AN EXTENSION OF THE FIEDLER-MOCHIZUKI THEORY FOR REGULATORY NETWORKS OF ODES TO TIME-DELAY SYSTEMS

ATSUSHI KONDO

Abstract. We consider the dynamics of a system of differential equations called a Regulatory Network, which represents complex regulatory relationships such as gene regulatory networks. The paper by Fiedler-Mochizuki et al.[3] showed that it is possible to identify a set of determining nodes that determines the asymptotic dynamics of the Regulatory Network from its network structure alone. We extend this theory to the case where the regulatory network contains time delays.

1. Introduction

In biology, it is widely believed that the interaction between many molecules such as genes produces the biological functions and properties. For example, a differentiated state of a cell is considered as a steady state of the dynamics of gene expression resulting from gene regulations. Such regulations can be represented as a directed graph such as gene regulatory network. For example, Figure 1 shows a reduced gene regulatory network for cell differentiation in the development of the ascidian Ciona intestinalis [3]. It is important to understand the relationship between the structure of a network and dynamics to study and control biological systems.

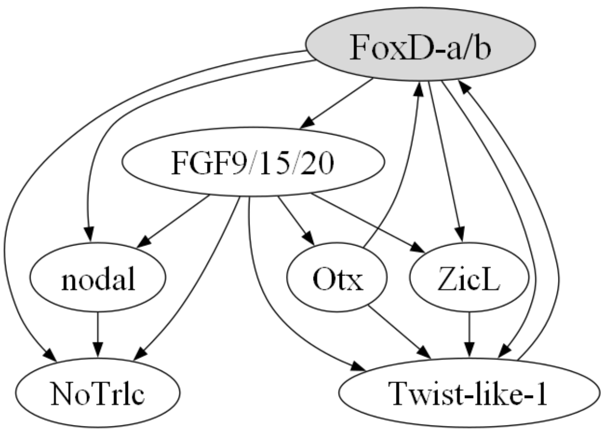

Figure 1. A gene regulatory network

Fiedler, Mochizuki et al [3] developed a theory, in this paper, which we call the Fiedler-Mochizuki theory, to analyze and control a long-term dynamical behavior in a system of ODEs on a network based on information of regulatory linkages alone [3]. They showed that it is sufficient to observe on only a feedback vertex set in order to determine the long-term dynamics of the entire system. See Theorem 8. A feedback vertex set is a subset of vertices whose removal becomes a graph without directed cycles. For example, the minimal feedback vertex set of the directed graph in Figure 1 is {FoxD-a/b}, so the dynamics of FoxD-a/b determines the dynamics of the others.

Their theory is applicable to a wide range of systems. Mathematically, they assumed that the dynamics can be obtained from ordinary differential equations and they considered the following class with some assumptions:

Definition 1. For given subsets $I_k \subset \{1,\ldots,N\}\backslash\{k\} \quad (k=1,\ldots,N)$, we call an ODE systems of the form

$$\dot{z}_k(t) = F_k(t, z_k(t), \{z_j(t)\}_{j\in I_k}),\ k=1,\ldots,N. \tag{1}$$

a *regulatory network (RN)* if the nonlinearities F_k satisfies the following assumptions:

Continuity (C): The nonlinearities $F_k : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}\to\mathbb{R}$ are C^1.
Absorbing (A): There exists a positive constant R such that for any solution $z(t)$ of (1), there exists a positive time T such that $|z(t)| \leqq R$, for all $t \geqq T$.
Boundedness (B): For all $k\in\{1,\ldots N\}$ and $j\in I_k$, there exists $B>0$ such that $|\frac{\partial F_k}{\partial z_j}(t, z_k, \{z_j\}_{j\in I_k})| \leqq B$ for all $t\geqq 0$, $z\in\mathbb{R}^N$ with $|z|\leqq R$.

Remark 2. From assumption (C), we have a unique solution of each initial value problem for (1). In assumption (A), we implicitly assume that the solution $z(t)$ exists for all $t \geqq 0$.

In some biological models such as gene transcription and translation, recent studies have shown that in order to capture the whole range of dynamics, time delay often plays an important role [10]. For example, in periodic somite segmentation, time delays in the timing of specific gene (Hes7) expression caused by the introns are required for oscillatory expression. However, Fiedler-Mochizuki theory is not formulated for time-delay systems. If we can extend their theory to a class of time-delay systems, we will be able to analyze broader and more biologically accurate models.

One thing to note in studying time-delay systems is that time-delay systems cannot be written in the form of (1). For example, the derivative of x at time t of the system

$$\dot{x}(t) = -x(t-\frac{\pi}{2})$$

depends on the past state, that is the value of x at time $t-\frac{\pi}{2}$. Time-delay systems contain wider range of equations than ordinary differential equations. Generally, we use the formulation of the functional differential equations. This is one of the difficulties in extending Fiedler-Mochizuki theory.

The other difficulty is related to the fact that even a simple one-dimensional time-delay system can show rich dynamical behaviors depending on the time-delay in contrast to an ODE without time-delay. For example, the solution x of $\dot{x}(t) = -x(t-\tau)$ can diverge, converge to 0 or possibly oscillate depending on τ.

To extend Fiedler-Mochizuki theory to time-delay systems, we assume delays are bounded among various cases of time-delay systems and use Hale's formulation[6].

In Section 2, we review Fiedler-Mochizuki theory for ordinary differential equations. In Section 3, we introduce basic concepts of delay differential equations, and in Section 4, we formulate delayed regulatory networks. Our main results is presented in Section 5. After preparing some tools and results in Sections 6 and 7, we give the proof of our result in Section 8.

2. Fiedler-Mochizuki theory

In this Section, we briefly review the Fiedler-Mochizuki theory based on [3]. We represent dependencies between variables of a (RN) by a directed graph. We admit self-loops.

Definition 3 (Decay condition)**.** We say a function $F_k(t, z_k, \{z_j\}_{j \in I_k})$ in the right hand side of RN (1) satisfies a *decay condition*, if the following holds for F_k:

> **(DC):** There exists a positive number a such that for all $t \geqq 0$ and $z \in \mathbb{R}^N$ with $|z| \leqq R$, $\frac{\partial F_k}{\partial z_k}(t, z_k, \{z_j\}_{j \in I_k}) \leqq -a < 0$

Definition 4 (Directed graph associated with RN)**.** For a given RN (1), the *associated directed graph* $\Gamma = (V, E)$ with $V := \{1, \ldots, N\}$ as vertices and $E \subset V \times V$ as edges is defined as follows:

- $(j, k) \in E \quad (j \neq k) \overset{\text{def}}{\iff} j \in I_k$
- $(k, k) \in E \overset{\text{def}}{\iff} F_k$ does not satisfy the decay condition.

Definition 5. Let $\Gamma = (\{1, \ldots, N\}, E)$ be a digraph. The system (1) is called a *regulatory network on* Γ if it satisfies assumptions (A),(B),(C), and Γ is the directed graph associated with (1)

In [3], they showed that it is possible to identify a subset of variables $\{z_1, \ldots, z_N\}$ that determines long-term dynamics of (1) from the structure of its associated directed graph. This is the concept of determining nodes and proven to be related a property of the associated digraph called the feedback vertex sets.

Definition 6 (Determining nodes)**.** We call a subset $I \subset \{1, \ldots, N\}$ a set of *determining nodes* for (1) if for any two solutions $\tilde{z}(t)$, $z(t)$ of (1) satisfying

$$\tilde{z}_j(t) - z_j(t) \xrightarrow[t \to \infty]{} 0 \text{ for all } j \in I$$

we have

$$\tilde{z}(t) - z(t) \xrightarrow[t \to \infty]{} 0.$$

Definition 7 (Feedback vertex set)**.** A subset $I \subset V$ is called a *feedback vertex set (abbreviated as FVS)* of a di-graph $\Gamma = (V, E)$ if $\Gamma \backslash I$ has no directed cycles.

Theorem 8 ([3] Lemma 3.2)**.** Let $\Gamma = (\{1, \ldots, N\}, E)$ be a given digraph and I be a FVS of Γ. Then, I is a set of determining nodes for any regulatory networks on Γ.

