Cluster construction and limit properties of renewal Hawkes processes

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1 Introduction

Roughly speaking, *point processes* are stochastic processes related to the distributions of random points in space. To make this idea more precise, we will rely on two different approaches for their study throughout this thesis. One of such approaches is to define point processes as random elements on a space of counting measures. This definition allows us to understand the counting properties of the process: the distribution of the number of points that fall in a given set. Meanwhile, another method that introduces a notion of temporal evolution and provides a more dynamic study by incorporating all the information collected up to a specific time is the martingale theory.

Firstly, following [19], we have established a construction of the *renewal Hawkes process* as an independent cluster process and computed its *probability generating functional.* Secondly, following [17], we have found convergence rate results for the Key Renewal Theorem for *spread out* distributions, and for the compensator and recurrence times of a *renewal process.* Finally, following [18], we have made use of the previous two results and of martingale theory to find a law of large numbers and a central limit theorem for the RHP.

1.1 Background: Classical Hawkes processes

Alan Hawkes [15] introduced *classical Hawkes processes* in 1971 as point processes with a self-exciting nature, in the sense that previous events facilitate the occurrence of future events. Later, in 1974, Hawkes–Oakes [16] showed that the classical Hawkes process could be understood as an *independent cluster process* in which the center process is given by a homogeneous Poisson process of immigrants and the satellite processes are given by branching processes formed by the offspring of those immigrants. For this, they rely on results established by Westcott [39], Kendall [22] and Lewis [25].

Classical Hawkes processes have been extensively used to model real-life phenomena that exhibit self-excitation. To cite some examples, Ogata [31] proposed a statistical model for aftershocks produced after a big earthquake, more recently Kim–Paini–Jurdak [23] proposed a model for transmission of a contagious disease within a population, and Bessy-Roland–Boumezoued–Hillairet [6] used the Hawkes process as a model for the prediction of cyberattacks.

Estimation for classical Hawkes processes has also been treated. Ogata studied the asymptotic properties of the maximum likelihood estimator for a wide variety of point processes including the classical Hawkes process [30]. Bacry–Dayri–Muzy [4] proposed a non parametric estimator for the classical multivariate Hawkes process using Fourier transform techniques. In addition, Bacry–Delattre–Hoffmann–Muzy [5] established limit theorems for the classical multivariate Hawkes process and this work is part of the inspiration for my research.

1.2 Background: Renewal processes and renewal Hawkes processes

As a generalization to the classical case, Wheatley–Filimonov–Sornette [40] introduced the *Renewal Hawkes process* (abbreviated as RHP) in which immigration is given by a renewal process. This generalization allows for more flexibility when fitting Hawkes processes to data sets as Stindl–Chen [34] did for modeling financial returns using Hawkes processes where the renewals were Weibull distributed. Chen–Stindl [8] studied the evaluation of the likelihood for the RHP and explained the challenges of computing the likelihood with respect to the natural filtration, and Chen–Stindl [9] refined the method of evaluation to improve the speed of the calculation.

Since renewal Hawkes processes are closely related to renewal processes, the study of the former will require the frequent use of renewal theory. Most of the necessary results are of rates of convergence of the key renewal theorem and convergence in distribution of processes obtained from a renewal process. The key renewal theorem states the limiting behavior of the solution to a *renewal equation*. Lindvall [26] found some of the first results on the rate of convergence between two differently delayed renewal measures with the aid of coupling techniques introduced by Athreya–Ney [3]. More recently, Willmot–Cai–Lin [41] have found certain sharp estimates for the asymptotics of such solutions under a variety of reliability-type assumptions for the inter-arrival distribution of the renewal process. Asmussen–Foss–Korshunov [2] considered this problem in the case of subexponential distributions. Yin–Zhao [42] studied non-exponential asymptotics in the case of *defective distributions*. Sgibnev [33] treated some cases where the solution to the renewal equation diverges to infinity.

A different approach to the study of asymptotic properties in renewal theory concerns the limiting behavior of processes constructed from a renewal process. Gakis–Sivazlian [12] obtained the limit distributions for the forward and backward recurrence times of a renewal process. Meanwhile, Godrèche–Luck [13] found results for occupation times of a renewal process using Laplace transform methods.

1.3 Structure of this thesis

The structure of this thesis is the following: Section 2 contains all the necessary ideas for the proof of our results, in particular Section 2.1 presents point processes as random counting measures, while Section 2.2 introduces them through martingale theory. Sections 2.3 and 2.4 treat renewal theory and regenerative processes respectively. Section 3 states the set of assumptions we made. Our results on the representation of the RHP as a cluster process are presented in Section 4. Section 5 contains our convergence rate results for renewal processes and Section 6 establishes our limit theorems for renewal Hawkes processes.

2 General theory of point processes

We present an overview of the random measure and martingale theories for point processes. Additionally, we include general theory for renewal and regenerative processes which are necessary for the proof of our main results. Whenever the proof of some result has been written in more detail than in the corresponding reference, it will be indicated.

2.1 Point processes as random counting measures

Some classical references for the study of point processes as part of the theory of random measures are Harris [14] and Moyal [28]. A more recent textbook on the subject is [21].

We define point processes in a general setting. Let $\mathcal{X}$ be a complete separable metric space (abbreviated as c.s.m.s.) and let $\mathcal{B}(\mathcal{X})$ denote its Borel σ-field. We denote by $\mathcal{N}^\sharp_\mathcal{X}$ the space of *counting measures* ν on $\mathcal{X}$ that are *locally finite*, i.e.,

$$\nu : \mathcal{B}(\mathcal{X}) \ni B \mapsto \nu(B) \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}\,, \tag{2.1}$$

$$\nu(B) < \infty, \quad \text{for all bounded } B \in \mathcal{B}(\mathcal{X}). \tag{2.2}$$

The space $\mathcal{N}^\sharp_\mathcal{X}$ can be equipped with the topology of *weak* $\sharp$ convergence (denoted by $w^\sharp$) defined as follows.

Definition 2.1 ([11, Section A2.6])**.** Let $\{\nu_n\}_{n\in\mathbb{N}}$ and ν be measures in $\mathcal{N}^\sharp_\mathcal{X}$. We say that ν_n converges to ν in the $w^\sharp$-topology if

$$\int_X f(s)\nu_n(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_X f(x)\nu(\mathrm{d}x), \tag{2.3}$$

for all bounded continuous functions f on $\mathcal{X}$ that vanish outside a bounded set.

We note that this is equivalent to *vague convergence* if $\mathcal{X}$ is locally compact. With the $w^\sharp$ topology, the space $\mathcal{N}^\sharp_\mathcal{X}$ is a c.s.m.s. (c.f. [11, Proposition 9.1.IV]) and we can think of its Borel σ-field, which we denote by $\mathcal{B}(\mathcal{N}^\sharp_\mathcal{X})$. Moreover, $\mathcal{B}(\mathcal{N}^\sharp_\mathcal{X})$ coincides with the σ-field generated by the maps

$$\mathcal{N}^\sharp_\mathcal{X} \ni \nu \mapsto \nu(B), \quad \text{for } B \in \mathcal{B}(\mathcal{X}). \tag{2.4}$$

With these elements, we introduce the notion of a *point process*:

Definition 2.2. A *point process* N on $\mathcal{X}$ is a random element $N(\omega, \cdot)$ of $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}))$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A consequence of Definition 2.2 is that, for fixed $\omega \in \Omega$, the set function $N(\omega, \cdot)$ is a counting measure on $\mathcal{X}$ and that, for fixed $B \in \mathcal{B}(\mathcal{X})$, $N(B) = N(\cdot, B)$ is an integer-valued random variable (the dependence on ω will sometimes be dropped from the notation and will just be implied).

Since N is a random element, we can speak of its distribution, defined as below.

Definition 2.3. The distribution of a point process N is the probability measure it induces on $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}))$, i.e.

$$\mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}) \ni B \mapsto \mathbb{P}(N \in B). \tag{2.5}$$

From the previous discussion, we can see that in order to specify a point process it is enough to give a probability measure on $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}))$. It is often the case that $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}), \Pi)$ is taken as the canonical probability space, where Π is a probability measure on $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}))$. Let us denote by $\mathcal{P}(\mathcal{N}^\sharp_\mathcal{X})$ the space of all probability measures on $\mathcal{N}^\sharp_\mathcal{X}$. One characteristic of the measures in $\mathcal{P}(\mathcal{N}^\sharp_\mathcal{X})$ is that they are completely determined by their *finite dimensional* (or *fidi*) distributions.

Definition 2.4. The *finite dimensional distributions* of a point process N are the joint distributions, for all finite families of bounded Borel sets $B_1, \dots, B_k \in \mathcal{B}(\mathcal{X})$ of the random variables $N(B_1), \dots, N(B_k)$, i.e.

$$\mathbb{P}_k(B_1, \dots, B_k; n_1, \dots n_k) = \mathbb{P}(N(B_1) = n_1, \dots, N(B_k) = n_k), \quad n_1, \dots n_k \in \mathbb{Z}^+. \tag{2.6}$$

We have the following characterization result (see for example [11, Proposition 9.2.III]).

Proposition 2.5. *The distribution of a point process is completely determined by its fidi distributions* (2.6) *for all finite families* $(A_1, \dots, A_k)$ *of disjoint sets from a semiring* $\mathcal{A}$ *of bounded sets generating* $\mathcal{B}(\mathcal{X})$.

Because of Proposition 2.5, a convenient tool for the characterization of the law of a point process is the *probability generating functional* (abbreviated p.g.fl.), which is defined on the class Ξ of measurable functions $z : \mathcal{X} \to [0, 1]$ such that $1 - z$ vanishes outside some bounded set.

Definition 2.6. Let N be a point process on $\mathcal{X}$. The *probability generating functional* G of the point process N is given as,

$$G[z] = \mathbb{E}\left[\exp\left(\int_\mathcal{X} \log z(t) N(\mathrm{d}t)\right)\right] = \mathbb{E}\left[\prod_{t \in N(\cdot)} z(t)\right], \quad \text{for } z \in \Xi. \tag{2.7}$$

Remark. Notice that for positive constants $\lambda_1, \dots, \lambda_k$, and disjoint bounded Borel sets $B_1, \dots, B_k$, by taking

$$z(\cdot) = \exp\left(-\sum_{i=1}^{k} \lambda_i 1_{B_i}(\cdot)\right), \tag{2.8}$$

we obtain

$$G[z] = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{k} \lambda_i N(B_i)\right)\right], \tag{2.9}$$

which corresponds to the Laplace transform of $(N(B_1), \dots, N(B_k))$.

Finally, we have a result for the convergence in distribution of point processes that involves the p.g.fl. Let Ξ_c denote the set of functions $z \in \Xi$ such that z is continuous. Recall that for every $\Pi \in \mathcal{P}(\mathcal{N}_{\mathcal{X}}^{\sharp})$ we have an associated point process with distribution Π and p.g.fl. G. Let $\xrightarrow{w}$ denote the weak convergence of measures, then we have the following fact (see for example [11, Proposition 11.1.VIII]).

Proposition 2.7. *Let $\Pi, \Pi_1, \Pi_2, \dots$ be in $\mathcal{P}(\mathcal{N}_{\mathcal{X}}^{\sharp})$ and let $G, G_1, G_2, \dots$ denote the p.g.fl. for their associated point processes. Then*

$$\Pi_n \underset{n\to\infty}{\xrightarrow{w}} \Pi \tag{2.10}$$

is equivalent to

$$G_n[z] \underset{n\to\infty}{\longrightarrow} G[z], \quad \textit{for all } z \in \Xi_c. \tag{2.11}$$

While the study of point processes as random counting measures is very wide, the elements presented in this section are sufficient for proving the corresponding results in this thesis, especially the ones in Section 4.

2.2 Martingale approach to point processes

In this section we introduce the basic notions of the martingale theory of point processes. This is especially important because the Hawkes processes which are the object of this thesis are defined in terms of their *intensity*. We will restrict ourselves to the case of *simple* point processes on $\mathbb{R}^+ = [0, \infty)$. In other words $\mathcal{X} = \mathbb{R}^+$ and $\mathbb{P}(N(\{x\}) = 0 \text{ or } 1 \text{ for all } x \in \mathbb{R}^+) = 1$.

We then introduce our framework. Let $\{T_n\}_{n\geq 0}$ be a sequence of random variables on $[0, \infty)$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that $T_0 = 0$ and, for all $i \geq 0$, we have $T_i < T_{i+1}$ on the event $\{T_i < \infty\}$, and $T_{i+1} = \infty$ on $\{T_i = \infty\}$. Notice that the random counting measure N given by

$$N(B) = \sum_i 1_{\{T_i \in B\}}, \quad B \in \mathcal{B}(\mathbb{R}^+), \tag{2.12}$$

is a point process as of Definition 2.2 and also it is simple. We then identify the sequence $\{T_n\}_{n\geq 0}$ with the associated counting process:

$$N(t) = \sum_i 1_{\{T_i \leq t\}}, \quad t \geq 0. \tag{2.13}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration to which N is adapted. It was first pointed out by Watanabe [38] that when N is a Poisson process with characteristic intensity $\lambda_p(t)$ that is locally integrable and deterministic, the *compensated process*

$$N(t) - \int_0^t \lambda_p(s)\mathrm{d}s, \quad t \geq 0, \tag{2.14}$$

is an $(\mathcal{F}_t)$-martingale. The martingale approach to point processes uses a generalization of this notion for the study of the evolution of N. To understand this, it is necessary to introduce first the notion of *predictability.*

Definition 2.8. We say that $X = (X_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-predictable if it is measurable with respect to the σ-field

$$\mathcal{P}(\mathcal{F}_\cdot) = \sigma((s,t] \times A; 0 \leq s \leq t < \infty, A \in \mathcal{F}_s)\,. \tag{2.15}$$

We now introduce the following Definition.

Definition 2.9. Let N be a point process and $(\mathcal{F}_t)_{t\geq 0}$ a filtration to which N is adapted. Let λ be a nonnegative, a.s. locally integrable process that is $(\mathcal{F}_t)$-progressive. We say that N admits the $(\mathcal{F}_t)$-*intensity* λ if the process given as

$$M(t) = N(t) - \int_0^t \lambda(s)\mathrm{d}s, \quad t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale, in which case M is called the *characteristic martingale* of N and the process

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s, \quad t \geq 0, \tag{2.16}$$

is called the *compensator* of N.

Remark. As a consequence of Definition 2.9, for any nonnegative process $C(t)$ that is predictable, it holds that

$$\mathbb{E}\left[\int_0^\infty C(s)N(\mathrm{d}s)\right] = \mathbb{E}\left[\int_0^\infty C(s)\lambda(s)\mathrm{d}s\right]. \tag{2.17}$$

One of the neat properties of the intensity of a point process is that one can always find a version that is predictable, in which case, it is essentially unique [7]. We present the proofs below.

