Essays on Strategic Information Transmission and Spreading Information

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Abstract

This dissertation investigates the transmission of information, exploring both intentional and unintentional aspects. Chapter 1 summarizes the background and provides an overview.

Chapter 2 analyzes truth-telling behavior in a dynamic cheap talk game with binary types. An expert receives signals and provides recommendations regarding the state of the world over time until the state is publicly revealed. The aim of the expert is to maximize the reputation of the information acquisition ability on which the precision of the signals depend. Under this circumstance, giving different recommendations may be seen as a sign of a poor information acquisition ability, but it can also work as a "safety net" that prevents the worst reputation. Focusing on equilibria where all signals are delivered promptly, I propose two truth-telling strategies. One necessitates telling the truth at any history, while the other demands truth-telling if it has happened before.

Chapter 3 shares a similar model with Chapter 2, but with more general information structure. It reaffirms the findings from Chapter 2 as robust to an extent. Notably, the sufficient condition that the expert prefers truth-telling than any other less informative strategy, continues to hold under games of finite periods with any information structure. Furthermore, it highlights that the challenge of truth-telling is not solely due to dynamicity but significantly involves multi-dimensionality. A game in which the expert does not send messages until the last period before the state reveals, is essentially identical with a static game in which the signal is multi-dimensional. In such games, truth-telling equilibria may not be feasible, in contrast to a static game with a unidimensional signal where truth-telling emerges as a dominant strategy.

Chapter 4 studies a rational bubble model, established on network environments. Players engage in trading a unit of indivisible good within a network where only one of them values it. They do not know which network they exactly belong; but they recognize their own neighbor set, which is used to infer the whole network. There arises information asymmetricity due to the difference in scope; and it would be reflected in the price. I define network bubbles, a trade at a positive price that occurs when every player knows that the good cannot eventually reach the player. I found a necessary condition for the probability space of networks to have an equilibrium with network bubble. It poses a severe restriction; otherwise, information asymmetricity collapses.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

By strategic and non-strategic information transfers, Bayesian individuals update their information and infer to the true state. Strategic information transfer include cheap-talk (Crawford and Sobel (1982)), persuasion (Milgrom (1981) and Milgrom and Roberts (1986)) and disclosure (Dye (1986) and Gigler (1994)). Non-strategic information transmission includes learning by experiments such as strategic experimentation[1] (Bolton and Harris (1999)). In this dissertation, I explore the multifaceted landscape of information transmission, where the transmission occurs both by design and by happenstance. I delve into the intricacies of how individuals, endowed with distinct types and preferences, choose to convey their private knowledge and attitudes, shaping the outcomes of diverse interactions.

This thesis is divided into two folds. The first fold is about strategic non-cost-incurring messages on non-verifiable information to enhance the expectation about type – reputational cheap talks. Canonical models on cheap talks about preference of Sender, drive equilibria by preference alingedness of Sender and Receiver. Because optimal targets of Sender and Receiver are somewhat close, they can settle on some amount of noise in equilibrium. In cheap talks about being well-informedness of Sender, Sender has monotonically increasing payoffs in the beliefs of Receiver. In such models, possibilities of outcomes that

[1]In the literature, an agent is both Sender and Receiver at the same time. When translating the agent as Receiver, then the Sender here, is nature and not strategic. When translating the agent as Sender, s/he strategically decides how to gather and release information.

lower Sender's reputation, lead to equilibrium. Chapters 2 and 3 center around truthful outcomes in reputational cheap talk about information acquisiton ability with a dynamic model based on Tajika (2021). In Chapter 2, I introduce two strategies, which lead to truth-telling on their respective paths. In Chapter 3, I generalize the results in Chapter 2 and consider the factors that makes truth-telling difficult.

The second fold is dedicated to individuals learning from observations. Wise wisdom says you cannot have bubble under a common ex-ante prior, when agents are rational anticipate and it is common knowledge (Milgrom and Stokey (1982)). Recent studies on rational bubble, thus focused on information asymmetricity, were initiated by Allen et al. (1993), and applied game-theoretically by Awaya et al. (2022) in a network. When trading, the value of the good depends on the state which also affects trade histories, price, and the willingness-to-pay of individuals. As time flows, individuals learn more and more about the state even when the initial information was asymmetric. Chapter 4 considers rational bubbles when there is uncertainty about the networks. In the model, prices coincide to the willingness-to-pay's due to private information reflected on the attitude, thus learnings during the game occur.

The rest of this thesis is organized as follows. Chapter 2 analyzes truth-telling outcomes in a two-period reputational cheap-talk model with binary types. An expert receives multiple pieces of singals, which become more precise over time. The expert strategically provides recommendations to the public, aiming to enhance the reputation through carefully crafted advice. At the end of the game, the true state, and thereby, whether the recommendations were correct or not, are revealed; and the expert is evaluated by the Bayesian public. The model is based on Tajika (2021) who focused on equilibria in which truth-telling occurs sometimes, and bubbling occurs other times on the paths. I focus on equilbria in which truth-telling occurs all the time. Those are sustained under certain conditions composed of two dimensions: the information structure of the signals and the ex ante prior on the ability of the expert.

The results show that the expert's chosen strategy can determine whether a society attains a truth-telling equilibrium. The commitment to behave truthfully across all histories is more difficult to be achieved than the one to behave truthfully after truthful histories, in the sense that the condition sustains the former implies the latter; and the reverse is not true. It sheds lights both on the importance of off-path behavior in cheap-talk environments and on a normative question: should an individual be truthful all the time. The expert ex-ante prefers truth-telling equilibria than ones in which some information is lost, if s/he is risk-loving. Furthermore, it is robust under monetary transfers that alleviate reputational bias.

Chapter 3 is built based on a similar model, with more general priors and information structure. I confirmed the robustness of the results in Chapter 2 to an extent. Especially, the condition for truth-telling to be the most favored by both Sender and Receiver remains unaltered under any information structure in dynamic reputational cheap-talk games with finite periods. Moreover, I derive a result regarding to robustness of a truth-telling equilibrium at a specific condition that is different from existing one in Tajika (2021). I alter the model to an essentially static one with a multi-dimensional signal space – the expert collects all available information before making a one-shot recommendation. The impact of the change is ambiguous; it relies on the strategy the expert was previously employing and which signal holds more significance in the assessment.

Chatper 4 formulates rational bubbles due to asymmetric information in networks. Players trade a unit of indivisible good between neighbors, while there is only one player who appreciates it. Due to their restricted scope, they do not exactly know which network they are in; but they infer that the network is one of those in which they have the same neighbor set. Using thier information, they form expectation about the value that good will bring to them, which will be reflected in the price. Despite that they learn through the accumulated public history, there may exist an equilibrium in which there is a state such that on path, a player buys the good at a positive price while all players know there is no feasible path from the player to the one who appreciates it, which I call, a network bubble. Necessary conditions for such equilibria are investicated; and an example is provided.

The results indicate that network bubbles require severe restrictions on the probability space. In any network bubbles it is essential there exists a state that the buyer can and the seller cannot distinguish from the bubble state. However, since the price reflects private information of trading parties, the next potential buyer would infer the willingness-to-pay of the current buyer. Nonetheless, I provide an example and an equilibrium with a network bubble, that satisfies neccesary conditions I suggested. In the model, the price rises because the states in which a buyer cannot find a next buyer are gradually excluded as time flows.

Chapter 2

Reputational Cheap-talk in a Dynamic Game with Binary Types

2.1 Introduction

How precise signals are experts receiving? – especially when the signals are unverifiable. Information acquisition abilities are often not public information. Sometimes it is not even private information, which implies that experts themselves do not exactly know their ability. This uncertainty may lead experts to wish to be perceived as having access to more precise information. One might naively assume that providing a truthful report is a best response for an expert when the signals are informative. This assumption stems from the belief that a correct report suggests the expert is more likely to possess high ability (cf. Milgrom (1981)). However, the reality may be more complex. Consider a scenario where an expert truthfully reports the first signal received. If the expert then obtains a second signal that contradicts the initial one, providing another truthful report might damage the expert's reputation. This outcome implies the expert's ability is not sufficiently high to generate two accurate signals. Would it be in the expert's best interest to truthfully report if the second signal aligns with the first one? Surprisingly, this also might not be the case. The expert may prefer to avoid the worst-case scenario – providing two incorrect signals – and opt for a more moderate, and hence safer, reputation.

Furthermore, when experts are reputation-concerned, they may want to control the timing to reveal their information. Sometimes the timing can be determined at infinitely later, which implies *never*.[1] In cheap-talk literature, this is translated to babbling equilibria, in which no information is delivered, and more importantly, which always exist. Under such circumstances, experts may want to choose the timing and contents of their advice strategically, while principals want transparent information.

The main question in this paper is: when we can expect truth-telling equilibria to exist. To answer these questions, I use a model from Tajika (2021): a two-period-two-state game with an expert who receives a series of noisy signals whose accuracy indicates the ability of the expert and increases over time. The state, on which signals depend, is determined at the very first and does not change through the game. The expert is assumed to be not aware of own ability, following Holmström (1999)[2], and to send a cheap-talk message after each signal. At the end of the game, the state is publicly revealed, and the expert is evaluated and receives the corresponding payoff.

In the model, I introduce two truth-telling strategies. In one, the expert always tells the truth; in the other, the expert tells the truth if there were no misreport. Under both strategies, non-empty set of priors that sustain the strategy are characterized. Moreover, if there exists an equilibrium under the former truth-telling strategy, there exists one under the latter. The reverse is not true. The inclusion relationship is strict, and robust under monetary transfer. Given a prior, there exist truth-telling equilibria under both strategy, when the increment in accuracy is sufficiently large. A similar inclusion relationship holds, but it may be weak.

This is because that, having the expert always tell the truth at any history requires having the expert tell the truth after lying in the previous period. This is more than needed to achieve on-path-truth-telling outcomes as, in such equilibrium, the histories

[1]For example, Gratton et al. (2018).
[2]This assumption keeps the calculation simple, not essentially changing the result.

where the expert lied in the first period never come, if played along the strategy. Notice that the latter strategy requires the minimum to achieve truth-telling outcomes. Removing this incentive compatibility condition allows truth-telling equilibria to exist under a more relaxed condition.

This paper analyzes behavior of expert who cares reputation on information acquisition ability (Holmstrom and Costa (1986); Scharfstein and Stein (1990)). When information is unverifiable, incentives for distortion may occur. However, many of which are thought driven by priors of senders and/or receivers, as in Scharfstein and Stein (1990), Levy (2004), and Gentzkow and Shapiro (2006). However, following Tajika (2021), the most related work, the priors are even; but there still are incentives to misreport, caused by statements made by expert him/herself. One of the keys of this paper is established on the difference in out-of-equilibrium behaviors. Most cases in which continuation game after out-of-equilibrium behavior lay in the center of consideration, are related to information selection (Banks and Sobel (1987), Kreps and Wilson (1982)). In contrast, in this paper, it brings substantial difference in equilibrium support. A representative study on cheap-talks regarding well-informedness Ottaviani and Sørensen (2006a,b) addresses the difficulties of fully truthful outcomes, while I focus on existence of such outcomes.

2.2 Model

This is a special case of Tajika (2021)'s model. In particular, I narrow our focus to binary-type cases where the payoff is solely determined by reputation.[3] Consider a two-period game where there exist an expert (he) and an evaluator (she). There are two states in this world, $\omega \in \Omega := \{x, y\}$, that are equally likely. At the start of the game, a state is drawn, and it remains fixed throughout the game. It is not known to both players, but the expert privately receives a series of noisy signals about the state over time. The extent of the noise varies depending on his type. Formally, in each period, he receives one piece

[3] I will cover the payoff including a monetary transfer that is considered in Tajika (2021) in Section 2.4.

of information from a signal space, $S_t = \Omega$, for $t = 1, 2$. The expert's type space is also binary, $\theta \in \Theta = \{\theta_H, \theta_L\}$, with $0.5 < \theta_L < \theta_H < 1$, which describes the accuracy of the signals. Abusing notation, θ indicates the accuracy of the signal expert of type θ in the first period, $Pr(s_1 = \omega|\omega, \theta) = \theta$. The signal becomes more precise in the second period, reaching an accuracy of $Pr(s_2 = \omega|s_1, r_1, \omega, \theta) = Pr(s_2 = \omega|\omega, \theta) = (1+\alpha)\theta/(1+\alpha\theta) \geq \theta$ with $\alpha \geq 0$. $Pr(s_2 = \omega|\omega, \theta)$ is weakly increasing in θ and α; is equal to θ if $\alpha = 0$ and converges to 1 if α increases to infinity. This specific form makes the model tractable. The probability of the expert being θ_H is denoted by $\pi \in (0, 1)$. All the information structure and the flow of the game are common knowledge among the players, except the type, state, and the realized signals. To keep it simple, I assume that the expert does not know his own type.

The expert makes a recommendation each period after observing a signal. The message space in each period is represented by $R_t = \{x, y\}$ for $t = 1, 2$.[4] I focus on pure strategies for the expert. Then, a strategy function is represented by $r_1 : S \to R_1$ and $r_2 : S_1 \times S_2 \times R_1 \to R_2$. After two rounds of recommendation, the true state becomes public. Players can observe the true state and whether the advice coincides with the state. The evaluator updates her beliefs, $\beta : R_1 \times R_2 \times \Omega \to \Delta\Theta$, about the type of the expert following Bayes' rule. The ex post expected type, perceived by the evaluator, is denoted by $\theta_{r_1 r_2 \omega}$, where r_t represents the message sent in period $t = 1, 2$. Occasionally, a superscript will be included to differentiate the belief system forming the expectation.

Given realized r_1, r_2 and ω, the expert receives the corresponding payoff. Specifically,

$$\Phi(\underbrace{E_\beta[\theta|r_1, r_2, \omega]}_{\theta^{(\beta)}_{r_1 r_2 \omega}}),$$

where Φ is a differentiable function with $\Phi'(\cdot) > 0$. The expert receives higher payoff if

[4] Although Tajika (2021) incorporated a message $\varnothing$, indicating "clamming up", I have omitted it. Typically, this message finds support within babbling equilibria. Its significance only arises when considering equilibrium refinements, which might be preferable to the expert. Instead, in Section 2.4, I show a sufficient condition for truth-telling to be expert-efficient.

the evaluator considers him as competent, or, having higher information acquisition ability. The reputation, however, depend only on the expected ability calculated upon the ex post distribution. For example, two different ex post distributions with the same mean, give the expert the same reputational payoffs.

2.3 Truth-telling equilibria

This section aims to present equilibria on whose path truth-telling occurs throughout periods, often considered the most socially desirable equilibria. A straightforward strategy for the expert resulting in such outcomes, is to be honest, in any occasions. That is, he reports the signal received in the period every time a new piece of signal arrives at any history. This strategy, denoted as fully truthful recommendation strategy (hereinafter, referred to as FT strategy; an FT equilibrium refers to an equilibrium where FT strategy is employed.), will serve as a benchmark.[5] In corresponding equilibrium, the evaluator forms her beliefs in accordance with Bayes' rule. Formally, FT strategy and the beliefs in FT equilibriums are given by $r_1^T(s_1) = s_1$, $r_2^T(s_1, s_2; r_1) = s_2$ and $\theta^T_{r_1 r_2 \omega} = E[\theta | s_1 = r_1, s_2 = r_2, \omega]$. By symmetricity, $\theta^T_{xxx} = \theta^T_{yyy}$, $\theta^T_{xxy} = \theta^T_{yyx}$, $\theta^T_{xyx} = \theta^T_{yxy}$ and $\theta^T_{xyy} = \theta^T_{yxx}$. Moreover, $\theta^T_{xyx} = \theta^T_{yxy} = \theta^T_{xyy} = \theta^T_{yxx}$. Notice that, given arbitrarily fixed θ and ω,

$$\frac{LR(s_1 = \omega|\theta, \omega)}{LR(s_1 \neq \omega|\theta, \omega)} = \frac{LR(s_2 = \omega|\theta, \omega)}{LR(s_2 \neq \omega|\theta, \omega)} = \frac{\theta_H}{\theta_L} / \frac{1-\theta_H}{1-\theta_L}. \tag{2.1}$$

The ratio of likelihood-ratio of correct message to that of incorrect message is held at the same level through periods. On the other hand, by definition of reputation, $\theta^T_{r_1 r_2 \omega}$ can be written as

$$\begin{aligned} \theta^T_{r_1 r_2 \omega} &= \theta_L + (\theta_H - \theta_L) Pr(H | s_1 = r_1, s_2 = r_2, \omega) \\ &= \theta_L + (\theta_H - \theta_L) \frac{1}{1 + LR(r_1|\omega)^{-1} LR(r_2|\omega)^{-1} (1-\pi)/\pi}. \end{aligned} \tag{2.2}$$

[5] See Tajika (2021).

Taking (2.1) into account, (2.2) implies that one correct and one incorrect messages give the same reputation no matter which period which message is sent. To summarize, under FT strategy, outcomes of reputations result in three different levels.

Let $p_{s_1 s_2} := Pr(\omega = x | s_1, s_2)$ be the posterior of the state being x after observing s_1 and s_2. Since $\alpha \geq 0$ and $.5 < \theta_L < \theta_H < 1$, $.5 < p_{yy} < p_{xy} \leq p_{yx} < p_{xx} < 1$. By symmetricity, $p_{yy} + p_{xx} = p_{xy} + p_{yx} = 1$. It follows that, $\frac{1-p_{xx}}{p_{xx}} < \frac{1-p_{yx}}{p_{yx}} < 1 < \frac{p_{yx}}{1-p_{yx}} < \frac{p_{xx}}{1-p_{xx}}$. Then,

$$\frac{p_{yx}}{1-p_{yx}} = \frac{p_{yx}}{p_{xy}} = \frac{Pr(s_1 \neq \omega) Pr(s_2 = \omega)}{Pr(s_1 = \omega) Pr(s_2 \neq \omega)} = 1 + \alpha \tag{2.3}$$

Lemma 2.1. *FT strategy and the corresponding beliefs form an equilibrium iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyy}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq 1 + \alpha \tag{2.4}$$

Proof. See Appendix 2.A. □

Proposition 2.1. *Suppose that Φ is differentiable and $\Phi(\cdot) > 0$. For any $\alpha \geq 0$ and $1/2 < \theta_L < \theta_H < 1$, there exists $\pi(\theta_L, \theta_H; \alpha)$ such that a distribution characterized by $(\theta_L, \theta_H, \pi(\theta_L, \theta_H; \alpha))$ satisfies* (2.4).

Proof. See Appendix 2.A. □

Lemma 2.1 above characterizes the necessary and sufficient condition for FT equilibria to exist. Not only that, but the next proposition shows that such a prior that satisfies the condition exists over a sufficiently wide range. Fixing α, the distribution of abilities is determined by 3 parameters, $(\theta_L, \theta_H, \pi)$. In other words, it states that it can be reduced down to a 2-dimensional manifold of (θ_L, θ_H). It can easily follow that, when $\alpha > 0$, one can find an open ball around $\pi(\theta_L, \theta_H; \alpha)$ that satisfies (2.4), given (θ_L, θ_H).

Truth-telling equilibria – not only FT equilibria – require $\frac{\Phi(\theta^T_{xyy}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})}$ sufficiently close to 1. Roughly speaking, a moderate – not the best nor the worst – reputation, $\theta^T_{xyx} = \theta^T_{yxy} = \theta^T_{xyy} = \theta^T_{yxx}$ can be seen as reserved payoff, that can be earned regardless of realized signals, by simply changing from the one in previous period. Notice that, $\frac{\Phi(\theta^T_{xyy}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})}$ increases with the reserved payoff. Suppose the expert recommended truthfully in the first period. If the second signal is consistent with the first one and the reserved payoff is relatively great, the expert may choose it, instead of taking a risky shot. If the second signal is inconsistent with the first one and the reserved payoff, in this case, the payoff from truth-telling, is relatively small, the expert may not choose to be truthful.

A sufficient condition for (2.4) to hold, is that $\Phi(\theta^T_{xyy}) - \Phi(\theta^T_{xxy}) = \Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})$. For simplicity, let $\Phi(\theta) = \theta$ and $\alpha = 0$. Given $.5 < \theta_L < \theta_H < 1$, there is a unique $\pi \in (0,1)$ that satisfies it, given by

$$\pi^* = \frac{\theta_H(1-\theta_H)}{\theta_H(1-\theta_H) + \theta_L(1-\theta_L)}. \tag{2.5}$$

Given a distribution, increments in signal accuracy, parameterized by α, also influence the existence of an FT equilibrium. As α approaches infinity, players in the first period anticipate a precise signal with few error arriving as the next signal. Intuitively speaking, this situation essentially reduces the game to one in which uncertainty lasts only for the first period. To see this, suppose $\Phi(\theta) = \theta$. It derives

$$\begin{aligned}
\theta_{xxx} - \theta_{xyx} &= \frac{\theta_L^2\kappa + \theta_H^2}{\theta_L\kappa + \theta_H} - \frac{\theta_L(1-\theta_L)\kappa + \theta_H(1-\theta_H)}{(1-\theta_L)\kappa + (1-\theta_H)} = \frac{\kappa Z(\theta_L,\theta_H)}{(\theta_L\kappa + \theta_H)((1-\theta_L)\kappa + (1-\theta_H))} \\
\theta_{xyy} - \theta_{xxy} &= \frac{\theta_L^2\nu + \theta_H^2}{\theta_L\nu + \theta_H} - \frac{\theta_L(1-\theta_L)\nu + \theta_H(1-\theta_H)}{(1-\theta_L)\nu + (1-\theta_H)} = \frac{\nu Z(\theta_L,\theta_H)}{(\theta_L\nu + \theta_H)((1-\theta_L)\nu + (1-\theta_H))}
\end{aligned} \tag{2.6}$$

and

$$\frac{\theta_{xyy} - \theta_{xxy}}{\theta_{xxx} - \theta_{xyx}} = \frac{(1-\theta_L)(1-\theta_H)}{\theta_L\theta_H} \frac{\theta_L^2(1+\alpha\theta_H)(1-\pi) + \theta_H^2(1+\alpha\theta_L)\pi}{(1-\theta_L)^2(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)^2(1+\alpha\theta_L)\pi} \tag{2.7}$$

where

$$\begin{aligned}
\kappa &= \frac{\theta_L(1+\alpha\theta_H)(1-\pi)}{\theta_H(1+\alpha\theta_L)\pi}, \\
\nu &= \frac{(1-\theta_L)(1+\alpha\theta_H)(1-\pi)}{(1-\theta_H)(1+\alpha\theta_L)\pi} = \frac{1-\theta_L}{1-\theta_H}\frac{\theta_H}{\theta_L}\kappa, \text{ and} \\
Z(\theta_L,\theta_H) &= \theta_L^2(1-\theta_H) + \theta_H^2(1-\theta_L) - \theta_H\theta_L(1-\theta_H) - \theta_H\theta_L(1-\theta_L) > 0
\end{aligned} \tag{2.8}$$

Then, it is clear Lemma 2.1 is satisfied when α is sufficiently large:

$$\begin{aligned}
&\lim_{\alpha\to\infty}\frac{1}{1+\alpha}=0\\
&<\lim_{\alpha\to\infty}\frac{\theta_{xyy}-\theta_{xxy}}{\theta_{xxx}-\theta_{xyx}}=(1-\theta_H)(1-\theta_L)\frac{\theta_L(1-\pi)+\theta_H\pi}{(1-\theta_L)^2\theta_H(1-\pi)+(1-\theta_H)^2\theta_L\pi}\\
&<\lim_{\alpha\to\infty}1+\alpha=\infty
\end{aligned}\tag{2.9}$$

Proposition 2.2. *Suppose $\Phi(\cdot)$ is differentiable with $\Phi'(\cdot)\geq 0$ and $\Phi''(\cdot)\leq 0$. For each (θ_L,θ_H,π), there exists $\alpha^F\geq 0$ such that for any $\alpha\geq\alpha^F$, distribution (θ_L,θ_H,π) satisfies* (2.4).

Proof of Proposition 2.2. It is sufficient to show that $\lim_{\alpha\to\infty}\frac{\Phi(\theta_{xyy})-\Phi(\theta_{xxy})}{\Phi(\theta_{xxx})-\Phi(\theta_{xyx})}\in(0,\infty)$. Since $\theta_{xxx}>\theta_{xyx}=\theta_{xyy}>\theta_{xxy}$ and $\Phi(\cdot)$ is concave, $\lim_{\alpha\to\infty}\frac{\Phi(\theta_{xyy})-\Phi(\theta_{xxy})}{\Phi(\theta_{xxx})-\Phi(\theta_{xyx})}\geq\lim_{\alpha\to\infty}\frac{\theta_{xyy}-\theta_{xxy}}{\theta_{xxx}-\theta_{xyx}}>0$. In addition, taking the limits of (2.6), $\lim_\alpha\theta_{xxx}>\lim_\alpha\theta_{xyx}>0$ and $\lim_\alpha\theta_{xyy}>\lim_\alpha\theta_{xxy}$. By continuity,

$$\lim_{\alpha\to\infty}\frac{\Phi(\theta_{xyy})-\Phi(\theta_{xxy})}{\Phi(\theta_{xxx})-\Phi(\theta_{xyx})}=\frac{\lim_{\alpha\to\infty}(\Phi(\theta_{xyy})-\Phi(\theta_{xxy}))}{\lim_{\alpha\to\infty}(\Phi(\theta_{xxx})-\Phi(\theta_{xyx}))}=\frac{\Phi(\lim_\alpha\theta_{xyy})-\Phi(\lim_\alpha\theta_{xxy})}{\Phi(\lim_\alpha\theta_{xxx})-\Phi(\lim_\alpha\theta_{xyx})}<\infty\tag{2.10}$$

□

2.3.1 On-Path Truth-telling Recommendation

FT equilibrium, however, is more demanding than it actually needs for truth-telling outcomes. This is because under FT strategy the expert always tell the truth even off the path. In this subsection, on-path truth-telling recommendation strategy is introduced (hereinafter, referred to as PT; PT equilibrium refers to equilibriums where PT strategy is played). PT strategy is the same with FT strategy on the path. Specifically, the expert recommends truthfully in the first period. In the following period, he recommends truthfully, if he did in the previous period. It is different only when the expert reported untruthfully in the first period. Formally, $r_1^*(s_1) = s_1$, $r_2^*(s_1, s_2; r_1 = s_1) = s_2$, and

$$r_2^*(x, x; y) = \begin{cases} x & \text{if } \frac{1-p_{xx}}{p_{xx}} \leq \frac{\Phi(\theta^T_{xyy})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})} \\ y & \text{otherwise} \end{cases} \tag{2.11}$$

$$r_2^*(x, y; y) = \begin{cases} y & \text{if } \frac{\Phi(\theta^T_{xyy})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})} \leq 1+\alpha \\ x & \text{otherwise} \end{cases} \tag{2.12}$$

$r_2^*(y, x; x)$ and $r_2^*(y, y; x)$ are defined analogously. The corresponding beliefs are given by $\theta^T_{r_1 r_2 \omega}$.