Example 9.

$$\begin{aligned} \dot{z}_1(t) &= z_1(t) - z_1(t)^3 \\ \dot{z}_2(t) &= -z_2(t) + z_1(t)^2 \end{aligned} \tag{2}$$

The system has two asymptotically stable equilibria $(-1, 1)$ and $(1, 1)$, and one saddle equilibrium $(0, 0)$. Any solution which starts from a point on z_2-axis converges to the origin, and any other solution converges to either $(-1, 1)$ or $(1, 1)$.

For this RN (2), the set $\{1\}$ is a set of determining nodes. To see this, let $\tilde{z}$ and z be two solutions. Then $\tilde{z}_1(t) - z_1(t) \xrightarrow[t \to \infty]{} 0$ implies $\lim_{t \to \infty} \tilde{z}_1(t) = \lim_{t \to \infty} z_1(t) = -1, 1$ or 0. If $\lim_{t \to \infty} \tilde{z}_1(t) = \lim_{t \to \infty} z_1(t) = -1$ or 1, then $\lim_{t \to \infty} \tilde{z}_2(t) = \lim_{t \to \infty} z_2(t) = 1$.

The case $\lim_{t\to\infty} \tilde{z}_1(t) = \lim_{t\to\infty} z_1(t) = 0$ occurs only when $\tilde{z}_1(t) = z_1(t) \equiv 0$, in which case $\lim_{t\to\infty} \tilde{z}_2(t) = \lim_{t\to\infty} z_2(t) = 0$. In any case, $\tilde{z}_1(t) - z_1(t) \xrightarrow[t\to\infty]{} 0$ implies $\tilde{z}_2(t) - z_2(t) \xrightarrow[t\to\infty]{} 0$.

The function $F_2(z) = -z_2 + z_1^2$ satisfies the decay condition, but $F_1(z) = z_1 - z_1^3$ does not satisfy the decay condition, hence the directed graph Γ associated with this RN (2) has a self-loop on vertex 1 as shown in Figre 2. Consequently the minimal FVS of the di-graph Γ is $I = \{1\}$. According to the above Theorem 8, I must be a set of determining node for any RN on Γ, and is indeed the case for RN (2).

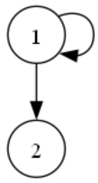

FIGURE 2. The directed graph Γ associated with RN (2)

3. TIME-DELAY SYSTEMS: BASIC CONCEPTS

In this section, we introduce a general class of time-delay systems and the definition of stability of the solution.

We explain some notations. Let $C([a, b], \mathbb{R}^m)$ denote the Banach space of continuous functions from the interval $[a, b]$ to $\mathbb{R}^m$ with the norm given by

$$\|\phi\| = \max_{s\in[a,b]} |\phi(s)|.$$

Let $\tau > 0$ be a given upper bound on the bounded time delay and let $C = C([-\tau, 0], \mathbb{R}^n)$. For a continuous function x defined on the interval $[\sigma - \tau, \sigma + d)$ $(\sigma \in \mathbb{R}, d \geqq 0)$ and for any $t \in [\sigma, \sigma + d)$, we let $x_t \in C$ be the function defined on $[-\tau, 0]$ by

$$x_t(\theta) = x(t + \theta), \ -\tau \leqq \theta \leqq 0.$$

For $\beta > 0$, let

$$C_\beta = \{\phi \in C \mid \|\phi\| < \beta\}.$$

Let $f : \mathbb{R} \times C \supset \mathrm{dom}(f) =: D \to \mathbb{R}^n$ be a given function . We say

$$\dot{x}(t) = f(t, x_t) \tag{3}$$

is a time-delay system on D (" $\cdot$ " represents the right-handed derivative). A function x is said to be a solution of equation (3) on $[\sigma - \tau, \sigma + d)$ if there are $\sigma \in \mathbb{R}$ and $d > 0$ such that $x \in C([\sigma - \tau, \sigma + d), \mathbb{R}^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies equation (3) for $t \in [\sigma, \sigma + d)$. For given $\sigma \in \mathbb{R}$, $\phi \in C$, we say $x(t; \sigma, \phi, f)$ is a *solution* of equation (3) *with initial value* ϕ *at* σ or simply a *solution through* (σ, ϕ) if there there is an $d > 0$ such that $x(t; \sigma, \phi, f)$ is a solution of equation (3) on $[\sigma - \tau, \sigma + d)$ and $x(\cdot; \sigma, \phi, f)_\sigma = \phi$. For notational purposes, we take the domain of definition of f to be $\mathbb{R} \times C$. We give the definition of stability of the solution $x = 0$.

Definition 10 (Stability). Suppose $f(t, 0) = 0$ for all $t \in \mathbb{R}$.

- The solution $x = 0$ of equation (3) is said to be *uniformly bounded* if for any $\alpha > 0$, there exists $\beta = \beta(\alpha) > 0$ such that for all $\sigma \in \mathbb{R}$, $\phi \in C_\alpha$, we have $\|x(\cdot;\sigma,\phi)_t\| \leqq \beta(\alpha)$ for all $t \geqq \sigma$.
- The solution $x = 0$ of equation (3) is said to be *stable* if for any $\sigma \in \mathbb{R}$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon,\sigma)$ such that for all $\phi \in C_\delta$, we have $x(\cdot;\sigma,\phi)_t \in C_\epsilon$ for $t \geqq \sigma$.
- The solution $x = 0$ of equation (3) is said to be *asymptotically stable* if it is stable and there is a $b_0 = b_0(\sigma) > 0$ such that for all $\phi \in C_{b_0}$, we have $x(t;\sigma,\phi) \to 0$ as $t \to \infty$.
- The solution $x = 0$ of equation (3) is said to be *uniformly stable* if it is stable and the number δ in the definition above is independent of σ.
- The solution $x = 0$ of equation (3) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is $\gamma > 0$ such that for any $q > 0$, there is a $t_0(q)$ such that for all $\phi \in C_\gamma$, we have $x(\cdot;\sigma,\phi)_t \in C_q$ for $t \geqq \sigma + t_0(q)$ for every $\sigma \in \mathbb{R}$.

We introduce a result of Yorke [11]. Let $C = C([-\tau,0],\mathbb{R})$. For $\phi \in C$, define

$$M(\phi) = \max\{0, \max_{s\in[-\tau,0]} \phi(s)\}.$$

We note that for all $\phi \in C$ and $s \in [-\tau, 0]$, $-M(-\phi) \leqq \phi(s) \leqq M(\phi)$ holds.

Theorem 11 ([11])**.** Let $\tau > 0$ and $F : \mathbb{R} \times C \to \mathbb{R}$ be continuous. Assume for some constant $\alpha \geqq 0$, the following three conditions hold:

(I) $-\alpha M(-\phi) \leqq -F(t,\phi) \leqq \alpha M(\phi)$, for sufficiently large $t \geqq 0$ and for all $\phi \in C$

(II) $\alpha\tau < \frac{3}{2}$

(III) for all sequences $t_n \to \infty$ and $\phi_n \in C$ converging to a constant nonzero function in C , $F(t_n,\phi_n)$ does not converge to 0.

Then, the zero solution $x = 0$ of the one-dimensional delay differential equation

$$\dot{x}(t) = F(t, x_t)$$

is uniformly asymptotically stable.

4. Formulation of delayed regulatory networks

Let $C(\tau) := C([-\tau,0],\mathbb{R})$ for $\tau \geqq 0$. For a continuous function x defined on the interval $[\sigma-\tau,\sigma+d)$ ($\sigma \in \mathbb{R}, d \geqq 0$) and for any $t \in [\sigma,\sigma+d)$, we let $x_t^{(\tau)} \in C(\tau)$ be the function defined on $[-\tau, 0]$ by

$$x_t^{(\tau)}(\theta) = x(t+\theta),\ -\tau \leqq \theta \leqq 0.$$

The norm of an element $\phi \in C(\tau)$ is $\|\phi\|^{(\tau)} = \max_{s\in[-\tau,0]} |\phi(s)|$. For $\phi \in C(\tau)$, define

$$M^{(\tau)}(\phi) = \max\{0, \max_{s\in[-\tau,0]} \phi(s)\}.$$

Let $I_k \subset \{1,\dots,N\}\backslash\{k\}$ be a given subset and $F_k : \mathbb{R}\times C(\tau_{k,k})\times\prod_{j\in I_k} C(\tau_{k,j}) \to \mathbb{R}$ be a given functional for each $k = 1,\dots,N$. In this paper, we study time-delay systems of the form below:

$$\dot{z}_k(t) = F_k(t,(z_k)_t^{(\tau_{k,k})},\{(z_j)_t^{(\tau_{k,j})}\}_{j\in I_k})\ ,\ k = 1,\dots,N \tag{4}$$

An upper bound on the delay of the influence from node j on node k is represented as $\tau_{k,j}$.