Theorem 2.10 ([7, Section II, Theorem T13])**.** *Let N be a point process adapted to $(\mathcal{F}_t)_{t\geq 0}$ that admits the $(\mathcal{F}_t)$-intensity λ. Then one can find an $(\mathcal{F}_t)$-intensity $\widetilde{\lambda}$ that is $(\mathcal{F}_t)$-predictable.*

Proof. Denote by $\widetilde{\lambda}(\omega,t)$ the Radon–Nikodym derivative of the restriction of $P(\mathrm{d}\omega)\lambda(\omega,t)\mathrm{d}t$ to $\mathcal{P}(\mathcal{F}_\cdot)$ with respect to the restriction of $P(\mathrm{d}\omega)\mathrm{d}t$ to $\mathcal{P}(\mathcal{F}_\cdot)$. Let $(C(t))_{t\geq 0}$ be a nonnegative $(\mathcal{F}_t)$-predictable process. Then, we have

$$\mathbb{E}\left[\int_0^\infty C(s)N(\mathrm{d}s)\right] = \mathbb{E}\left[\int_0^t C(s)\lambda(s)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^t C(s)\widetilde{\lambda}(s)\mathrm{d}s\right], \tag{2.18}$$

and therefore $\widetilde{\lambda}$ is $(\mathcal{F}_t)$-predictable. □

Theorem 2.11 ([7, Section II, Theorem T12])**.** *Let N be a point process adapted to the filtration $(\mathcal{F}_t)$, and λ, $\widetilde{\lambda}$ two $(\mathcal{F}_t)$-intensities of N which are $(\mathcal{F}_t)$-predictable. Then*

$$\lambda(\omega,t) = \widetilde{\lambda}(\omega,t) \quad P(\mathrm{d}\omega)N(\omega,\mathrm{d}t)\text{-}a.e. \tag{2.19}$$

In particular

$$\lambda(T_n) = \widetilde{\lambda}(T_n) \quad a.s.\ on\ \{T_n < \infty\},\ n \geq 1, \tag{2.20}$$

$$\lambda(\omega,t) = \widetilde{\lambda}(\omega,t) \quad \lambda(\omega,t)\mathrm{d}t\text{-}a.e.\ and\ \widetilde{\lambda}(\omega,t)\mathrm{d}t\text{-}a.e., \tag{2.21}$$

$$\lambda(T_n) > 0 \quad a.s.\ on\ \{T_n < \infty\},\ n \geq 1, \tag{2.22}$$

Proof. Let $a \geq 0$ and take $C(s) = 1_{(\lambda(s)>\widetilde{\lambda}(s))}1_{(s\leq a)}$. It holds that $C(s)$ is $(\mathcal{F}_t)$-predictable and that $\lambda(s)$ and $\widetilde{\lambda}(s)$ are predictable as well. Then

$$\mathbb{E}\left[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}\widetilde{\lambda}(s)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}N(\mathrm{d}s)\right] = \mathbb{E}\left[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}\lambda(s)\mathrm{d}s\right], \tag{2.23}$$

$$\mathbb{E}\left[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}\left(\lambda(s)-\widetilde{\lambda}(s)\right)\mathrm{d}s\right] = 0 \quad \text{for arbitrary } a \geq 0, \tag{2.24}$$

so $1_{(\lambda(s)>\widetilde{\lambda}(s))} = 0$ $P(\mathrm{d}\omega)\lambda(t)\mathrm{d}t$-a.e. or $P(\mathrm{d}\omega)\widetilde{\lambda}(t)\mathrm{d}t$-a.e. Similarly, we have $1_{(\lambda(s)<\widetilde{\lambda}(s))} = 0$ $P(\mathrm{d}\omega)\lambda(t)\mathrm{d}t$-a.e. or $P(\mathrm{d}\omega)\widetilde{\lambda}(t)\mathrm{d}t$-a.e. By definition of intensities

$$P(\mathrm{d}\omega)N(\omega,\mathrm{d}t) = P(\mathrm{d}\omega)\lambda(\omega,t)\mathrm{d}t = P(\mathrm{d}\omega)\widetilde{\lambda}(\omega,t)\mathrm{d}t \quad \text{on } \mathcal{P}(\mathcal{F}_\cdot). \tag{2.25}$$

Now, consider $C(t) = 1_{(\lambda(t)>\widetilde{\lambda}(t))}1_{(T_{n-1}<t\leq T_n)}$, $t \geq 0$. Then

$$0 = \mathbb{E}\left[\int_{T_{n-1}}^{T_n} 1_{(\lambda(s)>\widetilde{\lambda}(s))}\lambda(s)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^\infty 1_{(\lambda(s)>\widetilde{\lambda}(s))}1_{(T_{n-1}<s\leq T_n)}N(\mathrm{d}s)\right] \tag{2.26}$$

$$= \mathbb{E}\left[\int_{T_{n-1}}^{T_n} 1_{(\lambda(s)>\widetilde{\lambda}(s))}N(\mathrm{d}s)\right] \tag{2.27}$$

$$= \mathbb{E}\left[1_{(\lambda(s)>\widetilde{\lambda}(s))}1_{(T_n<\infty)}\right]. \tag{2.28}$$

Similarly, for $1_{(\lambda(s)<\widetilde{\lambda}(s))}$, and we get

$$\lambda(T_n) = \widetilde{\lambda}(T_n) \quad \text{a.s. on } \{T_n < \infty\},\ n \geq 1. \tag{2.29}$$

Next, observe that

$$\int_0^a \mathbb{E}\Big[1_{(\lambda(s)>\widetilde{\lambda}(s))}\Big]\,\lambda(s)\mathrm{d}s = \mathbb{E}\Big[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}\lambda(s)\mathrm{d}s\Big] \tag{2.30}$$

$$= \mathbb{E}\Big[\int_0^a 1_{(\lambda(s)>\widetilde{\lambda}(s))}\widetilde{\lambda}(s)\mathrm{d}s\Big] \tag{2.31}$$

$$= \int_0^a \mathbb{E}\Big[1_{(\lambda(s)>\widetilde{\lambda}(s))}\Big]\,\widetilde{\lambda}(s)\mathrm{d}s = 0, \tag{2.32}$$

so we conclude that $\lambda(\omega,t) = \widetilde{\lambda}(\omega,t)$, $\lambda(\omega,t)\mathrm{d}t$-a.e. and $\widetilde{\lambda}(\omega,t)\mathrm{d}t$-a.e.

Finally, take $C(t) = 1_{(\lambda(t)=0)}1_{(T_{n-1}<t\leq T_n)}$, $t \geq 0$.

$$\mathbb{E}\big[1_{(\lambda(t)=0)}1_{(T_n<\infty)}\big] = \mathbb{E}\Big[\int_{T_n-1}^{T_n} 1_{(\lambda(t)=0)}\lambda(t)\mathrm{d}t\Big] = 0. \tag{2.33}$$

Hence $\lambda(T_n) > 0$ a.s. on $\{T_n < \infty\}$. □

Remark. Since $P(\mathrm{d}\omega)\lambda(\omega,t)\mathrm{d}t = P(\mathrm{d}\omega)N(\omega,\mathrm{d}t)$ on $\mathcal{P}(\mathcal{F}_\cdot)$, we see that $\widetilde{\lambda}(t)$ is the $(\mathcal{F}_t)$-predictable process given by the Radon-Nikodym derivative

$$\widetilde{\lambda}(\omega,t) = \left(\frac{\mathrm{d}PN(\mathrm{d}u)}{\mathrm{d}P\mathrm{d}u}\right)(t,\omega). \tag{2.34}$$

The following result is known as Komatsu's Lemma and is useful for writing the explicit form of the intensity.

Lemma 2.12 (Komatsu [24, Lemma 1.4])**.** *Let $(X_t)_{t\geq 0}$ be a real-valued $(\mathcal{F}_t)$-progressive process such that X_T is integrable and $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ for all bounded $(\mathcal{F}_t)$-stopping times T. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-martingale.*

Proof. For any bounded stopping times S and T, we have

$$\mathbb{E}[X_T] = \mathbb{E}[X_S]\,. \tag{2.35}$$

We take then a particular choice of S and T. Fix $0 \leq s \leq t$ and take $A \in \mathcal{F}_s$ arbitrarily. Define T and S as

$$T(\omega) = t, \tag{2.36}$$

$$S(\omega) = \begin{cases} s & \text{if} \quad \omega \in A, \\ t & \text{if} \quad \omega \in A^c. \end{cases} \tag{2.37}$$

Clearly T is a stopping time, since it is deterministic. On the other hand, for any $v \geq 0$, we have

$$\{S \leq v\} = \begin{cases} \emptyset \in \mathcal{F}_v & \text{if} \quad 0 \leq v < s, \\ A \in \mathcal{F}_s \subset \mathcal{F}_v & \text{if} \quad s \leq v < t, \\ \Omega \in \mathcal{F}_v & \text{if} \quad t \leq v, \end{cases} \tag{2.38}$$

and thus S is an $(\mathcal{F}_t)$-stopping time. Now

$$\mathbb{E}[X_S] = \mathbb{E}[X_s 1_A] + \mathbb{E}[X_t 1_{A^c}], \tag{2.39}$$

$$\mathbb{E}[X_T] = \mathbb{E}[X_t 1_A] + \mathbb{E}[X_t 1_{A^c}], \tag{2.40}$$

$$\mathbb{E}[X_s 1_A] = \mathbb{E}[X_t 1_A], \quad \forall A \in \mathcal{F}_s. \tag{2.41}$$

Therefore, we conclude that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-martingale. □

Let us now introduce a predictable representation of the intensity.

Theorem 2.13. *Let N be a point process and let $(\mathcal{F}_t^N)_{t\geq 0}$ be its natural filtration. In order for the process (X_t) to be $(\mathcal{F}_t^N)$-predictable, it is necessary and sufficient that it admits the representation*

$$X_t(\omega) = \sum_{n\geq 1} f^{(n)}(t,\omega) 1_{(T_n < t \leq T_{n+1})} + f^{(\infty)}(t,\omega) 1_{(T_\infty < t < \infty)}, \tag{2.42}$$

where $(t,\omega) \mapsto f^{(n)}(t,\omega)$ are $\mathcal{B}_+ \otimes \mathcal{F}_{T_n}^N$-measurable for all $n \in \mathbb{N}$.

Proof. To show sufficiency, it is enough to show that for all $n \geq 0$, $f^{(n)}(t,\omega)1_{(T_n<t\leq T_{n+1})}$ is $(\mathcal{F}_t^N)$-predictable. We show it for

$$f^{(n)}(t,\omega) = 1_A(\omega) 1_B(t), \quad A \in \mathcal{F}_{T_n}^N, B \in \mathcal{B}_+. \tag{2.43}$$

Consider $T_n^A = +\infty \cdot 1_{A^c}(\omega) + T_n 1_A(\omega)$, then

$$f^{(n)}(t,\omega) 1_{(T_n<t\leq T_n)} = 1_B(t) 1_{\left(T_n^A < t \leq T_{n+1}^A\right)}, \tag{2.44}$$

and $1_B(t)$ is deterministic, hence $(\mathcal{F}_t^N)$-predictable. Moreover, the process $1_{\left(T_n^A<t\leq T_{n+1}^A\right)}$ is left-continuous and $\left\{T_n^A \leq t\right\} = A \cap \{T_n \leq t\} \in \mathcal{F}_t^N$ (by the definition of $\mathcal{F}_{T_n}^N$). This means that T_n^A is an $(\mathcal{F}_t^N)$-stopping time for all $n \geq 0$ and for all $A \in \mathcal{F}_{T_n}^N$. Also $A \in \mathcal{F}_{T_n}^N \subset \mathcal{F}_{T_n+1}^N$, so by definition of $\mathcal{F}_{T_n+1}^N$, $A \cap \{T_{n+1} \leq t\} \in \mathcal{F}_t^N$, which means that T_{n+1}^A is also an $(\mathcal{F}_t^N)$-stopping time. Therefore, $f^{(n)}(t)1_{(T_n<t\leq T_{n+1})}$ is $(\mathcal{F}_t^N)$-predictable.

For necessity, we know that $\mathcal{P}(\mathcal{F}_t)$ is generated by sets of the form $\{T \leq t\}$ where T are bounded stopping times. We can restrict ourselves to processes $X_t = 1_{(t\leq S)}$, where S is a finite $(\mathcal{F}_t^N)$-stopping time. If we stick to the convention $T_0 = 0$ and $T_{\infty+1} = \infty$, then

$$1_{(t\leq S)} = \sum_{n=0}^{\infty} 1_{(t\leq S)} 1_{(T_n<t\leq T_{n+1})} + 1_{(t\leq S)} 1_{(T_\infty<t<\infty)} \tag{2.45}$$

$$= \sum_{n=0}^{\infty} 1_{(T_n<t\leq S\wedge T_{n+1})} + 1_{(T_\infty<t\leq S\wedge T_{\infty+1})} \tag{2.46}$$

$$= \sum_{n=0}^{\infty} 1_{(T_n<t\leq (T_n+R_n)\wedge T_{n+1})} + 1_{(T_\infty<t\leq (T_\infty+R_\infty)\wedge T_{\infty+1})} \tag{2.47}$$

$$= \sum_{n=0}^{\infty} 1_{(t\leq T_n+R_n)} 1_{(T_n<t\leq T_{n+1})} + 1_{(t\leq T_\infty+R_\infty)} 1_{(T_\infty<t<\infty)}, \tag{2.48}$$

where each $1_{(t\leq T_n+R_n)}$ is $\mathcal{B}_+ \otimes \mathcal{F}_{T_n}^N$-measurable. □

We can now introduce the following result that allows us to compute the intensity explicitly. For a comprehensive exposition of this see for example [7, Ch.3, T7]. Let us denote $S_{n+1} := T_{n+1} - T_n$ and assume that the conditional distributions of S_{n+1} given $\mathcal{G}_{T_n} = \sigma(T_0, \dots, T_n)$ admit densities, i.e. for all $n \geq 0$ and $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{P}(S_{n+1} \in B \mid \mathcal{G}_{T_n})(\omega) = \int_B g^{(n+1)}(\omega, t)\mathrm{d}t = G^{(n+1)}(\omega, B), \tag{2.49}$$

where $(\omega, t) \mapsto g^{(n+1)}(\omega, t)$ are $\mathcal{G}_{T_n} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Also, for all $n \geq 0,\ B \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{P}(S_{n+1} \in B \mid \mathcal{G}_{T_n})(\omega) = \int_B g^{(n+1)}(\omega, t)\mathrm{d}t = G^{(n+1)}(\omega, B), \tag{2.50}$$

where $g^{(n+1)}(\omega, t) = \sum_{i=1}^m g^{(n+1)}(\omega, t, i)$.

Note that $\mathbb{P}(S_{n+1} = \infty \mid \mathcal{G}_{T_n}) = 1 - \int_{\mathbb{R}_+} g^{(n+1)}(t)\mathrm{d}t$.

Theorem 2.14. *Under the previous assumptions, if we define process*

$$\lambda(t) = \sum_{n \geq 0} \frac{g^{(n+1)}(t - T_n)}{G^{(n+1)}([t - T_n, \infty])} 1_{(T_n \leq t \leq T_{n+1})} \tag{2.51}$$

$$A(t) = \int_0^t \lambda(s)\mathrm{d}s, \tag{2.52}$$

then $N(t \wedge T_n) - A(t \wedge T_n)$ is a $\mathcal{G}_t$-martingale for all $n \geq 0$.

Proof. Notice that the processes defined are right-continuous and adapted, hence they are $\mathcal{G}_t$-progressive. We want to use Komatsu's Lemma 2.12. Let us check that the other hypothesis holds, i.e.

$$\mathbb{E}[N(S \wedge T_n)] = \mathbb{E}[A(S \wedge T_n)], \tag{2.53}$$

for all $n \geq 0$ and all finite $\mathcal{G}_t$-stopping times S. By [7, Theorem A2.T33], we know that S admits the representation

$$S \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1} \quad \text{on } \{S \leq T_n\}, \tag{2.54}$$

with R_n nonnegative $\mathcal{G}_{T_n}$-measurable random variables, and the convention $T_{\infty+1} = \infty$.

We have

$$\mathbb{E}[A(S\wedge T_n)] = \mathbb{E}\left[\int_0^{S\wedge T_n}\sum_{j\geq 0}\frac{g^{(j+1)}(u-T_j)}{G^{(j+1)}([\![u-T_j,\infty]\!])}1_{(T_j\leq u<T_{j+1})}\mathrm{d}u\right] \tag{2.55}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{S\wedge T_n}\frac{g^{(j+1)}(u-T_j)}{G^{(j+1)}([\![u-T_j,\infty]\!])}1_{(T_j\leq u<T_{j+1})}\mathrm{d}u\right] \tag{2.56}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{S\wedge T_{j+1}}\frac{g^{(j+1)}(u-T_j)}{G^{(j+1)}([\![u-T_j,\infty]\!])}1_{(T_j\leq u<T_{j+1})}\mathrm{d}u\ 1_{(S\geq T_j)}\right] \tag{2.57}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{T_j+R_j}\frac{g^{(j+1)}(u-T_j)}{G^{(j+1)}([\![u-T_j,\infty]\!])}1_{(T_j\leq u<T_{j+1})}\mathrm{d}u\ 1_{(S\geq T_j)}\right] \tag{2.58}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{R_j}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}1_{(0\leq u<S_{j+1})}\mathrm{d}r\ 1_{(S\geq T_j)}\right] \tag{2.59}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{R_j\wedge S_{j+1}}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\ 1_{(S\geq T_j)}\right] \tag{2.60}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\mathbb{E}\left[\int_0^{R_j\wedge S_{j+1}}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\ 1_{(S\geq T_j)}\middle|\mathcal{G}_{T_j}\right]\right] \tag{2.61}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1}\mathbb{E}\left[\int_0^{R_j\wedge S_{j+1}}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\middle|\mathcal{G}_{T_j}\right]\ 1_{(S\geq T_j)}\right]. \tag{2.62}$$

Notice now that given T_j, we can regard R_j as a constant, we then compute the conditional expectation, by integrating with respect to the conditional density of S_{j+1} given $\mathcal{G}_{T_j}$, as

$$\mathbb{E}\left[\int_0^{R_j\wedge S_{j+1}}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\middle|\mathcal{G}_{T_j}\right] = \int_0^\infty g^{(j+1)}(u)\int_0^{R_j\wedge u}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\mathrm{d}u \tag{2.63}$$

$$= \int_0^{R_j}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}r\int_r^\infty g^{(j+1)}(u)\mathrm{d}u \tag{2.64}$$

$$= \int_0^{R_j}\frac{g^{(j+1)}(r)}{G^{(j+1)}([\![r,\infty]\!])}\mathrm{d}rG^{(j+1)}([\![r,\infty]\!]) \tag{2.65}$$

$$= \int_0^{R_j}g^{(j+1)}(r)\mathrm{d}r. \tag{2.66}$$

This means that

$$\mathbb{E}[A(S\wedge T_n)] = \mathbb{E}\left[\sum_{j=0}^{n-1}\int_0^{R_j}g^{(n+1)}(r)\mathrm{d}r\ 1_{(S\geq T_j)}\right]. \tag{2.67}$$

Now, computing the left hand side of (2.53), we have

$$\mathbb{E}[N(S \wedge T_n)] = \mathbb{E}\left[\sum_{j=0}^{n-1} (N(S \wedge T_{j+1}) - N(S \wedge T_j))\, 1_{(S \geq T_j)}\right] \tag{2.68}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1} (N((T_j + R_j) \wedge T_{j+1}) - N(S \wedge T_j))\, 1_{(S \geq T_j)}\right]. \tag{2.69}$$

Here notice that if $N_{(T_j+R_j)\wedge T_{j+1}} - N_{S\wedge T_j} = 1$, then $T_j + R_j \geq T_{j+1}$implies $R_j \geq S_{j+1}$, so

$$\mathbb{E}\left[\sum_{j=0}^{n-1} \left(N_{(T_j+R_j)\wedge T_{j+1}} - N_{S\wedge T_j}\right) 1_{(S\geq T_j)}\right] \tag{2.70}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1} 1_{(R_j \geq S_{j+1})} 1_{(S\geq T_j)}\right] \tag{2.71}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[1_{(R_j\geq S_{j+1})} \mid \mathcal{G}_{T_j}\right] 1_{(S\geq T_j)}\right] \tag{2.72}$$

$$= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{P}\left(S_{j+1} \leq R_j \mid \mathcal{G}_{T_j}\right) 1_{(S\geq T_j)}\right]. \tag{2.73}$$

Therefore

$$\mathbb{E}[N(S \wedge T_n)] = \mathbb{E}\left[\sum_{j=0}^{n-1} \int_0^{R_j} g^{(n+1)}(r)\mathrm{d}r\ 1_{(S\geq T_j)}\right] = \mathbb{E}[A(S \wedge T_n)]. \tag{2.74}$$

Hence, by Komatsu's Lemma 2.12 $N_{t\wedge T_n} - A_{t\wedge T_n}$ is a $\mathcal{G}_t$-martingale, which finalizes the proof. □

Remark. Theorem 2.14 remains valid if

$$\lambda(t) = \sum_{n\geq 0} \frac{g^{(n+1)}(t - T_n)}{G^{(n+1)}([t - T_n, \infty])} 1_{(T_n\leq t\leq T_{n+1})}. \tag{2.75}$$

We have covered all the basic theory of the martingale approach to point processes that we require for this thesis. One particular kind of point process to which we apply the previous results is the *renewal process*, which we present in the following Section.