WLOG, suppose $s_1 = x$ and $r_1 = y$. When $\frac{\Phi(\theta^T_{xyy})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})} \notin [\frac{1-p_{xx}}{p_{xx}}, 1+\alpha]$, in histories where the expert previously lied, the following recommendation does not depends on the signal. If $\frac{\Phi(\theta^T_{xyy})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})} < \frac{1-p_{xx}}{p_{xx}}$, $r_2^*(x, x; y) = r_2^*(x, y; y) = y = r_1$. In this case, the expert simply repeats the previous recommendation. If $1+\alpha < \frac{\Phi(\theta^T_{xyy})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})}$, the opposite occurs: $r_2^*(x, x; y) = r_2^*(x, y; y) = x \neq r_1$. In this case, the expert simply recommends against his own previous recommendation. Recall that, r^T required truth-telling always, including in any of these cases.

Proposition 2.3. *Suppose Φ is differentiable and $\Phi(\cdot) > 0$. For any $\alpha \geq 0$ and $1/2 < \theta_L < \theta_H < 1$, there exists $\pi(\theta_L, \theta_H; \alpha)$ such that PT strategy and the corresponding beliefs form an equilibrium. A prior sustains PT equilibriums iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \leq \frac{p_{xx}}{1-p_{xx}} \tag{2.13}$$

Moreover, $1+\alpha < \frac{p_{xx}}{1-p_{xx}}$.

Proof. See Appendix 2.A. □

PT strategy, maintaining the same outcomes with FT strategy, requires (2.13), with the inequality of the right side of (2.4) replaced with a weaker condition. Hence, a distribution that sustains an FT equilibrium sustains a PT equilibrium. However, the reverse does not hold, in general. This especially outstands when α approaches to 0. The interval characterized by (2.4) converges to $\{1\}$, which pins down the type ratio by $\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) = \Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})$. On the other hand, the interval characterized by (2.13) has a positive length even if $\alpha = 0$.

Example 2.1. *Suppose that $\theta_H = 0.8$, $\theta_L = 0.5$ and $\pi = 0.9$. Suppose $\Phi(\theta) = \theta^2$ and set $\alpha = 1$. Then, there exists a PT but not an FT equilibrium.*

Proposition 2.4. *Suppose $\Phi(\cdot)$ is differentiable with $\Phi'(\cdot) \geq 0$ and $\Phi''(\cdot) \leq 0$. For each $(\theta_L, \theta_H, \pi)$, there exists α^F such that for any $\alpha \geq \alpha^P$, distribution $(\theta_L, \theta_H, \pi)$ satisfies (2.13). Moreover, under such Φ, $\underline{\alpha}^F(\theta_L, \theta_H, \pi) \geq \underline{\alpha}^P(\theta_L, \theta_H, \pi)$ where*

$$\begin{aligned} \underline{\alpha}^F(\theta_L, \theta_H, \pi) &:= \inf\left\{\alpha^F : \forall \alpha \geq \alpha^F, \text{ distribution } (\theta_L, \theta_H, \pi) \text{ satisfies } (2.4)\right\} \\ \underline{\alpha}^P(\theta_L, \theta_H, \pi) &:= \inf\left\{\alpha^P : \forall \alpha \geq \alpha^P, \text{ distribution } (\theta_L, \theta_H, \pi) \text{ satisfies } (2.13)\right\} \end{aligned} \tag{2.14}$$

The inequality hold strictly when

$$\frac{(1-\theta_L)(1-\theta_H)}{\theta_L\theta_H} \frac{\theta_L^2(1-\pi) + \theta_H^2\pi}{(1-\theta_L)^2(1-\pi) + (1-\theta_H)^2\pi} > 1 \tag{2.15}$$

Proof. See Appendix 2.A. □

2.4 Analysis

2.4.1 Expert-efficiency

There are equilibria with different paths, other than truthful ones. While truth-telling path is most favorable to the evaluator, the expert may prefer other equilibrium in which he receives higher expected payoff. However, if the expert does prefer truth-telling, Pareto-efficient equilibriums must be ones that deliver full information to the evaluator.

To explore the circumstances under which such equilibria are favored by the expert, this subsection compares the ex ante payoffs of the expert under different strategies. In particular, I contrast truth-telling with "waiting strategy" in which the expert sends an informative message only in the last period.[6] The beliefs of the evaluator are assumed to

[6] One can, of course, consider a strategy in which the expert sends an informative message only in the first period. Switching the periods results in a similar discussion.

be consistent with the expert's strategy.

Under waiting strategy, the expert maintains silence in the first period, and waits for a more accurate signal. Formally, $r_1^W(s_1) \in \{x, y\}$ and $r_2^W(s_1, s_2; r_1) = s_2$. The belief system is given by $\theta^W_{\varnothing r_2 \omega} = Pr(\theta_H | s_2 = r_2)$. A waiting equilibrium is an equilibrium in which waiting strategy is played. The proposition below provides a sufficient condition for the expert to ex-ante prefers truth-telling equilibria over waiting equilibria, that he is risk-loving.

Proposition 2.5. *The expected payoff of the expert under FT (or PT) strategy and belief $\theta^T_{r_1 r_2 \omega}$ is greater than one under waiting strategy and belief $\theta^W_{\varnothing r_2 \omega}$, if Φ is strictly convex.*

Proof. See Appendix 2.A. □

2.4.2 Monetary Transfer

So far, it has been assumed that the expert yields payoffs solely determined by reputation. In many cases, however, experts are employed by firms and paid for their expertise. In what follows, a monetary reward is given to the expert depending on the recommendations and the state. The payoff will be given by

$$K\mathbb{1}(r_2 = \omega) + \Phi(E_\beta[\theta | r_1, r_2, \omega])$$

The second term indicates reputational payoff. Assumptions regarding Φ remain unchanged. The first term indicates the monetary reward with $K \geq 0$.

The monetary reward is granted only when the final recommendation matches with the state. Consequently, the first recommendation affects the payoff through the reputational term. While the model does not explicitly illustrate the evaluator's action, she may take an action after receiving all the recommendations, resulting in a positive payoff if her action aligns with the true state. Assuming the recommendations are perceived as truthful, she would likely adhere to the final recommendation, given the increased precision of the second signal compared to the first. In this context, K can be interpreted as a contingent fee to the expert.

Denote by q_{cc}, q_{ci}, q_{ic}, and q_{ii},

$$
\begin{aligned}
q_{cc} &= Pr(\omega = s_2 = s_1|s_1) = \pi\frac{\theta_H^2(1+\alpha)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L^2(1+\alpha)}{1+\alpha\theta_L} \\
q_{ci} &= Pr(\omega = s_1 \neq s_2|s_1) = \pi\frac{\theta_H(1-\theta_H)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(1-\theta_L)}{1+\alpha\theta_L} \\
q_{ii} &= Pr(s_2 = s_1 \neq \omega|s_1) = \pi\frac{(1-\theta_H)^2}{1+\alpha\theta_H} + (1-\pi)\frac{(1-\theta_L)^2}{1+\alpha\theta_L}, \text{ and} \\
q_{ic} &= Pr(\omega = s_2 \neq s_1|s_1) = \pi\frac{\theta_H(1-\theta_H)(1+\alpha)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(1-\theta_L)(1+\alpha)}{1+\alpha\theta_L} \equiv (1+\alpha)q_{ci},
\end{aligned}
\tag{2.16}
$$

respectively. These are the probabilities of the signals with regard to the state, conditional on the first signal; the subscript i stands for "incorrect", and c stands for "correct." Also note that because the states are equally likely, the realization of the first signal does not change the ex ante probability of the second signal being correct. For example, q_{cc} is equal to $Pr(\omega = s_2 = s_1)$; and it is analogous for q_{ci}, q_{ic}, and q_{ii}. Additionally, let $q_c := q_{cc} + q_{ci} \equiv Pr(\omega = s_1|s_1)$ and $q_i := q_{ic} + q_{ii} \equiv Pr(\omega \neq s_1|s_1)$. These denote the probabilities of the first signal being correct and incorrect, respectively, given the first signal.

Proposition 2.6. *Suppose Φ is differentiable and $\Phi(\cdot) > 0$. FT strategy and the corresponding beliefs form an equilibrium iff*

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \leq 1+\alpha. \tag{2.17}$$

PT strategy and the corresponding beliefs form an equilibrium iff

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \leq \frac{p_{xx}}{1-p_{xx}} - \frac{q_c - q_i}{q_{ii}}\frac{K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K}. \tag{2.18}$$

Moreover,

$$1+\alpha < \frac{p_{xx}}{1-p_{xx}} - \frac{q_c - q_i}{q_{ii}}\frac{K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K}. \tag{2.19}$$

Proof. See Appendix 2.A. □

The impact of $K \geq 0$ in FT equilibriums is straight forward. As K increases, the (LHS) of the right inequality of (2.17) in the proposition monotonically approaches to 1. Although (2.18) may appear more intricate, but it addresses a similar intuition. The inequality on the right side of (2.18) is equivalent to

$$q_{cc}(\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})) + q_{ii}(\Phi(\theta^T_{xxy}) - \Phi(\theta^T_{xyx})) + \underbrace{(q_{ic} - q_{ci})}_{\geq 0} K \geq 0 \tag{2.20}$$

Since the (LHS) increases with K, if it holds at $K = 0$, then it holds for $K \geq 0$. Substituting $K = 0$ to (2.20) is equivalent, however, with the inequality on the right side of (2.13). Thus, if a distritubion satisfies (2.4) and (2.13), it satisfies (2.17) and (2.18), respectively. From (2.17), (2.18) and (2.20), it follows that, for any distribution, if K is sufficiently large and $\alpha > 0$, both FT and PT equilibria exist. This is consistent with the intuition.

The last inequality states that the inclusion relationship between the supports of FT and PT equilibria is robust to the monetary transfer. Even with $K > 0$, existence of FT equilibria implies that of PT equilibria; but the reverse does not necessarily hold.

Appendix of Chapter 2

2.A Proofs

Proof of Lemma 2.1. Suppose the expert played according to r_1. Then, the expected payoffs of playing r_2 are given as

$$p_{xx}\Phi(\theta^T_{xxx}) + (1-p_{xx})\Phi(\theta^T_{xxy}) \quad \text{if } s_1 = s_2,$$

$$p_{yx}\Phi(\theta^T_{xyy}) + (1-p_{yx})\Phi(\theta^T_{xyx}) \quad \text{if } s_1 \neq s_2,$$

If the expert lies in the second period, that is, $r_2 \neq s_1$ the expected payoffs are

$$\begin{aligned} &p_{xx}\Phi(\theta^T_{xyx}) + (1-p_{xx})\Phi(\theta^T_{xyy}) \quad \text{if } s_1 = s_2, \\ &p_{yx}\Phi(\theta^T_{xxy}) + (1-p_{yx})\Phi(\theta^T_{xxx}) \quad \text{if } s_1 \neq s_2 \end{aligned} \tag{2.21}$$

For the continuation strategy to recommend truthfully in the second period to be optimal, the following has to hold:

$$\frac{p_{xx}}{1-p_{xx}} \geq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx})} \geq \frac{1-p_{yx}}{p_{yx}} = \frac{1}{1+\alpha} \tag{2.22}$$

Now suppose the expert recommended the opposite signal from what he received in the first period. For the expert to recommend truthfully in the second period, the following two inequalities are required:

$$\begin{aligned} p_{xx}\Phi(\theta^T_{xyy}) + (1-p_{xx})\Phi(\theta^T_{xyx}) &\geq p_{xx}\Phi(\theta^T_{xxy}) + (1-p_{xx})\Phi(\theta^T_{xxx}) \\ p_{yx}\Phi(\theta^T_{xxx}) + (1-p_{yx})\Phi(\theta^T_{xxy}) &\geq p_{yx}\Phi(\theta^T_{xyx}) + (1-p_{yx})\Phi(\theta^T_{xyy}) \end{aligned} \tag{2.23}$$

It is now equal to

$$\frac{p_{xx}}{1-p_{xx}} \geq \left(\frac{\Phi(\theta^T_{xyx})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})}\right)^{-1} \geq \frac{1}{1+\alpha} \tag{2.24}$$

Combining (2.22) and (2.24), it has to be

$$\min\left\{\frac{p_{xx}}{1-p_{xx}}, 1+\alpha\right\} \geq \frac{\Phi(\theta^T_{xyx})-\Phi(\theta^T_{xxy})}{\Phi(\theta^T_{xxx})-\Phi(\theta^T_{xyx})} \geq \max\left\{\frac{1-p_{xx}}{p_{xx}}, \frac{1}{1+\alpha}\right\} \tag{2.25}$$

Considering $1+\alpha = p_{xy}/(1-p_{xy})$ and $p_{xx} > p_{xy} = 1-p_{yx}$, (2.25) results in (2.4).

It remains to show that, given $r^T_2(s_1, s_2; r_1) = s_2$, the optimal behavior for the expert in the first period is to recommend truthfully. As the monetary rewards are only determined by the second recommendation, only the reputational payoffs will be taken into account. It is obvious that there is no incentive to send $r_1 = \emptyset$, because the evaluator believes the expert is of type θ_L with probability 1.

For truthful recommendation to be the optimal behavior given the second period strategy, r^T_2, it needs to be satisfied that

$$\begin{aligned} &q_{cc}\Phi(\theta^T_{xxx}) + q_{ii}\Phi(\theta^T_{xxy}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) \\ &\geq q_{cc}\Phi(\theta^T_{xyx}) + q_{ii}\Phi(\theta^T_{xyx}) + q_{ci}\Phi(\theta^T_{xxy}) + q_{ic}\Phi(\theta^T_{xxx}). \end{aligned} \tag{2.26}$$

It is summarized as

$$\begin{aligned} &\underbrace{(1+\alpha)\left\{\pi\frac{\theta_H(2\theta_H-1)}{1+\alpha\theta_H} + (1-\pi)\frac{\theta_L(2\theta_L-1)}{1+\alpha\theta_L}\right\}}_{>0}(\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyy})) \\ \geq &\underbrace{\left\{\pi\frac{(1-\theta_H)(1-2\theta_H)}{1+\alpha\theta_H} + (1-\pi)\frac{(1-\theta_L)(1-2\theta_L)}{1+\alpha\theta_L}\right\}}_{<0}(\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy})). \end{aligned} \tag{2.27}$$

It always holds that, as $1 > \theta_H > \theta_L > 0.5$, the (LHS) is always positive and the (RHS) is always negative. □

Proof of Proposition 2.1. Lemma 2.1 shows a sufficient condition for a truthful equilibrium to hold, represented by

$$\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) = \Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) \tag{2.28}$$

It is obvious that, when there is only one type of accuracy, the ex post belief would not change. In other words, for any fixed θ_H and θ_L, if $\pi = 0$ or $\pi = 1$, $\theta^T_{xxx} = \theta^T_{xyx} = \theta^T_{xxy}$; hence, $\Phi(\theta^T_{xxx}) + \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx}) = 0$. On the other hand, as Φ is a differentiable increasing function, the derivatives of $\Phi(\theta^T_{xxx}) + \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx})$ at $\pi = 0$ and $\pi = 1$ is positive if and only if,

$$\left.\frac{\mathrm{d}\,\theta^T_{xxx}}{\mathrm{d}\,\pi}\right|_{\pi=0} + \left.\frac{\mathrm{d}\,\theta^T_{xxy}}{\mathrm{d}\,\pi}\right|_{\pi=0} - 2\left.\frac{\mathrm{d}\,\theta^T_{xyx}}{\mathrm{d}\,\pi}\right|_{\pi=0} = \frac{(\theta_H - \theta_L)^3(1+\alpha\theta_L)}{(1-\theta_L)^2\theta_L(1+\alpha\theta_H)} > 0$$
$$\left.\frac{\mathrm{d}\,\theta^T_{xxx}}{\mathrm{d}\,\pi}\right|_{\pi=1} + \left.\frac{\mathrm{d}\,\theta^T_{xxy}}{\mathrm{d}\,\pi}\right|_{\pi=1} - 2\left.\frac{\mathrm{d}\,\theta^T_{xyx}}{\mathrm{d}\,\pi}\right|_{\pi=1} = \frac{(\theta_H - \theta_L)^3(1+\alpha\theta_H)}{(1-\theta_H)^2\theta_H(1+\alpha\theta_L)} > 0.$$

Therefore, by the intermediate value theorem, there must exist $\pi^* \in (0,1)$ to make $\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xxy}) - 2\Phi(\theta^T_{xyx}) = 0$. It fulfills the sufficient condition for the existence of truthful equilibrium. And by continuity, if $\alpha > 0$, there is an interval with positive length around π^*. □

Proof of Proposition 2.3. The proof of Lemma 2.1 has already demonstrated the following: (i) if the continuation strategy in the second period is set as truth-telling, recommending truthfully in the first period is always a best response, (ii) if the expert deceived in the first period and received the same signal, $s_2 = s_1$, it is a unique best response to recommend truthfully, and (iii) Inequalities (2.13) are equivalent with the sufficient and necessary condition for the expert to maintain truthfulness in the second period after doing so in the first period. (ii) is derived by the fact that the inequality in the left side of (2.13) implies the strict inequality in the left side of (2.24).

(i) implies that, if there exists an optimal profitable deviation, there must exist $\hat{s} \in \{x, y\}$ on which the expert in the first period misreports r_1 observing. (ii) implies that, in such continuation game, if $s_2 = \hat{s}$, the expert must send $r_2 = s_2$. (iii) implies that, in such continuation game, if $s_2 \neq \hat{s}$, the expert must send $r_2 \neq s_2 = \hat{s}$. Otherwise, the continuation strategy becomes a truth-telling one, contradicting the observable (i). It follows that, profitable deviating expert recommends $r_1 \neq r_2$ when observing $\hat{s}$ in the first period.

On the other hand, from (2.1) and (2.2), sending different messages $(r_1, r_2) = (r, r')$ with $r \neq r'$ yields a fixed reputation, $\theta^T_{xyx} = \theta^T_{xyy} = \theta^T_{yxx} = \theta^T_{yxy}$. This implies that the expert receives the same payoff by recommending truthfully in the first period and recommends the opposite in the second period. However, it contradicts (i), concluding that, when (2.13) holds, any deviation strategy from FT strategy cannot be profitable. □

Proof of Proposition 2.4. Find $A, B > 0$ and $K > 0$ depending on $(\theta_L, \theta_H, \pi)$ so that (2.7) is written by in a form of

$$\frac{1}{A + B\alpha} + K \tag{2.29}$$

By (2.8), $A, B > 0$ and $K > 0$ are uniquely determined.

Claim 2.1. *$\frac{\theta_{xyy}-\theta_{xxy}}{\theta_{xxx}-\theta_{xyx}} > \frac{1}{1+\alpha}$ holds for $\alpha \geq 0$ under inequality* (2.15)*.*

Suppose Claim 2.1 holds for a while until proven. Since $\Phi(\cdot)$ is concave, it follows that

$$\frac{1}{1+\alpha} < \frac{\theta_{xyy} - \theta_{xxy}}{\theta_{xxx} - \theta_{xyx}} \leq \frac{\Phi(\theta_{xyy}) - \Phi(\theta_{xxy})}{\Phi(\theta_{xxx}) - \Phi(\theta_{xyx})} \tag{2.30}$$

Substitute (2.29) and take the limits as below.

$$0 < K = \lim_{\alpha} \frac{\theta_{xyy} - \theta_{xxy}}{\theta_{xxx} - \theta_{xyx}} \leq \lim_{\alpha} \frac{\Phi(\theta_{xyy}) - \Phi(\theta_{xxy})}{\Phi(\theta_{xxx}) - \Phi(\theta_{xyx})} < \infty. \tag{2.31}$$

The last inequality in (2.31) derives from the proof of Proposition 2.2, which guarantees the existence of α^* such that $\frac{\Phi(\theta_{xyy})-\Phi(\theta_{xxy})}{\Phi(\theta_{xxx})-\Phi(\theta_{xyx})}\Big|_{\alpha=\alpha^*} = 1+\alpha^*$. Although it may not be unique, but there exists infimum:

$$\begin{aligned}\hat{\alpha}^F &= \inf_{\alpha} \left\{ \alpha^* : \frac{\Phi(\theta_{xyy}) - \Phi(\theta_{xxy})}{\Phi(\theta_{xxx}) - \Phi(\theta_{xyx})}\bigg|_{\alpha=\alpha^*} = 1 + \alpha^* \right\} \\ &= \min_{\alpha} \left\{ \alpha^* : \frac{\Phi(\theta_{xyy}) - \Phi(\theta_{xxy})}{\Phi(\theta_{xxx}) - \Phi(\theta_{xyx})}\bigg|_{\alpha=\alpha^*} = 1 + \alpha^* \right\}\end{aligned} \tag{2.32}$$

Similarly define $\hat{\alpha}^P$ as follows.

$$\hat{\alpha}^F = \min_{\alpha} \left\{ \alpha^* \geq 0 : \frac{\Phi(\theta_{xyy}) - \Phi(\theta_{xxy})}{\Phi(\theta_{xxx}) - \Phi(\theta_{xyx})}\bigg|_{\alpha=\alpha^*} = \frac{p_{xx}}{1-p_{xx}}\bigg|_{\alpha=\alpha^*} \right\} \tag{2.33}$$

However, the set in (2.33) may be empty, that is, $\frac{\Phi(\theta_{xyy})-\Phi(\theta_{xxy})}{\Phi(\theta_{xxx})-\Phi(\theta_{xyx})} < \frac{p_{xx}}{1-p_{xx}}$ for $\alpha \geq 0$. In such cases, let $\hat{\alpha}^F = 0$.

By Claim 2.1, under (2.15), $\underline{\alpha}^F$ and $\underline{\alpha}^P$ are determined solely by the right side of inequality in (2.4) and (2.13), respectively. Then, $1 + \alpha < \frac{p_{xx}}{1-p_{xx}}$ for any α, by continuity, the proposition is proven if Claim 2.1 holds.

Notice that the equation below with respect to α has at most 2 solutions.

$$\frac{1}{A + B\alpha} + K = \frac{1}{1+\alpha}, \quad A, B, K > 0 \tag{2.34}$$

The (LHS) has its discontinuity point at $\alpha = -A/B$. Because

$$-A/B = \frac{(1-\theta_L)^2(1-\pi) + (1-\theta_H)^2(1-\pi)}{(1-\theta_L)^2\theta_H(1-\pi) + (1-\theta_H)^2\theta_L(1-\pi)} < -1, \tag{2.35}$$

it follows that

$$\lim_{\alpha \uparrow -A/B} \frac{1}{A+B\alpha} + K = -\infty < \lim_{\alpha \uparrow -A/B} \frac{1}{1+\alpha} \tag{2.36}$$

Since $\lim_{\alpha\to-\infty} \frac{1}{A+B\alpha} + K = K > \lim_{\alpha\to-\infty} \frac{1}{1+\alpha}$, there exist one solution of (2.34) such that smaller than -1. Besides,

$$\lim_{\alpha \downarrow -1} \frac{1}{A+B\alpha} + K < \infty = \lim_{\alpha \downarrow -1} \frac{1}{1+\alpha}. \tag{2.37}$$

If

$$\lim_{\alpha\to 0} \frac{1}{A+B\alpha} + K > \lim_{\alpha\to 0} \frac{1}{1+\alpha} = 1, \tag{2.38}$$

which is equivalent with (2.15), there exists another solution between -1 and 0. Since (2.15) is guaranteed by assumption, (2.34) has its maximum number of solutions; (LHS) of (2.34) is strictly greater than the (RHS) and there exists no $\alpha \geq 0$ that satisfies the equality. □

Proof of Proposition 2.5. Let $\tilde{\Phi}(x) := \Phi(\theta_L + (\theta_H - \theta_L)x)$. It is a different form of Φ such that $\Phi(\theta^T_{s_1 s_2 \omega}) = \tilde{\Phi}(Pr(H|s_1, s_2, \omega))$. The expected accuracy corresponding to waiting strategy will be denoted by $\theta^W_{\emptyset r_2 \omega} := Pr(\theta_H|s_2, \omega)$; and the expected reputational payoff under the strategy is given by

$$F^W(\alpha, \Phi) = Pr(\omega = s_2)\Phi(\theta^W_{\emptyset xx}) + Pr(\omega \neq s_2)\Phi(\theta^W_{\emptyset xy}) \tag{2.39}$$

On the other hand,

$$\begin{aligned}
\Phi(\theta^W_{\emptyset xx}) &= \Phi(\theta_L + (\theta_H - \theta_L)Pr(\theta_H|s_2 = \omega)) \\
&= \Phi(\theta_L + (\theta_H - \theta_L)[Pr(\theta_H, s_1 = \omega|s_2 = \omega) + Pr(\theta_H, s_1 \neq \omega|s_2 = \omega)]) \\
&= \Phi(\theta_L + \{(\theta_H - \theta_L)Pr(\theta_H|s_1 = \omega, s_2 = \omega)Pr(s_1 = \omega|s_2 = \omega)\} \\
&\quad + \{(\theta_H - \theta_L)Pr(\theta_H|s_1 \neq \omega, s_2 = \omega)Pr(s_1 \neq \omega|s_2 = \omega)\}) \\
&= \Phi(Pr(s_1 = \omega|s_2 = \omega)\{\theta_L + (\theta_H - \theta_L)Pr(\theta_H|s_1 = \omega, s_2 = \omega)\} \\
&\quad + Pr(s_1 \neq \omega|s_2 = \omega)\{\theta_L + (\theta_H - \theta_L)Pr(\theta_H|s_1 \neq \omega, s_2 = \omega)\})
\end{aligned} \tag{2.40}$$

The first equality derives from the definition, and the second and third derives from Bayes' rule. The last equality is from the fact that $Pr(s_1 = \omega | s_2 = \omega) + Pr(s_1 \neq \omega | s_2 = \omega) = 1$. Now, since we have assumed convexity in $\Phi(\cdot)$,

$$\begin{aligned}
\Phi(\theta^W_{\emptyset xx}) = \Phi(&Pr(s_1 = \omega | s_2 = \omega) \{\theta_L + (\theta_H - \theta_L) Pr(\theta_H | s_1 = \omega, s_2 = \omega)\} \\
&+ Pr(s_1 \neq \omega | s_2 = \omega) \{\theta_L + (\theta_H - \theta_L) Pr(\theta_H | s_1 \neq \omega, s_2 = \omega)\}) \\
&< Pr(s_1 = \omega | s_2 = \omega) \Phi(\theta_L + (\theta_H - \theta_L) Pr(\theta_H | s_1 = \omega, s_2 = \omega)) \\
&\quad + Pr(s_1 \neq \omega | s_2 = \omega) \Phi(\theta_L + (\theta_H - \theta_L) Pr(\theta_H | s_1 \neq \omega, s_2 = \omega)) \\
&= Pr(s_1 = \omega | s_2 = \omega) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 = \omega)) \\
&\quad + Pr(s_1 \neq \omega | s_2 = \omega) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 = \omega))
\end{aligned} \tag{2.41}$$

In a similar fashion,

$$\begin{aligned}
\Phi(\theta^W_{\emptyset xy}) <& Pr(s_1 = \omega | s_2 \neq \omega) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 \neq \omega)) \\
&+ Pr(s_1 \neq \omega | s_2 \neq \omega) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 \neq \omega))
\end{aligned} \tag{2.42}$$

Substituting (2.41) and (2.42) to (2.39),

$$\begin{aligned}
F^W(\alpha, \Phi) <& Pr(s_2 = \omega) Pr(s_1 = \omega | s_2 = \omega) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 = \omega)) \\
&+ Pr(s_2 = \omega) Pr(s_1 \neq \omega | s_2 = \omega) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 = \omega)) \\
&+ Pr(s_2 \neq \omega) Pr(s_1 = \omega | s_2 \neq \omega) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 \neq \omega)) \\
&+ Pr(s_2 \neq \omega) Pr(s_1 \neq \omega | s_2 \neq \omega) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 \neq \omega)) \\
=& Pr(s_1 = s_2 = \omega) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 = \omega)) \\
&+ Pr(s_1 \neq \omega = s_2) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 = \omega)) \\
&+ Pr(s_1 = \omega \neq s_2) \tilde{\Phi}(Pr(\theta_H | s_1 = \omega, s_2 \neq \omega)) \\
&+ Pr(s_1 = s_2 \neq \omega) \tilde{\Phi}(Pr(\theta_H | s_1 \neq \omega, s_2 \neq \omega)) = F^T(\alpha, \Phi)
\end{aligned} \tag{2.43}$$

□

Proof of Proposition 2.6. Start with the last claim. Re-arrange (2.19) to

$$(q_c - q_i) \frac{K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} < q_{cc} - (1 + \alpha) q_{ii}. \tag{2.44}$$

The fact that $q_{cc}/q_{ii} = p_{xx}/(1 - p_{xx})$ is used to derive (2.44). Using the definition of q_c and q_i, and the fact that $\Phi(\theta^T_{xxx}) > \Phi(\theta^T_{xyx})$, (2.44) holds if the following inequality holds which is always true.

$$\alpha q_{ii} < q_{ic} - q_{ci} \equiv \alpha q_{ci} \tag{2.45}$$

As the monetary payoff only depends on the recommendation in the final period, the first recommendation only affects the reputational term. Therefore, there is no deviation incentive in the first period as long as the continuation strategy is truth-telling, as seen in the proof of Lemma 2.1. In the second period, it can be readily shown that the incentive compatibility conditions in the same proof, (2.22) and (2.24), are the special cases with $K = 0$. With $K \geq 0$, they are re-written to

$$\frac{p_{xx}}{1-p_{xx}} \geq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \geq \frac{1}{1+\alpha} \tag{2.46}$$

and

$$\frac{p_{xx}}{1-p_{xx}} \geq \left(\frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K}\right)^{-1} \geq \frac{1}{1+\alpha}, \tag{2.47}$$

resulting in (2.17).