Definition 12. For given subsets $I_k \subset \{1,\dots,N\}\backslash\{k\}$ $(k = 1,\dots,N)$, we call a time-delay system of the form (4) a *delayed regulatory network (DRN)* if the nonlinearities F_k satisfies the following assumptions:

Continuity (C): For all $k \in \{1,\dots N\}$ and $j \in I_k \cup \{k\}$, F_k and D_jF_k are continuous.

Absorbing (A): There exists a positive constant R such that for any solution z of (4), there exists a positive time T such that $|z(t)| \leqq R$, for all $t \geqq T$.

Boundedness (B): For all $k \in \{1,\dots N\}$, $j \in I_k$, $\phi \in C(\tau_{k,j})$, there exists $B_{k,j}(\phi) > 0$ such that $|D_jF_k(t, \psi_k, \{\psi_s\}_{s\in I_k})(\phi)| \leqq B_{k,j}(\phi)$ for all $t \geqq 0$, $\psi_\ell \in C(\tau_{k,\ell})$ with $\|\psi_\ell\|^{(\tau_{k,\ell})} \leqq R$ (for all $\ell \in \{1,\dots N\}$).

Remark 13. The derivative D_jF_k denotes the partial Frechet derivative of $F_k(t, x_k, \{x_s\}_{s\in I_k})$ with respect to the variable x_j $(j \in I_k \cup \{k\})$. Let $\mathcal{B}(C(\tau_{k,j}),\mathbb{R})$ denotes the space of all bounded linear operators from $C(\tau_{k,j})$ to $\mathbb{R}$.

- $F_k : \mathbb{R} \times C(\tau_{k,k}) \times \prod_{j\in I_k} C(\tau_{k,j}) \to \mathbb{R}$
- $D_jF_k : \mathbb{R} \times C(\tau_{k,k}) \times \prod_{j\in I_k} C(\tau_{k,j}) \to \mathcal{B}(C(\tau_{k,j}),\mathbb{R})$

Remark 14. Let $\tau := \max_{k,j\in\{1,\dots,N\}} \tau_{k,j}$. We can think of F_k as a mapping from $\mathbb{R} \times C(\tau)^N$. The product space $C([-\tau,0],\mathbb{R})^N$ and $C([-\tau,0],\mathbb{R}^N)$ are isometrically isomorphic. By this isometric isomorphism, we regard the equations (4) as the general form of (3).

We represent dependencies between variables of a DRN by a directed graph in the like manner as a RN. However we have to modify the decay condition. We show you an example.

Example 15.

$$\begin{aligned} \dot{z}_1(t) &= z_1(t) - z_1(t)^3 \\ \dot{z}_2(t) &= -z_2(t - \frac{\pi}{2}) + z_1(t)^2 \end{aligned} \tag{5}$$

For this DRN (5), $I = \{1\}$ is not a set of determining node. Indeed, we have two solutions $(\tilde{z}_1(t), \tilde{z}_2(t)) = (0, \cos t)$ and $(z_1(t), z_2(t)) = (0,0)$, whose first component is the same, but $\tilde{z}_2(t) - z_2(t) = \cos t \underset{t\to\infty}{\not\to} 0$. Compared with Example 9, we will show the di-graph associated with this time-delay system has self loops not only on vertex 1 but also vertex 2 in Example 21.

Remark 16. The system (5) does not satisfy the assumption (A) of the Definition 12 because for any $A \in \mathbb{R}$, $(x_1(t), x_2(t)) = (0, A\cos t)$ is a solution. In order to fix this, we can enforce the assumption (A) by a modification

$$\dot{x}_2(t) = -x_2(t - \frac{\pi}{2}) + x_1(t)^2 - f(x_2(t))$$

where the smooth function $f(y)$ is 0 for $y^2 \leqq 1$, and y for large $|y|$.

Definition 17 (Delayed decay condition)**.** We say a functional $F_k(t, \psi_k, \{\psi_s\}_{s\in I_k})$ in the right hand of DRN (4) satisfies a *delayed decay condition*, if the following holds for F_k:

(DDC): Case 1: $\tau_{k,k} > 0$. There exists $\alpha_k > 0$ such that for all $\psi_\ell \in C(\tau_{k,\ell})$ with $\|\psi_\ell\|^{(\tau_{k,\ell})} \leqq R$ (for all $\ell \in \{1, \ldots N\}$), the following holds.

(I) $-\alpha_k M^{(\tau_{k,k})}(-\phi) \leqq -D_k F_k(t, \psi_k, \{\psi_s\}_{s \in I_k})(\phi) \leqq \alpha_k M^{(\tau_{k,k})}(\phi)$, for all $t \geqq 0$ and $\phi \in C(\tau_{k,k})$

(II) $\alpha_k \tau_{k,k} < \frac{3}{2}$

(III) for all sequences $t_n \to \infty$ and $\phi_n \in C(\tau_{k,k})$ converging to a constant nonzero function in $C(\tau_{k,k})$, $D_k F_k(t_n, \psi_k, \{\psi_s\}_{s \in I_k})(\phi_n)$ does not converge to 0.

Case 2: $\tau_{k,k} = 0$. In this case, (DDC) is coincide with (DC).

Definition 18 (Directed graph associated with DRN)**.** For a given DRN (4), the *associated directed graph* $\Gamma = (\{1, \ldots, N\}, E)$ with $\{1, \ldots, N\}$ as vertices and $E \subset V \times V$ as edges is defined as follows:

- $(j, k) \in E \quad (j \neq k) \overset{\text{def}}{\iff} j \in I_k$
- $(k, k) \in E \overset{\text{def}}{\iff} F_k$ does not satisfy the delayed decay condition (DDC).

These conditions in (DDC) are motivated by Theorem 11 [11].

Definition 19. Let $\Gamma = (\{1, \ldots, N\}, E)$ be a digraph. The system (4) is called a *delayed regulatory network on* Γ if it satisfies assumptions (A),(B),(C), and Γ is the directed graph associated with (4).

5. Main result

By using Yorke's theorem and defining the (DDC), we can extend the Fiedler-Mochizuki theory to DRN.

Theorem 20. Let $\Gamma = (\{1, \ldots, N\}, E)$ be a given digraph and I be a FVS of Γ. Then, I is a set of determining nodes for any delayed regulatory networks on Γ.

Example 21.

$$\begin{aligned} \dot{z}_1(t) &= z_1(t) - z_1(t)^3 \\ \dot{z}_2(t) &= -z_2(t - \frac{\pi}{2}) + z_1(t)^2 \end{aligned} \tag{6}$$

We consider the same system in Example 15 again. If we define $F_2 : C(\frac{\pi}{2}) \to \mathbb{R}$ by $F_2(\psi) := -\psi(-\frac{\pi}{2})$, then $\dot{z}_2(t) = F_2((z_2)_t) + z_1(t)^2$ and $DF_2(\psi)(\phi) = -\phi(-\frac{\pi}{2})$ for all $\psi, \phi \in C(\frac{\pi}{2})$. Thus, we can choose $\alpha_2 = 1$ such that $-\alpha_2 M^{(\frac{\pi}{2})}(-\phi) \leqq -DF_2(\psi)(\phi) \leqq \alpha_2 M^{(\frac{\pi}{2})}(\phi)$. This means F_2 satisfies (DDC)(I). However F_2 does not satisfy (DDC)(II) since $\alpha_2 \tau_{2,2} = \frac{\pi}{2}$, hence the directed graph associated with this DRN has a self-loop on vertex 2. The set {1,2} is the FVS and obviously the set of determining nodes.