2.3 Renewal Theory

We introduce now the basic definitions related to *renewal processes* and state some general results that we apply in subsequent sections. Consider $\tau_1, \tau_2, \ldots$ a sequence of i.i.d. random variables on $(0, \infty)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with *interarrival* *distribution* function $F(x) = \mathbb{P}(\tau_1 \leq x)$ for $x \geq 0$. Additionally, consider a random variable τ_0 on $[0, \infty)$ defined on the same space, which is independent of $\tau_1, \tau_2, \dots$, with *delay distribution* $F_0(x) = \mathbb{P}(\tau_0 \leq x)$, $x \geq 0$, not necessarily equal to F. We will work in the setting of the following definition.

Definition 2.15. Let $\{\widehat{S}_n\}_{n\geq 0}$ be the sequence of partial sums given as

$$\widehat{S}_n = \tau_0 + \tau_1 + \cdots + \tau_n, \quad n \geq 0, \tag{2.76}$$

to which we associate the counting measure

$$\widehat{N}(B) = \sum_{n\geq 0} 1_{\{\widehat{S}_n \in B\}}, \quad B \in \mathcal{B}([0, \infty)). \tag{2.77}$$

In particular, we identify the *renewals* $\{\widehat{S}_n\}_{n\geq 0}$ with the *delayed renewal process*

$$\widehat{N}(t) := \widehat{N}([0,t]) = \sum_{n\geq 0} 1_{\{\widehat{S}_n \leq t\}}. \tag{2.78}$$

If instead we consider the partial sums

$$S_0 = 0, \quad \textit{and} \quad S_n = \tau_1 + \cdots + \tau_n, \quad n \geq 1 \tag{2.79}$$

then

$$N(t) := N([0,t]) = \sum_{n\geq 0} 1_{\{S_n \leq t\}} \tag{2.80}$$

is called a *zero-delayed* or *pure renewal process.*

Notice that since the inter-arrival distributions do not have an atom at $x = 0$, then necessarily $S_{n+1} > S_n$ on $\{S_n < \infty\}$ for all $n \geq 0$, which implies that the renewal process is a *simple point process*, i.e. $\mathbb{P}\Big(\widehat{N}(\{x\}) = 0 \text{ or } 1 \text{ for all } x \geq 0\Big) = 1$. From the definition of N, the following processes can be derived.

Definition 2.16. Given a renewal process N, the *backward recurrence time* $\{A_t\}_{t\geq 0}$ and *forward recurrence time* $\{B_t\}_{t\geq 0}$ are given for $t \geq 0$ as

$$A_t = t - S_{N(t)-1}, \quad B_t = S_{N(t)} - t. \tag{2.81}$$

It is notable that $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are time-homogeneous strong Markov processes (c.f. [1, Proposition V.1.5]). Denote by $(\mathcal{F}_t^N)_{t\geq 0}$ the augmentation of the natural filtration generated by the renewal process, given for each $t \geq 0$ by $\sigma(N(s); 0 \leq s \leq t)$. The process N is increasing and predictable, hence, by the Doob–Meyer decomposition theorem, it has an a.s. finite *compensator* Λ such that the process

$$M := N - \Lambda, \tag{2.82}$$

is an $(\mathcal{F}_t^N)$-martingale.

If we assume that $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$, it is possible to make a delayed renewal process stationary, in the sense that for any $t > 0$ the distribution of the increments $\left\{\widehat{N}(t+s) - \widehat{N}(t)\right\}_{s\geq 0}$ does not depend on t, by choosing the delay distribution F_0 equals to $\Pi(\mathrm{d}t) = \pi(t)\mathrm{d}t$ with density π given as

$$\pi(t) := m\overline{F}(t) := m(1 - F(t)), \quad t \geq 0, \tag{2.83}$$

in which case, the corresponding backward and forward recurrence times, $\widehat{A}_t$ and $\widehat{B}_t$ respectively, have common distribution Π for all $t \geq 0$ (see for example [10, Proposition 4.2.I]).

Throughout this thesis, if f and g are both functions, we will denote their convolution as

$$f * g(t) = \int_0^t f(t-s)g(s)\mathrm{d}s, \tag{2.84}$$

whereas, if F is a measure on $[0,\infty)$ and g is a function, the convention that $F * g$ is a function is used, and we write

$$F * g(t) = \int_0^t g(t-s)F(\mathrm{d}s). \tag{2.85}$$

We sometimes identify the measure $F(\mathrm{d}s)$ with its cumulative distribution function $F(t) = \int_0^t F(\mathrm{d}s)$. If F and G are both measures, we will denote their convolution as

$$F * G(t) = \int_0^t F(t-s)G(\mathrm{d}s) = \int_0^t G(t-s)F(\mathrm{d}s). \tag{2.86}$$

The Key Renewal Theorem states the limiting behavior of the solution to the *renewal equation*

$$Z(t) = z(t) + \int_0^t Z(t-u)F(\mathrm{d}u), \quad t \geq 0, \tag{2.87}$$

where $z : [0,\infty) \to [0,\infty)$ is known, $Z : [0,\infty) \to [0,\infty)$ is unknown and F is a given probability distribution on $[0,\infty)$. If additionally z is bounded on finite intervals, i.e. $\sup_{0\leq t\leq T} |z(t)| < \infty$ for all $T < \infty$, then the only solution to (2.87) which is bounded on finite intervals is given by

$$Z(t) = \Phi * z(t) = \int_0^t z(t-u)\Phi(\mathrm{d}u), \quad t \geq 0, \tag{2.88}$$

where the increasing function $\Phi(t) = \Phi([0,t])$ is given as

$$\Phi = \sum_{n\geq 0} F^{*n}, \tag{2.89}$$

with

$$F^{*0} = \delta_0, \quad \text{and} \quad F^{*(n+1)}(t) = F^{*n} * F(t) := \int_0^t F^{*n}(t-s)F(\mathrm{d}s), \quad t \geq 0. \tag{2.90}$$

The function $\Phi(t)$ is called the *renewal function*, and its induced measure $\Phi(\mathrm{d}t)$ is called the *renewal measure*. One renewal equation of particular importance is that generated by the density π in (2.83). As we can see below, such renewal equations have linear solutions.

Lemma 2.17. *Assume that $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$ holds. Then, the renewal equation*

$$Z(t) = z(t) + \int_0^t Z(t-x)F(\mathrm{d}x), \tag{2.91}$$

has the linear solution $Z_0(t) = mt$, $t \geq 0$, if and only if

$$z(t) = m\int_0^t \overline{F}(x)\mathrm{d}x, \tag{2.92}$$

where $\overline{F} = 1 - F$.

Proof. Let us suppose that equation (2.87) has a linear solution $Z_0(t) = mt$, which entails

$$mt = z(t) + \int_0^t m(t-x)F(\mathrm{d}x). \tag{2.93}$$

Solving for $z(t)$ we get

$$z(t) = mt - \int_0^t m(t-x)F(\mathrm{d}x) \tag{2.94}$$

$$= mt - \int_0^t m\int_x^t \mathrm{d}uF(\mathrm{d}x) \tag{2.95}$$

$$= mt - \int_0^t m\int_0^u F(\mathrm{d}x)\mathrm{d}u \tag{2.96}$$

$$= \int_0^t m\mathrm{d}x - \int_0^t mF(x)\mathrm{d}x \tag{2.97}$$

$$= \int_0^t m(1-F(x))\mathrm{d}x \tag{2.98}$$

$$= \int_0^t m\overline{F}(x)\mathrm{d}x. \tag{2.99}$$

Reversing the order of the above steps we can see that the function $z(t) = \int_0^t m\overline{F}(x)\mathrm{d}x$ generates a renewal Equation with solution

$$mt = Z_0(t) = \int_0^t m\int_0^{t-u} \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u). \tag{2.100}$$

The proof is now complete. □

One of the properties that is often required from the function z to know the asymptotic behavior of Z is that of being *directly Riemann integrable* (abbreviated d.R.i.). This property is defined as follows.

Definition 2.18. Let z be a nonnegative measurable function, and let $h > 0$. Define

$$\overline{z}_h(x) = \sup_{y \in I_n^h} z(y), \quad \underline{z}_h(x) = \inf_{y \in I_n^h} z(y), \quad x \in I_n^h = (nh, (n+1)h]. \tag{2.101}$$

We say that z is directly Riemann integrable if $\int_0^\infty \overline{z}_h(d)\mathrm{d}x$ is finite for some h and

$$\int_0^\infty \overline{z}_h(x)\mathrm{d}x - \int_0^\infty \underline{z}_h(x)\mathrm{d}x \underset{h\to 0}{\longrightarrow} 0. \tag{2.102}$$

We the state the most typical form of the Key Renewal Theorem.

Theorem 2.19. *Suppose that* $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$ *and the function* z *in the renewal equation* (2.87) *is d.R.i. Then*

$$Z(t) = \Phi * z(t) \underset{t\to\infty}{\longrightarrow} m \int_0^\infty z(x)\mathrm{d}x. \tag{2.103}$$

For a proof of this Theorem see for example [1, Section V.5] for an analytic proof and [10, Theorem 4.4.I and 4.4.II] for a probabilistic proof.

While there are many sufficient conditions for a function being d.R.i., for a certain class of renewal processes such a condition can be dropped for a simpler one. This is the case when the inter-arrival distribution F of a renewal process is *spread out.*

Definition 2.20 ([1, Sec VII. p.186])**.** A distribution F on $[0, \infty)$ is called *spread out* if for some $n \geq 1$, there exists a nonnegative measure G such that $0 \neq G \leq F^{*n}$ and G is absolutely continuous with respect to the Lebesgue measure. The measure G is called an *absolutely continuous component* of F^{*n}.

It has been shown that when F is spread out, the renewal measure can be written as a sum of a finite and an absolutely continuous component (see Stone [35]). This decomposition is not unique, as it depends on the absolutely continuous component from Definition (2.20). For our purposes, it is convenient to select *uniform components*, i.e. measures of the form

$$G(\mathrm{d}x) = c1_{(a,b)}(x)\mathrm{d}x, \tag{2.104}$$

where $0 \leq a < b < \infty$ and c is a positive constant. This can always be done as shown in the following Lemma.

Lemma 2.21. (see e.g., VII.1.2 from [1]) *If* F *is spread out, then* F^{*n_0} *has a uniform component on* $(a, a+b)$ *for some* $a, b, n_0 > 0$.

Proof. Since F is spread out, there exists a measure G and an $n \geq 0$ such that $0 \neq G \leq F^{*n}$ and G has a density g with respect to the Lebesgue measure. Suppose, without loss of generality, that g is bounded with compact support. Choose continuous functions $g_k \in L_1$ with compact supports such that

$$\|g - g_k\|_1 = \int |g(t) - g_k(t)| \, \mathrm{d}t \underset{k\to\infty}{\longrightarrow} 0. \tag{2.105}$$

Then

$$\sup_{|x-x'|<\delta} |g_k * g(x) - g_k * g(x')| \leq \sup_{|z-z'|<\delta} |g_k(z) - g_k(z')| \int g(y)\mathrm{d}y \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.106}$$

So $g_k * g$ is uniformly continuous. Since

$$\left\|g^{*2} - g * g_k\right\|_\infty \leq \|g\|_\infty \|g - g_k\|_1 \underset{k\to\infty}{\longrightarrow} 0. \tag{2.107}$$

we see that g^{*2} is continuous as a uniform limit of continuous functions, hence there exist $a, b, \delta > 0$ such that $g^{*2}(x) \geq \delta$ for $x \in (a, a+b)$. Finally, take $n_0 = 2n$, and then

$$G_0(\mathrm{d}x) = \delta 1_{(a,a+b)}(x)\mathrm{d}x \tag{2.108}$$

is a uniform component of F^{*n_0}. □

We can then write Stone's decomposition for the renewal measure.

Theorem 2.22 (Stone [35]). *If the inter-arrival distribution F is spread out, then we can write $\Phi = \Phi_1 + \Phi_2$, where Φ_1 and Φ_2 are nonnegative measures on $[0, \infty)$, Φ_2 is bounded (i.e. $\|\Phi_2\|_{\mathrm{t.v.}} < \infty$) and Φ_1 has a bounded density $\varphi_1(x) = \mathrm{d}\Phi_1(x)/\mathrm{d}x$ satisfying $\varphi_1(x) \underset{x\to\infty}{\longrightarrow} m$.*

The following proof is taken from Asmussen [1] with a slight correction.

Proof. Let G_0 denote the uniform component of F^{*n_0} which was given in the proof of Lemma 2.21, and g_0 its density, given as

$$g_0(x) = \frac{\|G_0\|_{\mathrm{t.v.}}}{b} 1_{[a,a+b)}(x), \quad x \geq 0, \tag{2.109}$$

and define $H := F^{*n_0} - G_0$. We then note that

$$\Phi = \sum_{k=0}^{n_0-1} F^{*k} * \Phi_0, \tag{2.110}$$

where

$$\Phi_0 = \sum_{n=0}^{\infty} F^{*nn_0}. \tag{2.111}$$

We can check by induction that

$$F^{*nn_0} = (H + G_0)^{*n} = G_0 * \sum_{k=0}^{n-1} F^{*(n-k-1)n_0} * H^{*k} + H^{*n}, \tag{2.112}$$

so Φ_0 becomes

$$\begin{aligned}
\Phi_0 =& \sum_{n=0}^{\infty} \left(G_0 * \sum_{k=0}^{n-1} F^{*(n-k-1)n_0} * H^{*k} + H^{*n} \right) && (2.113) \\
=& G_0 * \sum_{k=0}^{\infty} H^{*k} * \sum_{n=k+1}^{\infty} F^{*(n-k-1)n_0} + \sum_{n=0}^{\infty} H^{*n} && (2.114) \\
=& G_0 * \sum_{n=0}^{\infty} H^{*n} * \Phi_0 + \sum_{n=0}^{\infty} H^{*n} && (2.115)
\end{aligned}$$

Then a decomposition of Φ_0 is given as $\Phi_0 = \Phi_0^{(1)} + \Phi_0^{(2)}$ where

$$\Phi_0^{(2)} = \sum_{n=0}^{\infty} H^{*n}, \quad \Phi_0^{(1)} = G_0 * \Phi_0^{(2)} * \Phi_0. \tag{2.116}$$

Since $\|H\|_{\text{t.v.}} = 1 - \|G_0\|_{\text{t.v.}} < 1$, we have that

$$\left\| \Phi_0^{(2)} \right\|_{\text{t.v.}} = \frac{1}{1 - \|H\|_{\text{t.v.}}} = \frac{1}{\|G_0\|_{\text{t.v.}}} < \infty. \tag{2.117}$$

Moreover, since G_0 is absolutely continuous, so is $\Phi_0^{(1)}$, with density $\varphi_0^{(1)} = \Phi_0^{(2)} * (\Phi_0 * g_0)$. From Blackwell's renewal theorem (c.f. [10, Theorem 4.4.I]) we have that

$$\Phi_0 * g_0(x) = \frac{\|G_0\|_{\text{t.v.}}}{b} \Phi_0((x-a-b, x-a]) \underset{x\to\infty}{\longrightarrow} \frac{m}{n_0} \|G_0\|_{\text{t.v.}}. \tag{2.118}$$

From the subadditivity of the renewal function (c.f. [1, Theorem V.2.4]):

$$\Phi_0((x-a-b, x-a]) \le \Phi_0(b) \quad \text{for all } x \ge 0, \tag{2.119}$$

and the total finiteness of $\Phi_0^{(2)}$, we have from the Dominated Convergence Theorem, (2.117) and (2.118) that

$$\varphi_0^{(1)}(x) = \int_0^x \Phi_0 * g_0(x-y) \Phi_0^{(2)}(\mathrm{d}y) \underset{x\to\infty}{\longrightarrow} \frac{m}{n_0} \|G_0\|_{\text{t.v.}} \left\| \Phi_0^{(2)} \right\|_{\text{t.v.}} = \frac{m}{n_0}. \tag{2.120}$$

Finally, using (2.110) and (2.115), we decompose Φ as $\Phi = \Phi_1 + \Phi_2$ with

$$\Phi_2 = \sum_{k=0}^{n_0-1} F^{*k} * \Phi_0^{(2)}, \quad \Phi_1 = \sum_{k=0}^{n_0-1} F^{*k} * G_0 * \Phi_0^{(2)} * \Phi_0 = G_0 * \Phi_0^{(2)} * \Phi, \tag{2.121}$$

where

$$\|\Phi_2\|_{\text{t.v.}} = \frac{n_0}{\|G_0\|_{\text{t.v.}}} < \infty, \tag{2.122}$$

and $G_0 * \Phi_0^{(2)} * \Phi$ has density $\varphi_1 = \Phi_0^{(2)} * (\Phi * g_0)$ such that

$$\varphi_1(x) = \int_0^x \Phi * g_0(x-y)\Phi_0^{(2)}(\mathrm{d}y) \underset{x\to\infty}{\longrightarrow} m \|G_0\|_{\text{t.v.}} \left\|\Phi_0^{(2)}\right\|_{\text{t.v.}} = m. \tag{2.123}$$

Moreover, this density is also bounded since from subadditivity we have

$$\Phi * g_0(x) = \frac{\|G_0\|_{\text{t.v.}}}{b}\Phi([x-a-b, x-a)) \leq \frac{\|G_0\|_{\text{t.v.}}}{b}\Phi(b), \tag{2.124}$$

from which we conclude that

$$\|\Phi * g_0\|_\infty \leq \frac{\|G_0\|_{\text{t.v.}}}{b}\Phi(b) < \infty \tag{2.125}$$

$$\|\varphi_1\|_\infty \leq \|\Phi * g_0\|_\infty \sup_{x\geq 0} \Phi_0^{(2)}((x-a-b, x-a]) < \infty, \tag{2.126}$$

which concludes the proof. □

In the remainder of this thesis we use the decomposition given by (2.121). We proceed now to state a version of Theorem 2.19 for the case of spread out distributions.