The optimality of truth-telling after previous truth-telling recommendation under PT strategy is guaranteed if

$$\frac{1}{1+\alpha} \leq \frac{\Phi(\theta^T_{xyx}) - \Phi(\theta^T_{xxy}) + K}{\Phi(\theta^T_{xxx}) - \Phi(\theta^T_{xyx}) + K} \leq \frac{p_{xx}}{1-p_{xx}}, \tag{2.48}$$

which is satisfied if (2.18) holds. Hence, provided the last claim into account, if there exists a deviation strategy, it implies that there is a signal $s_1 = \hat{s}$ such that, upon observing it, the expert chooses to deceive. By the same logic as with the proof of Proposition 2.3, profitable deviating expert recommends $r_1 \neq r_2$ when observing $\hat{s}$ in the first period. By this deviation, the expert receives the monetary reward iff $\omega = \hat{s}_1$. Then, truth-telling performs equally or better than this deviation if the following holds,

$$\begin{aligned} & q_{cc}\Phi(\theta^T_{xxx}) + q_{ii}\Phi(\theta^T_{xxy}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) + (q_{cc} + q_{ic})K \\ \geq & \Phi(\theta^T_{xyx}) + (q_{cc} + q_{ci})K \\ \equiv & q_{cc}\Phi(\theta^T_{xyx}) + q_{ii}\Phi(\theta^T_{xyx}) + q_{ci}\Phi(\theta^T_{xyx}) + q_{ic}\Phi(\theta^T_{xyx}) + (q_{cc} + q_{ci})K, \end{aligned} \tag{2.49}$$

which is equivalent to the right inequality of (2.18). □

Chapter 3

Reputational Cheap-talk in a Dynamic Game

3.1 Introduction

It may seem curious at a first glance that an expert may have incentive to fabricate their informative source to enhance reputation, even when priors are even. Consider two, independent and unverifiable experiments $\mathcal{A}$ and $\mathcal{B}$ about a binary, ex-ante equally likely state. The accuracy depends on the ability of an expert, who sequentially (wlog, in order of $\mathcal{A}$ and $\mathcal{B}$) receives the results and sends a message, each time after a result. If the second signal is inconsistent with the previous message, a new honest message would convince evaluators expecting truth-telling that, the expert is not talented enough to receive two correct signals. On the other hand, two same messages connote a risk of a state inconsistent with it, which will convince the evaluator to assign the lowest assessment. In either case, the expert may have incentive to misreport. Notice that, $\mathcal{A}$ (or $\mathcal{B}$) alone, truth-telling is a best response for the expert, being expected truthful.

The expert may prefer to send a message after collecting both $\mathcal{A}$ and $\mathcal{B}$. The situation is reminiscent of Blackwell (1953); consider another experiment $\mathcal{C}$ whose posterior distribution is the same with one generated by $\mathcal{B}$ after $\mathcal{A}$. As before, when $\mathcal{C}$ alone, one can eaily expect that truth-telling equilibrium exists. Suppose in an equilibrium, a correct signal followed by a wrong one is more valuated than a wrong one followed by a correct one. If experiment $\mathcal{B}$ is ex-ante more accurate than $\mathcal{A}$ is, the expert may interim better off by switching the results of $\mathcal{A}$ and $\mathcal{B}$.[1] The incentive to misreport may persist even in cases where it is feasible for the expert to truthfully report in sequential-message environment.

The main questions of this paper are when we can expect truth-telling in dynamic information structure and what makes it difficult. This study analyzes a simple dynamic model, following Tajika (2021). In the model, an expert predicts a state of the world multiple times before the true state reveals, based on a sequence of signals whose accuracy depends on time as well as the information acquisition ability of the expert. The more competent the expert is, the higher the accuracy of the signals s/he receives is. When the reputation is determined by the messages s/he has sent and the true state revealed out, I focus on the distributions of the abilities that enable truthful outcomes.[2] This is one of key differences between Tajika (2021), who focused on the equilibria in which the expert reports the interim belief only once through the whole game. The mentioned above question, whether the expert wants to be truthful, does not arise in such equilibria.

Two distinct truth-telling strategies and their corresponding equilibria are characterized. These strategies differ in off-path behavior, where the 'FT' strategy mandates reporting the true signal even after a misreport, while the 'PT' strategy permits lying after previous deception. The supports of each equilibrium are both non-empty, and will be compared. It is suggested that the flexibility in off-path behavior may expand the support, particularly noticeable when increments in signal accuracy uniformly approach zero.

Further analysis explores that the genuine factor that hampers truth-telling is more attributed to multi-dimensionality rather than dynamicity. In the model, the expert waiting for the complete collection of information results in a similar effect of multi-dimensionalizing

[1] Readers may refer to Ottaviani and Sørensen (2006a) who pointed out the difference between reporting posterior and reporting signal.

[2] In this paper, the signal set is assumed to be discrete. The generic impossibility theorem of Ottaviani and Sørensen (2006a,b) does not apply here.

signals and the messages. Whether this alternation promotes previously unattained truth-telling or impedes existing truth-telling depends on the strategy adopted by the expert, which is, however, non-observable in general. These insights shed light on the complex interplay between multi-dimensionality and dynamicity in communication dynamics, and highlight the challenges faced by society designers in promoting truthful outcomes.

A sufficient condition found in Woo (2022) for truth-telling to be ex-ante profitalbe, is generalized. The expert prefers the strategy that results in more informative posterior distribution, when risk-loving. This result holds for any finite and discrete time horizon, and any information structure. Not to say, it is also socially the most desirable outcome, in Blackwell (1953)'s sense.

The existence of ability distributions that sustain truth-telling outcomes does not depict the shape of such distributions. I reduced the model down while inheriting the essence of the original model, with a linear payoff function and binary types, to make an explicit example. This exploration sheds light on the nature of ability distributions that facilitate truth-telling outcomes. Moreover, through this example, I dispute one of the claims in Tajika (2021). See a counter example in Section 3.5.

Section 3.2 describes the model through the paper. Section 3.3 provides a static benchmark and truth-telling equilibria of the model. In section 3.4, I compare the ex ante payoffs for the expert in different equilibria and derives a condition for truth-telling equilibria to be favored. Then, I alter the timing of messages and staticize the model to assess its impact. Section 3.5 focuses on a binary-type model and produces a results differing from one in Tajika (2021). Here, I not only provide a counterexample to his claim but also offer an intuitive explanation of how the distribution's shape influences the expert's incentives. Section 3.6 concludes.

3.1.1 Related Literature

This paper is related to a large body of literature on reputation building on information acquisition abilities initiated by Holmstrom and Costa (1986). It has been shown in many previous researches that, information is distorted when the expert is a careerist. Hence, many of the literature consider contract schemes to have experts take a desirable action in view of the principal. Results in this paper share the insight that experts are reluctant to share their information when they are risk-averse. However, in cheap-talk environment, if there exists a truth-telling equilibrium, it seems reasonable to choose the most desirable one in equilibrium refinement.[3] I prove the existence of truth-telling equilibria by cheap talk to exist, which does not involve monetary transfer.

In this respect, growing literature of cheap-talks on being well-informed, is another vein this paper contribute. Existing works, such as Ottaviani and Sørensen (2006a,b), and Tajika (2021), have predominantly discussed the vulnerability of fully truth-telling equilibria. However, in this paper I show, besides the existence of such equilibria, their robustness. One type of truth-telling equilibria, which I call PT equilibria, are robust both in the prior and the growth of accuracy. Regarding another one, which I call FT equilibria, I derive a different result on robustness of it, other than one in Tajika (2021).

In sender-receiver games, settings of multi-dimensions in signal, are often a blessing for receivers. For instance, it may have cheap talk truthful (Battaglini (2002)), along with multiple-sender; even make it credible (Chakraborty and Harbaugh (2010)); or may enable receiver to infer additional information from the timing of disclosure (Guttman et al. (2014)). By contrast, in this paper, it is suggested that, multi-dimensional signal in a static model may fail to achieve truth-telling when it is always possible to achieve one with unidimensional signal. The result in this regard gives a different insight than Ottaviani and Sørensen (2006a) who argued that expert of a certain type has higher desires to misreport when reporting posterior than when reporting signal. Their result holds when the signal is

[3] For example, as do Crawford and Sobel (1982) and Alonso et al. (2008).

unidimensional but assumes the expert knows their ability.

Finally, this work can be seen as a literature about media bias of a monopolist. There are plenty of empirical studies on it in the century; and relatively a few of theoretical approaches. Baron (2006), Mullainathan and Shleifer (2005), and Gentzkow and Shapiro (2006) showed that the media would be biased to the prior of the public. While the first two focused on the demand from public and sales, the third viewed it partially as reputation building. Gentzkow and Shapiro (2006), however, considered reputation-driven bias arises because the expert would choose to predict the state that is more likely ex ante. They viewed bias lying outside of expert. In this paper, however, the states are equally likely. The incentives to misreport arise because of previous messages sent by expert themselves.

3.2 Model

Consider the following dynamic single-player game that lasts for two periods. There is a state of the world, $\omega \in \Omega := \{x, y\}$ whose probabilities are equally likely. It is drawn when the game starts and continues to be unknown until the last stage of the game. However, there is an expert in this world, who receives a series of informative but noisy signals about the state. The expert receives a signal each period from a signal space $S \equiv \Omega$, where the extent of the noise is determined by type. In particular, the accuracy of the first signal that an expert of type τ receives is given by θ_τ, or, $Pr(s_1 = \omega|\omega) = \theta_\tau$. Assume there is monotone likelihood ratio property in types, i.e., $\theta_{\tau'} > \theta_{\tau''}$ if and only if τ' is more competent than τ''. WLOG, let $T = [.5, 1]$ and $\theta_\tau = \tau$. The distribution of the type is commonly known by a (possibly discrete) p.d.f. f. Assume that f is bounded and $\operatorname{supp} f \geq 2$.

In the second period, the accuracy increases by $\Delta_\tau \geq 0$; $\{\Delta_\tau\}_{\tau \in T}$ preserves the order in types. In other words,

$$Pr(s_2 = \omega|\omega) = \theta_\tau + \Delta_\tau,$$

and there does not exist $\tau' > \tau''$ such that $\theta_{\tau'} + \Delta_{\tau'} < \theta_{\tau''} + \Delta_{\tau''}$. That is, $\tau' > \tau''$ if and only if $Pr(s_1 = \omega|\tau', \omega) > Pr(s_1 = \omega|\tau'', \omega)$ and $Pr(s_2 = \omega|\tau', \omega) \geq Pr(s_2 = \omega|\tau'', \omega)$. The increment in accuracies expresses the nature of information, typically becoming more accessible to the public as time elapses. It may be due to information leakage, rumors spreading, or advances in technologies. For example, more advanced technology for experiment may be adopted in the later period. The experiment with the new technology would be independent with the first experiment. However, it does not affect the proficiency in information acquisition ability that belongs to the individual.

Each period, after observing a signal, the expert publicly sends a message. Messages are transferred in cheap talks – by revelation principle, the message space will be denoted by $R \equiv S = \{x, y\}$. The expert's message strategy is denoted by $r_1 : S \to R$ and $r_2 : S \times S \times R \to R$. After the second message r_2 is sent, the state becomes public. Thus, it naturally reveals whether the messages match with the state. For notational convenience, define a switching mapping that returns the complement signal. That is, for $a \in \{x, y\}$, $\bar{a} \in \{x, y\} \setminus \{a\}$. I will refer a deterministic strategy as symmetric, if for any $s_1, s_2, \omega \in \{x, y\}$, $r_1(s_1) = \overline{r_1(\overline{s_1})}$ and $r_2(s_1, s_2; r_1) = \overline{r_2(\overline{s_1}, \overline{s_2}; \overline{r_1})}$.

The beliefs over the ability of the expert is updated by $\beta : R \times R \times \Omega \to \Delta T$ that follows Bayes rule. If any off-path message is sent, it puts probability 1 on $\inf\{\operatorname{supp} f\}$.[4] Reputation refers to the ex post expected accuracy of signal at $t = 1$ after observing r_1, r_2 and ω. The payoff for the expert is determined by the reputation:

$$\Phi(E_\beta[\theta|r_1, r_2, \omega]), \tag{3.1}$$

where $\Phi(\cdot)$ is a differentiable function with $\Phi'(\cdot) > 0$. Hence, the objective of the expert is to maximize his/her expected reputation. Notice that, the reputation term only depends on the ex post expected ability. For example, two different ex post distribution with a same mean gives the same reputational payoff. This setting can be seen in earlier researches

[4] A rationale of this off-path beliefs will be provided in the section 3.4.

including Gentzkow and Shapiro (2006) and Tajika (2021) while Ottaviani and Sørensen (2006a,b) considered a more gnenral expression.

3.3 Benchmarks and Equilibriums

Any cheap-talk communication have a babbling equilibrium. In an analogous sense, one can consider a period where babbling occurs, and some informative message is sent in the other periods. I consider here only deterministic and symmetric strategies, as it is sufficient to address the essence. Suppose, in an equilibrium, truth-telling is expected in a period $t \in \{1, 2\}$. In other words, the expert sends $r_t = s_t$ with a unit probability and β puts 1 on $s_t = r_t$ upon receiving r_t. Denote by $m_t \in \{c, i\}$, the event representing whether the message was equal to ω or not. $m_t = c$ if and only if $r_t = \omega$. 'c' stands for 'correct' and 'i' stands for 'incorrect'. If babbling is expected in a period $t \in \{1, 2\}$, r_t would not affect the reputation. In other words, for any r_t, $E_\beta[\theta|r_t, r_{-t}, \omega] = E_\beta[\theta|\bar{r}_t, r_{-t}, \omega]$ for any $r_t, r_{-t} \in \{x, y\}$, and $\omega \in \Omega$. In such cases, let $m_t = \varnothing$. Then, a tuple (r_1, r_2, ω) will be uniquely re-written in a form of (m_1, m_2, ω).

Let $\theta^{\beta}_{m_1 m_2 \omega}$ be the corresponding reputation after observing (m_1, m_2, ω). The superscription is used to explicitly stress the belief function under which the reputation is formed. By symmetricity of the model, if the message strategy is symmetric, $E[\theta_\tau|s_1, s_2, \omega] = E[\theta_\tau|\bar{s}_1, \bar{s}_2, \bar{\omega}]$. Hence, if there are no rooms for confusion, I will drop off the last argument in $\theta^{(\beta)}_{m_1 m_2 \omega}$ and simply write the reputation by $\theta^{(\beta)}_{m_1 m_2}$. It is also useful to define $p_{s_1 s_2} = Pr(\omega = x|s_1, s_2)$, the interim belief on $\omega = x$ after receiving (s_1, s_2).

In this section, two benchmark cases are presented. One is a static case, where the game lasts only for one period. It is essentially the same case with an equilibrium of dynamic case where the expert babbles in all the periods except in one. Another benchmark case is concerned with a truthful strategy, suggested by Tajika (2021), where the expert reports truthfully under any history. Then another strategy is proposed, which also induces truthful recommendation on path.

3.3.1 Static Benchmark

Suppose the game only has one period. From the perspective of Receiver, it is equivalent with the original game where only uninformative messages are conveyed in the second period. As the information structure reveals MLRP, a report that is matching with the state will give the expert favorable payoffs than a report that fails to match with the state.[5] For $\tau' > \tau''$,

$$E[s = \omega|\tau', \omega] > E[s = \omega|\tau'', \omega]. \tag{3.2}$$

In other words,

$$E_\tau[\theta_\tau|s = \omega] \geq E_\tau[\theta_\tau|s \neq \omega]. \tag{3.3}$$

The incentive constraint for the expert to report the true signal is

$$\begin{aligned} &\int (\theta\Phi(E[\theta|s = \omega]) + (1-\theta)\Phi(E[\theta|s \neq \omega]))f(\theta)\mathrm{d}\theta \\ \geq &\int ((1-\theta)\Phi(E[\theta|s = \omega]) + \theta\Phi(E[\theta|s \neq \omega]))f(\theta)\mathrm{d}\theta \end{aligned} \tag{3.4}$$

This inequality always holds because of $\theta > 1-\theta$ and (3.3). As a consequence, the incentives of the expert and the society (who wants truthful information) align, and the desire of the expert for constructing reputation is fulfilled by truth-telling. Therefore, under any prior distribution of θ, there exists an equilibrium where the expert makes reports only truthfully. The following subsection shows, in a dynamic setting, how the expert becomes biased.

It is worth noting that there always exists equilibria in which babbling occurs in one period and truth-telling occurs in the other period. Consider a following strategy such that the expert tells the signal truthfully in the first period and in the second, s/he merely repeats the first message. In a PBE, the second message does not convey any new information, as r_1, equivalently s_1, is already reflected in the interim beliefs. By definition, there babbling

[5] Milgrom (1981)

occurs in the second period. Then it is essentially equal to a static model, in which I argued above that there always exist truth-telling equilibria.

3.3.2 Fully Truthful Strategy

Suppose the expert adopts fully truthful strategy (hereinafter, FT strategy; FT equilibria are the equilibria where FT strategy is played). That is, $r_1^T(s_1) = s_1$ and $r_2^T(r_1, s_1, s_2) = s_2$. By superscription T, I will refer to the belief that puts 1 on $s_t = r_t$ upon receiving r_t. The reputations formed under such belief are given as $E[\theta|m_1, m_2, \omega]$ for $m_t \in \{i, c\}$ and $t = 1, 2$. Obviously, $\theta_{cc}^T > \theta_{ci}^T, \theta_{ic}^T >$ and $\theta_{ci}^T, \theta_{ic}^T > \theta_{ii}^T$. The two intermediate reputations (θ_{ci}^T and θ_{ic}^T) can be seen as reserved payoffs in a broad sense. Notice that, since the expert can earn one of those intermediate reputations for sure, by just sending two different messages, regardless of the true state. Sending the same messages is relatively more adventurous. In this sense, $\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T)$ is the gain from sending the first signal through two periods, conditional on that the signal is indeed correct. Analogously, $\Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)$ is the loss from sending the first signal through two periods, conditional on that the signal is indeed incorrect. Define ratio of loss to gain to describe further results.

$$E := \frac{\Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T)} \tag{3.5}$$

Note that E approaches 1 as the potential gain and loss become similar to each other. I introduces a similar notation,

$$E' := \frac{\Phi(\theta_{ci}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ic}^T)}, \tag{3.6}$$

representing the ratio of the potential gain and loss of deception in the first period, followed by the opposite recommendation in the second period.

Proposition 3.1. r^T *and the corresponding beliefs consist a PBE if and only if*

$$\frac{1 - p_{yx}}{p_{yx}} \leq E \leq \frac{p_{yx}}{1 - p_{yx}}, \tag{3.7}$$

where E is given by (3.5)*.*

Proof. See Appendix 3.A. □

Since $p_{yx} > 1/2$, Proposition 3.1 state that FT equilibrium requires E sufficiently close to 1. If loss from sending the same messages in a row is relatively large enough, the expert will have an anti-self-herding incentive, ignoring new, more convincing information. If the gain from sending the same messages in a row is relatively large enough, maintaining the original stance and pretending as if the signals were consistent will be a more attractive option, causing a self-herding incentive. For these reasons, to establish an FT equilibrium, the sizes of loss and gain from sending the same messages need to be similar. This presents the sufficient and necessary condition for an FT equilibria in Proposition 3.1, and Proposition 3.2 below demonstrates that there exist such priors that satisfy the condition.

Proposition 3.2. *For any differentiable $\Phi(\cdot)$ with $\Phi'(\cdot) > 0$, there always exist priors on abilities that sustain FT equilibrium.*

Proof. See Appendix 3.A. □

Corollary 3.1. *p_{yx} is weakly greater than 1/2 and converges to 1/2 if Δ_τ uniformly converges to 0.*

Proof. p_{yx} is weakly greater than 1/2 because the second signal is more informative. Since the type is independently determided with the true state, $f(\tau|r) = f(\tau)$. Additionally, for any bounded function h of τ, $\int h(\tau)\Delta_\tau \, \mathrm{d}\tau$ converges to 0. Therefore, both $Pr(m_1 = c, m_2 = i)$ and $Pr(m_1 = i, m_2 = c)$ converge to $\int \theta_\tau(1-\theta_\tau)f(\tau)\, \mathrm{d}\tau > 0$. By Bayes rule, p_{yx} converges to $Pr(x)$ which is equal to 1/2. □

It should be noticed from Corollary 3.1 that, when Δ_τ uniformly approaches 0, the existence of FT equilibrium becomes non-generic. Δ_τ uniformly approaching 0 also implies that the information arrivals in two peroids becomes similar. It can be interpreted as an identical experiment being executed independently twice. As seen in the static benchmark, however, when the experiment is run once, there always existed truthful outcomes, regardless of the distribution of abilities. This extension leaves few distributions that support FT equilibrium.

3.3.3 On-Path Truthful Strategy

I attempt to expand truth-telling outcomes in this dynamic model. Notice that FT strategy requires the expert to send a message truthfully in the second period, even after lying in the first period. This demands more than necessary to have a truth-telling outcome, because, if the expert played along the strategy in the first period, the histories where s/he lied in the past period never come with a positive probability in the equilibrium. The requirements contingent on such events may restrict the support of truth-telling outcomes.

Define $r^* = (r_1^*, r_2^*)$ by $r_1^*(s_1) = s_1$, $r_2^*(s_1, s_2; s_1) = s_2$ and

$$
\begin{aligned}
r_2^*(s_1, s_2 = s_1; r_1 \neq s_1) &= \begin{cases} s_2 & \text{if } \frac{1-p_{xx}}{p_{xx}} \leq E \\ \overline{s_2} & \text{otherwise} \end{cases} \\
r_2^*(s_1, s_2 \neq s_1; r_1 \neq s_1) &= \begin{cases} s_2 & \text{if } E \leq \frac{p_{yx}}{1-p_{yx}} \\ \overline{s_2} & \text{otherwise} \end{cases}
\end{aligned}
\tag{3.8}
$$

Despite that off-the-path continuation strategy sends a truthful message on some occasions, it may not convey any information about s_2. To see why, suppose the expert misreported in the first period, sending $r \neq s$ after observing $s_1 = s$. If E is sufficiently great, $r^*(s, \cdot; r_1 \neq s)$ will send s, regardless of realized s_2: when $s_2 = s$, $r^*(s, s_2; r_1 \neq s) = s_2 = s$ and when $s_2 \neq s$, $r^*(s, s_2; r_1 \neq s) = \overline{s_2} = \overline{\overline{s}} = s$. Recall that, $r_1 = \bar{s}$. In this case, the expert is reluctant to send the same message with the previous one, leading to an anti-self-herding phenomenon. If E is sufficiently small, $r^*(s, \cdot; r_1 \neq s)$ will send $\bar{s}$, regardless of realized s_2: when $s_2 = s$, $r^*(s, s_2; r_1 \neq s) = \overline{s_2} = \overline{s}$ and when $s_2 \neq s$, $r^*(s, s_2; r_1 \neq s) = s_2 = \bar{s}$. Similarly, in this case, the expert has a strong incentive to be consistent, leading to a self-herding phenomenon. However, these phenomena would not arise if the expert has played along with r^*.