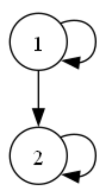

Figure 3. The directed graph associated with DRN (6)

Example 22.

$$(7)\qquad \begin{aligned}\dot{z}_1(t) &= z_1(t) - z_1(t)^3\\ \dot{z}_2(t) &= -z_2(t-1) + z_1(t)^2\end{aligned}$$

Compared to Example 21, $F_2(\psi) := -\psi(-1)$ satisfies (DDC)(I) for $\alpha_2 = 1$ and (DDC)(II) since $\alpha_2\tau_{2,2} = 1$. (DDC)(III) is fulfilled, so the directed graph associated with this DRN (7) is the same as Figure 2.

We prepare some necessary lemmas and theorems before we show the proof of Theorem 20.

6. Variation of constants formula

Throughout this and next sections, we fix $\tau > 0$ and let $C = C([-\tau, 0], \mathbb{R}^n)$.

Theorem 23 (Variation of constants formula [6])**.** For $(\sigma, \phi) \in \mathbb{R} \times C$, consider the linear system

$$(8)\qquad \begin{aligned}\dot{x}(t) &= \int_{-\tau}^{0} d_\theta[\eta(t,\theta)]x_t(\theta) + f(t),\ t \geqq \sigma,\\ x_\sigma &= \phi,\end{aligned}$$

where $f \in \mathcal{L}^1_{loc}([\sigma, \infty), \mathbb{R}^n)$. The $n \times n$ matrix function $\eta(t, s)$ is measurable in (t, s). We suppose that for every $t \in \mathbb{R}$, $\eta(t, s) = \eta(t, -\tau)$ for $s \leqq -\tau$, $\eta(t, s) = 0$ for $s \geqq 0$, and also assume that $\eta(t, s)$ is continuous from the left in s on $(-\tau, 0)$. We suppose that $[t \mapsto \eta(t, \cdot)]$ is continuous and $\eta(t, s)$ has bounded variation in s on $\mathbb{R}$ for each t. Let $k(t, s) := \eta(t, s - t)$ and $r(t, s)$ be the solution of the equation

$$r(t,s) + \int_s^t k(t,u)r(u,s)du = r(t,s) + \int_s^t r(t,u)k(u,s)du = k(t,s),\ \text{for all } t, s \in [\sigma, \infty).$$

We define

$$X(t,\sigma) = \begin{cases} E - \int_\sigma^t r(u,\sigma)du & t \geqq \sigma\\ 0 & t < \sigma,\end{cases}$$

where E is the $n \times n$ identity matrix.

Then there exists a unique solution $x(t; \sigma, \phi, f)$ defined and being continuous on $[\sigma - \tau, \infty)$ that satisfies (8) on $[\sigma, \infty)$. Furthermore, this solution is given by

(9)
$$x(t;\sigma,\phi,f) = X(t,\sigma)\phi(0) + \int_\sigma^t X(t,\alpha)\left(\int_{-\tau}^{\sigma-\alpha} d_\theta[\eta(\alpha,\theta)]\phi(\alpha-\sigma+\theta)\right)d\alpha + \int_\sigma^t X(t,\alpha)f(\alpha)d\alpha.$$

Remark 24. Using the representation (9), we have

$$(10)\qquad x(t;\sigma,\phi,f) = x(t;\sigma,\phi,0) + \int_\sigma^t X(t,\alpha)f(\alpha)d\alpha.$$

We later use the representation of this form.

Remark 25. The function $r(t, s)$ in Theorem 23 is called the *resolvent* of k. Existence and uniqueness of r is proved in [1], [4], [6]. The resolvent $r(t, s)$ is right continuous and Riemann integrable function with respect to t in any closed subinterval of $[\sigma, \infty)$.

Remark 26. For a fixed s, $X(t,s)$ is a solution as a function of t for $t \geqq s+\tau$ to (8) with $f \equiv 0$, that is

$$\dot{x}(t) = \int_{-\tau}^{0} d_\theta[\eta(t,\theta)]x_t(\theta) \tag{11}$$

with respect to the initial data

$$X_0 := X(\cdot, s)_{s+\tau} \tag{12}$$

since if $t \geqq s+\tau$,

$$\begin{aligned}
\frac{\partial X(t,s)}{\partial t} &= -r(t,s) \\
&= \int_s^t k(t,u)r(u,s)du - k(t,s) \\
&= \int_s^t \eta(t,u-t)r(u,s)du - \eta(t,s-t)X(s,s) + \eta(t,0)X(t,s) \\
&= \int_s^t d_u[\eta(t,u-t)]X(u,s) \\
&= \int_{s-t}^0 d_\theta[\eta(t,\theta)]X(\theta+t,s) \\
&= \int_{s-t}^{-\tau} d_\theta[\eta(t,\theta)]X(\theta+t,s) + \int_{-\tau}^0 d_\theta[\eta(t,\theta)]X(\theta+t,s) \\
&= \int_{-\tau}^0 d_\theta[\eta(t,\theta)]X(\theta+t,s).
\end{aligned}$$

We used integration by parts and the assumption that $\eta(t,\theta) = \eta(t,-\tau)$ for $\theta \leqq -\tau$. When $s \leqq t \leqq s+\tau$, $X(t,s)$ satisfies (11) if we think of the right-hand side of (11) as the Lebesgue-Stieltjes integral. We call the matrix solution $X(t,s)$ the *fundamental matrix solution* of (11). We note that the fundamental matrix solution $X(t,s) = E - \int_s^t r(u,s)du$ is continuous with respect to $t \in [s,\infty)$ because $r(\cdot,s)$ is integrable.

Remark 27. For an interval $[-\tau,0]$, $\mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$ denotes the space of bounded variation matrix functions $\eta : [-\tau,0] \to M_n(\mathbb{R})$ such that $\eta(0) = 0$ and $\eta(s)$ is continuous from the left on $(-\tau,0)$. $\|\eta\|_{\mathrm{BV}} := Var_{s\in[-\tau,0]}\eta(s)$ defines a norm on $\mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$ with which $\mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$ forms a Banach space. The restriction $\eta(t,\cdot)|_{[-\tau,0]}$ used in Theorem 23 is in $\mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$.

Remark 28. From the Riesz representation theorem, for any linear mapping $L(t) : C \to \mathbb{R}^n$, there exists an $\eta : \mathbb{R}\times[-\tau,0] \to M_n(\mathbb{R})$ such that $\eta(t,\cdot) \in \mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$, $\|\eta(t,\cdot)\|_{\mathrm{BV}} = \|L(t)\|$ and we can obtain the expression of $L(t)$ by a Riemann-Stieltjes integral

$$L(t)\phi = \int_{-\tau}^{0} d_\theta[\eta(t,\theta)]\phi(\theta)$$

for all t. We can get the continuity of the mapping $\mathbb{R} \ni t \mapsto \eta(t,\cdot) \in \mathrm{NBV}([-\tau,0],M_n(\mathbb{R}))$ if we assume $t \mapsto L(t)$ is continuous.

7. STABILITY

In this section we consider the homogenous linear equation (11) where η satisfies the assumptions of Theorem 23. Let $x(t;\sigma,\phi)$ denote the solution of (11) through (σ,ϕ).

Definition 29. A collection of two-parameter family of bounded linear operators $T(t,s)$ $(t \geqq s)$ on a real Banach space $\mathcal{B}$ is called an *evolutionary system* on $\mathcal{B}$ if

- $T(s,s) = I$
- $T(u,t)T(t,s) = T(u,s)$, $u \geqq t \geqq s$.

A *solution operator* of Equation (11) is defined to be an evolutionary system T on C given by $T(t,s)\phi := x(\cdot;s,\phi)_t$. Let $T(t,s)$ be the solution opeartor of (11) and $X(t,s)$ be the fundamental solution of (11).

Lemma 30 ([6])**.** The following statements are equivalent:

(I) The solution $x = 0$ is uniformly bounded.
(II) The solution $x = 0$ is uniformly stable.
(III) There is a constant $B_0 > 0$ such that for all $\sigma \in \mathbb{R}$, $|T(t,\sigma)| \leqq B_0$, $t \geqq \sigma$.

