email address: stsuji@kurims.kyoto-u.ac.jp