Proposition 3.3. *Suppose that $\theta_{ic}^T > \theta_{ci}^T$. r^* and the corresponding beliefs consist a PBE* *if*

$$\frac{1-p_{yx}}{p_{yx}} \leq E \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.9}$$

and

$$E' \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.10}$$

where E and E' are given by (3.5) *and* (3.6)*, respectively.*

Proof. See Appendix 3.A. □

Corollary 3.2. *If $E' \leq E$, PT equilibrium exists when FT equilibrium does. The inclusion relationship is strict. If $E' > E$, the inclusion relationship is ambiguous.*

Recall that range of E in (3.7) that sustain FT equilibrium converges to a singleton, $\{1\}$, as Δ_τ uniformly approaches 0. However, p_{xx} is strictly greater than p_{yx}, as the first signal is also ex-ante informative. Indeed, for a sequence $\{\Delta_{\tau,n}\}^n$ such that uniformly converges to 0, there exists the limit of $\frac{p_{xx}}{1-p_{xx}}$, which is given as

$$\lim_{n\to\infty} \frac{p_{xx}}{1-p_{xx}} = \frac{\int \frac{\theta_\tau^2}{\theta_\tau^2+(1-\theta_\tau)^2} f(\tau|s_1=s_2)\,\mathrm{d}\,\tau}{\int \frac{(1-\theta_\tau)^2}{\theta_\tau^2+(1-\theta_\tau)^2} f(\tau|s_1=s_2)\,\mathrm{d}\,\tau} > 1 \tag{3.11}$$

3.4 Analysis

Thus far, I have analyzed equilibria where information is conveyed in both periods. Slightly alter the message set so that it contains a neologism, 0, which implies the expert chooses to keep silent. However, these are out-of-equilibrium messages of both FT and PT strategies, and as assumed, the expert will be considered of type $\inf\{\operatorname{supp} f\}$ for sure. Under this assumption, it is clear that the expert does not have an incentive to choose 0. One might argue that the existence of the truthful equilibria is contributed by such severe off-path belief assumption. They might claim: if it had not been the assumption, the expert might strategically choose when to reveal their information. For example, Guttman et al. (2014) claimed that, the expert will benefits from revealing information in the last period and keep silent in the first period compared to earlier revelation. In contrast, Tajika (2021) claimed that the expert would reveal the information in earlier period and ignore the information that arrives later. However, in the first subsection, I argue that it seems to be natural for truthful equilibria to arise under a certain condition even without such an assumption.

On the other hand, I have investigated the factors that make truth-telling difficult even under this simple, symmetric environment. Section 3.3 suggested dynamicity as one of such factors by comparing a static benchmark to dynamic models. But another class they fall into, is multi-dimensionality – one would agree that any series of signals can be perceived as having multiple dimensions.

The second subsection considers an adjusted variation in which the expert sends a message only at $t = 2$ about (s_1, s_2) after collecting all the information. It can be essentially seen as a static model where the signal space is 2-dimensional, whose elements have different accuracies. The support of truth-telling equilibria in this version differs from those in the original model under both FT and PT strategies.

3.4.1 Expert-efficient Equilibrium

Consider a following strategy: the expert sends a message truthfully in the first period and repeat it regardless of the second signal. Formally, $r_1^C(s_1) = s_1$ and $r_2^C(\cdot, \cdot; r_1) = r_1$. For simplicity, a corresponding belief, β^C, will put $\theta^C_{m_1 m_2} = \inf\{suppf\}$ for any off-path messages, $r_1 \neq r_2$. Then $r^C = (r_1^C, r_2^C)$ the corresponding beliefs consist an equilibrium. To see why, observe that the second message does not deliver information about s_2; instead, it is only correlated with s_1 that is already conveyed. Bayes feasible beliefs will assign the same ex post distribution upon receiving $r_2 = r_1$. Then the second period is, by definition, a babbling period, thereby m_2 will be taken as $\varnothing$. With the second period where babbling occurs, it can be essentially seen as a single period model. It can be easily shown that a single period version of this game always has truth-telling equilibria.

It is obvious that, r^* is more informative than r^C. This implies the ex post distribution under r^* is more dispersed than that under r^C (cf. Blackwell (1953)). Define $\theta^C_{m_1 m_2}$ in an analogous manner.

$$\theta^C_{m_1\varnothing} = A_{m_1c}\theta^T_{m_1c} + A_{m_1i}\theta^T_{m_1i} \text{ for } m_1 \in \{c, i\}$$

where $A_{m_1c} = Pr(m_2 = c|m_1)$ and $A_{m_1i} = Pr(m_2 = i|m_1)$. The ex ante payoff under r^C will be calculated as

$$Pr(m_1 = c)\Phi(A_{cc}\theta^T_{cc} + A_{ci}\theta^T_{ci}) + Pr(m_1 = i)\Phi(A_{ic}\theta^T_{ic} + A_{ii}\theta^T_{ii}) \tag{3.12}$$

By Jensen's inequality and Bayes rule, if Φ is convex, (3.12) is weakly smaller than

$$\begin{aligned} &Pr(m_1 = c)\{A_{cc}\Phi(\theta^T_{cc}) + A_{ci}\Phi(\theta^T_{ci})\} + Pr(m_1 = i)\{A_{ic}\Phi(\theta^T_{ic}) + A_{ii}\Phi(\theta^T_{ii})\} \\ &= q_{cc}\Phi(\theta^T_{cc}) + q_{ci}\Phi(\theta^T_{ci}) + q_{ic}\Phi(\theta^T_{ic}) + q_{ii}\Phi(\theta^T_{ii}). \end{aligned} \tag{3.13}$$

The RHS of (3.13) is none other than the ex ante payoff under r^T (equivalently, that under r^*). Intuitively, a risk-loving expert would ex ante prefer more dispersed distributions of reputation. Likewise, less dispersed distributions of reputation would be more preferable to a risk-averse expert.

This discussion can go further. While this paper focuses on a two-period game, with a specific information structure, it can be extended to a T-period game, with an arbitrary information structure. r^C and β^C can be generalized so that the expert reports truthfully in period $t_1, \cdots, t_{T'}$ with $T' < T$.[6]

Proposition 3.4. *Consider an extended T-period game and two strategies and corresponding beliefs; (1) the expert always sends a message truthfully in every period, and (2) the expert sends a message truthfully only in some fixed periods, $t_1, \cdots, t_{T'}$ with $T' < T$ and babbles for the remaining period. For any information structure, the former overperforms the latter in terms of expected reputational payoffs for the expert if Φ is convex.*

Proof. See Appendix 3.A. □

[6] An easy way to describe such strategy and belief is to assume the expert tells the truth only when $t = t_1, \cdots, t_{T'}$; and babbles otherwise. Proposition 3.4 does not necessarily require it constitutes an equilibrium.

3.4.2 Delayed Communication

In subsection 3.3.1, I compared a static benchmark with a dynamic model and showed how dynamicity in model makes truth-telling difficult, even under one of the most simple environments. The logic is that, when the expert makes additional announcement after one another, the past announcement causes incentives to misreport. Such incentive only occurs in the later period. One should be aware that, however, this does not necessarily imply that staticity always fosters truth-telling.

Suppose there is a society designer who naively conjectures one-shot communication makes the society truthful. She does not know which strategy the expert is adopting, but does observe that truth-telling in both period is not an equilibrium in the current society. She forces the expert to keep silence until the expert collects all the information at $t = 2$. Therefore, in new timeline, s_1 arrives at $t = 1$; the expert observes s_2 in addition to s_1 at $t = 2$ and sends two messages r_1 and r_2. Alternatively assume that, in another universe, the expert receives a multi-dimensional signal (s_1, s_2) and sends a multi-dimensional message (r_1, r_2) in a single period. These two situations are essentially the same. In other words, the social designer made a dynmaic game into a static one. How would this change affect truth-telling behavior?[7]

Due to the symmetricity in this model, it is sufficient to consider symmetric strategies. Assume that the expert received the same signals in both periods. Then the incentives to fabricate the one of those signals for safer reputation may arise. It is incentive compatible to report truthfully if

$$p_{xx}\Phi(\theta^T_{cc}) + (1 - p_{xx})\Phi(\theta^T_{ii}) \geq \max \begin{Bmatrix} p_{xx}\Phi(\theta^T_{ci}) + (1 - p_{xx})\Phi(\theta^T_{ic}), \\ p_{xx}\Phi(\theta^T_{ic}) + (1 - p_{xx})\Phi(\theta^T_{ci}), \\ p_{xx}\Phi(\theta^T_{ii}) + (1 - p_{xx})\Phi(\theta^T_{cc}) \end{Bmatrix} \tag{3.14}$$

The last inequality always holds because $p_{xx} > 1/2$ and $\theta^T_{cc} > \theta^T_{ii}$. The first inequality is no

[7] I appreciate Ishida Junichiro for the comment of this idea.

other than the condition under which the expert keeps being truthful after two same signals in the original model, corresponding to the right side of inequalities in (3.9). The second inequality is new, a constraint that has not existed in the timely message protocol. This corresponds to the incentive that the expert pretends to have received a different signal in the past, while reporting truthfully in regard with the signal in the current period. Then, (3.14) is equivalent with

$$\max\left\{\frac{\Phi(\theta^T_{ic})-\Phi(\theta^T_{ii})}{\Phi(\theta^T_{cc})-\Phi(\theta^T_{ci})},\frac{\Phi(\theta^T_{ci})-\Phi(\theta^T_{ii})}{\Phi(\theta^T_{cc})-\Phi(\theta^T_{ic})}\right\}\leq\frac{p_{xx}}{1-p_{xx}} \tag{3.15}$$

Assume now that the expert received the different signals in two periods. If the expert received different signals, the expert would believe the second signal is more likely. Then, the incentives to fabricate the first signal in accordance with the second signal so that s/he can pretend to be more competent may arise. Additionally, there is another incentive, of switching s_1 and s_2. These two incentives are new, that has not been existed in the timely message protocol. It is incentive compatible to report truthfully if

$$p_{yx}\Phi(\theta^T_{ic})+(1-p_{yx})\Phi(\theta^T_{ci})\geq\max\begin{Bmatrix}p_{yx}\Phi(\theta^T_{ci})+(1-p_{yx})\Phi(\theta^T_{ic}),\\ p_{yx}\Phi(\theta^T_{cc})+(1-p_{yx})\Phi(\theta^T_{ii}),\\ p_{yx}\Phi(\theta^T_{ii})+(1-p_{yx})\Phi(\theta^T_{cc})\end{Bmatrix} \tag{3.16}$$

The last inequality holds if the second holds because $p_{yx}>1/2$ and $\theta^T_{cc}>\theta^T_{ii}$. The second inequality is the condition under which the expert does not fabricate the past signal. The first inequality indicates the incentive constraint that the expert does not switch the signals. Indeed, it is equivalent with $\theta^T_{ic}\geq\theta^T_{ci}$. Then, (3.16) is equivalent with $\theta^T_{ic}\geq\theta^T_{ci}$ and

$$\frac{p_{yx}}{1-p_{yx}}\leq\frac{\Phi(\theta^T_{ci})-\Phi(\theta^T_{ii})}{\Phi(\theta^T_{cc})-\Phi(\theta^T_{ic})} \tag{3.17}$$

It immediately follows the proposition below.

Proposition 3.5. *If $\theta^T_{ci}>\theta^T_{ic}$, there cannot be a truth-telling equilibrium in the delayed communication game. If $\theta^T_{ci}\leq\theta^T_{ic}$, there exists a truth-telling equilibrium in the delayed* *communication game if*

$$E \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.18}$$

and

$$\frac{p_{yx}}{1-p_{yx}} \leq E' \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.19}$$

where E and E' are given by (3.5) *and* (3.6)*, respectively.*

When $\theta_{ci}^T > \theta_{ic}^T$, truth-telling is not feasible at all. This is because signals are more accurate in the second period. When two different signals arrive, the expert would believe the second signal is more likely. If $\theta_{ci}^T > \theta_{ic}^T$, it is strictly better for the expert to tell the more likely signal arrived first.

For intuitive explanation, in the rest of this subsection, assume $\theta_{ci}^T = \theta_{ic}^T$ and thereby $E = E'$: one correct and one incorrect messages give the same reputations, no matter which period. Then, the condition is reduced to

$$\frac{p_{yx}}{1-p_{yx}} \leq E = E' \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.20}$$

The right side of inequalities in (3.20) is the condition under which the expert would not lie when observed two same signals. It corresponds with the condition about anti-self-herding phenomenon in the original setting.

On the other hand, self-herding incentive, corresponding to the left side of inequalities in (3.20), becomes stronger, compared with the original with timely messages. This incentive had been prevented by a relatively mild condition. Note that, a sufficient condition was that E is greater than 1. This is because the second signal is more accurate. To see this, suppose you recevied two different signals. As soon as you received s_2, you realize that $\omega = s_2$ is more likely. In the original model, since you already have sent $r_1 = s_1 = \bar{s}_2$, your options are limited. In this alternative setting, there has not been a message sent yet for s_1, providing an opportunity to potentially fabricate it. The expert can align the messages with the most likely state, pretending to be more competent. $E > 1$ is no longer a sufficient condition, and we need a stronger constraint.

In the society designer example, she changed the communication timing without knowing which strategy the expert was adopting. Suppose the expert was employing FT strategy and failed to achieve truth-telling because E was relatively great. In such cases, delayed communication may help lead to a truth-telling equilibrium. However, suppose the expert was employing PT strategy and failed to achieve truth-telling because E was relatively small. In such cases, altering the timing of communication does not resolve the problem.

3.5 Back to Fully Truthful Equilibrium with a Binary Case

In this section, I focus on a specific information structure, suggested by Tajika (2021). In particular, $Pr(s_2 = \omega|\omega, \tau)$ is given by

$$\frac{(1+\alpha)\theta_\tau}{1+\alpha\theta_\tau},$$

for some $\alpha \geq 0$ where α is common across the types. Note that it is an increasing function of α, equivalent with an identity function if and only if $\alpha = 0$. Furthermore, it satisfies the monotonicity in θ_τ. Hence, this is a specific form of the main model with $\Delta_\tau = \frac{1-\theta_\tau}{1+\alpha\theta_\tau}\alpha\theta_\tau$ and α is interpreted as the degree of increments in accuracies.

To understand this structure better, consider a binary type space, $\{L, H\}$ with probability π on H and $\theta_L < \theta_H$. Notice that

$$\frac{LR(s_1 = \omega|\tau,\omega)}{LR(s_1 \neq \omega|\tau,\omega)} = \frac{LR(s_2 = \omega|\tau,\omega)}{LR(s_2 \neq \omega|\tau,\omega)} = \frac{\theta_H}{\theta_L} / \frac{1-\theta_H}{1-\theta_L}. \tag{3.21}$$

The ratio of likelihood-ratio of correct message to that of incorrect message is held at the same level through periods. On the other hand, by definition of reputation, $\theta^T_{s_1 s_2}$ can be written as

$$\theta^T_{s_1 s_2} = \theta_L + (\theta_H - \theta_L) Pr(H|s_1, s_2, \omega) = \theta_L + \frac{(\theta_H - \theta_L)}{1 + LR(s_1|\omega)^{-1} LR(s_2|\omega)^{-1}(1-\pi)/\pi} \quad (3.22)$$

Taking (3.21) into account, (3.22) implies that one correct and one incorrect messages give the same reputation no matter which period which message is sent. Indeed, with this specific information structure, reputations are given by

$$\begin{aligned} \theta^T_{cc} &= \frac{\theta_L^3(1+\alpha\theta_H)(1-\pi) + \theta_H^3(1+\alpha\theta_L)\pi}{\theta_L^2(1+\alpha\theta_H)(1-\pi) + \theta_H^2(1+\alpha\theta_L)\pi}, \\ \theta^T_{ci} = \theta^T_{ic} &= \frac{(1-\theta_L)\theta_L^2(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)\theta_H^2(1+\alpha\theta_L)\pi}{(1-\theta_L)\theta_L(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)\theta_H(1+\alpha\theta_L)\pi}, \\ \theta^T_{ii} &= \frac{(1-\theta_L)^2\theta_L(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)^2\theta_H(1+\alpha\theta_L)\pi}{(1-\theta_L)^2(1+\alpha\theta_H)(1-\pi) + (1-\theta_H)^2(1+\alpha\theta_L)\pi}. \end{aligned} \quad (3.23)$$

Although the second signal is more informative with regard to the state ($p_{yx} > p_{xy}$), it does not mean it is more informative with regard to the type of the expert. With this property, the intermediate reputation $\theta^T_{ci} = \theta^T_{ic}$ can be seen as a reserved payoff.

Tajika (2021) asserted that, with all other held constant, an FT equilibrium cannot exist if α is sufficiently small and the payoffs are weakly convex in the reputation. However, this section shows that it is not the case, using a linear Φ and the information structure illustrated above.

Let $\Phi(\theta) = \theta$. A sufficient condition for (3.7) is

$$\theta_L(1-\theta_L)(1-\pi) = \theta_H(1-\theta_H)\pi. \quad (3.24)$$

For any fixed θ_L and θ_H, there always exists a well-defined distribution, denoted by $\pi^* \in (0,1)$ that satisfies (3.24).

$$\pi^* = \frac{\theta_H(1-\theta_H)}{\theta_H(1-\theta_H) + \theta_L(1-\theta_L)} \quad (3.25)$$

By design, at $\alpha = 0$, $(\theta_L, \theta_H, \pi)$ sustains an FT equilibrium.

Letting $\Phi(\theta) = \theta$ and substituting θ_{cc}^T, θ_{ii}^T and $\theta_{ci}^T = \theta_{ic}^T$ in (3.23) to (3.7) yields

$$\frac{1}{1+\alpha} \leq \frac{\alpha\theta_H^2(1-\theta_L) + \alpha\theta_L^2(1-\theta_H) + \theta_H + \theta_L - 2\theta_H\theta_L}{\alpha\theta_H\theta_L(\theta_H+\theta_L)(2-(\theta_H+\theta_L)) + \theta_H + \theta_L + 2\theta_H\theta_L} \leq 1+\alpha \tag{3.26}$$

Note that, when $\alpha = 0$, the inequalities hold with the equalities and all equal to 1, as π^* is designed as such. As $\mathrm{d}(1/(1+\alpha))/\mathrm{d}\,\alpha = -1$ and $\mathrm{d}(1+\alpha)/\mathrm{d}\,\alpha = 1$, to demonstrate the inequalities hold when α is sufficiently small, it is sufficient to show that the derivative when $\alpha = 0$ is between -1 and 1.

$$\frac{\mathrm{d}}{\mathrm{d}\,\alpha}\left(\frac{\alpha\theta_H^2(1-\theta_L) + \alpha\theta_L^2(1-\theta_H) + \theta_H + \theta_L - 2\theta_H\theta_L}{\alpha\theta_H\theta_L(\theta_H+\theta_L)(2-(\theta_H+\theta_L)) + \theta_H + \theta_L + 2\theta_H\theta_L}\right)\Bigg|_{\alpha=0} = -\frac{(\theta_H-\theta_L)^2}{\theta_H+\theta_L-2\theta_H\theta_L}, \tag{3.27}$$

which is in $(-1, 0)$ because $\theta_L < \theta_H < 1$. It provides the following proposition.

Proposition 3.6. *Let $\Phi(\theta) = \theta$ and suppose the distribution is characterized by $(\theta_H, \theta_L, \pi^*)$ with $\theta_L < \theta_H$ and π^* given by (3.25). An FT equilibrium exists when α is sufficiently small.*

Although his proposition was not correct, the intuition does mean something. Certainly, it is true that when $\alpha = 0$, an FT equilibrium exists only if $\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T) = \Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)$. In PT equilibria, the threshold of the left side in (3.27) is the same and the right side threshold is replaced with $p_{xx}/(1-p_{xx})$, which also decreases with α. This implies that when the accuracy remains the same or the increment is negligible, truth-telling may not persist.

The simplified environment effectively helps us derives an explicit and unique solution of $E = 1$. Thus, we will explore further to see an observation below.

Remark 3.1. *π^* given in (3.25) decreases in θ_H and increases in θ_L.*

Intuitively speaking, in the second period, the expert has to choose whether to report truthfully. To behave truthfully, the expert has to be confident enough with the second signal. The confidence increases in θ_H, the accuracy conditional on being a more competent type. If θ_H decreases the confidence would decrease; to compensate this, the expert has to believe more strongly that s/he is of type θ_H. On the other hand, if θ_L increases, or, if less competent type receives more accurate signal than before, receiving correct signal would be a weaker evidence that the expert is competent. Thus, it requires higher ex-ante probability of being of type θ_H.

3.6 Conclusion Remarks

This paper analyzes a dynamic cheap talk game where the sender cares about his/her reputation for being well-informed. The structure of the model is so simple that if there was only one period, the existence of truth-telling equilibrium is obvious. However, extending the game by adding one period causes incentives for self-herding and anti-self-herding, depending on histories. If the realized signals were aligned to each other, truth-telling behavior may be interfered by anti-herding incentives. The incentive may arise because two incorrect messages will give the expert the worst reputation. If the realized signals are opposite to each other, truth-telling behavior may be interfered by herding incentives. The incentive may arise because two incompatible messages will make the sender less competent. To build truth-telling equilibria, it is important to manage both incentives properly. Following Woo (2022), I considered two different truth-telling strategies and corresponding equilibria.

The truth-telling strategies are different in off-path behavior. In FT strategy, the sender is required to report the true signal even after a misreport. On-path truthful strategy loosen this constraint and allow the sender to lie if there has been a lie in the past. If $E' \leq E$, as in the binary example in the last section, this tweak strictly expands the set of distributions compatible with truth-telling outcomes.

In section 3.4, I analyzed the results more deeply. Firstly, it demonstrated a condition for truth-telling all the time to be, in turn, beneficial to the expert as well. As being a cheap talk game, there always exists a babbling equilibrium. However, I showed that more informative equilibrium performs better for the expert than ones less informative, such as a babbling equilibrium, if the payoffs function is convex in the reputation. Therefore, one can assume that truth-telling outcome is the one that arises naturally under such payoff functions that reveal convexity. It also obviously maximizes the payoffs of principal. This result implies that managers in industries where agents have career concerns may consider incorporating reputation-based elements into contract terms to incentivize agents to reveal their inverifiable expertise.

Secondly, the impact of delayed communication has been investigated. In the original model, the sender receives multiple signals over time and must send messages after each signal. However, from the results above, one may naively assume dynamicity is the factor that hinders truth-telling. In the alternative setting, the sender still receives multiple signals over time, but must wait until all information is collected before sending messages. It can be interpreted that the change staticized the original model. The analysis concluded that altering the timing of message transmission could promote truth-telling in certain scenarios, while in others, it may not. Even worse, it may potentially undermine already established truth-telling behavior. These results imply that changing the timing of communication, not knowing the current status the society, for examples, the strategy the people employs or the distribution in the society, may be dangerous.

Section 3.5 analyzed a stylized version of the model and provided an explicit solution of distribution. Using the solution, I present a counter example of a proposition in Tajika (2021). More specifically, I showed the existence of the ability distribution of the first period that sustains FT equilibria when the degree of accuracy increment is sufficiently small, under a weakly convex payoff function. Moreover, the solution suggests a characteristic of distributions that facilitate truth-telling equilibria.