Theorem 2.23 ([1, Corollary VII.1.3])**.** *Let $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$ and let z be bounded and Lebesgue integrable with $z(x) \longrightarrow 0$ as $x \to \infty$. Then*

$$\Phi * z(t) \underset{t\to\infty}{\longrightarrow} m \int_0^\infty z(x)\mathrm{d}x, \tag{2.127}$$

provided F is spread out.

Proof. From Stone's decomposition, we can write $\Phi = \Phi_1 + \Phi_2$ as in (2.121). Then

$$Z(x) = \Phi_1 * z(x) + \Phi_2 * z(x). \tag{2.128}$$

Using the boundedness of z and the finiteness of the measure Φ_2, we have from the Dominated Convergence Theorem that

$$\Phi_1 * z(x) + \Phi_2 * z(x) = \int_0^x z(y)\varphi_1(x-y)\mathrm{d}y + \int_0^x z(x-y)\Phi_2(\mathrm{d}y) \tag{2.129}$$

$$\underset{x\to\infty}{\longrightarrow} m \int_0^\infty z(y)\mathrm{d}y + \int_0^\infty 0 \cdot \Phi_2(\mathrm{d}y). \tag{2.130}$$

The result is proved. □

To finalize this Section, let us utilize renewal equations to derive some other important tools for the prove of our results. The first of those is the distribution of the forward recurrence time, which can be found through a renewal argument as follows.

Lemma 2.24. *The distribution function of the forward recurrence time is given for $x \geq 0$ as*

$$\mathbb{P}(B_t \leq x) = \int_0^t F((t-u, t+x-u])\, \Phi(\mathrm{d}u). \tag{2.131}$$

Proof. Write $Z_x(t) = \mathbb{P}(B_t \leq x)$, and $z_x(t) = \mathbb{P}(B_t \leq x; \tau_1 > t)$. Then,

$$\begin{aligned}
Z_x(t) =& z_x(t) + \mathbb{P}(B_t \leq x; \tau_1 \leq t) & (2.132)\\
=& z_x(t) + \int_0^t \mathbb{P}(B_t \leq x \mid \tau_1 = u)\, F(\mathrm{d}u) & (2.133)\\
=& z_x(t) + \int_0^t \mathbb{P}(B_{t-u} \leq x)\, F(\mathrm{d}u) \quad \text{(by the Strong Markov property of } B_t\text{)} & (2.134)\\
=& z_x(t) + \int_0^t Z_x(t-u) F(\mathrm{d}u) = \int_0^t z_x(t-u)\Phi(\mathrm{d}u). & (2.135)
\end{aligned}$$

Now, analyzing $z_x(t)$, we have,

$$\begin{aligned}
z_x(t) =& \mathbb{P}(B_t \leq x; \tau_1 > t) & (2.136)\\
=& \mathbb{P}(\tau_1 - t \leq x; \tau_1 > t) & (2.137)\\
=& F(t+x) - F(t). & (2.138)
\end{aligned}$$

Therefore, we obtain,

$$Z_x(t) = \int_0^t F((t-u, t+x-u])\, \Phi(\mathrm{d}u), \tag{2.139}$$

which concludes the proof. ☐

Finally, we can find the following results for Φ whenever the inter-arrival distribution has a finite second moment.

Lemma 2.25. *Suppose that the distribution F has finite second moment $\sigma^2 + \frac{1}{m^2}$. Then*

$$\Phi(t) - mt \underset{t\to\infty}{\longrightarrow} \frac{1}{2}\left(m^2\sigma^2 + 1\right). \tag{2.140}$$

Proof. Set $z(t) = m \int_t^\infty \overline{F}(y)\mathrm{d}y$ in (2.87). We know then that the solution to this equation is given by

$$\begin{aligned}
Z_0(t) =& \int_0^t m \int_{t-u}^\infty \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u) & (2.141)\\
=& \int_0^t m \int_0^\infty \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u) - \int_0^t m \int_0^{t-u} \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u). & (2.142)
\end{aligned}$$

From Lemma 2.17, we know that the second summand is equal to mt. Let us look at the other integral,

$$\int_0^\infty \overline{F}(x)\mathrm{d}x = \int_0^\infty \int_x^\infty F(\mathrm{d}y)\mathrm{d}x = \int_0^\infty \int_0^y \mathrm{d}x F(\mathrm{d}y) = \int_0^\infty yF(\mathrm{d}y) = \frac{1}{m} \tag{2.143}$$

$$\Rightarrow \int_0^t m \int_0^\infty \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u) = \int_0^t \frac{1}{m} m\Phi(\mathrm{d}u) = \int_0^t \Phi(\mathrm{d}u) = \Phi(t). \tag{2.144}$$

Using these two arguments, we obtain

$$Z_0(t) = \Phi(t) - mt. \tag{2.145}$$

We can immediately see that Z_0 is nonnegative. Now, observe that $z(t)$ is d.R.i. In effect, it is monotonically decreasing, then it is enough to show that it is integrable.

$$\int_0^\infty z(t)\mathrm{d}t = m \int_0^\infty \int_t^\infty \overline{F}(x)\mathrm{d}x\mathrm{d}t \tag{2.146}$$

$$= m \int_0^\infty \int_0^x \overline{F}(x)\mathrm{d}t\mathrm{d}x \tag{2.147}$$

$$= m \int_0^\infty x\overline{F}(x)\mathrm{d}x \tag{2.148}$$

$$= m \int_0^\infty x \int_x^\infty F(\mathrm{d}y)\mathrm{d}x \tag{2.149}$$

$$= m \int_0^\infty x \int_0^y x\mathrm{d}x F(\mathrm{d}y) \tag{2.150}$$

$$= m \int_0^\infty \frac{y^2}{2} F(\mathrm{d}y) = \frac{m}{2}\left(\sigma^2 + \frac{1}{m^2}\right). \tag{2.151}$$

This shows that $z(t)$ is d.R.i., hence we can apply the Key renewal Theorem to find the limit

$$0 \le \Phi(t) - mt = \int_0^t m \int_{t-u}^\infty \overline{F}(x)\mathrm{d}x\Phi(\mathrm{d}u) \underset{t\to\infty}{\longrightarrow} m \int_0^\infty z(t)\mathrm{d}t \tag{2.152}$$

$$\Rightarrow 0 \le \Phi(t) - mt \underset{t\to\infty}{\longrightarrow} \frac{1}{2}\left(m^2\sigma^2 + 1\right) \tag{2.153}$$

□

Lemma 2.26. *In the same context as Lemma 2.25,*

$$mt \le \Phi(t) \le mt + m^2\sigma^2 + 1 \tag{2.154}$$

Proof. The left side of the inequality has already been proved in the previous Lemma. For the right side, consider T_1 and T_2 independent random variables with the same distribution $\mathbb{P}(T_1 > t) = m \int_t^\infty \overline{F}(x)\mathrm{d}x$. Let $t \ge 0$, from the subadditivity of Φ we get,

$$\Phi(t) = \mathbb{E}[\Phi(t)] = \mathbb{E}[\Phi(t + T_1 - T_1 + T_2 - T_2)] \tag{2.155}$$

$$\le \mathbb{E}[\Phi(t + T_1 - T_2)] + \mathbb{E}[\Phi(T_2 - T_1)] \tag{2.156}$$

$$\Rightarrow \Phi(t) \le \mathbb{E}[\mathbb{E}[\Phi(t + T_1 - T_2) \mid T_1]] + \mathbb{E}[\mathbb{E}[\Phi(T_2 - T_1) \mid T_2]]. \tag{2.157}$$

From the independence of T_1 and T_2,

$$\begin{aligned}
\mathbb{E}[\Phi(t+T_1-T_2)\mid T_1] &= \int_0^\infty \Phi(t+T_1-s)F_{T_2}(\mathrm{d}s) && (2.158)\\
&= \int_0^\infty \int_0^{t+T_1-s} \Phi(\mathrm{d}u)F_{T_2}(\mathrm{d}s) && (2.159)\\
&= \int_0^{t+T_1} \int_0^{t+T_1-u} F_{T_2}(\mathrm{d}s)\Phi(\mathrm{d}u) && (2.160)\\
&= \int_0^{t+T_1} F_{T_2}(t+T_1-u)\Phi(\mathrm{d}u) && (2.161)\\
&= \int_0^{t+T_1} m \int_0^{t+T_1-u} \overline{F}(s)\mathrm{d}s\Phi(\mathrm{d}u) \quad \text{(from definition of } F_{T_2}\text{)} && (2.162)\\
&=m(t+T_1) \quad \text{(from Lemma 2.17)} && (2.163)
\end{aligned}$$

Similarly, $\mathbb{E}[\Phi(T_2-T_1)\mid T_2]=mT_2$. Then,

$$\Phi(t) \le \mathbb{E}[m(t+T_1)] + \mathbb{E}[mT_2] = mt + 2m\mathbb{E}[T_1]\,. \tag{2.164}$$

Let us then compute $\mathbb{E}[T_1]$,

$$\begin{aligned}
\mathbb{E}[T_1] &= \int_0^t ms(1-F(s))\,\mathrm{d}s && (2.165)\\
&= \int_0^t ms \int_s^\infty F(\mathrm{d}u)\mathrm{d}s && (2.166)\\
&= \int_0^\infty \int_0^u ms\mathrm{d}sF(\mathrm{d}u) && (2.167)\\
&= \int_0^\infty \frac{mu^2}{2}F(\mathrm{d}u) = \frac{m}{2}\left(\sigma^2+\frac{1}{m^2}\right). && (2.168)
\end{aligned}$$

So we ultimately get,

$$mt \le \Phi(t) \le mt + m^2\sigma^2 + 1. \tag{2.169}$$

□

From the previous Lemma, it is easy to check that

$$0 \le \lim_{t\to\infty} \frac{(\Phi(t)-mt)^k}{t} \le \lim_{t\to\infty} \frac{(m^2\sigma^2+1)^k}{t} = 0, \tag{2.170}$$

which implies that for all $k>0$,

$$\lim_{t\to\infty} \frac{(\Phi(t)-mt)^k}{t} = 0. \tag{2.171}$$

This concludes the treatment of renewal processes. However, in the next Section we review some basic results for a more general class of processes.

2.4 Regenerative processes

We now turn to the notion of *regenerative processes*. Intuitively, such processes evolve in cycles between the epochs of a renewal process. Formally, we have the following definition.

Definition 2.27 (Sec VI. p.169 in [1]). Let $\{X_t\}_{t\geq 0}$ be a stochastic process with state space E. We call $\{X_t\}_{t\geq 0}$ *regenerative* (pure or delayed) if there exists a renewal process (pure or delayed) $\left\{\widehat{S}_n\right\} = \{\tau_0 + \tau_1 + \cdots + \tau_n\}$ such that for each $n \geq 0$, the *post-$\widehat{S}_n$ process*

$$\theta_{\widehat{S}_n} X := \left(\tau_{n+1}, \tau_{n+2}, \ldots, \left\{X_{\widehat{S}_n+t}\right\}_{t\geq 0}\right), \tag{2.172}$$

is independent of $\tau_0, \ldots, \tau_n$, and its distribution does not depend upon n.

Now, let $\{X_t\}_{t\geq 0}$ be a regenerative process. It will be useful to compute the distribution of the maximum of $\overline{X}_T := \max_{0\leq t\leq T} X_t$ for any $T > 0$. To do this, we use the following result of Rootzén and give its proof in slightly more detail than presented in [1, Proposition VI.4.7].

Theorem 2.28 (Rootzén [32]). *Assume that E is a real interval and define*

$$G(x) := \mathbb{P}_0\left(\overline{X}_{\tau_1} \leq x\right), \quad F_T(x) = \mathbb{P}\left(\overline{X}_T \leq x\right). \tag{2.173}$$

Then,

$$\lim_{T\to\infty} \left\|F_T - G^{mT}\right\|_\infty = 0. \tag{2.174}$$

In the delayed case with G having finite support we need additionally,

$$\mathbb{P}\left(\xi_0 > \max_{k=1,\ldots,n} \xi_k\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{2.175}$$

Proof. First we look at the function $z(1 - z^\gamma)$, $\gamma \in \mathbb{R}_+$, $z \in [0, 1]$.

$$\frac{\mathrm{d}}{\mathrm{d}z} z(1 - z^\gamma) = 1 - z^\gamma - \gamma z^\gamma = 0, \tag{2.176}$$

$$z^\gamma = \frac{1}{1+\gamma}, \tag{2.177}$$

which means that

$$z_0 = \left(\frac{1}{1+\gamma}\right)^{1/\gamma} \tag{2.178}$$

is a critical point. Computing the second derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} z(1 - z^\gamma) = -\gamma z^{\gamma-1} - \gamma^2 z^{\gamma-1} = -z^{\gamma-1}\left(\gamma + \gamma^2\right) < 0 \text{ for all } z \in (0, 1). \tag{2.179}$$

Since the function vanishes at the endpoints of the interval, we can conclude that z_0 corresponds to a maximum, with value,

$$(1+\gamma)^{-1/\gamma}\big(1-(1+\gamma)^{-1}\big) \tag{2.180}$$

$$=\frac{1+\gamma-1}{(1+\gamma)^{1+1/\gamma}} \tag{2.181}$$

$$=\frac{\gamma}{(1+\gamma)^{1+1/\gamma}} \leq \gamma. \tag{2.182}$$

Therefore, the function is bounded by γ. Hence, for all T, x, and ϵ,

$$\left|G(x)^T - G(x)^{T(1+\epsilon)}\right| \leq \epsilon. \tag{2.183}$$

Define $k_T^{\pm} = \lfloor mT(1 \pm \delta) \rfloor$ for some $\delta > 0$. Then

$$F_T(x) \geq \mathbb{P}\Big(\overline{X}_{S_{N(T)+1}} \leq x\Big) \tag{2.184}$$

$$\geq \mathbb{P}\left(N(T)+1 \leq k_T^+, \max_{0\leq k\leq k_T^+} \xi_k \leq x\right), \tag{2.185}$$

because $\left\{N(T)+1 \leq k_T^+, \max_{0\leq k\leq k_T^+} \xi_k \leq x\right\} \subset \left\{S_{N(T)+1} \leq x\right\}$. Further,

$$\mathbb{P}\left(N(T)+1 \leq k_T^+, \max_{0\leq k\leq k_T^+} \xi_k \leq x\right) \tag{2.186}$$

$$=\mathbb{P}\left(\max_{0\leq k\leq k_T^+} \xi_k \leq x\right) - \mathbb{P}\left(N(T)+1 > k_T^+, \max_{0\leq k\leq k_T^+} \xi_k \leq x\right) \tag{2.187}$$

$$\geq \mathbb{P}\left(\max_{0\leq k\leq k_T^+} \xi_k \leq x\right) - \mathbb{P}\big(N(T)+1 > k_T^+\big) \tag{2.188}$$

$$=\mathbb{P}\left(\max_{1\leq k\leq k_T^+} \xi_k \leq x\right) - \mathbb{P}\left(\max_{1\leq k\leq k_T^+} \xi_k \leq x, \xi_0 > \max_{k=1,\ldots,n} \xi_k\right) - \mathbb{P}\big(N(T)+1 > k_T^+\big) \tag{2.189}$$

$$\geq \mathbb{P}\left(\max_{1\leq k\leq k_T^+} \xi_k \leq x\right) - \mathbb{P}\left(\xi_0 > \max_{k=1,\ldots,n} \xi_k\right) - \mathbb{P}\big(N(T)+1 > k_T^+\big). \tag{2.190}$$