Proof. (I⇒III) From the uniform boundedness, for $\alpha = 1$, we can take $B_0 > 0$ such that for all $\sigma \in \mathbb{R}$ and $\phi \in C_\alpha$, we have $\|T(t,\sigma)\phi\| = \|x(\cdot;\sigma,\phi)_t\| \leqq B_0$ for $t \geqq \sigma$. Hence $|T(t,\sigma)| \leqq B_0$ for $t \geqq \sigma$.

(III⇒II) For any $\epsilon > 0$, we choose $\delta(\epsilon)$ so that $0 < \delta(\epsilon) < \frac{\epsilon}{B}$. Then, for any $\phi \in C_{\delta(\epsilon)}$, $\sigma \in \mathbb{R}$ and $t \geqq \sigma$,

$$\begin{aligned} \|x(\cdot;\sigma,\phi)_t\| &\leqq |T(t,\sigma)|\|\phi\| \\ &\leqq B_0\delta(\epsilon) \\ &< \epsilon. \end{aligned}$$

(II⇒I) From the uniform stability, for $\epsilon = 1$, we can take $\delta(\epsilon = 1) > 0$ such that for any $\phi \in C_{\delta(1)}$, $\sigma \in \mathbb{R}$ and $t \geqq \sigma$, we have $\|x(\cdot;\sigma,\phi)_t\| < \epsilon = 1$. For any $\alpha > 0$, we choose $\beta(\alpha) = \frac{\alpha}{\delta(1)}$. Then, for all $\phi \in C_\alpha$, $\sigma \in \mathbb{R}$ and $t \geqq \sigma$, we have $\|\frac{\delta(1)}{\alpha}x(\cdot;\sigma,\phi)_t\| = \|x(\cdot;\sigma,\frac{\delta(1)}{\alpha}\phi)_t\| < 1$ since $\frac{\delta(1)}{\alpha}\phi \in C_{\delta(1)}$. Therefore, $\|x(\cdot;\sigma,\phi)_t\| < \frac{\alpha}{\delta(1)} = \beta(\alpha)$. ☐

Lemma 31 ([6])**.** The following statements are equivalent:

(I) The solution $x = 0$ is uniformly asymptotically stable.
(II) There is a constant $B > 0$, $\beta > 0$ such that for all $s \in \mathbb{R}$, $|T(t,s)| \leqq Be^{-\beta(t-s)}$, $t \geqq s$.

Proof. (I⇒II) From the uniform asymptotic stability, there is $\gamma > 0$ such that for any $q > 0$, there is a $t_0(q) > 0$ such that for all $\phi \in C_\gamma$, we have $T(t,\sigma)\phi \in C_{\gamma q}$ for $t \geqq \sigma + t_0(q)$ for every $\sigma \in \mathbb{R}$. We choose $0 < q < 1$ and take a $t_0 = t_0(q) > 0$. Then for all $\phi \in C_1$, we have $\|T(t,\sigma)(\gamma\phi)\| < \gamma q$ for all t and σ satisfying $t - \sigma \geqq t_0$ since $\gamma\phi \in C_\gamma$. Thus, for all t and σ with $t - \sigma \geqq t_0$, $|T(t,\sigma)| \leqq q$ holds. We note that uniform asymptotic stability implies uniform stability in general. By using B_0 from Lemma 30, we define $\beta := -t_0^{-1}\log(q)$ and $B := B_0e^{\beta t_0}$. We fix $s \in \mathbb{R}$ and $t \geqq s$, and take a $n \in \mathbb{N}_{\geqq 0}$ so that $nt_0 \leqq t - s < (n+1)t_0$. Then,

$$\begin{aligned} |T(t,s)| &\leqq |T(t, s+nt_0)||T(s+nt_0, s)| \\ &\leqq B_0|T(s+nt_0, s)| \\ &\leqq B_0|T(s+nt_0, s+(n-1)t_0)||T(s+(n-1)t_0, s)| \\ &\leqq B_0 q|T(s+(n-1)t_0, s)| \\ &\leqq B_0 q^n \\ &= B_0 e^{-\beta n t_0} \\ &= B_0 e^{\beta t_0} e^{-\beta(n+1)t_0} \\ &\leqq B e^{-\beta(t-s)}. \end{aligned}$$

(II⇒I) For any $\epsilon > 0$, we choose $\delta(\epsilon)$ so that $0 < \delta(\epsilon) < \frac{\epsilon}{B}$. Then, for any $\phi \in C_{\delta(\epsilon)}$, $\sigma \in \mathbb{R}$ and $t \geqq \sigma$,

$$\begin{aligned} \|x(\cdot;\sigma,\phi)_t\| &\leqq |T(t,\sigma)|\|\phi\| \\ &\leqq Be^{-\beta(t-\sigma)}\delta(\epsilon) \\ &\leqq c\delta(\epsilon) \\ &< \epsilon. \end{aligned}$$

Therefore, the solution $x = 0$ is uniformly stable.

For any $q > 0$, we choose $t_0(q) > 0$ so that $t_0(q) > -\frac{1}{\beta}\log(\frac{q}{B})$. Then, for any $\phi \in C_1$, $\sigma \in \mathbb{R}$ and $t \geqq \sigma + t_0(q)$,

$$\begin{aligned} \|x(\cdot;\sigma,\phi)_t\| &\leqq |T(t,\sigma)|\|\phi\| \\ &\leqq Be^{-\beta(t-\sigma)} \\ &\leqq Be^{-\beta t_0(q)} \\ &< q. \end{aligned}$$

□

Lemma 32 ([6])**.** Assume that there exists a constant m_1 such that

$$\int_t^{t+\tau} \|\eta(u,\cdot)\|_{\mathrm{BV}} du \leqq m_1$$

holds for all $t \in \mathbb{R}$ and assume there is a constant $B_0 > 0$ such that for all $\sigma \in \mathbb{R}$, $|T(t,\sigma)| \leqq B_0$, $t \geqq \sigma$. Then there is a constant $A_0 > 0$ such that for all $s \in \mathbb{R}$, $|X(t,s)| \leqq A_0$, $t \geqq s$

Proof. The fundamental solution $X(t,s)$ is a solution of (11), thus

$$X(t,s) = X(s,s) + \int_s^t \int_{-\tau}^0 d_\theta[\eta(u,\theta)]X(\theta+u,s)du, \ t \geqq s.$$

Therefore, for $s \leqq t \leqq s+\tau$,

$$\begin{aligned}|X(t,s)| &\leqq c_0 + \int_s^t \left| \int_{-\tau}^0 d_\theta[\eta(u,\theta)]X(\theta+u,s)\right| du \\ &= c_0 + \int_s^t \left| \int_{s-u}^0 d_\theta[\eta(u,\theta)]X(\theta+u,s)\right| du \\ &\leqq c_0 + \int_s^t Var_{\theta\in[s-u,0]}\eta(u,\theta)\|X(\cdot,s)_u\|du \\ &\leqq c_0 + \int_s^t \|\eta(u,\cdot)\|_{\mathrm{BV}}\|X(\cdot,s)_u\|du,\end{aligned}$$

where $c_0 := |X(s,s)| = |E|$. The same inequality holds for $t \geqq s+\tau$.

When $s-\tau \leqq k \leqq s$, we have $|X(k,s)| \leqq c_0$. When $s \leqq k \leqq t$, we have

$$\begin{aligned}|X(k,s)| &\leqq c_0 + \int_s^k \|\eta(u,\cdot)\|_{\mathrm{BV}}\|X(\cdot,s)_u\|du \\ &\leqq c_0 + \int_s^t \|\eta(u,\cdot)\|_{\mathrm{BV}}\|X(\cdot,s)_u\|du.\end{aligned}$$

Thus, if $s-\tau \leqq k \leqq t$,

$$|X(k,s)| \leqq c_0 + \int_s^t \|\eta(u,\cdot)\|_{\mathrm{BV}}\|X(\cdot,s)_u\|du$$

holds. Using this, we have

$$\begin{aligned}\|X(\cdot,s)_t\| &= \sup_{\theta\in[-\tau,0]} |X(t+\theta,s)| \\ &= \sup_{k\in[t-\tau,t]} |X(k,s)| \\ &\leqq \sup_{k\in[s-\tau,t]} |X(k,s)| \\ &\leqq c_0 + \int_s^t \|\eta(u,\cdot)\|_{\mathrm{BV}}\|X(\cdot,s)_u\|du.\end{aligned}$$

From Grönwall's inequality, we have

$$\|X(\cdot,s)_t\| \leqq M \exp \int_s^t \|\eta(u,\cdot)\|_{\mathrm{BV}}du,\ t \geqq s.$$

Consequently, if $s \leqq t \leqq s+\tau$,

$$\|X(\cdot,s)_t\| \leqq M \exp \int_s^{s+\tau} \|\eta(u,\cdot)\|_{\mathrm{BV}}du \leqq Me^{m_1}. \tag{13}$$

If $t \geqq s+\tau$,

$$\begin{aligned}\|X(\cdot,s)_t\| &\leqq |T(t,s+\tau)|\|X(\cdot,s)_{s+\tau}\| \\ &\leqq B_0Me^{m_1}.\end{aligned}$$

In either case, the conclusion holds. □

We recall an important lemma from [6], to which we give a proof using an argument in [8].