Appendix of Chapter 3

3.A Proofs

Proof of Proposition 3.1. Suppose the expert played according to r_1^T. Then the expected payoffs of playing r_2^T are given as

$$\begin{aligned} & p_{xx}\Phi(\theta_{cc}^T) + (1-p_{xx})\Phi(\theta_{ii}^T) \quad \text{if } s_1 = s_2, \\ & p_{yx}\Phi(\theta_{ic}^T) + (1-p_{yx})\Phi(\theta_{ci}^T) \quad \text{if } s_1 \neq s_2. \end{aligned} \tag{3.28}$$

If the expert lies in the second period, that is, $r_2 \neq s_2$ the expected payoffs are

$$\begin{aligned} & p_{xx}\Phi(\theta_{ci}^T) + (1-p_{xx})\Phi(\theta_{ic}^T) \quad \text{if } s_1 = s_2, \\ & p_{yx}\Phi(\theta_{ii}^T) + (1-p_{yx})\Phi(\theta_{cc}^T) \quad \text{if } s_1 \neq s_2. \end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), for r_2^T to be an optimal, the following inequalities have to hold:

$$\frac{1-p_{yx}}{p_{yx}} \leq \frac{\Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T)} \leq \frac{p_{xx}}{1-p_{xx}} \tag{3.30}$$

Now suppose the expert has sent a message that is the opposite to the signal that arrived in the first period. In the second period, it requires the following two inequalities to send a truthful message.

$$\begin{aligned} & p_{xx}\Phi(\theta_{ic}^T) + (1-p_{xx})\Phi(\theta_{ci}^T) \geq p_{xx}\Phi(\theta_{ii}^T) + (1-p_{xx})\Phi(\theta_{cc}^T) \\ & p_{yx}\Phi(\theta_{cc}^T) + (1-p_{yx})\Phi(\theta_{ii}^T) \geq p_{yx}\Phi(\theta_{ci}^T) + (1-p_{yx})\Phi(\theta_{ic}^T) \end{aligned} \tag{3.31}$$

It is summarized by

$$\frac{1-p_{xx}}{p_{xx}} \leq \frac{\Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T)} \leq \frac{p_{yx}}{1-p_{yx}}. \tag{3.32}$$

Given the continued strategy in the second period as $r_2(s_1, s_2; r_1) = s_2$, the expert has no incentive to lie in the first period. To see this, let $q_{m_1 m_2} = Pr(m_1, m_2|s_1)$ be the conditional probability of the matching results at the interim point after receiving the first signal. For truthful recommendation to be an optimal behavior given the second period strategy is to do so, it needs to be satisfied that

$$\begin{aligned} &q_{cc}\Phi(\theta_{cc}^T) + q_{ii}\Phi(\theta_{ii}^T) + q_{ci}\Phi(\theta_{ci}^T) + q_{ic}\Phi(\theta_{ic}^T) \\ &\geq q_{cc}\Phi(\theta_{ic}^T) + q_{ii}\Phi(\theta_{ci}^T) + q_{ci}\Phi(\theta_{ii}^T) + q_{ic}\Phi(\theta_{cc}^T) \end{aligned} \tag{3.33}$$

By re-arranging (3.33),

$$(q_{cc} - q_{ic})(\Phi(\theta_{cc}^T) - \Phi(\theta_{ic}^T)) + (q_{ci} - q_{ii})(\Phi(\theta_{ci}^T) - \Phi(\theta_{ii}^T)) \geq 0, \tag{3.34}$$

which always holds. Hence, the only effective restrictions are (3.30) and (3.32). To put it together,

$$\max\left\{\frac{1-p_{yx}}{p_{yx}}, \frac{1-p_{xx}}{p_{xx}}\right\} \leq \frac{\Phi(\theta_{ic}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ci}^T)} \leq \min\left\{\frac{p_{xx}}{1-p_{xx}}, \frac{p_{yx}}{1-p_{yx}}\right\} \tag{3.35}$$

It can be easily shown that (3.7) is equivalent with (3.35) because $p_{yx} < p_{xx}$. □

Proof of Proposition 3.2. Arbitrarily fix two (possibly degenerate) p.d.f. functions denoted by $\overline{f}$ and $\underline{f}$, having different means $\overline{\theta}$ and $\underline{\theta}$, respectively. WLOG, $\overline{\theta} > \underline{\theta}$. For each $m = (m_1, m_2)$, let $\overline{q_m}$ and $\underline{q_m}$ be q_m calculated under $\overline{f}$ and $\underline{f}$, respectively. Define a p.d.f. $f(a)$ over the type space as a function of $a \in \mathbb{R}$. In particular, $f(\tau) = a\overline{f}(\tau) + (1-a)\underline{f}(\tau)$. Consider reputation θ_m^T, calculated under f. The derivative with respect to a at $a = 1$ yields below.

$$\begin{aligned} \left.\frac{\partial\theta_m^T}{\partial a}\right|_{a=1} &= \frac{\partial}{\partial a}\left.\frac{\int \theta_\tau q_m(\tau)\{a\overline{f}(\tau) + (1-a)\underline{f}(\tau)\}d\tau}{\int q_m(\tau)\{a\overline{f}(\tau) + (1-a)\underline{f}(\tau)\}d\tau}\right|_{a=1} \\ &= \frac{\int \theta_\tau q_m(\tau)\{\overline{f}(\tau) - \underline{f}(\tau)\}d\tau}{\int q_m(\tau)\overline{f}(\tau)d\tau} - \frac{\int q_m(\tau)\{\overline{f}(\tau) - \underline{f}(\tau)\}d\tau \int \theta_\tau q_m(\tau)\overline{f}(\tau)d\tau}{\left(\int q_m(\tau)\overline{f}(\tau)d\tau\right)^2} \\ &= (\overline{\theta} - \underline{\theta})\frac{\underline{q_m}}{\overline{\overline{q_m}}} \end{aligned} \tag{3.36}$$

Similarly,

$$\left.\frac{\partial\theta_m^T}{\partial a}\right|_{a=0} = (\overline{\theta} - \underline{\theta})\frac{\overline{q_m}}{\underline{q_m}} \tag{3.37}$$

Now, using the structure of the model, the following properties hold: (1) when type and state are fixed, signals in different periods are independently drawn, and (2) the signal in the first period does not contain information about the accuracy of itself and information structure is symmetric with regard to the states, i.e., $q_{m_1m_2} = Pr(m_1, m_2|s_1) = Pr(m_1, m_2)$. Then,

$$\frac{\overline{q_{m_1m_2}}}{\underline{q_{m_1m_2}}} = \frac{Pr(m_1, m_2|\overline{\theta})}{Pr(m_1, m_2|\underline{\theta})} = \frac{Pr(m_1|\overline{\theta})}{Pr(m_1|\underline{\theta})}\frac{Pr(m_2|\overline{\theta})}{Pr(m_2|\underline{\theta})} \equiv LR(m_1)LR(m_2), \tag{3.38}$$

where $LR(m_t) = Pr(s_t = \omega|\overline{\theta}, \omega)/Pr(s_t = \omega|\underline{\theta}, \omega)$. The first equality derives from the second property and second derives from the first property. The third is a re-statement in terms of likelihood ratio.

On the other hand, given a prior, a sufficient condition for the truthful equilibrium is $\Phi(\theta^T_{cc}) - \Phi(\theta^T_{ci}) = \Phi(\theta^T_{ic}) - \Phi(\theta^T_{ii})$. Denote by $\theta^T_{m_1m_2}(a)$ be the reputation corresponding to $m = (m_1, m_2)$ calculated under $f(a) = a\overline{f} + (1-a)\underline{f}$. Solutions of following function satisfy the sufficient condition.

$$g(a) := \Phi(\theta^T_{cc}(a)) - \Phi(\theta^T_{ci}(a)) - (\Phi(\theta^T_{ic}(a)) - \Phi(\theta^T_{ii}(a))) = 0 \tag{3.39}$$

Notice first that, if f is a degenerate distribution, this condition always holds: for any outcome, f and ex-post posterior put probability 1 on the same type. Additionally, (3.39) is continuous in a. Letting $\overline{f}$ and $\underline{f}$ be p.d.f. that put probability 1 on $\overline{\theta} \in T$ and $\underline{\theta} \in T$, respectively, $f(a)$ is a degenerate distribution at $a = 1$ and $a = 0$. At $a = 1$, $\theta^T_{cc}(1) = \theta^T_{ci}(1) = \theta^T_{ic}(1) = \theta^T_{ii}(1) = \overline{\theta}$ and $a = 0$, $\theta^T_{cc}(0) = \theta^T_{ci}(0) = \theta^T_{ic}(0) = \theta^T_{ii}(0) = \underline{\theta}$. It immediately follows that $g(1) = g(0) = 0$. Additionally,

$$\begin{aligned}\left.\frac{dg(a)}{da}\right|_{a=1} &= \Phi'(\overline{\theta})\frac{d}{da}\left\{\theta^T_{cc}(a) + \theta^T_{ii}(a) - \theta^T_{ci}(a) - \theta^T_{ic}(a)\right\}\Big|_{a=1}, \text{ and} \\ \left.\frac{dg(a)}{da}\right|_{a=0} &= \Phi'(\underline{\theta})\frac{d}{da}\left\{\theta^T_{cc}(a) + \theta^T_{ii}(a) - \theta^T_{ci}(a) - \theta^T_{ic}(a)\right\}\Big|_{a=0}.\end{aligned} \tag{3.40}$$

By assumption, $\Phi'(\cdot) > 0$. Using (3.37),

$$\left.\frac{d(\theta^T_{cc} + \theta^T_{ii} - \theta^T_{ci} - \theta^T_{ic})}{da}\right|_{a=0} = (\overline{\theta} - \underline{\theta})\left\{\frac{\overline{q_{cc}}}{\underline{q_{cc}}} + \frac{\overline{q_{ii}}}{\underline{q_{ii}}} - \frac{\overline{q_{ci}}}{\underline{q_{ci}}} - \frac{\overline{q_{ic}}}{\underline{q_{ic}}}\right\} \tag{3.41}$$

Substituting (3.38),

$$
\begin{aligned}
&\frac{\overline{q_{cc}}}{\underline{q_{cc}}} + \frac{\overline{q_{ii}}}{\underline{q_{ii}}} - \frac{\overline{q_{ci}}}{\underline{q_{ci}}} - \frac{\overline{q_{ic}}}{\underline{q_{ic}}} \\
&= LR(s_1 = \omega|\omega)LR(s_2 = \omega|\omega) + LR(s_1 \neq \omega|\omega)LR(s_2 \neq \omega|\omega) \\
&\quad - LR(s_1 = \omega|\omega)LR(s_2 \neq \omega|\omega) - LR(s_1 \neq \omega|\omega)LR(s_2 = \omega|\omega) \\
&= (LR(s_1 = \omega|\omega) - LR(s_2 = \omega|\omega))(LR(s_2 = \omega|\omega) - LR(s_2 \neq \omega|\omega)) > 0
\end{aligned}
\tag{3.42}
$$

In an analogous manner,

$$
\begin{aligned}
&\left.\frac{d(\theta^T_{cc} + \theta^T_{ii} - \theta^T_{ci} - \theta^T_{ic})}{da}\right|_{a=1} \\
&= (\overline{\theta} - \underline{\theta})(LR(s_1 = \omega|\omega)^{-1} - LR(s_2 = \omega|\omega)^{-1})(LR(s_2 = \omega|\omega)^{-1} - LR(s_2 \neq \omega|\omega)^{-1}) > 0
\end{aligned}
\tag{3.43}
$$

To wrap up, $g(0) = g(1) = 0$ and $g'(0) = g'(1) > 0$. By continuity, there exists $a^* \in (0,1)$ such that $a^*\overline{f} + (1-a^*)\underline{f}$ satisfies (3.39). □

Proof of Proposition 3.3. It is shown in the proof of Proposition 3.1 that, those inequalities in (3.9) are equivalent to the condition of the expert sending a truthful message in the second period, if s/he did in the first period. In regard with that, if there exists any profitable deviation strategy, there must exist some $a \in \{x, y\}$ such that the expert sends $r_1(a) = \bar{a}$ upon receiving $s_1 = a$. Consider an optimal strategy and the continuation game after $s_1 = a$ and $r_1 = \bar{a}$.

From the same proof, it follows that truth-telling is a unique best reponse when $s_2 = a$, because the inequality in the left side of (3.9) implies that the left side inequality in (3.32) holds strictly. If the expert tells the truth when $s_2 \neq a$, s/he always behaves truthfully in the continuation game. However, I have shown in the proof that if the continuation behavior is truth-telling, it is always a best response to do the same in the first period. Hence, in the continuation game, a profitable deviation strategy has to be $r_2(a) = a$ and $r_2(\bar{a}) = a$.

The interim expected payoff after receiving $s_1 = a$ is given by

$$
\begin{aligned}
&q_{cc}\Phi(\theta^T_{ic}) + q_{ii}\Phi(\theta^T_{ci}) + q_{ci}\Phi(\theta^T_{ic}) + q_{ic}\Phi(\theta^T_{ci}) \\
&> q_{cc}\Phi(\theta^T_{cc}) + q_{ii}\Phi(\theta^T_{ii}) + q_{ci}\Phi(\theta^T_{ci}) + q_{ic}\Phi(\theta^T_{ic}),
\end{aligned}
\tag{3.44}
$$

where the (RHS) is the expected payoff under truth-telling. Comparing the first two terms on each side, and then the second two terms on each side individually, (3.44) does not hold if

$$\frac{\Phi(\theta_{ci}^T) - \Phi(\theta_{ii}^T)}{\Phi(\theta_{cc}^T) - \Phi(\theta_{ic}^T)} \leq \frac{q_{cc}}{q_{ii}} \tag{3.45}$$

and

$$(q_{ic} - q_{ci})\Phi(\theta_{ic}^T) \geq (q_{ic} - q_{ci})\Phi(\theta_{ci}^T) \tag{3.46}$$

Since

$$\frac{q_{cc}}{q_{ii}} = \frac{Pr(s_1 = s_2 = \omega | s_1)}{Pr(s_1 = s_2 \neq \omega | s_1)} = \frac{Pr(s_1 = s_2 = \omega)}{Pr(s_1 = s_2 \neq \omega)} = \frac{Pr(s_1 = s_2 = \omega | s_1 = s_2)}{Pr(s_1 = s_2 \neq \omega | s_1 = s_2)} = \frac{p_{xx}}{1 - p_{xx}}, \tag{3.47}$$

(3.45) is equivalent with (3.10). Additionally, because the second signal is more accurate, $q_{ic} > q_{ci}$. Then the assumption $\theta_{ic}^T \geq \theta_{ci}^T$ guarantees that (3.46) holds. □

Proof of Proposition 3.4. In T-period games with truthful strategies, there are $n := 2^T$ possible message profiles. Without loss of generality, arbitrarily fix a state $x \in \{x, y\}$. Let $\{h_1, \cdots, h_n\}$ be the sequence of realized signals. Let $\{z_i\}_{i=1}^n$ denote the expected ability corresponds with $\{h_i\}_{i=1}^n$, assessed after the true state is realized. The ex ante probability of $h_i \in \{h_1, \cdots, h_n\}$ for type τ will be written by $b_i(\tau)$. Then the expected reputational payoff in the equilibrium under FT or PT strategy is given by

$$\int \left(\sum_{i=1}^{n} b_i(\tau)\Phi(z_i) \right) f(\tau) \, \mathrm{d}\, \tau \tag{3.48}$$

Consider a strategy where the expert sends messages truthfully in only specific T' ($T' < T$) periods, and let $m := 2^{T'}$. For the rest of the periods, the messages are babbling. Focus on the periods where truth-telling occurs and let $\{\tilde{o}_1, \cdots, \tilde{o}_m\}$ be the sequence of realized messages. Analogously, let $\{\tilde{z}_j\}_{j=1}^m$ denote the expected ability based on the messages sent and the true state, and let $\tilde{b}_j(\tau)$ denote the ex ante probability of $\tilde{o}_j$ for type τ. Then the expected reputational payoff in the equilibrium under this strategy is given by

$$\int \left(\sum_{j=1}^{n} \tilde{b}_i(\tau)\Phi(\tilde{z}_i) \right) f(\tau) \, \mathrm{d}\, \tau \tag{3.49}$$

For each $\tilde{o}_j \in \{\tilde{o}_j\}_{j=1}^m$, there exists a corresponding subset of $\{h_i\}_{i=1}^n$ whose each element is compatible with $\tilde{o}_j$. Let $\{h_{j_1}, \cdots, h_{j_k}\}$ be the set, where $\{j_1, \cdots, j_k\} \subseteq \{1, \cdots, n\}$.

$\{h_{j_1}, \cdots, h_{j_k}\}$ is the set of sequences of signals whose elements might have been happened upon observing $\tilde{o}_j$. The conditional probability of h_i upon observing $\tilde{o}_j$, will be denoted by

$$c(j, i) = \Pr(h_i | \tilde{o}_j) \tag{3.50}$$

for $i = j_1, \cdots, j_k$. Note that, the collection of the subset, $\{\{h_{j_\gamma}\}_\gamma\}_{j=1}^m$, is disjoint and its union equals to $\{h_1, \cdots, h_n\}$. That is, $\{\{h_{j_\gamma}\}_\gamma\}_{j=1}^m$ is a partition of $\{h_1, \cdots, h_n\}$. On the other hand, $(\tilde{o}_j, \tilde{b}_j)_{j=1}^m$ and $(h_i, b_i)_{i=1}^n$ can be seen as ex ante and ex post distributions of expected abilities, respectively, before and after an experiment. Using (3.50), each $\tilde{z}_j$ can be expressed as below.

$$\tilde{z}_j = \sum_{\gamma=1}^{k} c(j, j_\gamma) z_{j_\gamma} \tag{3.51}$$

Substituting $\tilde{z}_j$ in (3.51) to (3.49), the expected payoff under the strategy where the expert does not tell the truth in some period, (3.49), is written by

$$\int \left(\sum_{j=1}^{n} \tilde{b}_i(\tau) \Phi \left(\sum_{\gamma=1}^{k} c(j, j_\gamma) z_{j_\gamma} \right) \right) f(\tau) \,\mathrm{d}\,\tau \tag{3.52}$$

By Jensen's inequality, Φ is convex only if

$$\int \left(\sum_{j=1}^{n} \tilde{b}_i(\tau) \Phi \left(\sum_{\gamma=1}^{k} c(j, j_\gamma) z_{j_\gamma} \right) \right) f(\tau) \,\mathrm{d}\,\tau \leq \int \left(\sum_{j=1}^{n} \tilde{b}_i(\tau) \sum_{\gamma=1}^{k} c(j, j_\gamma) \Phi(z_{j_\gamma}) \right) f(\tau) \,\mathrm{d}\,\tau \tag{3.53}$$

The (RHS) of (3.53) equals with

$$\begin{aligned}
&\int \left(\sum_{j=1}^{m} \tilde{b}_j(\tau) \sum_{\gamma=1}^{k} c(j, j_\gamma)\Phi(z_{j_\gamma}) \right) f(\tau)\, \mathrm{d}\, \tau \\
&= \int \left(\sum_{j=1}^{m} \sum_{\gamma=1}^{k} \tilde{b}_j(\tau) c(j, j_\gamma)\Phi(z_{j_\gamma}) \right) f(\tau)\, \mathrm{d}\, \tau \\
&= \int \left(\sum_{j=1}^{m} \sum_{\gamma=1}^{k} Pr(\tilde{o}_j) Pr(h_{j_\gamma} | \tilde{o}_j)\Phi(z_{j_\gamma}) \right) f(\tau)\, \mathrm{d}\, \tau \qquad (3.54) \\
&= \int \left(\sum_{j=1}^{m} \sum_{\gamma=1}^{k} Pr(h_{j_\gamma})\Phi(z_{j_\gamma}) \right) f(\tau)\, \mathrm{d}\, \tau \\
&= \int \left(\sum_{i=1}^{n} Pr(h_i)\Phi(z_i) \right) f(\tau)\, \mathrm{d}\, \tau
\end{aligned}$$

The last equation derives from the fact that $\{\{h_{j_\gamma}\}_\gamma\}_{j=1}^m$ is a partition of $\{h_1, \cdots, h_n\}$. Since the (RHS) is equal to (3.48), this completes the proof. □

Chapter 4

Rational Bubble in Resell Markets in Networks

4.1 Introduction

The rational bubble literature derived from Allen et al. (1993) considers a bubble as a trade of a good at a price that is above everyone's valuation. It is often assumed that the market is efficient so that the good goes to the agent with the highest valuation. What they do *not* consider is the chance that the good may not reach the one. Most researches in the branch are modeled upon Walrasian market, where all agents have the same investment opportunities. The assumption may fit with large enough markets such as markets for public stocks, however, may not fit with small, specific market, such as markets for private equities and tickets (cf. Boyer et al. (2023) and Leslie and Sorensen (2014)). Research in search theory literature assumes that intermediaries may only probabilistically encounter the next buyer (See Rubinstein and Wolinsky (1987) and Wright and Wong (2014)).

In smaller markets with relatively fewer potential buyers available for resale, this may be particularly due to ambiguity surrounding who knows whom. When you decided a purchase to "flip", your next step is to look for someone who will buy it next. You may know an intermediary, but do not know if the one who appreciates the good is in his customer list. It goes vice versa: your customer may not know if you reached her because you do not know other agents who would pay more or because you expect them not buying your good, as they do not have the next one to sell. This kind of information asymmetricity may generate higher-order uncertainty, even when assuming ex ante common prior over the structures.

It has long been shown that higher order uncertainty is necessary for a bubble if players' rationality is common knowledge (See Tirole (1982), Milgrom and Stokey (1982) and Allen et al. (1993)). Allen et al. (1993) introduced the concept of expected and strong bubbles among rational players, assuming heterogeneous information structures and heterogeneous marginal state-dependent utilities. In their model, it is interchangeable with heterogeneous priors. They considered that a state exhibits strong bubble in an equilibrium, if every player certainly knows a good is traded at a price strictly above anyone's valuation when the state is realized.

In the realm of rational bubble literature, however, as far as I know, few are interested in why such information asymmetricity takes place. In this regard, Awaya et al. (2022), who modeled rational bubbles in a network, attributed it to information loss during non-strategic communication. Of course, it is difficult to confirm if something is really known as common knowledge (See Rubinstein (1989)). Still, I believe it is useful to provide another tractable structure leading to such uncertainty that naturally arise in human network.

In this paper model, the asymmetricity arises because of different scopes of vision. I formulated such situations with networks on which trades occur. In the model, state is given by a graph in which nodes correspond to players. Players can trade with adjacent players, but cannot trade with ones who are not in their neighborhood. There is uncertainty about the networks, since, when the game starts, players can tell different states apart iff they have different neighbor sets in it. Their information structure is given by partitions which can be refined as the game flows, by the trade offers, whether the offer has been accepted, and the price determined. It is assumed that the price reflects the willingness to pay of the trading party.

Let us more simplify the model. There is only one player who appreciates a good; any other players put zero value on it. Trades between the rest of players with a positive price, are based on a belief of buyer that the next buyer may believe s/he can sell it to someone who believes ... to the player who appreciate and would pay fair value for it. If there exists a state, in a given equilibrium, in which every player *know* they are not connected to the player, but in which trade occur at a positive price, the state is referred to as network bubble state.

I embedded bubbles defined in Allen et al. (1993) into the model; and defined network bubble which satisfies the revised definitions. Under the circumstance, I investigated necessary conditions that a minimal bubble must have, following Liu and Conlon (2018) and Liu et al. (2023). Both of which are in the same vein derived from Allen et al. (1993) in whch heterogeneous marginal state-dependent utilities are assumed. In the former it is claimed that it needs at least 5 states between 2 players, and that when the number of the states coincides with 5, the information structure is unique. In the latter, it requires three states with strictly risk-averse players. It is found that if the equilibrium has a state where the player finally buys the good on the path, it needs at least 5 players and 4 states. Moreover, players are risk-neutral, thus have same marginal utilities, following Awaya et al. (2022). On the contrary, the minimal condition for network bubble does not impose uniqueness not only on the graphs but also on the information structure.

Furthermore, any perfect Bayes equilibrium (PBE) with a network bubble has a severe condition on probability space, which keeps the buyers believing that there may be a next buyer. In this sense, it has the same limitation pointed out by Doblas-Madrid (2012) and Awaya et al. (2022) that the bubbles need knife-edge parameter restrictions. It is because, however, in their model, the price does not reflect players' private valuations. In Doblas-Madrid (2012), borrowing constraints and liquidity shocks are assumed; in Awaya et al. (2022), price function is exogenously given. Liu and Conlon (2018) and Liu et al. (2023) share the same limitation, since prices have to coincide to the willingness-to-pay of the buyer. However, there exists a subtle difference between this model and theirs: in their model, the price reflects the WTP, due to market clearing condition. In this model, there would be no significant difference if a well-defined price function different from WTP were given exogenously as long as it is incentive compatible and it keeps the next buyers uncertain.

Section 4.2 provides the model. Section 4.3 considers a single-state setting which will be a thought-experimental base of definition of network bubble and assumptions which Section 4.4 will provide. Section 4.5 proposes necessary conditions for a probability set of states and argues that network bubbles require a severe probability constraint. Section 4.6 shows an example.

4.2 Model

4.2.1 Players and Market

Consider a market with finitely many $K + 1$ players, $A_1, \cdots, A_K$, and A_0. They are able to identify other players but may and may not know each other. In particular, players are in a network; if a couple of players are adjacent to each other, they know each other. If a player has more than one neighbor, the player is able to tell one neighbor from the others. While they are precisely aware of their own neighbors, they face uncertainty about k-th order neighbors when $k > 1$. In other words, they do not precisely know which network they belong to.

Formally, the state of this world is represented by a graph that indicates the network between the players. Let $G = \{g_1, \cdots, g_M\}$ be a finite set of graphs with $K + 1$ nodes and $\mathcal{G}$ be a probability space with π as its probabilistic measure such that $\pi_m := \pi(\{g_m\})$ can be defined and is known as common knowledge for each $m = 1, \cdots, M$. Assume that $\pi_m > 0$ for every $m = 1, \cdots, M$. In graphs, each node corresponds to a player. A pair of adjacent nodes in a graph implies that the corresponding players know each other if the state is realized. Let N_k^g be the set of neighbors of A_k in graph g. Say, $g_n \sim_k g_m$ if and only if $N_k^{g_n} = N_k^{g_m}$. This equivalence relationship $\sim_k$ dividing G forms an information partition for A_k. Let $\mathcal{S}_k^0 := G/\sim_k$ be the information partition generated by $\sim_k$. The superscription indicates that the partition is formed at the very first history where the game started. It will sometimes be dropped when there are no rooms for confusion. Abusing notation, $\mathcal{S}_k^0(N) \in \mathcal{S}_k^0$ denotes the element in $\mathcal{S}_k^0$ such that player A_k has a neighbor set N. This may not be defined if there is no such $g \in G$. For any partition of G, $\mathcal{S}$, $\mathcal{S}(g)$ denotes the element in $\mathcal{S}$ such that contains g. This is a well-defined mapping.

There is a market for an indivisible good; and A_0 is the initial owner of the good at the start of the game. Players are risk-neutral and not wealth-constrained. The ways the players valuate the good are not the same, so that the utility from consuming the good generally varies. For $k = 0, \cdots, K$, the valuation for the good of A_k will be denoted by $v_k \geq 0$. Not only do the players identify other players, they also have $\{v_k\}_{k=0}^K$ as common knowledge. The player who owns the good can either consume it on his/her own or try to sell it to someone else through a day-long trade. If the current owner chose to not consume today; s/he can also save it to try to sell it tomorrow. However, players do not have access to all other players; they can only reach out within their neighborhood to offer a trade. Say, A_i, who currently owns the good, has reached A_j. If A_j accepts the trade, a mediator, who can extract their private information and thus their willingness to pay without errors, comes in and decides the price. For simplicity, I assume that the mediator is in favor of the seller so that the price decided exactly matches with the willingness to pay of the buyer. Details about willingness to pay are discussed later. Trade offers can be made at most once a day. Let $\delta \in (0, 1]$ be a common discount rate.

Following Wright and Wong (2014), I do not consider those situations where a player who sold it to another player buys it back. To put it differently, the good is not traded along any cycle: there are no a finite sequence of players, $\{k_n\}_{n=1}^m$, such that A_{k_n} sells the good to $A_{k_{n+1}}$ for $n = 1, \cdots, m-1$ and A_{k_m} sells the good to A_{k_1}. Although it may also mathematically be a solution, but is not our interest. To simplify the problem, I pose another assumption that, a player who rejected a trade offer from current owner cannot buy the good from the same player. It does not require that the player cannot buy the good ever. The owner can sell it to another one, who will (possibly indirectly through a path) deliver the good to the player.

4.2.2 Strategies and Willingness-to-pay

Past trade offers, trades that accepted, the price, and the timings are publicly observed and comprise a public history. Denote the set of all public histories by H. Neighbor set that each player faces is only private information that they have in this model. It also can be seen as a history that is determined at the beginning of the game and is never updated.

$$\mathcal{X}_k = \{(h, S) \in H \times \mathcal{S}_k^0 | \exists g \in S \text{ s.t. } h \text{ is feasible in } g\} \tag{4.1}$$

Strategies of A_k must be a function from $\mathcal{X}_k$ to the action set. Suppose $x = (h, S)$ is a history where A_k received an offer of a trade; and has not accepted. Let $x' = (h', S)$ be the history that succeeded x where A_k accepted the trade; and has yet to consume it or decide whom to try to sell it. Define a mapping $N_k(\cdot) : (h, S) \mapsto S$ for $(h, S) \in \mathcal{X}_k$, an inverse function in a sense. For $k = 0, \cdots, K$, a strategy of A_k as a seller is given by $\sigma_k^S(x') = \sigma_k^S(h', N) \in A(x')$ where $A(x')$ is the set of players A_k can reach. In particular,

$$A_k(x') := (N_k(x') \setminus N^1(x') \setminus N^2(x')) \cup \{A_k\} \tag{4.2}$$

where $N^1(x')$ is the set of players who has owned the good and $N^2(x')$ is the set of players that current owner, A_k, had offered a trade to be rejected at h'. $\sigma_k^S(x') = \sigma_k^S(h', N) = A_l \neq A_k$ implies that A_k offers a trade to A_l at history (h', N); and $\sigma_k^S(x') = \sigma_k^S(h', N) = A_k$ implies that A_k consumes the good at history (h', N). The strategy of A_k as a buyer is given by $\sigma_k^B(x) = \sigma_k^B(h, N) \in \{0, 1\}$. $\sigma_k^B(x) = 0$ or $\sigma_k^B(x) = 1$ implies A_k rejects or accepts the offer under x, respectively. This game ends either when a player consumes the good by her/himself or when $A(x)$ becomes a singleton.