We can see from the assumptions that the second term on the r.h.s. tends to zero as $T \to \infty$. In the case of the third term on the r.h.s. convergence to zero also occurs due to

$$\lim_{T\to\infty} \frac{N(T)+1}{\lfloor mT(1+\delta)\rfloor} = \frac{1}{1+\delta} < 1 \quad a.s., \tag{2.191}$$

so by the Dominated Convergence Theorem

$$\mathbb{P}\left(\frac{N(T)+1}{k_T^+} > 1\right) \underset{T\to\infty}{\longrightarrow} 0. \tag{2.192}$$

From the independence of the ξ_k, we get,

$$\mathbb{P}\left(\max_{1\le k\le k_T^+}\xi_k\le x\right)=\mathbb{P}\Big(\xi_1\le x,\dots,\xi_{k_T^+}\le x\Big) \tag{2.193}$$

$$=\mathbb{P}\left(\max_{t<Y_1}X_t\le x\right)^{k_T^+}=G(x)^{k_T^+}, \tag{2.194}$$

so

$$F_T(x)\ge G(x)^{k_T^+}+o(1), \tag{2.195}$$

independently of x. We can say,

$$F_T(x)-G(x)^{mT}\ge G(x)^{k_T^+}-G(x)^{mT}+o(1). \tag{2.196}$$

Now, $G(x)\le 1$ and $\lim_{T\to\infty}\frac{k_T^+}{mT}=1+\delta$, so there exists $T_1\in\mathbb{T}$ s.t. for all $T>T_1$,

$$mT<k_T^+<mT(1+2\delta), \tag{2.197}$$

$$G^{k_T^+}-G^{mT}<G^{(mT)(1+2\delta)}-G^{mT}<2\delta\underset{\delta\to 0}{\longrightarrow}0\quad\text{uniformly in } x. \tag{2.198}$$

Therefore,

$$\liminf_{T\to\infty}\left(\inf_{x\ge 0}\left[F_T(x)-G(x)^{mT}\right]\right)\ge 0. \tag{2.199}$$

Now we look for an upper estimate. We have,

$$F_T(x)=\mathbb{P}\big(N(T)+1>k_T^-,\overline{X}_T\le x\big)+\mathbb{P}\big(N(T)+1\le k_T^-,\overline{X}_T\le x\big) \tag{2.200}$$

$$\le\mathbb{P}\left(N(T)+1>k_T^-,\max_{k=0,1,\dots,k_T^-}\xi_k\le x\right)+\mathbb{P}\big(N(T)+1\le k_T^-\big). \tag{2.201}$$

This last argument hold for the following reason: on the event $N(T)+1>k_T^-$, $N(T)$ cannot be any smaller than k_T^-. This entails $T\ge S_{N(T)}\ge S_{k_T^-}$, and hence,

$$\big\{N(T)+1>k_T^-,\overline{X}_T\le x\big\}\subset\left\{N(T)+1>k_T^-,\max_{k=0,1,\dots,k_T^-}\xi_k\le x\right\}. \tag{2.202}$$

Moreover,

$$\mathbb{P}\left(N(T)+1>k_T^-,\max_{k=0,1,\dots,k_T^-}\xi_k\le x\right)+\mathbb{P}\big(N(T)+1\le k_T^-\big) \tag{2.203}$$

$$\le\mathbb{P}\left(\max_{k=1,\dots,k_T^-}\xi_k\le x\right)+\mathbb{P}\big(N(T)+1\le k_T^-\big). \tag{2.204}$$

By a Dominated Convergence argument, as before, we have

$$\mathbb{P}\big(N(T)+1 \leq k_T^-\big) \underset{T\to\infty}{\longrightarrow} 0, \tag{2.205}$$

which gives us the upper estimate

$$\limsup_{T\to\infty}\left(\sup_{x\geq 0}\big[F_T(x) - G(x)^{mT}\big]\right) \leq 0. \tag{2.206}$$

We conclude that

$$\lim_{T\to\infty}\sup_{x\geq 0}\big|F_T(x) - G(x)^{mT}\big| = 0. \tag{2.207}$$

The proof is complete. □

With this, we have covered all the general theory that is necessary for the proof of our results. We now proceed to state the assumptions under which said results were obtained.

3 Assumptions

Now that the necessary theoretical framework for the proofs of our results has been established, we proceed to introduce the Hawkes processes and the set of assumptions under which our results hold. Hawkes processes are defined through their intensity.

Definition 3.1. A point process N is called a *classical (univariate) Hawkes process* if N admits an $(\mathcal{F}_t)$-intensity given as

$$\lambda(t) = \mu + \int_0^t h(t-u)N(\mathrm{d}u), \quad t \geq 0, \tag{3.1}$$

where μ is a positive constant and h is a nonnegative measurable function on $[0,\infty)$ satisfying $\int_0^\infty h(t)\mathrm{d}t < 1$.

In (3.1), the intensity λ of N contains not only the constant rate μ at which immigrants arrive following a Poisson process, but also an additional term relative to N. For this reason, Hawkes processes are also known as *self-exciting* processes.

The notion of classical Hawkes processes can then be generalized in the following way. Let $\{T_i\}_{i\geq 0}$ be a point process on $[0,\infty)$ with counting process N and $\{D_i\}_{i\geq 0}$, be a sequence of $\{0,1\}$-valued random variables. For any index j such that $D_j = 0$, we say that the point T_j represents an *immigrant*, and if j is such that $D_j = 1$, we say that T_j represents an offspring. Define for any $t \geq 0$, $I(t) := \max\{i;\ T_i \leq t, D_i = 0\}$, i.e. the index of the last immigrant up to time t. Consider a filtration $(\mathcal{F}_t)_{t\geq 0}$ to which N and I are adapted. Additionally, consider a function h and a probability distribution F satisfying the assumptions:

(A0) h is a nonnegative measurable function on $[0,\infty)$ satisfying $\alpha := \int_0^\infty h(t)\mathrm{d}t < 1$.

(B0) F is a probability distribution on $[0,\infty)$ with finite mean $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$ and density f, i.e. $F(x) = \int_0^x f(s)\mathrm{d}s$.

Then, we can define the RHP through its intensity.

Definition 3.2. A point process N is called a *renewal Hawkes process* if N admits the $(\mathcal{F}_t)$-intensity,

$$\lambda(t) = \mu\big(t - T_{I(t)}\big) + \int_0^t h(t-u)N(\mathrm{d}u), \tag{3.2}$$

where h satisfies **(A0)** and μ is a function on $[0,\infty)$ satisfying

$$\mu(t) = \frac{f(t)}{1 - \int_0^t f(s)\mathrm{d}s} \tag{3.3}$$

for the probability density function f in **(B0)**. The function μ is often called the *hazard function*.

Additionally, we introduce the following assumptions:

(A1) The function h is bounded and $h(t) \underset{t\to\infty}{\longrightarrow} 0$.

(C0) F is a spread-out probability distribution on $[0,\infty)$ with finite mean $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$.

Remark. Note that **(B0)** implies **(C0)**.

In the following, Section 4 makes use of assumptions **(A0)** and **(B0)** only. Since Section 5 only treats with renewal processes, assumptions will be limited to **(B0)** and **(C0)**. Section 6 will make use of assumptions **(A0)**,**(B0)** and **(A1)**. Whenever any additional assumptions are needed, it will be indicated accordingly.

4 Cluster representation for renewal Hawkes processes

It was shown in Hawkes–Oakes [16] that the process with intensity (3.1) can be represented as a cluster process on $[0,\infty)$ with an homogeneous Poisson center process of intensity μ and satellite processes given by generalized branching processes. These branching processes consist of inhomogeneous Poisson processes of characteristic intensity h that start at each one of the previous points of the process up to time t. We want to generalize this to the case of the RHP.

The goal of this section is to obtain a cluster representation for the RHP and indicate explicitly what the center and satellite processes are. Our proof for existence of the cluster process is based on a result of Westcott [39] and we will use the uniqueness of predictable intensities of Theorem 2.11 to verify that the proposed construction indeed represents an RHP. Finally, we find the limit process for the RHP at long times and compute its probability generating functional.

Definition 4.1. In the same context as Definition 3.2, let N be a renewal Hawkes process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. The $(\mathcal{F}_t)$-predictable intensity for N is given as

$$\lambda(t) = \mu\big(t - T_{I(t-)}\big) + \int_0^{t-} h(t-u)N(\mathrm{d}u). \tag{4.1}$$

Note that we still have the property (2.17) with the predictable version (4.1) because the integral with respect to the Lebesgue measure stays unaltered by adding one point at t.

Before continuing with the construction of the RHP, it is necessary to review some basic theory of *Galton–Watson processes*.

4.1 Galton–Watson processes

Galton–Watson processes are a particular kind of branching processes that provide simple models for population dynamics. All of the results in this section were consulted in [14].

Definition 4.2. Let $(Z_n)_{n\geq 0}$ be a sequence of integer-valued random variables recursively defined by

$$Z_n = \sum_{k=1}^{Z_{n-1}} X_{n,k}, \quad n \geq 1 \tag{4.2}$$

where $\{X_{n,k} : n, k \geq 1\}$ forms a family of i.i.d. nonnegative integer-valued random variables with common distribution $(p_n)_{n\geq 0}$ and independent of Z_0. For each $n \geq 0$, the random variable Z_n is interpreted as the size of the n-th generation of a given population, and $(Z_n)_{n\geq 0}$ is called a Galton-Watson Process (abbreviated GWP) with offspring distribution $(p_n)_{n\geq 0}$.

Remark. Throughout this thesis, we will assume that in the GWP $Z_0 = 1$ with probability one.

Proposition 4.3. *The probability generating function (p.g.f) of Z_n, $\mathbb{E}\big[s^{Z_n}\big]$, is given by the n-fold composition $\theta^{\circ n}(s) = (\theta \circ \cdots \circ \theta)(s)$, where θ is the p.g.f. of the offspring distribution $(p_n)_{n\geq 0}$. Moreover, if $\mathbb{E}[Z_1] = \rho$, then $\mathbb{E}[Z_n] = \rho^n$.*

Proof. In effect, let $|s| \leq 1$, notice that

$$\mathbb{E}[s^{Z_0}] = s, \quad \mathbb{E}[s^{Z_1}] = \theta(s) = \sum_{n=0}^{\infty} p_n s^n. \tag{4.3}$$

For $n \geq 1$ we have

$$\mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}\left[s^{\sum_{k=0}^{Z_n} X_{n+1,k}}\right] \tag{4.4}$$

$$= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\mathbb{E}\left[s^{\sum_{k=0}^{j} X_{n+1,k}}\right] \tag{4.5}$$

$$= \sum_{j=0}^{\infty} \mathbb{P}(Z_n = j)\left(\theta(s)\right)^j \tag{4.6}$$

$$= \mathbb{E}\left[\theta(s)^{Z_n}\right]. \tag{4.7}$$

By induction we then see that

$$\mathbb{E}\left[s^{Z_{n+1}}\right] = \mathbb{E}\left[\theta(s)^{Z_n}\right] = \theta^{\circ n}\left(\theta(s)\right) \tag{4.8}$$

$$\Rightarrow \mathbb{E}\left[s^{Z_{n+1}}\right] = \theta^{\circ(n+1)}(s) \tag{4.9}$$

The expectation can be obtained by differentiation, for $n \geq 2$,

$$\theta_n'(s) = \mathbb{E}\left[Z_n s^{Z_{n-1}}; Z_n \geq 1\right], \tag{4.10}$$

and $Z_n s^{Z_{n-1}}$ is nonnegative and increasing as $s \uparrow 1$. By the monotone convergence theorem,

$$\theta'(1) = \mathbb{E}\left[Z_n\right] \leq \infty \tag{4.11}$$

On the other hand,

$$\mathbb{E}\left[Z_n\right] = \lim_{s\uparrow 1} \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}\left[s^{Z_n}\right] \tag{4.12}$$

$$= \lim_{s\uparrow 1} \frac{\mathrm{d}}{\mathrm{d}s}\theta^{\circ(n)}(s) \tag{4.13}$$

$$= \lim_{s\uparrow 1} \frac{\mathrm{d}}{\mathrm{d}s}\theta^{\circ(n-1)}(\theta(s)) \tag{4.14}$$

$$= \lim_{s\uparrow 1} \left(\theta^{\circ(n-1)}(\theta(s))\right)' \theta'(s) \tag{4.15}$$

$$= \left(\theta^{\circ(n-1)}(1)\right)' \theta'(1) \tag{4.16}$$

$$= \rho\mathbb{E}\left[Z_{n-1}\right]. \tag{4.17}$$

By induction, we obtain $\mathbb{E}\left[Z_n\right] = \rho^n$. □

Proposition 4.4. *The multivariate joint p.g.f. for $Z_1, \ldots, Z_n$ in the GWP is given by*

$$\mathbb{E}\left[s_1^{Z_1} \cdots s_n^{Z_n}\right] = \theta(s_1\theta(s_2 \cdots \theta(s_{n-1}\theta(s_n)) \cdots)) \tag{4.18}$$

Proof. Consider the filtration

$$\mathcal{F}_n = \sigma\left(Z_0, X_{m,k} : m \leq n, k \in \mathbb{N}\right), \tag{4.19}$$

and notice that Z_n is $\mathcal{F}_n$-measurable $forall$ $n \geq 0$. Computing the expectation

$$\begin{aligned}
\mathbb{E}\left[s_1^{Z_1} \cdots s_n^{Z_n}\right] =& \mathbb{E}\left[\mathbb{E}\left[s_1^{Z_1} \cdots s_{n-1}^{Z_{n-1}} s_n^{Z_n} \,\middle|\, \mathcal{F}_{n-1}\right]\right] &(4.20)\\
=& \mathbb{E}\left[s_1^{Z_1} \cdots s_{n-1}^{Z_{n-1}} \mathbb{E}\left[s_n^{Z_n} \,\middle|\, \mathcal{F}_{n-1}\right]\right] &(4.21)\\
=& \mathbb{E}\left[s_1^{Z_1} \cdots s_{n-1}^{Z_{n-1}} \mathbb{E}\left[s_n^{\sum_{k=0}^{Z_{n-1}} X_{n,k}} \,\middle|\, \mathcal{F}_{n-1}\right]\right] &(4.22)\\
=& \mathbb{E}\left[s_1^{Z_1} \cdots s_{n-1}^{Z_{n-1}} \theta^{Z_{n-1}}(s_n)\right] &(4.23)\\
=& \mathbb{E}\left[s_1^{Z_1} \cdots s_{n-2}^{Z_{n-2}} \left(s_{n-1}\theta(s_n)\right)^{Z_{n-1}}\right] &(4.24)\\
=& \mathbb{E}\left[s_1^{Z_1} \cdots s_{n-2}^{Z_{n-2}} \theta\left(s_{n-1}\theta(s_n)\right)\right] = \ldots &(4.25)\\
=& \theta(s_1\theta(s_2 \cdots \theta(s_{n-1}\theta(s_n)) \cdots)), &(4.26)
\end{aligned}$$

where the last equality was obtained by taking conditional expectation iteratively. □

The total size of the GWP is the sum of the number of individuals from all generations, including the original ancestor. The following proposition allows us to find the p.g.f. for the size of the GWP.