Lemma 33 ([6]). Assume that there exists a constant m_1 such that

$$\int_t^{t+\tau} \|\eta(u,\cdot)\|_{\mathrm{BV}} du \leqq m_1$$

holds for all $t \in \mathbb{R}$ and assume the zero solution of (11) is uniformly asymptotically stable. Then there is a constant $A > 0$, $\beta > 0$ such that for all $s \in \mathbb{R}$, $|X(t,s)| \leqq Ae^{-\beta(t-s)}, t \geqq s$.

Proof. From Lemma 31, we can choose a constant $B > 0$ and $\beta > 0$ such that $|T(t,s)| \leqq Be^{-\beta(t-s)}$ for all $t \geqq s$. We fix $\xi \in \mathbb{R}^n$ and define $y(t;s,\xi) := X(t,s)\xi$. Let $\psi_\xi := y(\cdot;s,\xi)_{s+\tau} \in C$. $y(t;s,\xi)$ is the solution of (11) for $t \geqq s+\tau$ with respect to the initial function ψ_ξ.

If $t \geqq s+\tau$,

$$\begin{aligned}
|X(t,s)\xi| &= |y(t;s,\xi)| \\
&\leqq \|y(\cdot;s,\xi)_t\| \\
&= \|T(t,s+\tau)y(\cdot;s,\xi)_{s+\tau}\| \\
&= |T(t,s+\tau)|\|\psi_\xi\| \\
&\leqq Be^{-\beta(t-s-\tau)}\|\psi_\xi\| \\
&= Be^{\beta\tau}e^{-\beta(t-s)} \sup_{\theta\in[-\tau,0]} |y(\theta+s+\tau;s,\xi)| \\
&= Be^{\beta\tau}e^{-\beta(t-s)} \sup_{u\in[s,s+\tau]} |X(u,s)\xi| \\
&= Be^{\beta\tau} \sup_{u\in[s,s+\tau]} |X(u,s)|e^{-\beta(t-s)}|\xi|
\end{aligned}$$

so, we have

$$\begin{aligned}
|X(t,s)| &\leqq Be^{\beta\tau} \sup_{u\in[s,s+\tau]} |X(u,s)|e^{-\beta(t-s)} \\
&\leqq A_0 Be^{\beta\tau}e^{-\beta(t-s)}.
\end{aligned}$$

We used the evaluation

$$\sup_{u\in[s,s+\tau]} |X(u,s)| \leqq A_0$$

from Lemma 32.

If $s \leqq t \leqq s+\tau$,

$$\begin{aligned}
|X(t,s)| &\leqq \left(\sup_{t\in[s,s+\tau]} |X(t,s)| \right) e^{\beta(t-s)}e^{-\beta(t-s)} \\
&\leqq \left(\sup_{t\in[s,s+\tau]} |X(t,s)|e^{\beta\tau} \right) e^{-\beta(t-s)} \\
&\leqq A_0 e^{\beta\tau}e^{-\beta(t-s)}.
\end{aligned}$$

In either case, the conclusion holds. □

We have finished the preparation for the proof of Theorem 20.

8. PROOF OF THE MAIN RESULT

We prove Theorem 20 in a similar way as the proof of Theorem 8 in [3]. Let $\tau := \max_{k,j\in\{1,\ldots,N\}} \tau_{k,j}$.

Lemma 34 (labeling order [3])**.** Let $\Gamma = (\{1,\ldots,N\},E)$ be a di-graph and I be a FVS of Γ. We can relabel the vertices of Γ such that

$$\begin{aligned} J &= \{1,\cdots,N\}\backslash I = \{1,\cdots,N'\}\ (N' = N - |I|) \\ I &= \{N'+1,\cdots,N\} : FVS \end{aligned}$$

and for all $k \in J$, we have

- $I_k \subset I \cup \{1,\cdots,k-1\}\ (k \geqq 2)$
- $I_1 \subset I$

Our proof will be based on this labeling order.

Lemma 35. Let z and $\tilde{z}$ be arbitrary two solutions of (4) and w be the difference between z and $\tilde{z}$, that is, $w(t) = \tilde{z}(t) - z(t)$. Then, $w(t)$ satisfies

$$\dot{w}_k(t) = L_k(t)\,(w_k)_t^{(\tau_{k,k})} + \sum_{j\in I_k} h_j(t) \quad (k = 1,\cdots,N) \tag{14}$$

where we define $L_k(t) : C(\tau_{k,k}) \to \mathbb{R}\ \ (t \in \mathbb{R}),\ \ h_j : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} L_k(t)\phi &= \int_0^1 D_k F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)(\phi)\,d\theta, \\ h_j(t) &= \int_0^1 D_j F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)\left((w_j)_t^{(\tau_{k,j})}\right)d\theta. \end{aligned}$$

Proof.

$$\begin{aligned} \dot{w}_k(t) &= \dot{\tilde{z}}_k(t) - \dot{z}_k(t) \\ &= \left[F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)\right]_{\theta=0}^{\theta=1} \\ &= \int_0^1 \frac{d}{d\theta} F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)d\theta \\ &= \int_0^1 D_k F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)\left((w_k)_t^{(\tau_{k,k})}\right)d\theta \\ &+ \sum_{j\in I_k}\int_0^1 D_j F_k\left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)\left((w_j)_t^{(\tau_{k,j})}\right)d\theta. \end{aligned}$$

□

Remark 36. When we fix z and $\tilde{z}$, the mapping $L_k(t)$ is linear. However the equation (14) in itself is not a linear equation in terms of w_k. In fact, $L_k(t)$ depends on w_k.

Now fix z and $\tilde{z}$, and consider the linear nonhomogeneous equation in terms of x

$$\begin{aligned} &\dot{x}_k(t) = L_k(t)\,(x_k)_t^{(\tau_{k,k})} + \sum_{j\in I_k} h_j(t),\ t \geqq \sigma, \\ &(x_k)_\sigma^{(\tau_{k,k})} = (w_k)_\sigma^{(\tau_{k,k})}, \end{aligned} \tag{15}$$

for each $k = 1, \dots, N$. We take an initial time $\sigma \geqq T + \tau$ arbitrarily. By Lemma 35, w_k is the solution of (15). To solve (15), we use Theorem 23 (the variation of constants formula [6]).

Remark 37. Regarding the equation (15), from the Riesz representation theorem, we can obtain the expression of $L_k(t)$ by a Riemann-Stieltjes integral

$$L_k(t)\phi = \int_{-\tau_{k,k}}^{0} d_\theta[\eta_k(t,\theta)]\phi(\theta), \tag{16}$$

where $\eta_k(t,\cdot)|_{[-\tau_{k,k},0]} \in \mathrm{NBV}([-\tau_{k,k},0],\mathbb{R})$ and $\|\eta_k(t,\cdot)\|_{\mathrm{BV}} = \|L_k(t)\|$. From the assumotion (C), the mapping $[t \mapsto L_k(t)]$ is continuous. Consequently, $t \mapsto \eta_k(t,\cdot) \in \mathrm{NBV}([-\tau_{k,k},0],\mathbb{R})$ is continuous.