To consider willingness-to-pay, focus on a player, say, A_k. At a history $x = (h, N) \in \mathcal{X}_k$ where A_k accepted a trade offer. The behavior plan can equivalently be re-written as a finite sequence in $A_k(h, N)$ that does not allow repetitions and ends with A_k. Let h^1 be the history where A_k accepted the offer and h^2 be the history that follows h^1 such that A_k offered a trade to $\sigma_k^S(h^1, N)$ to be rejected and continued to offer a trade to $\sigma_k^S(h^2, N)$. Inductively define history h^{l+1} following h^l such that A_k has been rejected by $\sigma_k^S(h^l, N)$ and tried with $\sigma_k^S(h^{l+1}, N)$. As the graph is finite, there must be a history h^L such that A_k will give up offering trades and consume the good by her/himself, or, $\sigma_k^S(h^L, N) = A_k$. One can define a sequence $\langle \sigma_k^S(x^l) \rangle_{l=1}^L$. Convert $\langle \sigma_k^S(h^l, N) \rangle_{l=1}^L$ into a sequence in the set of indices of players, $\langle a_l \rangle_{l=1}^L$, so that $A_{a_l} = \sigma_k^S(h^l, N)$ for $l = 1, \cdots, L$. If $\sigma_k^S(x^1) = A_k$, i.e., A_k consumes it right after the purchase at x^1, the sequence will be of length 1, $\langle \sigma_k^S(x^l) \rangle_{l=1}^1 = \langle A_k \rangle$. In such cases the willingness-to-pay of A_k is obvious. Hence, assume that $L \geq 2$.

Let G^x be the set of graphs that A_k puts a positive probability at x. Given a strategy profile $\sigma = (\sigma_0, \cdots, \sigma_K)$ fixed, where $\sigma_k = (\sigma_k^S, \sigma_k^B)$ for $k = 0, \cdots, K$, we can expect whether players in $\langle \sigma_k^S(x^l) \rangle_{l=1}^L$ will accept or decline the offer when their turn came, when a state (that is consistent with x) is realized. In other words, when a state and a strategy profile are fixed, the trade path realization, can be deterministically anticipated. Let $\mathbb{1}_l(g) = \sigma_{a_l}^B(h^l, N_{a_l}^g)$ and let $\mathbb{1}^l(g)$ be equal to

$$\sigma_{a_l}^B(h^l, N_{a_l}^g) \prod_{n=1}^{l-1} (1 - \sigma_{a_n}^B(h^n, N_{a_n}^g)) \equiv \mathbb{1}_l(g) \prod_{n=1}^{l-1} (1 - \mathbb{1}_n(g))$$

$\mathbb{1}^l(g)$ is an indicator function that returns 1 if and only if A_k has all failed to sell the good until A_k reach A_l. The willingness-to-pay of A_k under x, $W_k(h, N; \sigma)$, will be given by

$$\underset{g \in G^x}{E} \left[\sum_{l=1}^{L-1} \left(\delta^l \underbrace{B(W_k(h^l, N; \sigma), W_{a_l}(h^l, N_{a_l}^g; \sigma))}_{\text{the price determined}} \mathbb{1}^l(g) \right) + \delta^{L-1} v_k \mathbb{1}^L(g) \,\middle|\, \sigma, x^1 \right], \tag{4.3}$$

where $B(w^S, w^B)$ is the price that mediator decides when w^S (w^B) is the willingness-to-pay of seller (buyer). $\delta^l B(W_k, W_{a_l})$ captures the present value of the price that A_{a_l} will pays conditional on the purchase of A_{a_l} on the path. Recall that the mediator is assumed to be on behalf of the seller. Therefore, it is equal to

$$\underset{g \in G^x}{E}\left[\sum_{l=1}^{L-1} \delta^l W_{a_l}(h^l, N^g_{a_l}; \sigma)\mathbb{1}^l(g) + \delta^{L-1} v_k \mathbb{1}^L(g) \middle| \sigma, x^1 \right] \tag{4.4}$$

The second term in the expectation captures the payoff of consuming the good after either failed to or not trying to sell it. The present value when consuming it, will be as less discounted as one day. This is because, A_k can consume the good on the day of purchase or of rejection, while trades must wait until the next day. I will say, A_k purchasing the good at x is *speculative* or the purchase is made in a *speculative motive*, if $W_k(x; \sigma) > v_k$.

4.2.3 Information Updates

The expectations in (4.3) or (4.4) are operated in the set of graphs that A_k puts a positive probability at x, not under $N_k(x^1)$. This is because the information partition that A_k has may have been changed from $\mathcal{S}^0_k$ that is given by the private information. Players can learn from public history, at each step, as well under an equilibrium. In particular, there are three kinds of source of information: Whom the current owner of the good reach out to try to sell it, whether the player who received the trade offer accepted it, and the price determined in a case of acception.

To elaborate this, fix a strategy profile σ. Let h_1 be a public history where it is A_i's turn to take an action a from an action set. Let h_2 be the public history after a is taken. The action set is a subset of $\{A_1, \cdots, A_K\} \cup \{0, 1\}$. If $a \in \{A_1, \cdots, A_K\}$, A_i is the current owner at h who is looking for a player whom to sell it, including him/herself. If $a \in \{0, 1\}$, A_i is a potential buyer who just received a trade offer and has not decided yet. Let $\mathcal{S}^h_k$ denote the information partition of A_k after observing public history h. After observing a, A_k will learn that, the action is feasible. There is another thing that A_k will learn, that the action is chosen. Therefore, for $k \neq i$, $\mathcal{S}_k^{h_2}$ is determined by the join of $\mathcal{S}_k^{h_1}$,

$$\{\{g : a \in N_i^g\}, \{g : a \notin N_i^g\}\} \tag{4.5}$$

and

$$\{\{g : \sigma_i^S(h, N_i^g) = a \text{ or } \sigma_i^B(h, N_i^g) = a\}, \{g : (\sigma_i^S(h, N_i^g) \neq a \text{ and } \sigma_i^B(h, N_i^g)) \neq a\}\} \tag{4.6}$$

If A_i acted based on σ_i, (4.5) is implied by (4.6). However, if A_i deviated and a was not supposed to occur under σ, then, players will only learn from (4.5). Note that, this update would return the same information partition if $A_k = A_i$.

Now suppose $a = 1$, which implies A_i was a potential buyer in history h_1 and decided to buy. The price, q, will be determined publicly. Let h_3 denote the public history after observing q. Suppose that A_j is the seller. For all $k = 0, \cdots, K$, $\mathcal{S}_k^{h_3}$ is determined by the join of $\mathcal{S}_k^a$ and

$$\big\{\{g : B(W_j(h, N_j^g), W_i(h, N_i^g)) = q\}, \{g : B(W_j(h, N_j^g), W_i(h, N_i^g)) \neq q\}\big\} \tag{4.7}$$

Notice that under the assumption that the mediator makes the price equal to the willingness-to-pay of the buyer, (4.7) will be equal to

$$\{\{g : W_i(h, N_i^g) = q\}, \{g : W_i(h, N_i^g) \neq q\}\} \tag{4.8}$$

and A_i will learn nothing from this update, again. However, if not, A_i will learn some information about A_j, thus about the information that A_j has.

Suppose the state is given by g, which is not directly observable from players. Let h be the public history observed. Player A_k whose information partition is given by $\mathcal{S}_k^h$ at h, would observe an information cell, $\mathcal{S}_k^h(g)$. The interim belief of A_k at h will be given by

$$Prob(g'|g) \equiv \frac{\pi(g')}{\sum_{g_m \in \mathcal{S}_k^h(g)} \pi_m} \tag{4.9}$$

if $g' \in \mathcal{S}_k^h(g)$ and 0, otherwise.

4.3 Bubbles in networks

How to define a bubble when players do not share a same prior, has been suggested by Allen et al. (1993) and continues to be used in related researches. They proposed two classes of bubble, expected bubbles and strong bubbles, that can be adopted in Walrasian markets. In the former, it refers to a situation where a price is set strictly above the expected *fundamental value*, calculated by each player. In the latter, it refers to a situation where all players know that the price is set strictly above *the fundamental value* even though they may not know the exact value of it. But they set an upper bound of the value high enough that readers could be easily convinced.

It is necessary to modify those concepts to apply to the model in this paper, however, because of different environments. I suggest a way to inherit the spirits of those concepts using the concept of maximum valuation I will define later. Furthermore, another concept of bubbles will be provided, which I call a network bubble, that is defined in a stylized environment. It fits in existing concepts of bubbles I embedded as well. Before proceeds, it may be helpful to see an alternative model, to understand how they can be embraced in the current model with network, and how the definition embedded in the model inherits Allen et al. (1993). Then in the following subsection, definitions and the relationship between them will be explained.

4.3.1 Single-state Setting

Hypothetically assume that there is no uncertainty on the structure of society: the network is fixed at commonly known g. In this world, all players have the same information partition. The price is determined by bilateral bargaining. Let $\alpha \in [0, 1]$ be the bargaining power of the seller. WLOG, assume that g is a connected graph.

Suppose, A_0 can tell other all players how to choose their strategy. Since the game focuses on the pure strategies, observe that the flow of the good is fixed uniquely once strategies are determined. In other words, this assumption implies that A_0 can choose the flow of the good so that it maximizes her payoff. Let $\langle p_0 \rangle$ be the path of length $L = |\langle p_0 \rangle|$ and re-label the players so that the good is handed from A_l to A_{l+1} for $l = 0, \cdots, |\langle p_0 \rangle| - 1$. Then, $\{A_0, A_1, \cdots, A_L\}$ is the finite sequence of players A_0 chose, that describes whom the good goes through in $\langle p_0 \rangle$.

Consider the last player in the path, A_L, who will end up consuming the good by him/herself, not flipping it to another player, by assumption. Therefore, A_L must buy the good for own use and the willingness to pay equals to the consumption value. Given this, when A_{L-1} tries to sell the good to A_L, the bargaining solution will be determined by

$$q_L^* := \arg\max_q (q - v_{L-1})^\alpha (v_L - q)^{1-\alpha} = \alpha v_L + (1-\alpha) v_{L-1} \tag{4.10}$$

where $\alpha \in [0,1]$ is the bargaining power. Expecting A_L paying q_L^* next day, A_{L-1} would want to pay at most δq_L^*. It is time discounted because A_{L-1} will go to A_L only after a day from he paid. Replacing v_L and v_{L-1} with δq_L^* and v_{L-2}, respectively, $q_{L-1}^* = \alpha \delta q_L^* + (1-\alpha) v_{L-2} = \delta \alpha^2 v_L^2 + \alpha(1-\alpha)\delta v_{L-1} + (1-\alpha) v_{L-2}$ is the amount A_{L-1} will pay. By sequential induction, A_l will pay

$$q_l^* = (1-\alpha) \sum_{m=l-1}^{L-1} (\alpha\delta)^{m-l+1} v_m + \alpha^{L-l+1} \delta^{L-l} v_L. \tag{4.11}$$

$q_l^* = (1-\alpha) \sum_{m=l-1}^{L-1} (\alpha\delta)^{m-l+1} v_m + \alpha^{L-l+1} \delta^{L-l} v_L$. Then the problem is reduced to choosing the path that maximizes the equation below. As g is assumed to be finite, there must exist a solution.

$$q_1^* = \max_L \max_{\langle p_0 \rangle \in P_{0,L}^g} (1-\alpha) \sum_{m=0}^{L-1} (\alpha\delta)^m v_m + \alpha^L \delta^{L-1} v_L \tag{4.12}$$

Proposition 4.1. *Suppose G is a singleton, $\{g\}$ where g is a connected graph. For $0 \leq \alpha \leq 1$, there exists a subgame perfect equilibrium that maximizes A_0's payoffs amongst all* *feasible path realizations.*

Proof. See Appendix 4.A. □

The proposition above states that, when $G = \{g\}$ such that g is a connected graph, there is a subgame perfect equilibrium where the payoff of initial owner of the good equals to (4.11). Roughly speaking, if other players can earn higher payoffs by deviating from $\langle p_0 \rangle$, A_0 would have already taken advatage of it. In a graph g, $P^g_{i,j}$ denotes the set of paths from node i to j. It is an empty set if they are not connected, or, there is no path between them. For $p \in P^g_{i,j}$, $|p|$ denotes the length of the path. Given g, define a distance function d^g of the set as follows

$$d(i,j;g) = d^g(i,j) = \begin{cases} \min_{p \in P^g_{i,j}} |p| & \text{if } |P^g_{i,j}| > 0, \\ \infty & \text{if } P^g_{i,j} = \varnothing, \end{cases} \tag{4.13}$$

where $|p|$ is the length of the path. In words, it returns the length of the shortest path between i and j if there is any, and returns ∞ if A_i and A_j are not connected. It can be readily shown that this is a well-defined distance function.[1]

Corollary 4.1. *Suppose G is a singleton, $\{g\}$ where g is a connected graph. In an SPE, the profit of A_0 cannot exceed*

$$\max\left\{ v_0, \max_{k \neq 0}, \{\delta^{d(0,k;g)-1} v_k\} \right\} \tag{4.14}$$

In the SPE, the willingness-to-pay of A_i when receiving a trade offer at h cannot exceed

$$\max_k \{\delta^{d(i,k;\tilde{g})} v_k\} \tag{4.15}$$

where $\tilde{g}(h, N^g_i)$ is an induced graph of g whose set of nodes equal to $A_i(h, N^g_i)$.

[1] Symmetry, identity and non-negativity hold trivially. Fix $i, j, k \in \{0, \cdots, K\}$ and arbitrarily pick two pairs. If at least one pair is not connected, the sum of $d^g(\cdot,\cdot)$ of those pairs are weakly greater than the $d^g(\cdot,\cdot)$ of the remaing pair, as $\infty \leq \infty$. If the two pairs are connected, so is the remaining pair. Choose paths $p_1 \in P^g_{i,j}$ and $p_2 \in P^g_{j,k}$ such that satisfies $d^g(i,j) + d^g(j,k) = \min_{P^g_{i,j}} |p| + \min_{P^g_{j,k}} |p|$. Consider a walk from i to k that is composed of p_1 and p_2 whose length exactly equals to $\min_{P^g_{i,j}} |p| + \min_{P^g_{j,k}} |p|$. One can derive a path whose length is weakly less than the walk by eliminating repeated edges if exist, contained in $P^g_{i,k}$.

Proof of Corollary 4.1. The profit of A_0 monotonically increases in α, as long as $\max_k v_k > 0$. $\max_{k\neq 0}, \{\delta^{d(0,k;g)-1} v_k\}$ is the amount she can get from trade today in an equilibrium that maximizes her payoff when $\alpha = 1$ and v_0 is her consumption value. When A_i undertakes the good, A_i can be seen as an initial owner of the good in the game with $\tilde{g}$. □

It is worth noting that, if there is only one player who appreciates the good, that is, $v_0 = v_1 = \cdots = v_{K-1} = 0$ and $v_K = v > 0$, A_0 will maximize her payoff if and only if the path $\langle p_0 \rangle$ is the shortest path from A_0 to A_K. To see this, notice that, the amount that l-th player pays in (4.11) will be reduced down to $\alpha^{d^g(l,K)+1} \delta^{d^g(l,K)} v_K = (\alpha\delta)^{d^g(l,K)} \alpha v$. Since her consumption value is also pinned down at 0, her payoff is no more than the amount A_1 pays, or $(\alpha\delta)^{d^g(1,K)} \alpha v$. Under this setting, A_K will sometimes be called *end user* as there is no further buyer once he has the good. It also immediately follows that, since only A_K puts positive value on the good, if players other than A_K buy it, it must be from a speculative motive, i.e., in belief that she can re-sell it to others.

When $\alpha = 1$, the willingness to pay of A_i for $i = 0, \cdots, K$, cannot exceed $\max_j \delta^{d(i,j;g)} v_j$. It is also possible that $i = \arg\max_j \delta^{d(i,j;g)} v_j$. If this is the case, A_i would not look for a potential buyer and rather consume it. When such A_i faces a trade offer, the willingness-to-pay equals to A_i's own valuation for the good. Note that, it does not necessarily mean that $v_i = \max_k v_k$. Even when there are some players who appreciate the good more than A_i, A_i may choose not to deliver it to them, because the price must be discounted. In this sense, one may interpret δ as a distance discount factor, not only a time discounter.[2] In a same vein, $\max_j \delta^{d(i,j;g)} v_j$ is *the maximum present value* of A_i for the good.

4.3.2 Definitions

Fix a PBE and consider A_i who faces a trade offer from A_j at a public history h. $\{\mathcal{S}_k^h\}_{k=0}^K$ is the information partitions of players at h. Let $\{S_k\}_{k=0}^K$ be the information cells that players

[2] $d(\cdot, \cdot, g) <$ when focusing on connected graphs. It will be a hassle, however, when there are graphs with more than one component and $\delta = 1$. Considering the context of this research, 1^∞ will be treated as 0, taking the left limit of $\lim_{a\to 1}(\lim_{b\to\infty} a^b)$ in a.

belong at the time. Let $V_i(S_k)$ be the expected maximum valuation of A_i calculated based on S_k, given as below.

$$V_i(S_k) := E\left[\max_j \left\{\delta^{d(i,j;\tilde{g}(h,N_i^g))} v_j\right\} \middle| g \in S_k\right] \tag{4.16}$$

The equilibrium has *an expected bubble (in networks)* if there are a state, a player, and a history on the path where the player faces a trade offer such that the player undertakes the offer at a price strictly above everyone's expected maximum valuation at the history. Let g^b be the state where expected bubble occurs at the trade offered at h and let q be the amount that A_i pays. Then, the definition of expected (network) bubble requires that

$$q > \max_k V_i(\mathcal{S}_k^h(g^b)) = \max_k E\left[\max_j \left\{\delta^{d(i,j;\tilde{g}(h,N_i^g))} v_j\right\} \middle| g \in S_k^h(g^b)\right]. \tag{4.17}$$

The equilibrium has *a strong bubble (in networks)* if there are a state, a player, and a history on the path where the player faces a trade offer such that the player undertakes the offer at a price strictly above each player's any possible maximum valuation at the history. Let g^b be the state where strong bubble occurs at the trade offered at h and let q be the amount that A_i pays. Then, the definition of strong (network) bubble requires that for any $k \in \{0, \cdots, K\}$ and $g \in \mathcal{S}_k^h(g^b)$,

$$q > \max_j \left\{\delta^{d(i,j;\tilde{g}(h,N_i^g))} v_j\right\}. \tag{4.18}$$

In other words, when g^b is realized, every player knows that the good is traded at a price strictly above maximum valuation no matter which state in their information cell is the true state.

Suppose $v_0 = v_1 = \cdots = v_{K-1} = 0$ and $v_K = v > 0$. Under this setting, the conditions for bubbles in (4.17) and (4.18) can be simplified. In particular, a state g^b exhibits an expected bubble at A_i and h if

$$q/v > \max_k E\left[\delta^{d(i,K;\tilde{g}(h,N_i^g))}\middle| g \in \mathcal{S}_k^h(g^b)\right]. \tag{4.19}$$

A state g^b exhibits a strong bubble at A_i and h if $q/v > \delta^{d(i,K;g)}$ for each k and $g \in \mathcal{S}_k^h(g^B)$. Notice that the (RHS) of (4.19) approaches to the maximum probability that A_i is connected to A_K calculated based on each player's knowledge, as δ approaches to 1.

Under the same setting, an equilibrium has *a network bubble* if there are a state, a player, and a history on the path where the player faces a trade offer such that the player undertakes the offer at a strictly positive price; and that all players know the player is not connected to the end user. Let g^b be the state where network bubble occurs at the trade offered at h and let q be the amount that A_i pays. Then, the definition of network bubble requires that $q > 0$ and for any $k \in \{0, \cdots, K\}$ and $g \in \mathcal{S}_k^h(g^b)$,

$$d(i, K; g) = \infty \tag{4.20}$$

Remark 4.1. *If a state g^b exhibits a network bubble at a history h with a player A_i in an equilibrium, then $P_{i,K}^g = \varnothing$ in any state $g \in \cup_k \mathcal{S}_k^h(g^b)$.*

Verbally speaking, when g^b is realized, every player knows that A_i is not connected to A_K, i.e., there is no path from A_i to A_K in any states they put a positive probability on. Nonetheless, a network bubble requires A_i to undertake the good at a positive price. Clearly, a network bubble satisfies the definition of strong bubble: the distance between A_i and A_K is ∞ in an induced subgraph if it is in the original one.

4.4 Assumptions

I have assumptions to make to avoid both trivial and non-trivial cases. Firstly, it is not hard to imagine a world that everyone does not want the good, i.e., $v_0 = \cdots = v_K = 0$, but still it is being trade at price 0. It does not hurt anyone's incentive compatibility, but is not of our interest. I assume that if willingness-to-pay equals to 0, the player should not buy the good, even if the good is free. It is formally described in Assumption 4.1 below.

Assumption 4.1. *Let x be a history where A_k faces a trade offer. Given σ_{-k}, $\sigma_k^B(x) = 0$ if $W_k(x) = 0$.*

Secondly, because the mediator absorbs all the willingness-to-pay of buyers, buyers will be indifferent between buying and not buying. It is also a PBE that all players choose not buying, that is, $\sigma^B(\cdot) = 0$. The next assumption is to prevent this sort of equilibriums from arising. Readers may see that these assumptions are not to restrict the model but to refine equilibria. However, before directly proceeding, I want readers to recall the alternative setting in the previous section with a single state of common knowledge.

Corollary 4.2. *Suppose that G is a singleton, $\{g\}$ where g is a connected graph and that $v_0 = \cdots = v_{K-1} = 0$, $v_K > 0$ and $\delta, \alpha \in (0, 1)$. An SPE σ maximizing A_0's payoff that satisfies Assumption 4.1 has the following properties.*

P1 $(\sigma_K^B(x), \sigma_K^S(x)) = (1, A_K)$ *for any feasible history x where A_K faces a trade offer.*

P2 $(\sigma_k^B(x), \sigma_k^S(x)) = (1, A_K)$ *for any feasible history x where A_k faces a trade offer if $A_K \in A_k(x)$.*

P3 $\sigma_k^B(x) = 0$ *for any feasible history x where A_k faces a trade offer if $A_k(x) = \{A_k\}$ and $k \neq K$*

The first property states that A_K, the only player appreciates the good, will buy and consume it whenever receiving a trade offer. Otherwise, even if the end user may receive another trade offer later, the utility from consumption will be time-discounted, while the price remains the same. As long as $v_K > 0$, A_K would not reject any trade. The second property states that a player must accept any trade and try to sell it to A_K if the player has A_K as a potential buyer. Not buying the good may be a best response for some strategy profile where there is a possibility that A_K may not undertake it. However, by immediately preceding this property, A_K will undertake it as long as $A_K \in A_k(x)$. A_k will not sell it to someone else because, even if there is no uncertainty, the price must reflect the time discount to reach A_K. The third property requires non-end-user players not to buy the good if there is no feasible future buyers. In such cases, the highest price they would pay must be 0, because the consumption value is also 0. In other words, they can still accept the trade and pay 0, which is forbidden by Assumption 4.1. In fact, the third property is implied by the assumption.

Consider the original model. Once both parties agree on a trade, the price is determined by mediation not by bargaining. As mediator, as mentioned above, behaves on behalf of the seller, the buyer has to pay as much as the willingness-to-pay. In the alternative setting, it corresponds with the special case where $\alpha = 1$, i.e., the seller has the whole bargaining power so that the buyer pays no less than the willingness-to-pay. It puts buyers in an indifferent position – they will be indifferent between buying the good paying the expected value from it, and not buying it. However, in the alternative setting, a unique best response of A_k to σ_{-k} when $\alpha \in (0, 1)$, still remains as a best response when $\alpha = 1$ because of continuity in the payoff. On the other hand, in the original model, there may be a balanced mediator, in a sense that the price is between the willingness-to-pay's of buyers and seller. It becomes not easy to predict the behaviors of players, from which the prediction of the behaviors in special case may derive, especially when the support of beliefs of both parties does not coincide, generally. Nonetheless, in the special case I set, where the mediator extracts all the expected benefits of the buyer, the decision-making problem buyers are facing is similar to ones in the alternative setting with $\alpha = 1$. It may justify that properties P1 to P3 apply on the strategies of original model.

Assumption 4.2. *A strategy profile satisfies P1 to P3.*

4.5 Simplest Network Bubble and Analysis

Proposition 4.2. *Suppose* $v_0 = \cdots = v_{K-1} = 0$, $v_K > 0$. *Consider an equilibrium that has a network bubble state, and satisfies Assumption 4.1. In such an equilibrium,* $K \geq 4$. *Moreover, there exists at least 4 different states if (i)* $K = 4$ *or if (ii) there exists a state* *the good arrives in A_K.*

Proof. Players whose index is even will be referred as female and others will be referred as male. Let g_1 be the state that exhibits a network bubble. As the number of the players are finite, there must be the last player who buys the good in g_1. Re-label the players so that A_0 and A_1 are the seller and the buyer in the trade. Let h^0 be the public history observed when A_0 offers a trade to A_1 at g_1. Indices of players will be attached in the order that aquire the good. Public histories following h^0 will be called h^k where A_k is the player who just acquired it.

Because it is a network bubble state, by definition, A_1 does not appreciate the good. Therefore, the purchase is speculative, or, A_1 bought it intending to sell it to someone. But she cannot be certain that she will be able to, because she is the last owner of the good in g_1. The amount she will pay to purchase it must be strictly lower than v. It implies that neither is A_0 the player who sees the value in the good, because A_0 will earn v if he consumes the good today. Hence, $v_0 = v_1 = 0$.