Proposition 4.5. *Let $\phi_n(s)$ be the p.g.f. for the size of the GWP up to the n-th generation $Z_0 + Z_1 + \cdots + Z_n$, in other words,*

$$\phi_n(s) = \mathbb{E}\left[s^{Z_0 + Z_1 + \cdots + Z_n}\right]. \tag{4.27}$$

Then the following recurrence relation is satisfied

$$\phi_{n+1}(s) = s\theta(\phi_n(s)), \quad n \geq 0. \tag{4.28}$$

Moreover, the p.g.f. $\phi(s)$ for the total size of the GWP $\overline{Z} = Z_0 + Z_1 + \ldots$, i.e.

$$\phi(s) = \mathbb{E}\left[s^{\overline{Z}}\right], \tag{4.29}$$

satisfies the relation

$$\phi(s) = s\theta(\phi(s)). \tag{4.30}$$

Proof. Once again, consider the filtration (4.19). Computing the expectation

$$
\begin{aligned}
\phi_{n+1}(s) =& \mathbb{E}\left[s^{Z_0+Z_1+\cdots+Z_{n+1}}\right] && (4.31)\\
=& \mathbb{E}\left[s^{Z_0}\right]\mathbb{E}\left[s^{Z_1+\cdots+Z_{n+1}}\right] && (4.32)\\
=& s\ \mathbb{E}\left[\mathbb{E}\left[s^{Z_1+\cdots+Z_{n+1}}\ \middle|\ \mathcal{F}_n\right]\right] && (4.33)\\
=& s\ \mathbb{E}\left[s^{Z_1}\cdots s^{Z_n}\mathbb{E}\left[s^{Z_{n+1}}\ \middle|\ \mathcal{F}_n\right]\right] && (4.34)\\
=& s\ \mathbb{E}\left[s^{Z_1}\cdots s^{Z_n}\theta^{Z_n}(s)\right] = \dots && (4.35)\\
=& s\ \mathbb{E}\left[\left[s\theta(s\theta(s\cdots\theta(s\theta(s))\cdots))\right]^{Z_1}\right], && (4.36)
\end{aligned}
$$

where the p.g.f. $\theta(s)$ inside the expectation is iterated n times. Comparing this to the p.g.f. from Proposition 4.4, we notice that it corresponds to the joint p.g.f of

$$\phi_n(s) = \mathbb{E}\left[s^{Z_0+Z_1+\cdots+Z_n}\right] = s\mathbb{E}\left[s^{Z_1}\cdots s^{Z_n}\right] = s\,\underbrace{\theta(s\theta(s\cdots\theta(s\theta(s))\cdots))}_{n \text{ times}}, \tag{4.37}$$

thus

$$\phi_{n+1}(s) = s\mathbb{E}\left[\phi_n(s)^{Z_1}\right] = s\theta(\phi_n(s)). \tag{4.38}$$

Notice now that the sequence

$$s^{Z_0+\cdots+Z_n} \tag{4.39}$$

is decreasing in n, dominated by 1 since $|s| \leq 1$, and converges pointwise

$$\lim_{n\to\infty} s^{Z_0+\cdots+Z_n} = s^{Z_0+Z_1+\cdots} = s^{\overline{Z}}. \tag{4.40}$$

Then, by the Dominated Convergence Theorem

$$
\begin{aligned}
\lim_{n\to\infty}\phi_n(s) = \lim_{n\to\infty}& \mathbb{E}\left[s^{Z_0+Z_1+\cdots+Z_n}\right] && (4.41)\\
=& \mathbb{E}\left[\lim_{n\to\infty} s^{Z_0+Z_1+\cdots+Z_n}\right] && (4.42)\\
=& \mathbb{E}\left[s^{\overline{Z}}\right] = \phi(s) && (4.43)
\end{aligned}
$$

Taking limits on both sides of (4.38) and using the continuity of $\theta(s)$

$$
\begin{aligned}
\lim_{n\to\infty}\phi_{n+1}(s) &= \lim_{n\to\infty} s\theta(\phi_n(s)), && (4.44)\\
\phi(s) &= s\theta\left(\lim_{n\to\infty}\phi_n(s)\right), && (4.45)\\
\phi(s) &= s\theta\left(\phi(s)\right), && (4.46)
\end{aligned}
$$

which is the result we wanted. □

If the generation n has zero offspring, we say that the population went extinct at generation n. Let $T := \inf\{n : Z_n = 0\}$, we call T the extinction time. The events $\{Z_n = 0\}$ is the same as the event $\{T \leq n\}$. Define

$$E_n := \theta_n(0) = \mathbb{P}(Z_n = 0) = \mathbb{P}(T \leq n). \tag{4.47}$$

The sequence of events $\{T \leq n\}$ is increasing and $\cup_{n\in\mathbb{N}}\{T \leq n\} = \{T < \infty\}$. From the monotonicity and continuity of the probability measure it is obvious that the sequence $\{E_n\}$ is non decreasing and converges,

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \mathbb{P}(T \leq n) = \mathbb{P}(T < \infty) =: E. \tag{4.48}$$

Proposition 4.6. *Let $E = \mathbb{P}(T < \infty)$, then $E = 1$ for $\rho \leq 1$, while $E < 1$ for $\rho > 1$, where E is the unique root of $\theta(s) = s$ strictly between 0 and 1.*

Proof. There are two trivial cases. One of them is $p_0 = 0$, in which $E = 0$ because Z_n can never become zero. The second case is when $0 < p_0 \leq p_0 + p_1 = 1$, so that $\theta(s)$ is a linear function and $\theta(1) = 1$, so $E = 1$. We assume then that $0 < p_0 \leq p_0 + p_1 < 1$, which means that $\theta(x)$ is strictly convex. The extinction probabilities for the n-th generation satisfy the following recurrence relation

$$E_{n+1} = \theta_{n+1}(0) = \theta^{\circ(n+1)}(0) = \theta(\theta^{\circ(n)}(0)) = \theta(E_n) \tag{4.49}$$

$$\Rightarrow E_{n+1} = \theta(E_n). \tag{4.50}$$

Taking the limit as $n \to \infty$ in (4.50) and using the continuity of θ we obtain

$$E = \theta(E), \tag{4.51}$$

therefore E should be a root of $g(s) := \theta(s) - s$. Let x_0 be the smallest positive root in $(0, 1]$. $\theta(s)$ is increasing for $s > 0$, thus

$$0 < E_1 = \theta(0) < \theta(x_0) = x_0. \tag{4.52}$$

From $E_1 < x_0$ and θ being increasing, we get by induction that

$$0 < E_n < x_0 \quad \text{for all } \ n \in \mathbb{N} \tag{4.53}$$

$$\Rightarrow \lim_{n\to\infty} E_n = x_0. \tag{4.54}$$

Computing some values of g and its derivatives

$$g(1) = 0, \quad g(0) = \theta(0) > 0 \tag{4.55}$$

$$g'(1) = \theta'(1) - 1 = \rho - 1. \tag{4.56}$$

Since θ is convex, θ' is increasing and $g'' = \theta'' \geq 0$. Considering the cases for ρ we have that if $\rho \leq 1$, from (4.56), $g'(1) \leq 0$, thus

$$g'(s) = \theta'(s) - 1 \leq \theta'(1) - 1 \quad \text{for all } \ s \in [0, 1], \tag{4.57}$$

so g will be decreasing, and hence the only possible root is $x_0 = 1$.

If on the other hand $\rho > 1$, $g'(1) > 0$, and by continuity of g', there exists $\delta_1 > 0$ such that $g'(s) > 0 \quad \text{for all} \quad s \in (1-\delta_1, 1]$. In this neighborhood, g is increasing, and since $g(1) = 0$, that means that in the neighborhood $g(x) < 0$. Since $g(0)$ is nonnegative, by the Intermediate Value Theorem, there should be a root s_1 of $g(s)$ in $(0, 1]$. Let us suppose there is another root s_2 in $(0, 1]$ and without loss of generality assume that $s_1 < s_2$. Then we have $g(s_1) = g(s_2) = g(1) = 0$, hence, by Rolle's Lemma, there exist $a, b \in (0, 1]$ with $s_1 < a < s_2 < b < 1$ and $g'(a) = g'(b) = 0$, which implies $\theta'(a) = \theta'(b)$, contradicting the strict convexity of θ. Thus, the root x_0 should be unique in $(0, 1]$, and this finalizes the proof. □

With the last Proposition we finish the exposition of the necessary results from the Galton–Watson theory. We now proceed with the introduction of *cluster processes*.

4.2 Cluster processes

Cluster processes are defined in a general setting. Let $\mathcal{X}$ and $\mathcal{Y}$ be complete separable metric spaces. As in Section 2.1, $(\mathcal{N}^\sharp_\mathcal{X}, \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}))$ denotes the measurable space of counting measures ν on $\mathcal{X}$ which are locally finite with its Borel σ-field $\mathcal{B}(\mathcal{N}^\sharp_\mathcal{X})$. Let also $\mathcal{P}(\mathcal{N}^\sharp_\mathcal{X})$ denote the space of probability measures on $\mathcal{N}^\sharp_\mathcal{X}$. The *convolution* of Π and $\Pi' \in \mathcal{P}(\mathcal{N}^\sharp_\mathcal{X})$ is defined as

$$(\Pi * \Pi')(U) = \int\limits_{\mathcal{N}^\sharp_\mathcal{X} \times \mathcal{N}^\sharp_\mathcal{X}} 1_{(\nu+\nu' \in U)} \Pi(\mathrm{d}\nu)\Pi'(\mathrm{d}\nu') \quad \text{for all } U \in \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X}). \tag{4.58}$$

If we have two independent point processes $N(\cdot)$ and $N'(\cdot)$ on $\mathcal{X}$, we can then use (4.58) to write the law of their sum. Let $\Pi(\cdot) = \mathbb{P}(N \in \cdot)$ and $\Pi'(\cdot) = \mathbb{P}(N' \in \cdot)$, then

$$\Pi * \Pi' = \mathbb{P}(N + N' \in \cdot). \tag{4.59}$$

We can also write an expression for the p.g.fl. If we denote

$$G_\Pi[z] = \mathbb{E}\left[\exp \int \log z(t) N(\mathrm{d}t)\right], \quad \text{and} \quad G_{\Pi'}[z] = \mathbb{E}\left[\exp \int \log z(t) N'(\mathrm{d}t)\right], \tag{4.60}$$

where the expectation is taken w.r.t. Π and Π' respectively, then the p.g.fl. of $N + N'$ is given as

$$G_{\Pi*\Pi'}[z] = \mathbb{E}\left[\exp \int \log z(t)(N + N')(\mathrm{d}t)\right] \tag{4.61}$$

$$= G_\Pi[z] G_{\Pi'}[z]. \tag{4.62}$$

Definition 4.7 ([10, Ch.6 p.165])**.** A (symbolic) *measurable family of point processes* on $\mathcal{X}$ is a family $\{N(\cdot \mid y) : y \in \mathcal{Y}\}$ where for all $y \in \mathcal{Y}$, $N(\cdot \mid y)$ is a point process on $\mathcal{X}$, and for all $U \in \mathcal{B}(\mathcal{N}^\sharp_\mathcal{X})$ the function

$$y \longmapsto \mathbb{P}(N(\cdot \mid y) \in U) \tag{4.63}$$

is $\mathcal{B}(\mathcal{Y})$-measurable.

The construction of a cluster process involves two components: a point process N_c of cluster centers whose realization consists of the points $\{y_i\}_{i\geq 0} \subset \mathcal{Y}$, and a family of point processes on $\mathcal{X}$, namely $\{N_s(\cdot \mid y); y \in \mathcal{Y}\}$, whose superposition constitute the observed process. We formalize this idea through the convolution in $\mathcal{P}(\mathcal{N}^\sharp_{\mathcal{X}})$.

Definition 4.8. Let N_c be a point process on $\mathcal{Y}$ and $\{N_s(\cdot \mid y) : \ y \in \mathcal{Y}\}$ a measurable family of point processes on $\mathcal{X}$. (The family $\{N_s(\cdot \mid y) : \ y \in \mathcal{Y}\}$ is considered to be mutually independent and to be independent of N_c.) Then, the *independent cluster process* on $\mathcal{X}$, with *center process* N_c and *satellite processes* $\{N_s(\cdot \mid y) : y \in \mathcal{Y}\}$, which we denote by

$$N(\cdot) = \int_{\mathcal{Y}} N_s(\cdot \mid y) N_c(dy) = \sum_{y \in N_c(\cdot)} N_s(\cdot \mid y), \tag{4.64}$$

is defined in law as

$$\mathbb{P}(N \in U) = \int_{N^\sharp_{\mathcal{Y}}} \mathbb{P}(N_s(\cdot \mid \mu) \in U)\, \mathbb{P}(N_c \in \mathrm{d}\mu)\,, \quad U \in \mathcal{B}(\mathcal{N}^\sharp_{\mathcal{X}}), \tag{4.65}$$

where $\mathbb{P}(N_s(\cdot \mid \mu) \in U)$ for $\mu = \sum_i \delta_{y_i} \in \mathcal{N}^\sharp_{\mathcal{Y}}$ is defined as the infinite convolution

$$\mathbb{P}(N_s(\cdot \mid \mu) \in U) = (\Pi_{y_1} * \Pi_{y_2} * \cdots)\,(U) \quad \text{for } U \in \mathcal{B}(\mathcal{N}^\sharp_{\mathcal{X}}), \tag{4.66}$$

with $\Pi_y(U) = \mathbb{P}(N_s(\cdot \mid y) \in U)$ for $U \in \mathcal{B}(\mathcal{N}^\sharp_{\mathcal{X}})$ and $y \in \mathcal{Y}$.

We now give an expression for the p.g.fl. of the independent cluster process in the following Theorem.

Theorem 4.9. *Let N be an independent cluster process with center process N_c and satellite processes $\{N_s(\cdot \mid y) : \ y \in \mathcal{Y}\}$. Let $G_c[z]$ denote the p.g.fl. of the center process and $G_s[z \mid y]$ the p.g.fl. of $N_s(\cdot \mid y)$. Then $G[z]$, the p.g.fl. of $N(\cdot)$, is given by*

$$G[z] = G_c[G_s[z \mid \cdot]] \quad \textit{for any } z \in \Xi. \tag{4.67}$$

Proof. In the definition of the p.g.fl. we take expectation with respect to the law of $N(\cdot)$ given by (4.65), and we obtain

$$G[z] = \mathbb{E}\left[\exp \int \log z(x) N(\mathrm{d}x)\right] \tag{4.68}$$

$$= \int_{\mathcal{N}^\sharp_{\mathcal{Y}}} \left(\mathbb{E}\left[\exp \int_{\mathcal{X}} \log z(x) N_s(\mathrm{d}x \mid \mu)\right]\right) \mathbb{P}(N_c \in \mathrm{d}\mu) \tag{4.69}$$

$$= \int_{\mathcal{N}^\sharp_{\mathcal{Y}}} \left(\prod_{y \in \mu(\cdot)} G_s[z \mid y]\right) \mathbb{P}(N_c \in \mathrm{d}\mu) \tag{4.70}$$

$$= \mathbb{E}\left[\prod_{y \in N_c(\cdot)} G_s[z \mid y]\right] \tag{4.71}$$

$$= G_c[G_s[z \mid \cdot]], \tag{4.72}$$

concluding the proof. □

A necessary and sufficient condition for the superposition (4.64) to define a point process on $\mathcal{X}$, in which case we say that the independent cluster process exists, is that for every bounded set $B \in \mathcal{B}(\mathcal{X})$,

$$N(B) = \int_{\mathcal{Y}} N_s(B \mid y) N_c(dy) = \sum_{y \in N_c(\cdot)} N_s(B \mid y) < \infty \quad \text{a.s.} \tag{4.73}$$

Under the assumption that the process has almost surely finite clusters, i.e.

$$N_s(\mathcal{X} \mid y) < \infty \quad \text{a.s. for all } y \in \mathcal{Y}, \tag{4.74}$$

equivalent conditions for verifying (4.73) were presented in Westcott [39], namely the following theorem and its corollary.

Theorem 4.10 ([39, Theorem 3])**.** *The independent cluster process N exists if and only if for every bounded set $B \in \mathcal{B}(\mathcal{X})$,*

$$\int_{\mathcal{Y}} \mathbb{P}(N_s(B \mid y) > 0)\, N_c(\mathrm{d}y) < \infty \quad a.s. \tag{4.75}$$

We give a more detailed proof than that in [39].

Proof. Let $B \in \mathcal{B}(\mathcal{X})$ bounded, then the probability generating function (p.g.f.) for the random variable $N(B)$ is obtained by evaluating the p.g.fl. of N in the function $\xi(u) = 1 - (1 - z)1_B(u)$ where z is a constant in $(0, 1)$. We obtain

$$\mathbb{E}\left[z^{N(B)}\right] = \mathbb{E}\left[\exp\left\{-\int_{\mathcal{Y}} Q_B(z \mid y) N_c(\mathrm{d}y)\right\}\right], \tag{4.76}$$

where we defined $Q_B(z \mid y) := -\log \mathbb{E}\left[z^{N_s(B|y)}\right]$. Considering now the sequence $\{z_n\}_{n\in\mathbb{N}}$ given by $z_n = 1 - \frac{1}{n}$ we can compute

$$\begin{aligned}
\mathbb{P}(N(B) < \infty) &= \lim_{n\to\infty} \mathbb{E}\left[z_n^{N(B)}\right] && (4.77)\\
&= \lim_{n\to\infty} \mathbb{E}\left[\exp\left\{-\int_{\mathcal{Y}} Q_B(z_n \mid y) N_c(\mathrm{d}y)\right\}\right] && (4.78)\\
&= \mathbb{E}\left[\exp\left\{-\lim_{n\to\infty}\int_{\mathcal{Y}} Q_B(z_n \mid y) N_c(\mathrm{d}y)\right\}\right]. && (4.79)
\end{aligned}$$

The last equality is the result of applying the Monotone Convergence Theorem to the increasing sequence $\{-Q_B(z_n \mid y)\}_{n\in\mathbb{N}}$ and using the continuity of the exponential function. Hence $\mathbb{P}(N(B) < \infty) = 1$ if and only if

$$\lim_{n\to\infty} \int Q_B(z_n \mid y) N_c(\mathrm{d}y) = 0 \quad \text{a.s.} \tag{4.80}$$

Since $0 \le Q_B(z_n \mid y) \le 1$ and $Q_B(z_n \mid y) \downarrow 0$ as $n \to \infty$, this is equivalent, by the Dominated Convergence Theorem, to

$$\int Q_B(z \mid y) N_c(\mathrm{d}y) < \infty \quad \text{for some } 0 < z < 1 \quad \text{a.s.} \tag{4.81}$$

Notice that if $0 < a \le x \le 1$,

$$1 - x \le -\log x \le c(a)(1-x) \quad \text{for } c(a) := \frac{-\log a}{1-a}. \tag{4.82}$$

Since z is a constant in $(0,1)$,

$$\mathbb{E}z^{N_s(B|y)} \ge \mathbb{E}\left[\prod_{y' \in N_c(\cdot)} z^{N_s(B|y')}\right] = \mathbb{E}z^{N(B)}, \tag{4.83}$$

and $Q_B(z \mid y) := -\log \mathbb{E}z^{N_s(B|y)}$, we have from (4.82)

$$1 - \mathbb{E}z^{N_s(B|y)} \le Q_B(z \mid y) \le \widetilde{c}(z)\left(1 - \mathbb{E}z^{N_s(B|y)}\right), \tag{4.84}$$

for $\widetilde{c}(z) := c\left(\mathbb{E}z^{N(B)}\right)$. Hence, condition (4.81) holds if and only if

$$\int \left(1 - \mathbb{E}z^{N_s(B|y)}\right) N_c(\mathrm{d}y) < \infty \quad \text{for some } 0 < z < 1 \quad \text{a.s.} \tag{4.85}$$

Additionally,

$$\sum_{n=0}^{\infty} \mathbb{P}(N_s(B \mid y) > n) z^n = \sum_{n=0}^{\infty} z^n \sum_{m=n+1}^{\infty} \mathbb{P}(N_s(B \mid y) = m) \tag{4.86}$$

$$= \sum_{m=1}^{\infty} \mathbb{P}(N_s(B \mid y) = m) \sum_{n=0}^{m-1} z^n \tag{4.87}$$

$$= \sum_{m=1}^{\infty} \mathbb{P}(N_s(B \mid y) = m) \frac{1 - z^m}{1 - z} \tag{4.88}$$

$$= \frac{1}{1-z}\left(1 - \mathbb{E}z^{N_s(B|y)}\right), \tag{4.89}$$

since $1 - z^0 = 0$. Hence condition (4.85) holds if and only if

$$\sum_{n=0}^{\infty} \left\{ \int \mathbb{P}(N_s(B \mid y) > n) N_c(\mathrm{d}y) \right\} z^n < \infty \quad \text{for some } 0 < z < 1 \quad \text{a.s.} \tag{4.90}$$

Since $\mathbb{P}(N_s(B \mid y) > n)$ is decreasing in n, we see that condition (4.90) holds if and only if

$$\int \mathbb{P}(N_s(B \mid y) > 0) N_c(\mathrm{d}y) < \infty \quad \text{a.s.} \tag{4.91}$$

□

Corollary 4.11 ([39, Corollary 3.3]). *Let $\mathcal{X}, \mathcal{Y} = \mathbb{R}$ and assume the following conditions:*

(i) $\sup_t \mathbb{E}[N_c(I - t)] < \infty$ *for all bounded interval I.*

(ii) $N_s(\cdot \mid t) \overset{\mathrm{d}}{=} N_s(\cdot - t \mid 0)$ *for all* $t \in \mathbb{R}$.
(iii) $\mathbb{E}[N_s(\mathbb{R} \mid 0)] < \infty$.