Example 38. We define $L : C(\tau) \to \mathbb{R}$ by $L\phi := a\phi(0) + b\phi(-\tau)$ for $a, b \in \mathbb{R}$. If we define $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(\theta) = \begin{cases} -a-b & (\theta \leqq -\tau) \\ -a & (-\tau < \theta < 0) \\ 0 & (0 \leqq \theta), \end{cases}$$

then we have $L\phi = \int_{-\tau}^{0} d\eta(\theta)\phi(\theta)$.

We now prove the theorem 20 by induction based on the labeling order.

Proof of Theorem 20. We want to show that the FVS I is a set of determining nodes, so we assume that for any fixed two solutions of (4) z and $\tilde{z}$,

$$\tilde{z}_j(t) - z_j(t) \xrightarrow[t\to\infty]{} 0 \text{ for all } j \in I = \{N'+1, \dots, N\} \tag{17}$$

and show

$$\tilde{z}_k(t) - z_k(t) \xrightarrow[t\to\infty]{} 0 \text{ for all } k \in J = \{1, \dots, N\} \backslash I = \{1, \dots, N'\} \tag{18}$$

by induction over k. As an induction hypothesis, we assume, for fixed $k \in \{2, \dots, N'\}$,

$$\tilde{z}_j(t) - z_j(t) \xrightarrow[t\to\infty]{} 0 \text{ for all } j \in \{1, \dots, k-1\}. \tag{19}$$

Using the convergences (17) and (19) for a fixed $k \in \{2, \dots, N'\}$, we shall show

$$\tilde{z}_k(t) - z_k(t) \xrightarrow[t\to\infty]{} 0. \tag{20}$$

We assume $\tau_{k,k} > 0$. In case $\tau_{k,k} = 0$, we can prove (20) in the same way as in [3]. Using the representation (10), we have the solution of (15)

$$x_k(t;\sigma,\phi,\sum_{j\in I_k} h_j(t)) = x_k(t;\sigma,\phi,0) + \sum_{j\in I_k} \int_\sigma^t X_k(t,s)h_j(s)ds, \tag{21}$$

where $X_k(t,s)$ is the fundamental solution of

$$\dot{x_k}(t) = L_k(t)\,(x_k)_t^{(\tau_{k,k})}. \tag{22}$$

We show each term of (21) converges to 0. The first term on the right hand of (21) is the solution of (22). By the assumption (A), $\|(z_\ell + \theta w_\ell)_t\| \leqq R$ for all ℓ and $t \geqq \sigma (\geqq T + \tau)$, and the nonlinearity F_k satisfies (DDC). According to the Theorem 11, the zero solution of (22) is uniformly asymptotically stable. The equation (22) is a linear equation, so from Lemma 31, $x_k(t;\sigma,\phi,0)$ converges to 0.

The nonlinearity F_k satisfy (DDC)(I). Thus, we have

$$-\alpha_k M^{(\tau_{k,k})}(-\phi) \leqq -D_k F_k \left(t, (z_k+\theta w_k)_t^{(\tau_{k,k})}, \left\{(z_j+\theta w_j)_t^{(\tau_{k,j})}\right\}_{j\in I_k}\right)(\phi) \leqq \alpha_k M^{(\tau_{k,k})}(\phi)$$

for $t \geqq \sigma$ and $\phi \in C(\tau_{k,k})$. Hence, we obtain

$$-\alpha_k M^{(\tau_{k,k})}(-\phi) \leqq L_k(t)\phi \leqq \alpha_k M^{(\tau_{k,k})}(\phi),$$

hence

$$\begin{aligned}|L_k(t)\phi| &\leqq \alpha_k \max\{M^{(\tau_{k,k})}(-\phi), M^{(\tau_{k,k})}(\phi)\}\\ &\leqq \alpha_k \|\phi\|^{(\tau_{k,k})}.\end{aligned}$$

By the Riesz theorem, the norm of $L_k(t)$ is equivalent to the total variation of η_k defined by (16). Hence,

$$Var_{s\in[-\tau_{k,k},0]}\eta_k(t,s) = \|L_k(t)\| \leqq \alpha_k.$$

Therefore, η_k satisfies the assumption of Lemma 33. Lemma 33 gives

$$|X_k(t,s)| \leqq C_k e^{-\gamma_k(t-s)}. \tag{23}$$

By the assumption (B) and the uniform boundedness principle, for all $k \in \{1,\dots N\}$, $j \in I_k$, there exists a positive constant $B_{k,j}$ such that $|D_j F_k(t, \psi_k, \{\psi_s\}_{s\in I_k})(\phi)| \leqq B_{k,j}\|\phi\|^{(\tau_{k,j})}$ for all $t \geqq 0$, $\phi \in C(\tau_{k,j})$ and $\psi_\ell \in C(\tau_{k,\ell})$ with $\|\psi_\ell\|^{(\tau_{k,\ell})} \leqq R$ (for all $\ell \in \{1,\dots,N\}$). Thus, we have

$$|h_j(s)| \leqq B_{k,j}\left\|(w_j)_s^{(\tau_{k,j})}\right\|^{(\tau_{k,j})}, \ s \geqq \sigma. \tag{24}$$

Therefore, we can estimate the second term on the right hand of (21) as follows:

$$\left|\int_\sigma^t X_k(t,s)h_j(s)ds\right| \leqq C_k B_{k,j}\int_\sigma^t e^{-\gamma_k(t-s)}\left\|(w_j)_s^{(\tau_{k,j})}\right\|^{(\tau_{k,j})} ds. \tag{25}$$

We show that the integral on the right-hand side converges to 0 by the Lebesgue dominated convergence theorem in the same manner as Fiedler-Mochizuki theory, namely

$$\begin{aligned}\int_\sigma^t e^{-\gamma_k(t-s)}\left\|(w_j)_s^{(\tau_{k,j})}\right\|^{(\tau_{k,j})} ds &= \int_0^{t-\sigma} e^{-\gamma_k u}\|(w_j)_{t-u}^{(\tau_{k,j})}\|^{(\tau_{k,j})}du\\ &= \int_{\mathbb{R}\geq 0} e^{-\gamma_k u}\mathbb{1}_{[0,t-\sigma]}(u)\|(w_j)_{t-u}^{(\tau_{k,j})}\|^{(\tau_{k,j})}du\end{aligned} \tag{26}$$

Note that $\|(w_j)_t^{(\tau_{k,j})}\|^{(\tau_{k,j})}$ converges to 0 for $t \to 0$ since $j \in I_k \subset I \cup \{1,\dots,k-1\}$ and we supposed (17) and (19). Therefore, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|(w_j)_s^{(\tau_{k,j})}\|^{(\tau_{k,j})} < \epsilon$ holds for $s > \delta$. Owing to the continuity of $s \mapsto \|(w_j)_s^{(\tau_{k,j})}\|^{(\tau_{k,j})}$, $\|(w_j)_s^{(\tau_{k,j})}\|^{(\tau_{k,j})}$ is bounded on $[0,\delta]$. Thus, $\|(w_j)_s^{(\tau_{k,j})}\|^{(\tau_{k,j})}$ is bounded on $[0,\infty)$. Let M be an upper bound of $\|(w_j)_s^{(\tau_{k,j})}\|^{(\tau_{k,j})}$, then, for all $t \geqq 0$ and $u \geqq 0$, we have

$$e^{-\gamma_k u}\mathbb{1}_{[0,t-\sigma]}(u)\|(w_j)_{t-u}^{(\tau_{k,j})}\|^{(\tau_{k,j})} \leqq Me^{-\gamma_k u}.$$

Observe that $e^{-\gamma_k u}$ is integrable and $e^{-\gamma_k u}\mathbb{1}_{[0,t-\sigma]}(u)\|(w_j)_{t-u}^{(\tau_{k,j})}\|^{(\tau_{k,j})} \to 0$ as $t \to 0$, and hence, the integral (26) converges to 0. □

9. Other results in Fiedler, Mochizuki et al [3]

In [3], another theorem about a global attractor of a regulatory network was proved. We briefly review the theorem and give some remark for extending this theorem to time-delay systems.