Also, the fact that speculative A_1 who is the last owner of the good in g_1 bought the good when there is a positive probability that the state is g_1, imply there must be a state that A_1 cannot distinguish from g_1 where there is a player who would accept trade and pay a positive price, which will not happen in g_1. Otherwise, A_1 will not pay a positive price in g_1. Let g_2 be the state, and A_2 be the player who buys the good from A_1 at g_2. For notational convenience, let $N_k^m := N_k^{g_m}$. Since A_1 cannot distinguish g_1 and g_2 until the trade offer is rejected from A_2, A_1 must have the same neighbor set. That is, $A_0, A_2 \in N_1^1 \cap N_1^2$. For the same reason, players before A_2 have played in the same way in those states and prices must have been decided at a same level. In particular, in both g_1 and g_2, A_1 buys the good from A_0. On the other hand, A_2 takes different actions at h^1 in both states. This implies that she must have different neighborsets in those states, that is, $N_2^1 \neq N_2^1$.

By the definition of network bubble, A_1 knows for certain that he is not connected to the end user, the player who appreciates the good, when accepting the trade. That is, in g_1 and in all states that he cannot distinguish from g_1, A_1 and the end user is in different component. Let A^* be the player. Since $g_2 \in \mathcal{S}_1^{h^0}(g_1)$ and $A_2 \in N_1^2$, $A_2 \neq A^*$, which implies $v_2 = 0$ as well and A_2 buying the good in g_2 is also speculative. A_2 has a neighbor, A_3, who will buy the good with a positive probability in $\mathcal{S}_2^{h^1}(g_2)$. A_3 cannot sell it to A_0 nor A_1, as they have owned the good in previous history. Not can he be A^* either, A_1 is not connected to A^* in g_2, while A_3 is adjacent to A_2 who is adjacent to A_1. We have reached the minimum number of players in the statement: A_0, A_1, A_2, A_3 whose valuations equal to 0 and A^* who appreciates the good.

(i) Let $K = 4$ and $A_4 = A^*$. As mentioned above, A_2 believe there are some states in $\mathcal{S}_2^{h^2}(g_2)$ where A_3 would buy the good after observing h^2, in the equilibrium. However, g_2 is not one of those states: the purchase of A_3 at h^2 must be of speculative motive, because $v_3 = 0$. In g_2, there are 4 players in the game, except A_3 himself. Among those, A_0, A_1 and A_2 had had the good in h^2. A_4 must not be a neighbor of A_3 in g_2, because A_1 and A_3 are already connected in the state. A_4 adjacent to A_3 is also connected to A_1, which is a contradiction. Therefore, there must be a state that A_2 cannot distinguish from g_2 where A_3 would accept the trade, while she would not in state g_2. Let the state $g_3 \in \mathcal{S}_2^{h^2}(g_2)$.

As A_2 cannot distinguish states in $\mathcal{S}_2^{h^2}(g_2)$ after observing h^2, she behaves in the same way in those states, by asking A_3 for a trade. By the same logic, A_3 would not accept the trade after h_2, as long as he does not have A_4 as his neighbor. However, A_3 buys the good in g_3 after h_2 as the state is designed so, $A_4 \in N_3^3$. Let h^3 be the public history following h^2. Pick a state from $\mathcal{S}_2^{h^2}(g_2)$ such that A_4 accepts the trade after h^3. If there is no such state, another player needs to be introduced, to buy the good from A_3 after h^3, which implies $K > 4$. Therefore, there must exits such a state in $\mathcal{S}_2^{h^2}(g_2)$. WLOG, let g_3 be such a state and h^4 be the public history following h^3.

The fact h^4 is feasible in g_3 implies that A_k is adjacent to A_{k-1}, for $k = 1, 2, 3, 4$. That is, A_1 is connected to A_4 in g_3. Remark 4.1 implies that $g_3 \notin \mathcal{S}_1^{h^0}(g_1)$. Now, suppose there are only three states, g_1, g_2 and g_3. Under information partition $\mathcal{S}_1^{h^0}$, A_1 can tell g_3 from the others but cannot tell g_1 and g_2 apart. $\mathcal{S}_1^{h^0}$ is fully determined by $\mathcal{S}_1^{h^0} = \{S_1^1, S_1^2\}$ where $S_1^1 = \{g_1, g_2\}$ and $S_1^2 = \{g_3\}$. Similarly, the information partition for A_2 after h^1, $\mathcal{S}_2^{h^1}$, can be determined by $\{\{g_1\}, \{g_2, g_3\}\}$.

Worth noting that, players have the same amount of willingness-to-pay for the states that are in the same cell. In equilibrium path, A_2 does not buy the good after h^1 when the state is g_1, and does when the state is either g_2 or g_3, as the states are labelled so. As the mediator make A_2 pays the whole willingness-to-pay, letting q be the price A_2 pays on path, $q = W_2(h^2, N_2^2) = W_2(h^2, N_2^3)$. On the other hand, in g_3, A_2 sells the good to A_3 who will sell the good to A_4. Since $v_4 > 0$, A_2 will receive a positive price from A_3, which makes $W_2(h^2, N_2^3) > 0$. Thus, A_1 will receives a positive price q, if the state is g_2 or g_3, and 0, otherwise.

Consider the willingness-to-pay's of A_1. A_1 will be able to sell the good to A_2 in S_1^1 on path, iff the state is g_2. His willingness-to-pay for $S_1^1 = \{g_1, g_2\}$ has to match with

$$\frac{\pi_2}{\pi_1 + \pi_2}\delta W_2(h_2, N_2^2) = \frac{\pi_2}{\pi_1 + \pi_2}\delta q, \tag{4.21}$$

the present value of expected payoff. Notice this is strictly smaller than δq. In state g_3, the willingness-to-pay of A_1 for $S_1^2 = \{g_3\}$ is equal to $\delta W_2(h_2, N_2^3) = \delta q$, since he knows that he will be paid $W_2(h_2, N_2^3)$ tomorrow. Thus, A_1 has to have higher willingness-to-pay for S_1^2 than one for S_1^1, because A_2 cannot distinguish g_2 from g_3 at the moment A_1 offers a trade after h^1. However, it contradicts the assumption: higher willingness-to-pay for S_1^2 than one for another, must have been reflected in the price in the equilibrium. Hence, there must exist extra states, adjusting the expected payoffs of A_1 so that the willingness-to-pay's for g_2 and g_3 becomes the same.

(ii) Suppose there exists a state where A_K purchases the good. Let g_M be the state and A_{K-1} the player who is penultimate buyer of the good in g_M. Replacing g_M and A_{K-1} with g_3 and A_3, the proof holds analogously with (i). □

The proposition above states the smallest numbers of players and states, under a single-collector model. And not only that, it provides a part of information structure that an equilibrium with a network bubble must have. Let A_1 be the player who is the last player who buys at a history h the good in the bubble state on the equilibrium path. Let g^b be the state and h^* be the public history after the purchase. There must exist states, g^B and g^G, widely-used notations in the literature. The subscriptions expressse the posture of A_1 as a seller under the information he has. g^B indicates that A_1 behaves as a 'bad' seller in the state, because he knows there is 0 probability that the good can reach A_K. g^G indicates that A_1 behaves as a 'good' seller in the state, because he believes there is a probability that the good reaches A_K. The next buyer pays the same amount for the good in g^B and g^G, while A_1 cannot sell the good in g^b. To sell the good, it is important for him to keep A_2 in uncertainty about g^G and g^B. However, since A_1 can distinguish g^G and g^B, if his willingness-to-pay is higher at one than at the other, it is bounded to be revealed out, since the attitude is reflected in the price.

Corollary 4.3. *Suppose $v_0 = \cdots = v_{K-1} = 0$, $v_K > 0$. Consider an equilibrium that has a network bubble state under Assumption 4.1, in Proposition 4.2 and its proof. The willingness-to-pay of A_1 for the information cell containing g_1 is equal to that for the information cell containing g_3. That is,*

$$W_1(h^0, N_1^1) = W_1(h^0, N_1^3) \tag{4.22}$$

Proposition 4.3. *Suppose $v_0 = \cdots = v_{K-1} = 0$, $v_K > 0$. Suppose there exists an equilibrium with a network bubble under Assumption 4.1, and let g_1 be the bubble state. Consider a probability distribution $\pi^1 = (\pi_1, \tilde{\pi}_2/\pi_1, \cdots, \tilde{\pi}_M/\pi_1)$, where $\pi(g_m) = \tilde{\pi}_m/\pi_1$ for $m = 2, \cdots, M$. Up to re-labelling the states and players, π_1 is uniquely determined.*

Proof. Similarly with the proof of the Proposition 4.2, let A_1 be the last player who buys the good in g_1, A_0 be the seller, h^1 be the history at the trade. By the way A_1 is chosen, for any on path history that succeeding h where any player buys the good from A_1, players do not put a positive probability on g_1. On the other hand, the willingness-to-pay, described in (4.3) and (4.4), is iteratively calculated on the supports of the beliefs of the players. In none of those supports, g_1 is not included. In particular, changes in π_1 of π^1 does not change the willingness-to-pay of the players who has not owned the good at the time A_1 acquires it.

Consider the willingness-to-pay of A_1. Following the notation of the proof of the Proposition 4.2, the same proof shows that, $\mathcal{S}_0^{h^1}$ has at least two elements in it. $\mathcal{S}_1^{h^0}(g_1)$ and $\mathcal{S}_1^{h^0}(g_3)$. Changes in π_1 does not change the willingness-to-pay of A_1 in each state in $\mathcal{S}_1^{h^0}(g_3)$. It also does not change the expected payment from other players on the path in each state in $\mathcal{S}_1^{h^0}(g_1) \setminus \{g_1\}$. It does not change the expected payment from other players on the path in state g_1 as well, which is fixed at 0. The consumption value for A_1 is 0. Additionally, the willingness-to-pay of A_1 for both cells must be the same; and $\mathcal{S}_1^{h^0}(g_3) > 0$. This implies that increases in π_1 strictly increases the willingness-to-pay for $\mathcal{S}_1^{h^0}(g_1)$, while $\mathcal{S}_1^{h^0}(g_3)$ remains the same. Since I assumed that existence of an equilibrium that has g_1 as the network bubbble state, there must exist a π_1 and π^1 that makes (4.22) hold. □

Proposition 4.4. *Suppose $K = 4$, $M = 4$, $v_0 = \cdots = v_3 = 0$ and $v_4 > 0$. In a equilibrium with a network bubble that satisfies Assumption 4.1 and 4.2, the last player who buys it in the bubble state, has information partition $\{\{g_1, g_2\}, \{g_3, g_4\}\}$ when faces the offer. Moreover,*

$$\frac{\pi_2}{\pi_1 + \pi_2} = \frac{\pi_3}{\pi_3 + \pi_4} \tag{4.23}$$

Proof. See Appendix 4.A. □

Corollary 4.3 emphasizes a conclusion derived from the proof of Proposition 4.2, that the willingness-to-pay of A_1 for the cell he will be a 'good' seller, must coincide with that for the cell he will be a 'bad' seller. Proposition 4.3 showed that it requires a strict restriction over the prior set; and Proposition 4.4 gave us a specific example. There must be a forth state, where A_1 cannot distinguish from g_3 until he fails to sell it. In this way, A_1 has two information cells in each of which there are one paying state and one non-paying state. However, this is not sufficient to make A_1 indifferent between those cells. Because there will be the same amount of payoffs across paying states, the interim probability on paying state when each cell is realized, must be the same. Therefore, the priors that sustain a network bubble equilibrium, must be degenerated. It is not surprising that, an equilibrium that has a bubble state requires strict conditions.

In Allen et al. (1993) the relationship between the price and the willingness-to-pay is not explicitly discussed. In Liu and Conlon (2018) and Liu et al. (2023), the price has to coincide with the higher willingness-to-pay's in both parties in bilateral trades, to satisfy market clearing condition. Both parties can learn from the price and the position of themselves in the trade. This model not only has this feature but also introduced it as an instrument to determine the price. In Awaya et al. (2022) who analyzed rational bubble using e-mail game, their results hold under any prior as long as it has full-support. This may be partially due to the exogenous price function they adopted that increases exponentially and irrelavantly with willingness-to-pay's.

4.6 An Example of Network Bubble

The previous section provided some necessary conditions of an equilibrium with network bubble, if it exists. In this section, I will show a simple example of probability space of 4 graphs with 5 nodes. It exactly coincides with the necessary condition for simplest bubble in Proposition 4.2. Indices of players and states follow those in the proof of the proposition. I start by drawing edges that are required to be included, through Figures 4.6.1 and 4.6.2, to draw a true probability space with $G = \{g_1, g_2, g_3, g_4\}$. Let $E(g)$ be the set of edges in graph g. $(i, j) \in E(g)$ implies that in graph g, nodes i and j are adjacent. m-th graph in Figure 4.6.1 (4.6.2), g^1_m, has edge set $E(g^1_m)\,(E(g^2_m)) \subseteq E(g_m)$.

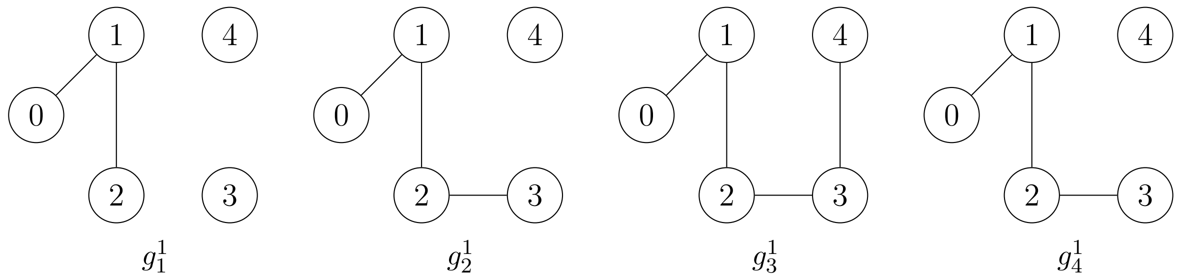

Figure 4.6.1: Edges required for trade offers

Consider the flows of the good in each state on path, described in the proofs of Proposition 4.2. In g_1, the good is handed in order of indices of players until A_2. In g_2, the good is handed in the same order until A_3. In g_3, the good is handed in order of indices of players over and over and it reaches the end-user, A_4. Proposition 4.4 implies that, in g_4, the flows of the good is the same with one in g_1. The flows imply that $(0, 1), (1, 2) \in E(g_2)$, $(0, 1), (1, 2), (2, 3), (3, 4) \in E(g_2)$, and $(0, 1) \in E(g_1), E(g_4)$. Not only that, in g_2, A_2 asks A_3 for a trade, although it is supposed to be rejected. In g_1, A_1 asks A_2 for a trade to be rejected; it happens in g_4 as well. Since trade offers are only made between neighbors, it implies that $(2, 3) \in E(g_2)$, and $(1, 2) \in E(g_1), E(g_4)$. It is described in Figure 4.6.1.

Consider the information partition of each player when they had their turn. From the fact that A_2 cannot distinguish g_2 and g_3 when receiving an offer from A_1, the public history at the timing must be the same, h^1. Since h^1 succeeds h^0, the public histories when A_0 offers a trade in g_2 and g_3 must be the same as well. However, it is shown that, when A_1 receives an offer from A_0, the information partition has to equal with $\mathcal{S}_1^{h^0} = \{\{g_1, g_2\}, \{g_3, g_4\}\}$. It follows that A_1 has different neighbor sets in states g_2 and g_3: $N_1^2 \neq N_1^3$. In other words, the difference set of N_1^2 and N_1^3 is not empty. However, Figure 4.6.1 requires that

$\{A_0, A_2\} \in N_1^2, N_1^3$. To summarize, it must satisfy the following relationship.

$$\varnothing \neq (N_1^2 \cup N_1^3) \setminus \overbrace{(N_1^2 \cap N_1^3)}^{\{0,2\}\subseteq} \subseteq \{3, 4\} \tag{4.24}$$

But by Remark 4.1, $A_4 \notin N_1^2$. It can neither be $A_4 \notin N_1^3$, by P2: considering A_1 after h^1 in g_3, A_1 will directly go to A_4 if $A_4 \in N_1^3$, which action is not feasible in g_2. This will enable A_2 to distinguish g_2 from g_3. By a similar logic, $A_3 \notin N_1^3$. Consider A_1 after h^1 in $\{g_3, g_4\}$. If the state is g_4, A_1 will not be able to sell it on the equilibrium which will gives him the lower bound payoff. If the state is g_3, on path, A_1 would sell it to A_2 who will pay him $\delta^2 v$. If $A_3 \in N_1^3$ and theryby, $A_3 \in N_1^4$, A_1 would deviate to offer a trade to A_3, who will accept it (P2) and pay δv, if the state is g_3. Therefore,

$$\begin{aligned} N_1^1 = N_1^2 &= \{A_0, A_2, A_3\} \\ N_1^3 = N_1^4 &= \{A_0, A_2\}, \end{aligned} \tag{4.25}$$

which is reflected in Figure 4.6.2.

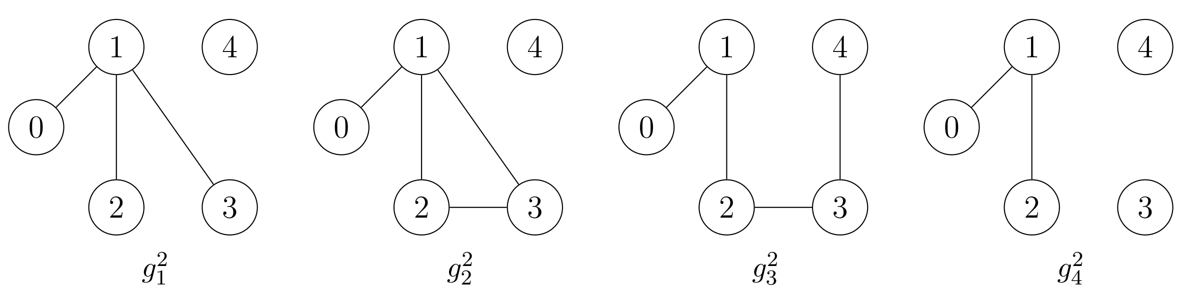

Figure 4.6.2: Edges required for information structure of A_1

By the definition of network bubble, when A_1 accepts his trade, every player must know that $d(1, 4; g) = \infty$ for any g they put positive probabilities on.[3] However, A_0 in $G^2 = \{g_1^2, g_2^2, g_3^2, g_4^2\}$ has the same neighborhood in any state, that is, $N_0^1 = N_0^2 = N_0^3 =$

[3] One may see this too strict and attempt to loosen the definition so that this requirement only applies on the very parties of the transaction. Although the Allen et al. (1993) imposed on all players, that is because, in their model, all players had access to the trade, any time they want. In such circumstance, it is natural to require it for all market participants. However, in this model, players has to wait until one of their neighbors acquire the good and offer a trade to them. Nonetheless, I keep my definition to be prudent.

$N_0^4 = \{A_1\}$. Then, G^2 generates an information partition for A_0, $\{\{g_1^2, g_2^2, g_3^2, g_4^2\}\}$. Since A_0 is the initial owner and has only one neighbor, there is no information updates by the time A_0 trades with A_1. If $G = G^2$, when g_1 is realized, A_0 expects there is a positive probability that A_1 may be able to reach A_4, which does not yet satisfy the requirements for network bubble. There are several ways to modify G^2 to resolve it. One of them is to simply add an edge $(0, 3)$ to g_1^2, separating g_1 from other states. The set of graphs, and the information partitions generated by the set of the players under this solution, are given in Figure 4.6.3 and (4.26), respectively.

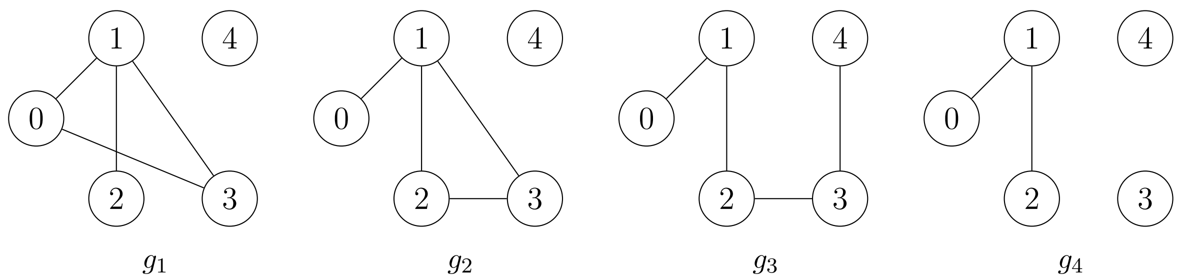

Figure 4.6.3: An example of state set G

$$
\begin{aligned}
\mathcal{S}_0 &= \{\{g_1\}, \{g_2, g_3, g_4\}\} \\
\mathcal{S}_1 &= \{\{g_1, g_2\}, \{g_3, g_4\}\} \\
\mathcal{S}_2 &= \{\{g_1, g_4\}, \{g_2, g_3\}\} \\
\mathcal{S}_3 &= \{\{g_1\}, \{g_2\}, \{g_3\}, \{g_4\}\} \\
\mathcal{S}_4 &= \{\{g_1, g_2, g_4\}, \{g_3\}\}
\end{aligned} \tag{4.26}
$$

I will show there is an equilibrium with a network bubble in the proposition below. However, in the game, only A_0 can get a positive payoff, and others have to pay the exact amount they expect to earn upon the purchase. Thus, it is not difficult to show that, there is also an equilibrium such in which no trade occurs at all. Also, there may be multiple options for a player after a trade offer is rejected. The player may either consume it or look for another player to offer a trade, if any. To narrow down our focus, I add two assumptions below.

Assumption 4.3. *Let x be a history where A_k is the current owner of the good. Given σ_{-k}, $\sigma_k^S(x) = A_k$ if $W_k(x) = 0$.*

Assumption 4.4. *Let x be a history where A_k faces a trade offer. Given σ_{-k}, $\sigma_k^B(x) = 1$ if $W_k(x) > 0$.*

Assumption 4.3 requires that, if A_k, who currently owns the good, is certain that the willingness-to-pay of his potential buyers equals to 0, he/she chooses to consume it by him/herself. Consider a history where the A_3 buys the good from A_0 and sold it to A_1, in g_1 in G, in Figure 4.6.3. Under this history, both A_1 and A_2 may be aware, in equilibrium, that A_2 has 0 willingness-to-pay. However, technically A_1 has $\{A_1, A_2\}$ as his action set and may choose A_2, knowing it will be rejected. Assumption 4.4, together with Assumption 4.1, require that a trade offer is accepted if and only if the buyer's willingness-to-pay is positive.

Proposition 4.5. *Suppose G and π is given by Figure 4.6.3 and Table 4.6.1. There exists an equilibrium with network bubble. If the equilibrium strategy profile satisfies Assumptions 4.1 to 4.4, it is unique.*

Proof. See Appendix 4.A. □

Table 4.6.1: Probabilities over states

states	π_1	π_2	π_3	π_4
probabilities	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Proposition 4.5 establishes an equilibrium with a network bubble. It is easy to confirm that, in the equilibrium, the statements in Propositions 4.2, 4.3 and 4.4 are satisfied. Moreover, it is shown that, under G, prior in Table 4.6.1 and Assumptions 4.1 to 4.4, if an equilibrium has a network bubble, it is unique. In the equilibrium constructed in the proposition, the good flows from hand to hand of players in the order of their index. Under

Assumptions 4.1 to 4.4, if a player turns down an offer, there is no further trades or trade offers. The flows of the good and the prices in each state on the path are given in Appendix 4.B.

Unlike Liu and Conlon (2018), the simplest structure to acquire a network bubble is not unique; they showed that a strong bubble in their model needs at least 5 states; and the information structure of players is essentially unique. However, in this model, this is not the case. Figure 4.6.4 represents another version of example, G', earned by adding an edge $(3, 4)$ to g_4 in G. Under G', the information partition of A_4 changes to $\mathcal{S}'_4 = \{\{g_1, g_2\}, \{g_3, g_4\}\}$. But it is still possible, with prior in Table 4.6.1, to construct a similar equilibrium with a network bubble.

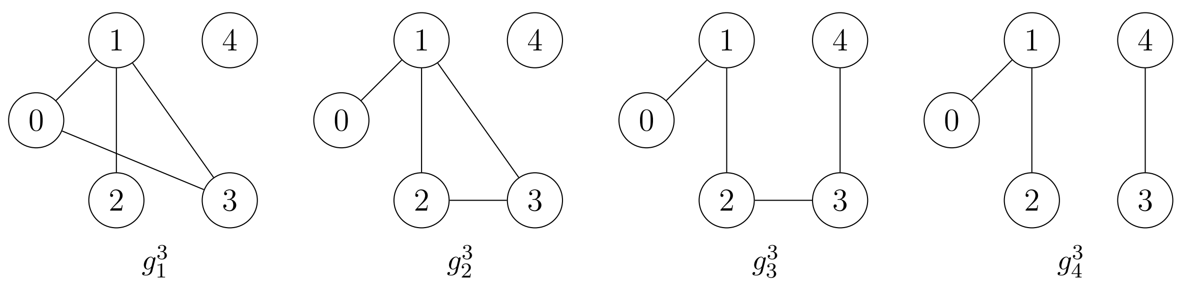

Figure 4.6.4: An alternative example of state set

Appendix of Chapter 4

4.A Proofs

Proof of Proposition 4.1. Suppose that A_0, a female, can tell all other players how to choose their strategy to maximize her (expected) payoff. The strategy profile, $\sigma^0 = (\sigma^0_0, \cdots, \sigma^0_K)$, draws a path $\langle p_0 \rangle$ in g, equivalently given by a sequence of players, $\{A_0, \cdots, A_L\}$, upon renaming players. Consider A_l, a male, who received an offer and has yet to decide whether to take it at history h_l and who accepts the offer at history $h_{l+0.5}$ following h_l. It can easily be proven that A_l does not have incentives to deviate at $h_{l+0.5}$. Suppose he can profitably deviate, resulting in the flow of the good drawing a path $\langle p_{l+0.5} \rangle$ instead of $\langle p_0 \rangle$. This implies that A_l would have paid more at h_l if A_0 chose $\langle p_0 \rangle$ instead of $\langle p_0 \rangle$, which increases the payoff of A_0. It is a contradiction to the assumption that $\langle p_0 \rangle$ maximizes her payoff.