Then (4.75) *is satisfied.*

We give a simpler proof than that in [39].

Proof. Notice that

$$\mathbb{E}[N_s(I \mid t)] \geq \mathbb{E}[N_s(I \mid t); N_s(I \mid t) > 0] \geq \mathbb{P}(N_s(I \mid t) > 0)\,, \tag{4.92}$$

because $N_s(I \mid t)$ is integer-valued. From (ii), we have

$$\mathbb{E}[N_s(I \mid t)] = \mathbb{E}[N_s(I - t \mid 0)]\,. \tag{4.93}$$

Taking expectation in (4.75) and using (4.92) and (4.93) we have

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{P}(N_s(I \mid t) > 0)\mathbb{E}[N_c(\mathrm{d}t)] &\leq \int_{\mathbb{R}} \mathbb{E}[N_s(I - t \mid 0)]\,\mathbb{E}[N_c(\mathrm{d}t)] && (4.94)\\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{I-t}(u)\mathbb{E}[N_s(\mathrm{d}u \mid 0)] \right) \mathbb{E}[N_c(\mathrm{d}t)] && (4.95)\\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{I-u}(t)\mathbb{E}[N_c(\mathrm{d}t)] \right) \mathbb{E}[N_s(\mathrm{d}u \mid 0)] && (4.96)\\
&= \int_{\mathbb{R}} \mathbb{E}[N_c(I - u)]\,\mathbb{E}[N_s(\mathrm{d}u \mid 0)] && (4.97)\\
&\leq \mathbb{E}[N_s(\mathbb{R} \mid 0)] \sup_t \mathbb{E}[N_c(I - t)] < \infty, && (4.98)
\end{aligned}$$

from (i) and (iii), which implies (4.75). □

4.3 The center process

In the RHP, immigration is given by a renewal process, which naturally we take as our center process. Let $\tau, \tau_1, \tau_2, \ldots,$ be positive i.i.d. random variables whose probability distribution function

$$F(t) := \mathbb{P}(\tau \leq t), \tag{4.99}$$

satisfies the assumption **(B0)**, and let $\tau_0 = 0$.

The epochs of this zero-delayed renewal process correspond to the partial sums $S_0 = 0$, $S_n = \tau_1 + \cdots + \tau_n$, and we denote its associated counting process as $N_R(t) := \sum_i 1_{\{S_i \leq t\}}$. For each $n \geq 0$ and $x \geq 0$, the distribution of S_n is $\mathbb{P}(S_n \leq x) = F^{*n}(x)$, where

$$F^{*0}(x) = \delta_0(x), \quad F^{*(n+1)}(x) = \int_0^x F^{*n}(x - y)F(\mathrm{d}y). \tag{4.100}$$

The mean number of events up to time t is given by the corresponding renewal function,

$$\Phi(t) = \sum_{n \geq 0} F^{*n}(t). \tag{4.101}$$

Since the distribution function F has a density f, the *renewal measure* $\Phi(\mathrm{d}t)$ is absolutely continuous w.r.t. the Lebesgue measure with density $\varphi(t) = \sum_{n\geq 1} f^{*n}(t)$ for all $t \geq 0$ [1, Proposition V.2.7].

Denote by $(\mathcal{F}_t^R)_{t\geq 0}$ the augmentation of the natural filtration of the renewal process $\sigma(N_R(s); 0 \leq s \leq t)$, and take the hazard function $\mu(t)$ as in (3.3). We use Theorem 2.14 to see that

$$\lambda_R(t) := \mu(t - S_{N_R(t-)-1}), \tag{4.102}$$

is a predictable $(\mathcal{F}_t^R)$-intensity of N_R (a more detailed proof is given later in Section 6).

We will also want to consider a delayed renewal process where S_0 is replaced by a positive random variable $\widehat{S}_0$ independent of $\tau_1, \tau_2, \dots$, with distribution function F_0 not necessarily equal to F. The partial sums $\widehat{S}_n = \widehat{S}_0 + \tau_1 + \cdots + \tau_n$ have the associated counting process $\widehat{N}_R(t) = \sum_i 1_{\{\widehat{S}_i \leq t\}}$, where the distribution of $\widehat{S}_n$ for $n \geq 1$ is given by $\mathbb{P}(\widehat{S}_n \leq x) = F_0 * F^{*n}(x)$ for $x \geq 0$.

4.4 The satellite processes

One feature of the RHP is that in addition to immigrants being able to generate offspring, these offspring themselves can generate further offspring. Thus, offspring points could be described as higher-level center processes. In order to represent this structure, we use a notation similar to Neyman–Scott [29]. The renewal process will be named a zero-th order center process $N_c^{(0)}(\cdot) := N_R(\cdot)$.

Let $\left\{N_s^{(n)}(\cdot \mid t);\ t \in [0,\infty)\right\}_{n\geq 1}$ be a sequence of measurable families of point processes which is considered to be i.i.d. and to be independent of $N_c^{(0)}$, such that $N_s^{(n)}(\cdot \mid t)$ has the same law as $N_s(\cdot \mid t)$ whose p.g.fl. is given by

$$\mathbb{E}\left[\prod_{x\in N_s(\cdot|t)} z(x)\right] = \exp\left(\int_0^\infty (z(x+t)-1)\, h(x)\mathrm{d}x\right), \tag{4.103}$$

with h satisfying **(A0)**. If we take $z(x) = e^{-\lambda}$ for $\lambda > 0$, we have

$$\mathbb{E}\left[e^{-\lambda N_s([0,\infty)|t)}\right] = \exp\left((e^{-\lambda}-1)\int_0^\infty h(x)\mathrm{d}x\right) = e^{-\alpha\left(1-e^{-\lambda}\right)}, \tag{4.104}$$

which shows that $N_s([0,\infty) \mid t) \overset{\mathrm{d}}{=} \mathrm{Poi}(\alpha)$. More generally, if we substitute the function

$$z(x) = \begin{cases} e^{-\lambda_i} & (x \in (a_i, b_i]), \\ 1 & \left(x \notin \cup_{i=1}^k (a_i, b_i]\right), \end{cases} \tag{4.105}$$

into (4.103), we obtain

$$\mathbb{E}\left[e^{-\lambda_1 N_s((a_1,b_1]|t)}\cdots e^{-\lambda_k N_s((a_k,b_k]|t)}\right] = \prod_{i=1}^{k}\exp\left((e^{-\lambda_i}-1)\int_{a_i}^{b_i} h(x-t)\mathrm{d}x\right). \qquad (4.106)$$

This shows that $\{N_s([a_i,b_i)\mid t)\}_{i=1}^k$ is mutually independent and

$$N_s([a_i,b_i)\mid t) \overset{\mathrm{d}}{=} \mathrm{Poi}\left(\int_{a_i}^{b_i} h(x-t)\mathrm{d}x\right), \qquad (4.107)$$

or in other words, $N_s(\cdot\mid t)$ is an inhomogeneous Poisson process with intensity $h(x-t)\mathrm{d}x$, where we understand $h(x)=0$ for $x<0$. In particular, we see that $\{N_s(\cdot\mid t): t\geq 0\}$ is a measurable family of point processes.

Given that there is a center at $t_0 \geq 0$, we construct higher-level center processes $N_c^{(n)}$, for $n\geq 1$, from a superposition of the processes $N_s^{(n)}$ with the following recursive structure:

$$N_c^{(0)}(\cdot\mid t_0) := \delta_{t_0}, \quad N_c^{(n+1)}(\cdot\mid t_0) = \sum_{t\in N_c^{(n)}(\cdot|t_0)} N_s^{(n+1)}(\cdot\mid t), \qquad (4.108)$$

where $N_c^{(0)}(\cdot\mid t_0)$ is the original immigrant at t_0 and $N_c^{(n)}(\cdot\mid t_0)$ represents its n-th generation offspring. We define as well some processes of interest, namely, the total number of n-th generation descendants,

$$N_c^{(n)}(\cdot) = \sum_{t_0\in N_R(\cdot)} N_c^{(n)}(\cdot\mid t_0), \qquad (4.109)$$

and the complete offspring of the immigrant at t_0 (including the immigrant),

$$N_c(\cdot\mid t_0) = \sum_{n\geq 0} N_c^{(n)}(\cdot\mid t_0). \qquad (4.110)$$

We take the processes defined as in (4.110) as the satellite processes of our construction for a center located at t_0. Finally, the RHP is given by the superposition:

$$N(\cdot) = \int_0^\infty N_c(\cdot\mid t)N_R(\mathrm{d}t) = \sum_{t_0\in N_R(\cdot)}\sum_{n\geq 0} N_c^{(n)}(\cdot\mid t_0). \qquad (4.111)$$

Note that (4.111) can also be written as

$$N(\cdot) = \sum_{n\geq 0}\sum_{t_0\in N_R(\cdot)} N_c^{(n)}(\cdot\mid t_0) = N_R(\cdot) + \sum_{n\geq 1} N_c^{(n)}(\cdot). \qquad (4.112)$$

4.5 Validity of the construction

We are concerned with whether our construction of the RHP in fact represents a valid cluster process. In the following Theorem we show that with the assumptions made for its construction, it is a valid definition.

Theorem 4.12. *If h is a function satisfying* **(A0)**, *and $N_R(\cdot)$ is a renewal process satisfying* **(B0)**, *then the cluster process defined as in* (4.111) *exists and has a.s. finite clusters.*

Proof. Using Corollary 4.11 , the existence of the cluster process is proved if we can show that conditions (i)-(iii) hold.

Claim (i) can be obtained by using the fact that

$$\mathbb{E}[N_R(t+a) - N_R(t)] \leq \Phi(a), \quad \text{for all } a > 0, \tag{4.113}$$

(see for example [1, Sec. V, Theorem 2.4, p.146]). Let I be any bounded interval in $[0,\infty)$ and write $|I|$ for its Lebesgue measure. Then, from (4.113) we can see that $\mathbb{E}[N_R(I)] \leq \Phi(|I|)$ and we can conclude that,

$$\sup_{t\geq 0} \mathbb{E}[N_R(I-t)] \leq \sup_{t\geq 0} \Phi(|I-t|) = \Phi(|I|) < \infty \tag{4.114}$$

Claim (ii) follows from the construction of the satellites as superposition of inhomogeneous Poisson processes with p.g.fl. (4.103) that originate at previous points of the process and the observation (4.107).

Finally, to prove claim (iii), let $t_0 \geq 0$ and consider $N_c(\cdot \mid t_0)$ a cluster with center at t_0. Let us call $Z_n := N_c^{(n)}([0,\infty) \mid t_0)$ for $n \geq 0$ and $\overline{Z} := N_c([0,\infty) \mid t_0)$ so that Z_n and $\overline{Z}$ represent respectively the number of n-th generation points and the total number of points in the cluster. These random variables satisfy:

$$Z_n = \sum_{k=0}^{Z_{n-1}} X_{n,k},$$

$N_c^{(n)}(\cdot \mid t_0) = \{Y_1^{(n)}, \ldots, Y_{Z_n}^{(n)}\}$ for $n \geq 0$, and $X_{n,k} := N_s^{(n)}([0,\infty) \mid Y_k^{(n-1)})$. This shows that the variables Z_n form a Galton–Watson process with offspring density h, and that the number of points per generation follows a Poisson distribution of parameter α. Then $\overline{Z}$ is given by the total size of the Galton–Watson process. From the Galton–Watson theory [14, Theorem 6.1, p.7] we know that $\mathbb{P}(\overline{Z} < \infty) = 1$ if $\alpha < 1$, and the p.g.f. of the cluster size $\pi(u) = \mathbb{E}\left[u^{\overline{Z}}\right]$, $0 < u < 1$ [14, Section 13.2, p.32] satisfies,

$$\pi(u) = u \exp\{\alpha[\pi(u) - 1]\},$$

from which we can conclude that $\pi'(1-) = \mathbb{E}\left[\,\overline{Z}\,\right] = \frac{1}{1-\alpha}$. We then see that the three assumptions needed are satisfied, which proves the existence of the cluster process representation for the RHP. □

4.6 Verification of the intensity

In this section we show that the process constructed in the previous section admits the desired intensity of an RHP. Let us consider a cluster process $N(\cdot) = \{T_1, T_2, \ldots\}$ given by (4.111). Define the random variables

$$D_i = \begin{cases} 1 & (\text{if } T_i \in N_R(\cdot)), \\ 0 & (\text{otherwise}), \end{cases} \tag{4.115}$$

and the function $I(t) = \max\{i;\ T_i \le t, D_i = 0\}$ as before. We construct the filtration $(\mathcal{F}_t)_{t\ge0}$ by the augmentation of the natural filtration $(\mathcal{F}_t^0)_{t\ge0}$ defined as

$$\mathcal{F}_t^0 = \sigma\big(N_c^{(n)}((a,b])\,;\ 0 \le a \le b \le t,\ n = 0,1,2,\dots\big), \quad t \ge 0. \tag{4.116}$$

Notice that in the definition of the intensity (4.1), the term $\mu\big(t - T_{I(t-)}\big)$ is only affected by the terms that come from the renewal process. This is because the points $T_{I(t-)}$ all correspond to immigrants. Since $N_R(t) = \min\{i : S_i \le t\}$, let us then denote $N_R(\cdot) = \{T_i : D_i = 0\} = \{S_1, S_2, \dots\}$, and notice then that

$$\mu(t - T_{I(t-)}) = \mu(t - S_{N_R(t-)-1}), \quad t \ge 0. \tag{4.117}$$

Consider now an arbitrary $(\mathcal{F}_t)$-predictable process $C(u) = 1_A 1_{(r,t]}(u)$ for $A \in \mathcal{F}_r$ and $0 \le r \le t$. Then

$$\mathbb{E}\left[\int C(u) N(\mathrm{d}u)\right] = \mathbb{E}\left[1_A \int_r^t N_R(\mathrm{d}u)\right] + \mathbb{E}\left[1_A \int_r^t \sum_{n\ge0} N_c^{(n+1)}(\mathrm{d}u)\right]. \tag{4.118}$$

Since $\lambda_R(t)$ is an $(\mathcal{F}_t)$-intensity of N_R, the first term on the R.H.S. equals

$$\mathbb{E}\left[1_A \int_r^t N_R(\mathrm{d}u)\right] = \mathbb{E}\left[1_A \int_r^t \mu(u - S_{N_R(u-)-1})\mathrm{d}u\right] = \mathbb{E}\left[1_A \int_r^t \mu(u - T_{I(u-)})\mathrm{d}u\right]. \tag{4.119}$$

We also have,

$$\mathbb{E}\left[1_A \int_r^t \sum_{n\ge0} N_c^{(n+1)}(\mathrm{d}u)\right] = \mathbb{E}\left[1_A \int_r^t \sum_{n\ge0} \sum_{x\in N_c^{(n)}(\cdot)} N_s^{(n+1)}(\mathrm{d}u \mid x)\right] \tag{4.120}$$

$$= \sum_{n\ge0} \mathbb{E}\left[1_A \sum_{x\in N_c^{(n)}(\cdot)} N_s^{(n+1)}((r,t] \mid x)\right]. \tag{4.121}$$

Because of the independence property of a Poisson point process, we have independence of $\left\{N_s^{(n+1)}((r,t] \mid x);\ t>r, x\geq 0\right\}$ from $\mathcal{F}_r \vee \sigma\left(N_c^{(n)}\right)$, and by (4.107) we obtain

$$\begin{aligned}\sum_{n\geq 0}\mathbb{E}\left[1_A \sum_{x\in N_c^{(n)}(\cdot)} N_s^{(n+1)}((r,t]\mid x)\right] &= \sum_{n\geq 0}\mathbb{E}\left[1_A \sum_{x\in N_c^{(n)}(\cdot)} \int_r^t h(u-x)\mathrm{d}u\right] &(4.122)\\ &= \mathbb{E}\left[1_A \int_0^\infty \left(\int_r^t h(u-x)\mathrm{d}u\right)\sum_{n\geq 0} N_c^{(n)}(\mathrm{d}x)\right] &(4.123)\\ &= \mathbb{E}\left[1_A \int_0^\infty \left(\int_r^t h(u-x)\mathrm{d}u\right) N(\mathrm{d}x)\right] &(4.124)\\ &= \mathbb{E}\left[1_A \int_r^t \left(\int_0^u h(u-x)N(\mathrm{d}x)\right)\mathrm{d}u\right]. &(4.125)\end{aligned}$$

From the chain of equalities we obtain the identity

$$\mathbb{E}\left[\int C(u)N(\mathrm{d}u)\right] = \mathbb{E}\left[\int C(u)\left\{\mu(u-T_{I(u-)}) + \int_0^{u-} h(u-x)N(\mathrm{d}x)\right\}\mathrm{d}u\right], \quad (4.126)$$

which may be extended to all $(\mathcal{F}_t)$-predictable processes C. Thus, we can say that N admits the predictable $(\mathcal{F}_t)$-intensity

$$\lambda(t) := \mu(t-T_{I(t-)}) + \int_0^{t-} h(t-x)N(\mathrm{d}x), \quad t\geq 0. \quad (4.127)$$

4.7 Probability generating functional for the RHP

In this section we investigate the p.g.fl. of the RHP. The difficulty of finding the complete p.g.fl. lies on the fact that the renewal process is not a finite point process. We begin with the p.g.fl. of the renewal process.