For any subset $J \subset \{1, \dots, N\}$ and $z = (z_1, \dots z_N) \in \mathbb{R}^N$, let $z_J := (z_k)_{k \in J} \in \mathbb{R}^{|J|}$. For $z_0 \in \mathbb{R}^N$, $z(t; z_0)$ denotes the solution of an autonomous regulatory network (1) with an initial condition $z(0) = z_0$. For any subset $D \subset \mathbb{R}^N$, $z(t; D)$ denotes the set of solutions $z(t; z_0)$ for $z_0 \in D$, namely, $z(t; D) := \{z(t, z_0) | z_0 \in D\}$. For any subset $A, B \subset \mathbb{R}^N$, $\mathrm{dist}(A, B)$ denote the Hausdorff semidistance between A and B, which is defined as

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

Definition 39 (Global attractor)**.** A compact set $\mathcal{A} \subset \mathbb{R}^N$ is the global attractor for an autonomous regulatory network (1) if $z(t; \mathcal{A}) = \mathcal{A}$ for all t, and for each bounded subset $B \subset \mathbb{R}^N$, $\mathrm{dist}(z(t; B), \mathcal{A}) \to 0$ as $t \to \infty$.

Theorem 40 ([3] Theorem 1.6)**.** Consider an autonomous regulatory network. Let $\Gamma = (\{1, \dots, N\}, E)$ be a given digraph, I be a FVS of Γ and $\mathcal{A}$ be the global attractor. Then the continuous projection

$$\mathcal{P}_I : \mathcal{A} \to BC^2(\mathbb{R}, \mathbb{R}^{|I|})$$
$$z_0 \mapsto z_I(\cdot; z_0)$$

is injective.

Remark 41. For the (unique) existence of a global attractor, the assumption (A') below is sufficient [2]:

(A'): There exists a nonempty compact subset $K \subset \mathbb{R}^N$ such that for all bounded subset $D \in \mathbb{R}^N$, there exists a positive time $t_1(D)$ such that $z(t; D) \subset K$ for all $t \geqq t_1(D)$.

This assumption (A') seems different from the assumption (A) for RN:

(A): There exists a positive constant R such that for any $z_0 \in \mathbb{R}^N$, there exists a positive time $T(z_0)$ such that $|z(t; z_0)| \leqq R$, for all $t \geqq T(z_0)$.

In case of RN in [3], (A') implies (A), and actually the converse is also true since the phase space is $\mathbb{R}^N$ and the system we consider is autonomous [9]. Thus, the existence of the global attractor in Theorem 40 is guaranteed.

Remark 42. If there exists a global attractor $\mathcal{A}$, it can be characterized as the collection of globally defined and bounded solutions [2], namely,

$$\mathcal{A} = \{z_0 \in \mathbb{R}^N | \text{there is a bounded global solution } z(t; z_0)\}.$$

We use this characterization when we prove the Theorem 40.

In extending Theorem 40 to time-delay systems (4), we have to note that the phase space of time-delay system is in general an infinite dimensional space $C([-\tau, 0], \mathbb{R})^N \simeq C([-\tau, 0], \mathbb{R}^N)$. For $\phi \in C([-\tau, 0], \mathbb{R}^N)$, $z(t; \phi)$ denotes the solution of an autonomous DRN (4) with an initial condition $z_0 = \phi$. For any subset $D \subset C([-\tau, 0], \mathbb{R}^N)$, $z(\cdot; D)_t$ denotes the set of $z(\cdot; \phi)_t$ for $\phi \in D$, namely, $z(\cdot; D)_t := \{z(\cdot; \phi)_t | \phi \in D\}$. For any subset $A, B \subset C([-\tau, 0], \mathbb{R}^N)$, $\mathrm{dist}(A, B)$ denote the Hausdorff semidistance between A and B, which is defined as

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|.$$

Definition 43 (Global attractor in time-delay systems)**.** A compact set $\mathcal{A} \subset C([-\tau, 0], \mathbb{R}^N)$ is the global attractor for an autonomous delayed regulatory network (4) if $z(\cdot; \mathcal{A})_t = \mathcal{A}$ for all t, and for each bounded subset $B \subset C([-\tau, 0], \mathbb{R}^N)$, $\mathrm{dist}(z(\cdot; B)_t, \mathcal{A}) \to 0$ as $t \to \infty$.

Most importantly, any closed ball is non-compact in an infinite dimensional space $C([-\tau, 0], \mathbb{R}^N)$. Thus, the assumption (A) for an autonomous DRN does not seem to be appropriate for the existence of a global attractor. Therefore, we use the assumption (A') for an autonomous DRN.

(A'): There exists a nonempty compact subset $K \subset C([-\tau, 0], \mathbb{R}^N)$ such that for all bounded subset $D \in C([-\tau, 0], \mathbb{R}^N)$, there exists a positive time $t_1(D)$ such that $z(\cdot; D)_t \subset K$ for all $t \geqq t_1(D)$.

There are other hypotheses to obtain the existence of a global attractor in an infinite dimensional phase space [2], [5], [7]. For now, we do not carefully examine what assumption we should use. If there exists a global attractor $\mathcal{A}$ for an autonomous DRN, we have

$$\mathcal{A} = \{\phi \in C([-\tau, 0], \mathbb{R}^N) | \text{there is a bounded global solution } z(t; \phi)\}.$$

When we assume (A'), there exists $R' > 0$ such that $\mathcal{A} \subset K \subset C_{R'}$, and $R > 0$ in (DDC) should be replaced for R'. Then, we may be able to prove that the projection $\mathcal{P}_I$ is injective without the assumption (B).

Theorem 44. Consider an autonomous delayed regulatory network. We do not assume (B). Let $\Gamma = (\{1, \dots, N\}, E)$ be a given digraph, I be a FVS of Γ and $\mathcal{A}$ be the global attractor. Then the continuous projection

$$\begin{aligned} \mathcal{P}_I : \mathcal{A} &\to BC^2(\mathbb{R}, \mathbb{R}^{|I|}) \\ z_0 &\mapsto z_I(\cdot; z_0) \end{aligned}$$

is injective.

The proof is the same as that of Theorem 20.

Acknowledgements

I would like to thank my advisor, Prof.Hiroshi Kokubu, for helping to progress my research and writing, and Dr.Tomoyuki Miyaji for useful discussions. I am deeply grateful to Dr.Atsushi Mochizuki for giving a lecture about Fiedler-Mochizuki theory and suggesting the topic in this research. I would like to express my gratitude to Dr.Junya Nishiguchi. He shared his knowledge of variation of constants formula and the fundamental solution for time-delay systems.

References

[1] J. Blot and M. I. Koné. Resolvent of nonautonomous linear delay functional differential equations. *Nonautonomous Dynamical Systems*, 2(1), 2015.

[2] A. Carvalho, J. A. Langa, and J. Robinson. *Attractors for infinite-dimensional non-autonomous dynamical systems*, volume 182. Springer Science & Business Media, 2012.

[3] B. Fiedler, A. Mochizuki, G. Kurosawa, and D. Saito. Dynamics and control at feedback vertex sets. I: Informative and determining nodes in regulatory networks. *Journal of Dynamics and Differential Equations*, 25(3):563–604, 2013.

[4] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*. Number 34. Cambridge University Press, 1990.

[5] J. K. Hale. *Asymptotic behavior of dissipative systems*. Number 25. American Mathematical Soc., 2010.

[6] J. K. Hale and S. M. V. Lunel. *Introduction to Functional Differential Equations*, volume 99. Springer Science & Business Media, 1993.

[7] P. Kloeden and M. Rasmussen. *Nonautonomous Dynamical Systems*. American Mathematical Society, 2011.

[8] J. Nishiguchi. Mild solutions, variation of constants formula, and linearized stability for delay differential equations. *Electronic Journal of Qualitative Theory of Differential Equations*, (32):1–77, 8 2023.

[9] J. C. Robinson. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, volume 28. Cambridge University Press, 2001.

[10] Y. Takashima, T. Ohtsuka, A. González, H. Miyachi, and R. Kageyama. Intronic delay is essential for oscillatory expression in the segmentation clock. *Proceedings of the National Academy of Sciences*, 108(8):3300–3305, 2011.

[11] J. A. Yorke. Asymptotic stability for one dimensional differential-delay equations. *Journal of Differential equations*, 7(1):189–202, 1970.