Additionally, every player $A_l \in \{A_0, \cdots, A_L\}$ will receive a weakly positive payoff. For $l = 0, \cdots, L-1$, the fact that they do not deviate at $h_{l+0.5}$ implies that it is weakly better to obey σ^0_l, than consuming the good by themselves, which gives them $v_l \geq 0$. For $l = L$, the fact that A_{L-1} does not deviate implies that $v_{L-1} \leq \delta q^*_L \leq q^*_L$ where q^*_L is given in (4.10). This gives $q^*_L \leq v_L$. Using this, it can also be readily shown that A_l does not have incentives to deviate at h_l: not buying at all historeis that follows h_l cannot be a profitable deviation which gives him 0 payoff. Hence, a profitable deviation strategy for A_l must involve a history h'_l that follows h_l in which A_l turns down the offer at h_l and faces another offer of trade that he will accept. The later offer must be from an other player than A_{l-1}, A_{l-1} cannot offer a trade more than once to the same player. Moreover, at history $h'_{l+0.5}$ where A_l accepts the offer at h'_l, he must sell it to someone else – if consuming it at $h'_{l+0.5}$ is a profitable deviation, doing the same at $h'_{0.5}$ should also be profitable, but it has been proven not to be the case.

Let $\langle p'_l \rangle$ be the path caused by the deviation of A_l at h_l and the sequence of players who owned the good is given as below,

$$\{A_0, \cdots, A_{l-1}, A_{l_1}, \cdots, A_{l_n}, A_l, A_{l_{n+1}}, \cdots, A_{l_N}\},$$

where $\{A_{l_1}, \cdots, A_{l_n}\}$ is the squence of players who had owned the good between h_l and h'_l, and $\{A_{l_{n+1}}, \cdots, A_{l_N}\}$ is the squence of players who will own the good after $h'_{0.5}$. Recall that the payoff of A_l depends on $v_{l_{n+1}}, \cdots, v_{l_N}$, and not on $A_{l_1}, \cdots, A_{l_n}$ neither on $v_0, \cdots, v_{l-1}$. Define another path, $\langle p_l \rangle$, that is identified by a sequecne

$$\{A_0, \cdots, A_l, A_{l_{n+1}}, \cdots, A_{l_N}\}.$$

$\langle p_l \rangle$ is $1/\delta^n$ times more profitable than $\langle p_0 \rangle$, because he will receive the same amount of payment in earlier peirod. However, since $\langle p_{l+0.5} \rangle$ was chosen arbitrarily, the same logic applies to prove that $\langle p'_l \rangle$ is neither more profitable than $\langle p_0 \rangle$ for A_l. The strategies of other players, $\sigma_{-\langle p_0 \rangle}$ can be arbitrarily chosen within the incentive compatibility conditions. □

Lemma 4.1. *Suppose $K = 4$, $M = 4$, $v_0 = \cdots = v_3 = 0$ and $v_4 > 0$. Suppose there is an equilibrium that satisfies Assumption 4.1 and 4.2. Consider a history h on path in which A_2 rejected a trade offer from A_1 who bought the good from A_0. If A_1 offers a trade to A_3 next after h, A_3 will accept the good if and only if A_3 has A_4 as his neighbor. The statement also holds when switching A_2 and A_3.*

Proof of Lemma 4.1 . From the fact that A_2 did not accept the trade and try to re-sell it to A_4, players know that A_2 does not have A_4 in the state (P2). If A_3 accepts the trade after h, it must be of speculative motive (Assumption 4.1). Let x be the history where A_3 accepted the trade from A_1. Conditional on x, $\{A_0, A_1, A_3\}$ is the set who had owned the good in the history. The set of possible buyer of A_3, $A_3(x)$ must be a subset of $\{A_2, A_3, A_4\}$. If $A_4 \in A_3(x)$, A_3 will accept it (P2). If $A_4 \notin A_3(x)$, $A_3(x)$ must be a subset of $\{A_2, A_3\}$. However, A_3 knows that A_2 will not accept the trade in the equilibrium (P3). Consuming the good by himself will give him 0 profit as well. Thus, if $A_4 \notin A_3(x)$, buying at h is not on equilibrium path. □

Proof of Proposition 4.4 . I use the notations used in the proof of Proposition 4.2. In the proof, it was shown that there need more than 3 states. Let there be a fourth state, g_4. A_1 must not distinguish g_4 precisely at h^0; there must be states in the same cell in $\mathcal{S}_1^{h^0}$. If not, recall that g^2 and g^3 are paying states for A_1 at h^0, which makes the willingness-to-pay for $\mathcal{S}_1^{h^0}(g_3)$ strictly higher than $\mathcal{S}_1^{h^0}(g_1)$. The problem remains unsolved. Thus, g_4 must be either in $\mathcal{S}_1^{h^0}(g_1)$ or in $\mathcal{S}_1^{h^0}(g_3)$.

Suppose $g_4 \in \mathcal{S}_1^{h^0}(g_1)$, that is, $\mathcal{S}_1^{h^0} = \{\{g_1, g_2, g_4\}, \{g_3\}\}$. A_1 must offer a trade to A_2 first as he does in the states in the same cell. If g_4 makes the willingness-to-pay's of A_1 for both cells equal, the payment in g_4 must be higher than the payment in g_3.

Suppose $\sigma_2^B(h^1, N_2^4) = 0$. If $A_3 \notin N_1^4$, A_0 will receive 0, because there is no further players to ask for a trade and $v_1 = 0$. Even if $A_3 \in N_1^4$, A_3 would not take it, because of Lemma 4.1. In either way, A_0 will receive 0 if $\sigma_2^B(h^1, N_2^4) = 0$. Suppose $\sigma_2^B(h^1, N_2^4) = 1$. A_2 must be able to distinguish g_4 precisely at h^1. If $g_4 \in \mathcal{S}_2^{h^1}(g_1)$, A_2 will not buy the good, since not buying at g_1 is on equilibrium path and A_2 cannot distinguish both states. By the same logic, if $g_4 \in \mathcal{S}_2^{h^1}(g_2) = \mathcal{S}_2^{h^1}(g_3)$, A_2 would pay the same amount with that she would pay in g_2 or in g_3. The problem remains unsolved. Thus, $\mathcal{S}_2^{h^1} = \{\{g_1\}, \{g_2, g_3\}, \{g_4\}\}$. However, A_2 cannot pay more than she does in g_2 or in g_3 either. $g_4 \in \mathcal{S}_1^{h^0}(g_1)$, including A_0, A_1 and A_2 any players connected to them cannot be connected to A_4. If $A_3 \notin N_2^4$, $A_2(h^1, N_2^4) = \{A_2\}$. A_2 will not pay more than 0 (P3). If $A_3 \in N_2^4$, suppose A_2 offers a trade to A_3 after accepting and let h' be the history. However, given the set of past owners at h' is $\{A_0, A_1, A_2\}$, A_3 would not buy the good at h' unless $A_4 \in N_3^4$ (P3), which cannot be the case. Since A_2 precisely knows the state is g_4, A_2 can foresee that A_3 will not buy the good. This makes the willingness-to-pay of A_2 0; and she will not buy at h^1 in the state (Assumption 4.1). Therefore, $g_4 \notin \mathcal{S}_1^{h^0}(g_1)$.

Suppose $g_4 \in \mathcal{S}_1^{h^0}(g_3)$, or, $\mathcal{S}_1^{h^0} = \{\{g_1, g_2\}, \{g_3, g_4\}\}$. Re-label S_1^1 and S_1^2 so that $S_1^1 = \{g_1, g_2\}$ and $S_1^2 = \{g_3, g_4\}$. A_2 is the first player whom A_1 offers a trade, since $g_4 \in \mathcal{S}_1^{h^0}(g_3)$. Normalize the timeline and let day 0 be the day A_1 buys the good from A_0 in h^1. That is, A_1 offers a trade to A_2 on day 1 and A_2 pays q in states g_2 and g_3. Consider the information partition of A_2 after observing h^1. In equilibrium, A_1 receives strictly less in g_4 than he does in g_2 or g_3. This excludes $\mathcal{S}_2^{h^1} = \{\{g_1\}, \{g_2, g_3, g_4\}\}$, in which A_2 accepts the trade at h^1 and pays the same amount for $\{g_2, g_3, g_4\}$. The problem remains unsolved. It must be either $\mathcal{S}_2^{h^1} = \{\{g_1, g_4\}, \{g_2, g_3\}\}$ or $\mathcal{S}_2^{h^1} = \{\{g_1\}, \{g_2, g_3\}, \{g_4\}\}$.

It is convenient that, in either way, g_2 and g_3 are isolated in $\mathcal{S}_2^{h^1}$, in the sense that $\mathcal{S}_2^{h^1}(g_2) = \mathcal{S}_2^{h^1}(g_3) = \{g_2, g_3\}$. This enables us to explicitly calculate q. On the equilibrium path, in g_2, the good flows until A_2 and A_2 will receives nothing from A_3; and she does not have further potential buyers, as A_2 is not connected to A_4 in the state. However, in state g_3, A_3 will pay a positve amount because A_3 expects A_4 to pay her consumption value to him. In other words, A_4 pays A_3 v, on day 3; A_3 pays A_2 δv, on day 2 expecting

v on next day. On day 1, when A_2 is to buy the good and pay A_1, she calculates the willingness-to-pay. If the state is g_2, she will have 0; if the state is g_3, she will expect δv paid on the next day. Therefore, the willingness-to-pay of A_2 for $\{g_2, g_3\}$ on day 1 equals to

$$q = \frac{\pi_3}{\pi_2 + \pi_3}\delta(\delta v) = \frac{\pi_3}{\pi_2 + \pi_3}\delta^2 v \tag{4.27}$$

In what follows, I will show that the (discounted) amount A_1 receives in g_4 is not only strictly less than (discounted) q, but also is 0.

If $\mathcal{S}_2^{h^1} = \{\{g_1, g_4\}, \{g_2, g_3\}\}$ is the case, A_2 will reject the trade as she would do the same in the other state in the same cell. Let h'' be the history following h^1 after rejection. Note that, if this is on the path, then, $A_4 \notin N_2^4$. Not to mention, $A_4 \notin N_1^4$ as well (P2). After rejection, if A_1 does not have possible buyers, A_1 will receive 0 in g_4. If A_1 has a player left in $A_k(h'', N_1^4)$ to ask for a trade on day 2, it must be A_3. He will accept iff $A_4 \in N_3^4$ (Lemma 4.1). However, if this happens, A_3 will pay δv on day 2, which makes S_1^2 more desirable than S_1^2 for A_1 at h^0. Thus, $A_4 \notin N_3^4$, A_3 reject the trade, and A_1 will earn 0 in g_4.

If $\mathcal{S}_2^{h^1} = \{\{g_1\}, \{g_2, g_3\}, \{g_4\}\}$, A_2 can precisely know the state when it is g_4. Consider four cases: (i) $A_4 \in N_2^4$, (ii) $A_4 \notin N_2^4$ and $A_3 \notin N_2^4$ (iii) $A_4 \notin N_2^4$ and $A_3 \in N_2^4$. If (i) is the case, A_2 will accept the trade at h^0 and pay δv, the time discounted amount that A_4 will pay tomorrow, on the same day the trade is offered. This is strictly higher than q, which makes willingness-to-pay of A_1 for S_1^2 strictly higher than S_1^1. Thus, (i) cannot be the case. If (ii) is the case, A_2 will not accept the trade (P3). This goes back to the analogous problem discussed in the previous paragraph. The proof can be done similarly when (iii) is the case and A_2 reject the offer on the path. If (iii) is the case, A_2 has only one possible player whom she can ask for a trade, upon accepting the trade offer from A_1: A_3. Upon observing a history where the set of the previous and current owners is given by $\{A_0, A_1, A_2\}$, A_3 will accept the good if and only if $A_4 \in N_3^4$ (P2 and P3). Since $\mathcal{S}_2^{h^1}(g_4)$ is a singleton, A_2 knows whether $A_4 \in N_3^4$ or not. If not, A_2 will not accept the trade (P3), and the discussion in the previous paragraph is repeated. If $A_4 \in N_3^4$, she will know that A_3 will pay δv on day 2 for sure. If A_2 accepts the trade, she will pay $\delta^2 v$, which is higher than q. This completes the proof. □

Proof of Proposition 4.5. The strategy of A_4 in an equilibrium can be characterized easily. Since A_4 is isolated from other players in all states except g_3. g_3 is the only state A_4 is able to have an opportunity to buy the good. If A_4 chose buying, she has no choice but to consume it. Let $\sigma_4(\cdot,\cdot) = (1, A_4)$ be the strategy of A_4 (P1). On the other hand, A_3, who is the only possibly connected player to A_4, recognizes all the states precisely. Let $\sigma_3(\cdot, N_3^3) = (1, A_4)$ be the strategy of A_3 in state g_3[4] (P2). In the case A_4 rejects it, the game ends. It is readily shown that there is no incentive for A_4 to deviate. Given the strategy of A_4, A_3 does not have deviation incentives from $\sigma_3(\cdot, N_3^3)$. It is straightforward to let $\sigma_2^B(\cdot, N_2^1) = \sigma_2^B(\cdot, N_2^4) = 0$ (P3) be the strategy of A_2 whose neighbor set is given by $\{A_1\}$. If A_2 faces an offer, the seller must be A_1, the only neighbor, which implies A_2 does not have any further buyer. Since the willingness-to-pay equals to 0, A_2 has no deviation incentives from $\sigma_2(\cdot, N_2^1)$ nor from $\sigma_2(\cdot, N_2^4)$. In the case A_2 accepts it, the game ends. I will re-visit $\sigma_2(\cdot, N_2^2)$ and $\sigma_2(\cdot, N_2^3)$ later.

To consider $\sigma_3(\cdot, N_3^2)$, the strategy of A_3 in g_2, notice that A_1, A_2 and A_3 form a cycle. Let ζ be a history that contains a trade offer from A_1 to A_3. If A_3 accepts any offer from A_1, he will have only A_2 as his available buyer. At the start of the game, A_2 has $\mathcal{S}_2 = \{\{g_1, g_4\}, \{g_2, g_3\}\}$ and cannot distinguish g_2 and g_3. However, since ζ is not feasible with g_3, $\mathcal{S}_2^\zeta$ must be a refinement of $\{\{g_1, g_4\}, \{g_2\}, \{g_3\}\}$. If A_3 accepts an offer from A_2, there is no available buyers for him, whether or not the history contains a trade offer from A_1 to A_3. Thus, no matter whomever A_3 accepts any offer from, A_3 has either no buyers, or a buyer whose willingness-to-pay equals with 0. Let $\sigma_3(\cdot, N_3^2) = (0, A_3)$ be the strategy of A_3 in g_2 (Assumption 4.1 and 3). It follows that, $\sigma_2(\zeta, N_2^2) = (0, A_2)$ is the strategy of A_2 for (ζ, N_2^2) (Assumption 4.1 and 3). Now given $\sigma_2^B(\cdot, N_2^1) = 0$ and $\sigma_2(\zeta, N_2^2) = (0, A_2)$, A_1 expects there is no further buyer with a positive willingness-to-pay in $\{g_1, g_2\}$, if he offered a trade to A_3. Consequently, $\sigma_1(\zeta, N_1^1) = \sigma_1(\zeta, N_1^2) = (0, A_1)$ is the strategy of A_1 for $(\zeta, \{A_0, A_2, A_3\})$ (Assumption 4.1 and 3).

In g_1, given $\sigma_2^B(\cdot, N_2^1) = 0$, A_0, A_1 and A_3 form a cycle again. Let $\sigma_3(\cdot, N_3^1) = (0, A_3)$ be the strategy of A_3 in g_1 (Assumption 4.1 and 3): if A_3 receives a trade offer from A_1, it is obvious that there is no further buyer who had not had it in his neighborhood. Suppose A_3 receives a trade offer from A_0. A direct trade offer between A_0 and A_3 is only feasible in g_1. If A_3 accepts, he will have only A_1 as his available buyer, whose information partition is updated to a refinement of $\{\{g_1\}, \{g_2\}, \{g_3, g_4\}\}$; and whose willingness-to-pay is therefore 0. Thus, no matter whomever A_3 accepts any offer from, A_3 has either no buyers, or a buyer whose willingness-to-pay equals with 0. This determines $\sigma_3(\cdot, N_3^1) = (0, A_3)$ and $\sigma_1(\zeta', N_1^1) = (0, A_1)$ where ζ' is a history that contains a trade offer from A_0 to A_3

[4]Strictly speaking, it is the strategy of A_3 upon a history where he faces a trade offer. However, recall that a player cannot offer to another more than once. Since players are aligned in a line in g_3 and A_3 precisely distinguish g_3 from others, it is sufficient to describe $\sigma_3(\cdot, N_3^3)$.

(Assumption 4.1 and 3).

Consider a history ξ that contains a trade offer from A_0 to A_3, which is only feasible in g_1. Then, the information partitions of every player will be refined so that $\mathcal{S}_k^{\xi}(g_1)$ is a singleton. Given $\sigma_2(\cdot, N_2^1)$, players who might participate in trades are A_0, A_1 and A_3 in a cycle again. In a similar manner, it turns out that $\sigma_1^B(\xi, N_1^1) = \sigma_3^B(\xi, N_3^1) = 0$ is the strategy for A_1 and A_3 at ξ respectively, and that both A_1 and A_3 do not have incentive to deviate. In histories where they bought the good, they consume it by themselves. That is, $\sigma_1^S(\xi, N_1^1) = A_1$ and $\sigma_3^S(\xi, N_3^1) = A_3$.

It remains to determine the strategies of A_0, A_1 and A_2 for the histories that do not contain a trade offer from A_0 or A_1 to A_3. If A_2 receives a trade offer in such histories, the histories of trade offer and whether it was accepted is determined uniquely: A_0 offers a trade to A_1 who accepts it and now offers a trade to A_2. There are two types of equilibriums to consider. In one, A_2 can distinguish g_2 and g_3 by the time of trade offer from A_1. In the other, A_2 cannot distinguish those states.

In what follows, suppose that, A_2 cannot distinguish those states, under histories not containing a trade offer from A_0 or A_1 to A_3 in the equilibrium. This is possible only if A_1 pays the same price at g_2 and g_3, which will be confirmed later. Let h^1 be the history that A_2 receives a trade offer from A_1 who was the first player whom A_0 offered a trade. Given $\sigma_3(\cdot, N_3^2)$ and $\sigma_3(\cdot, N_3^3)$, A_2 knows that A_2 will be paid back some positive price if and only if the state is g_3 in $\{g_2, g_3\}$. The willingness-to-pay of A_2 for $\{g_2, g_3\}$ will be calculated by

$$W_2(h^1, \{A_1, A_3\}) = \frac{\pi_3}{\pi_2 + \pi_3}\delta q_3 > 0 \tag{4.28}$$

where q_3 is given by $q_3 = \delta q_4 = \delta v > 0$. Let $\sigma_2(h^1, \{A_1, A_3\}) = (1, A_3)$ be the strategy of A_2 under the situation (Assumption 4). In case of rejection, the game ends. Before accepting the trade, the expected payoff is equal to 0; and after accepting the trade, the interim expected payoff of offering a trade to A_3 is strictly positive. Not buying or consuming by himself will give 0 payoff, thus, there is no incentive to deviate. It characterizes the strategy of A_2.

Since histories containing a trade offer from A_0 to A_3 have been excluded, it leaves histories that A_1 receives an offer from A_0 as soon as the game starts, given G. Denote the history by h^0. Notice that, if A_1 accepts the offer and tries to sell it to A_2, it is no other than h^1. Because it is assumed that, A_2 cannot distinguish g_2 and g_3 after h^1, A_2 has a positive willingness-to-pay and would buy it in those states under the constructed strategy.

If the strategy satisfies Assumption 4, in the equilibrium, the strategy of A_1 is $(1, A_2)$ at h^0 in states in $\mathcal{S}_1^{h^0}(g_2) \cup \mathcal{S}_1^{h^0}(g_3)$. In the next paragraph, I will show that $\mathcal{S}_1^{h^0}$ must be $\{\{g_1, g_2\}, \{g_3, g_4\}\}$, and thereby $\mathcal{S}_1^{h^0}(g_2) \cup \mathcal{S}_1^{h^0}(g_3) = G$. Then, it follows $\sigma_1(h^0, \cdot) = (1, A_2)$. When rejected, A_1 consumes it by himself, if the game did not end. Although A_1 may have A_3 in the neighborhood, as discussed above, players would not buy it in any history A_1 had offered a trade to A_3.

Given $\sigma_1(h^0, N_1^2)$, $\sigma_1(h^0, N_1^3)$ and positive willingness-to-pay of A_1 for g_2 and g_3, a unique best response to A_0 is $\sigma_0(h, \{A_1\}) = A_1$. Combined with the equilibrium strategies, h^0 and h^1 will be an equilibrium path in state g_2 and g_3. Additionally, since A_1 cannot learn information that A_0 does not know upon h^0, it is either $\mathcal{S}_1^{h^0} = \mathcal{S}_1 \wedge \mathcal{S}_0 = \{\{g_1\}, \{g_2\}, \{g_3, g_4\}\}$ or $\mathcal{S}_1^{h^0} = \mathcal{S}_1$. Suppose in the equilibrium, $\mathcal{S}_1^{h^0} = \{\{g_1\}, \{g_2\}, \{g_3, g_4\}\}$. That is, $\sigma_0(h, \{A_1, A_3\}) \neq \sigma_0(h, \{A_1\}) = A_1$. However, this cannot be equilibrium. Given σ_2 and σ_3, not only A_2 and A_3, but A_1 would also not buy the good if he can isolate g_1. At the same time, A_1 in $\{g_1, g_2\}$ conjectures the state is g_2 if he observes A_0 offering the trade at the start of the game; and pays a price. A_0 has incentive to deviate, to fool A_1 by doing so, which is feasible because $A_1 \in N_0^1$. It gives us $\sigma_0(h, \cdot) = A_1$ (Assumption 4.4) and $\mathcal{S}_1^{h^0} = \mathcal{S}_1 = \{\{g_1, g_2\}, \{g_3, g_4\}\}$. When rejected, A_0 consumes it by herself, if the game did not end. Although A_0 may have A_3 in the neighborhood, as discussed above, players would not buy it in any history A_0 had offered a trade to A_3. For the same reason, there is no profitable devaition from $\sigma_0^S(h, \cdot) = A_0$.

I have constructed a strategy profile following Assumptions 4.1 to 4.4, and have confirmed there is no incentive to deviate for any players and any history. It is under the assumption that A_2 cannot distinguish g_2 and g_3 at h^1, under the strategy profile. It can be easily checked that A_1 has the same willingness-to-pay for $\{g_1, g_2\}$ and $\{g_3, g_4\}$ at h^0. Given the profile, A_2 will pay her willingness-to-pay for $\{g_2, g_3\}$, $W_2(h^1, \{A_1, A_3\})$ at h^1, which is given by (4.28). A_1 will be able to re-sell the good if the state is g_2, conditional on $\{g_1, g_2\}$ and if the state is g_3, conditional on $\{g_3, g_4\}$. In the rest of states, A_1 will not able to find a neighbor who will buy it. Under the prior given in Table 4.6.1,

$$\frac{\pi_2}{\pi_1 + \pi_2} \delta W_2(h^1, \{A_1, A_3\}) = \frac{\pi_3}{\pi_3 + \pi_4} \delta W_2(h^1, \{A_1, A_3\}). \tag{4.29}$$

The (LHS) of (4.29) coincides with the willingness-to-pay of A_1 for $\{g_1, g_2\}$ at h^0, while the (RHS) coincides with that for $\{g_3, g_4\}$. It is obvious that this equilibrium has a network bubble, considering that there is no information refinement until A_1 is rejected by A_2. By the time A_1 purchases the good in g_1, the information partitions remain as $\{\mathcal{S}_k\}_{k=0}^4$. In the information structure, for any k and $g \in \mathcal{S}_k(g_1)$, $d(1,4;g) = \infty$, while the payment of A_1 in g_1 is given by (4.29), which is strictly positive. □

4.B The Details of the Equilibrium in Proposition 4.5

The flows of the good in each state on path is given in the Figure 4.B.1. The edges in red reflect that a trade occurred between the players at the ends. The edges in blue reflect that a trade offer was made but rejected. Nodes in red reflect the corresponding player had had the good on the path, in the state. The good is traded in the same way in g_1 and g_4: in both states, A_1 buys the good and tries to sell it to A_2, but fails. In g_3, the good flows through until A_4, who is the end user. In g_2, A_2, who thinks the state could be g_3, buys the good and tries to sell it to A_3, who knows the exact state.

Table 4.B.1 represents the prices each player pays in each state. Rows represent the states and columns represent the players. It can be easily confirmed that as the time flows, the price goes up. However, this is not what I call bubble. The reasons price goes up here are: 1 time discount, 2 the probability that a potential buyer may reject trade offer. Note that, the state in which a potential buyer may reject an offer does not necessarily imply the state exhibits a bubble. This applies only on A_1 in $\{g_1, g_2\}$.

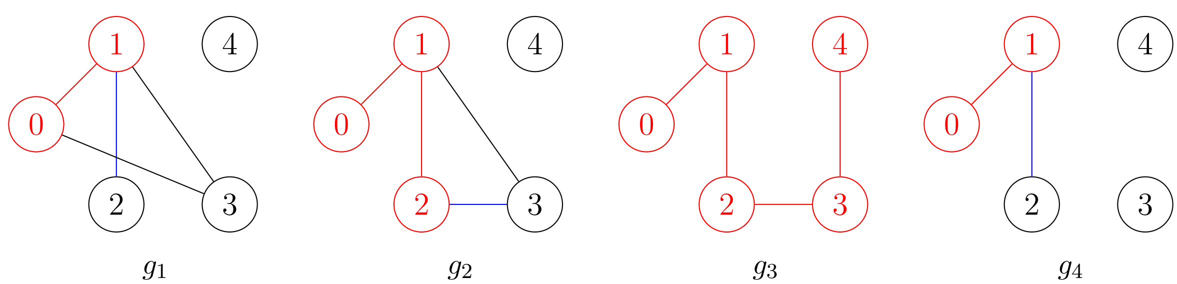

Figure 4.B.1: The flow of the good in each state on path of the equilibrium in Proposition 4.5

Table 4.B.1: Prices at each trade on path of the equilibrium in Proposition 4.5

	A_1	A_2	A_3	A_4
g_1	$\frac{\pi_2}{\pi_1+\pi_2}\frac{\pi_3}{\pi_2+\pi_3}\delta^3 v$	-	-	-
g_2	$\frac{\pi_2}{\pi_1+\pi_2}\frac{\pi_3}{\pi_2+\pi_3}\delta^3 v$	$\frac{\pi_3}{\pi_2+\pi_3}\delta^2 v$	-	-
g_3	$\frac{\pi_2}{\pi_1+\pi_2}\frac{\pi_3}{\pi_2+\pi_3}\delta^3 v$	$\frac{\pi_3}{\pi_2+\pi_3}\delta^2 v$	δv	v
g_4	$\frac{\pi_2}{\pi_1+\pi_2}\frac{\pi_3}{\pi_2+\pi_3}\delta^3 v$	-	-	-

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