Denote $p_n(T) := \mathbb{P}(N_R((0,T]) = n)$ for $T\geq 0$ and any nonnegative integer n, and for any $z\in\Xi$ and $T\geq 0$ define

$$z^T(t) := \begin{cases} z(t) & (t\leq T), \\ 1 & (t>T). \end{cases} \quad (4.128)$$

Since $\left\{z^T\right\}_{T\geq 0}$ is decreasing in T for $z\in\Xi$, and all z^T are dominated by the constant 1, it is a consequence of the Dominated Convergence Theorem that

$$G_R[z] = \lim_{T\to\infty} G_R[z^T], \quad (4.129)$$

where

$$G_R[z^T] = \sum_{n=0}^{\infty} \mathbb{E}\left[\prod_{x\in N_R(\cdot)} z^T(x) \,;N_R((0,T]) = n \right] \tag{4.130}$$

$$= p_0(T) + \sum_{n=1}^{\infty} \mathbb{E}[z(S_1)\cdots z(S_n)\,;N_R((0,T]) = n] \tag{4.131}$$

$$= p_0(T) + \sum_{n=1}^{\infty} p_n(T) \int_0^T \int_0^{T-s_1} \cdots \int_0^{T-s_{n-1}} z(s_1)\cdots z(s_n) f(s_1)\cdots f(s_n - s_{n-1})\mathrm{d}s_n \cdots \mathrm{d}s_1, \tag{4.132}$$

but this expression cannot be further simplified in general.

Now we focus on the p.g.fl. for the satellite processes. A cluster whose center is located at t_0, for some $t_0 \geq 0$, is formed by the immigrant that originated it together with all the generations of its offspring. The following relation has been already established in Hawkes–Oakes [16], and we provide a proof using our construction.

Theorem 4.13. *Let $t_0 \geq 0$. The p.g.fl. for a cluster starting at t_0, namely*

$$G_c[z \mid t_0] = \mathbb{E}\left[\exp \int_0^{\infty} \log z(t) N_c(\mathrm{d}t \mid t_0) \right], \tag{4.133}$$

satisfies the functional relation

$$G_c[z \mid t_0] = z(t_0) \exp\left\{ \int_0^{\infty} \left(G_c[z(x + \cdot) \mid t_0] - 1\right) h(x)\mathrm{d}x \right\}. \tag{4.134}$$

Proof. Let $t_0 \geq 0$. We want the p.g.fl. for the cluster

$$N_c(\cdot \mid t_0) = \sum_{n\geq 0} N_c^{(n)}(\cdot \mid t_0), \tag{4.135}$$

Let us call $G_c^{(n)}$ the p.g.fl. of the cluster up to generation n, namely

$$G_c^{(n)}[z \mid t_0] = \mathbb{E}\left[\exp \int_0^{\infty} \log z(t) \sum_{i=0}^{n} N_c^{(i)}(\mathrm{d}t \mid t_0) \right]. \tag{4.136}$$

Denote $\mathcal{F}_0 := \{\Omega, \emptyset\}$ and $\mathcal{F}_n := \sigma\Big(N_c^{(1)}(\cdot \mid t_0), \ldots, N_c^{(n)}(\cdot \mid t_0)\Big)$. We have,

$$G_c^{(n+1)}[z \mid t_0] = z(t_0)\mathbb{E}\left[\exp\left\{ \int \log z(x) \sum_{i=0}^{n} N_c^{(i+1)}(\mathrm{d}x \mid t_0) \right\} \right]. \tag{4.137}$$

Note that the above expectation can be written as,

$$\mathbb{E}\left[\exp\left\{ \int \log z(x) \sum_{i=0}^{n} N_c^{(i+1)}(\mathrm{d}x \mid t_0) \right\} \right] \tag{4.138}$$

$$= \mathbb{E}\left[\exp\left\{ \sum_{i=0}^{n} \sum_{y\in N_c^{(i)}(\cdot|t_0)} \int \log z(x) N_s^{(i+1)}(\mathrm{d}x \mid y) \right\} \right] \tag{4.139}$$

Using that the processes $N_s^{(i+1)}(\cdot \mid y)$ are an i.i.d. family and independent from $N_c^{(i)}(\cdot \mid t_0)$, we can rewrite the above expectation by recursively applying the tower property,

$$\mathbb{E}\left[\exp\left\{\sum_{i=0}^{n}\sum_{y\in N_c^{(i)}(\cdot|t_0)}\int \log z(x)N_s^{(i+1)}(\mathrm{d}x \mid y)\right\}\right] \tag{4.140}$$

$$=\mathbb{E}\left[\prod_{i=0}^{n}\prod_{y\in N_c^{(i)}(\cdot|t_0)}\mathbb{E}\left[\exp\left\{\int \log z(x)N_s^{(i+1)}(\mathrm{d}x \mid y)\right\}\middle|\mathcal{F}_i\right]\right] \tag{4.141}$$

$$=\mathbb{E}\left[\prod_{i=0}^{n}\prod_{y\in N_c^{(i)}(\cdot|t_0)}\mathbb{E}\left[\exp\left\{\int \log z(x)N_s^{(i+1)}(\mathrm{d}x \mid y)\right\}\right]\right] \tag{4.142}$$

$$=\mathbb{E}\left[\prod_{i=0}^{n}\prod_{y\in N_c^{(i)}(\cdot|t_0)}\mathbb{E}\left[\exp\left\{\int \log z(x+y)N_s^{(n+1)}(\mathrm{d}x \mid 0)\right\}\right]\right] \tag{4.143}$$

$$=\mathbb{E}\left[\prod_{i=0}^{n}\prod_{y\in N_c^{(i)}(\cdot|t_0)}\exp\left\{\int \log z(x+y)N_s^{(n+1)}(\mathrm{d}x \mid 0)\right\}\right] \tag{4.144}$$

$$=\mathbb{E}\left[\exp\left\{\sum_{i=0}^{n}\sum_{y\in N_c^{(i)}(\cdot|t_0)}\int \log z(x+y)N_s^{(n+1)}(\mathrm{d}x \mid 0)\right\}\right]. \tag{4.145}$$

Writing the summation above as an integral w.r.t. the counting measure $N_c^{(i)}(\cdot \mid t_0)$ and using Fubini's Theorem we get,

$$\mathbb{E}\left[\exp\left\{\sum_{i=0}^{n}\sum_{y\in N_c^{(i)}(\cdot|t_0)}\int \log z(x+y)N_s^{(n+1)}(\mathrm{d}x \mid 0)\right\}\right] \tag{4.146}$$

$$=\mathbb{E}\left[\exp\left\{\int\int \log z(x+y)\sum_{i=0}^{n}N_c^{(i)}(\mathrm{d}y \mid t_0)N_s^{(n+1)}(\mathrm{d}x \mid 0)\right\}\right] \tag{4.147}$$

$$=\mathbb{E}\left[\prod_{x\in N_s^{(n+1)}(\cdot|0)}\mathbb{E}\left[\exp\left\{\int \log z(x+y)\sum_{i=0}^{n}N_c^{(i)}(\mathrm{d}y \mid t_0)\right\}\right]\right] \tag{4.148}$$

$$=\mathbb{E}\left[\prod_{x\in N_s^{(n+1)}(\cdot|0)}G_c^{(n)}[z(x+\cdot) \mid t_0]\right] \tag{4.149}$$

$$=\mathbb{E}\left[\prod_{x\in N_s^{(n+1)}(\cdot|0)}G_c^{(n)}[z_x(\cdot) \mid t_0]\right], \tag{4.150}$$

where in the last expression we introduced $z_x(\cdot) := z(x+\cdot)$. We recognize the p.g.fl. of the process $N_s(\cdot \mid 0)$ and from equation (4.107), we have

$$G_c^{(n+1)}[z \mid t_0] = z(t_0) G_s\big[G_c^{(n)}[z_\cdot \mid t_0] \mid 0\big] \tag{4.151}$$

$$= z(t_0) \exp\left\{ \int_0^\infty \big(G_c^{(n)}[z_x \mid t_0] - 1\big)\, h(x)\mathrm{d}x \right\}. \tag{4.152}$$

Now, we want to take the limit as $n \to \infty$. First notice that for any measurable function z such that $0 \le z(y) \le 1$ for all $y \ge 0$ and $1 - z$ vanishes outside a bounded set, it holds that

$$\lim_{n\to\infty} \prod_{i=0}^{n} \prod_{x \in N_c^{(i)}(\cdot|t_0)} z(x) = \prod_{i\ge 0} \prod_{x \in N_c^{(i)}(\cdot|t_0)} z(x). \tag{4.153}$$

Since,

$$\mathbb{E}\left[\prod_{i\ge 0} \prod_{x \in N_c^{(i)}(\cdot|t_0)} z(x) \right] < \infty, \tag{4.154}$$

we have from the Dominated Convergence Theorem,

$$\lim_{n\to\infty} G_c^{(n)}[z \mid t_0] = \lim_{n\to\infty} \mathbb{E}\left[\prod_{i=0}^{n} \prod_{x \in N_c^{(i)}(\cdot|t_0)} z(x) \right] = \mathbb{E}\left[\prod_{i\ge 0} \prod_{x \in N_c^{(i)}(\cdot|t_0)} z(x) \right] = G_c[z \mid t_0], \tag{4.155}$$

this means that F_n converges to F pointwise. Now, since $0 \le G_c^{(n)}[z_x \mid t_0] \le 1$ for all $n \ge 0$ and $\left|G_c^{(n)}[z_x \mid t_0] - 1\right| h(x) \le h(x)$ with $\int_0^\infty h(x)\mathrm{d}x < \infty$, from the Dominated Convergence Theorem we get,

$$G_c[z \mid t_0] = \lim_{n\to\infty} G_c^{(n)}[z \mid t_0] = \lim_{n\to\infty} z(t_0) \exp\left\{ \int_0^\infty \big(G_c^{(n)}[z_x \mid t_0] - 1\big)\, h(x)\mathrm{d}x \right\} \tag{4.156}$$

$$= z(t_0) \exp\left\{ \int_0^\infty \left(G_c[z_x \mid t_0] - 1\right) h(x)\mathrm{d}x \right\}, \tag{4.157}$$

which concludes the proof. □

4.8 The stationary RHP

As pointed out in Lemma 2.17, under assumption **(B0)**, we can obtain a stationary version of the renewal process $\widehat{N}_R$ by considering an appropriate delay distribution F_0 which has a density given as

$$f_0(t) = m(1 - F(t)) \quad t \ge 0. \tag{4.158}$$

In the stationary case, the p.g.fl. for the renewal process can be computed on the entirety of $[0, \infty)$. For this, we use the general formula [10, (5.5.4) in Sec. V. p.146] to expand the p.g.fl. of the point process $\widehat{N}_R$ as

$$\widehat{G}_R[z] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^k} (z(x_1) - 1) \cdots (z(x_k) - 1) M_{[k]}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k), \tag{4.159}$$

where the *factorial moment measures* $M_{[k]}(\cdot)$ for $\widehat{N}_R$ are given for $A_1, \dots, A_r \in \mathcal{B}(\mathbb{R})$ and nonnegative integers $k_1, \dots, k_r$ such that $k_r \geq 1$ and $k_1 + \cdots + k_r = k$ as

$$M_{[k]}(A_1^{(k_1)} \times \cdots \times A_r^{(k_r)}) = \mathbb{E}\big[[N(A_1)]^{[k_1]} \cdots [N(A_r)]^{[k_r]}\big], \tag{4.160}$$

with the factorial powers defined as

$$n^{[k]} = \begin{cases} n(n-1)\cdots(n-k+1) & (k = 0, \dots, n), \\ 0 & (k > n), \end{cases} \tag{4.161}$$

for any nonnegative integer n. We use the formula [10, (5.4.15) in p.139] for the stationary renewal process and see that the factorial measures of $\widehat{N}_R$ have densities on $x_1 < \cdots < x_k$ given by [10, (5.4.15) in Sec. V. p.139]

$$M_{[k]}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k) = m\mathrm{d}x_1 \varphi(x_2 - x_1)\mathrm{d}x_2 \cdots \varphi(x_k - x_{k-1})\mathrm{d}x_k, \tag{4.162}$$

where we recall that $m^{-1} = \mathbb{E}[\tau]$ and $\varphi = \sum_{n \geq 1} f^{*n}$. We can rewrite the integral in (4.159) using the factorial densities to obtain

$$\begin{aligned} \widehat{G}_R[z] = 1+ &\sum_{k=1}^{\infty} \frac{m}{k!} \int_0^{\infty} \int_{x_1}^{\infty} \cdots \int_{x_{k-1}}^{\infty} [z(x_1) - 1] \cdots \\ &\cdots [z(x_k) - 1]\mathrm{d}x_1 \varphi(x_2 - x_1) \cdots \varphi(x_k - s_{k-1})\mathrm{d}x_k. \end{aligned} \tag{4.163}$$

Consider now a RHP in which we have replaced the center process N_R for its stationary version $\widehat{N}_R$. We denote this process by

$$\widehat{N}(\cdot) := \int_0^{\infty} N_c(\cdot \mid t)\widehat{N}_R(\mathrm{d}t) = \sum_{t_0 \in \widehat{N}_R(\cdot)} \sum_{n \geq 0} N_c^{(n)}(\cdot \mid t_0), \tag{4.164}$$

whose p.g.fl. $\widehat{G}[z] = \widehat{G}_R[G_c[z \mid \cdot]]$ is given as

$$\begin{aligned} \widehat{G}[z] = 1+ &\sum_{k=1}^{\infty} \frac{m}{k!} \int_0^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{k-1}}^{\infty} (G_c[z \mid t_1] - 1) \cdots \\ &\cdots (G_c[z \mid t_k] - 1)\mathrm{d}t_1 \varphi(t_2 - t_1) \cdots \varphi(t_k - t_{k-1})\mathrm{d}t_k. \end{aligned} \tag{4.165}$$

Since an RHP with p.g.fl. given by (4.165) has a stationary center process and its satellite processes satisfy $N_c(\cdot \mid t_0) \overset{\mathrm{d}}{=} N_c(\cdot - t_0 \mid 0)$ for $t_0 \geq 0$, from Vere-Jones [37] we can conclude that the process is stationary and we call it the *stationary renewal Hawkes process.*

Example 4.14. In particular, consider the renewal process $\widehat{N}_R(\cdot)$ to be an homogeneous Poisson process of constant intensity $\mu > 0$. In this case, the renewal density is constant $\varphi(t) = \mu$ for all $t \geq 0$, and $m = \mu$. We have

$$\widehat{G}[z] = 1 + \sum_{k=1}^{\infty} \frac{\mu^k}{k!} \int_0^{\infty} \cdots \int_0^{\infty} (G_c[z \mid s_1] - 1) \cdots (G_c[z \mid s_k] - 1) \mathrm{d}s_1 \cdots \mathrm{d}s_k \tag{4.166}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\mu^k}{k!} \left(\int_0^{\infty} (G_c[z \mid s] - 1) \mathrm{d}s \right)^k \tag{4.167}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_0^{\infty} \mu(G_c[z \mid s] - 1) \mathrm{d}s \right)^k \tag{4.168}$$

$$= \exp\left\{ \int_0^{\infty} \mu(G_c[z \mid s] - 1) \mathrm{d}s \right\}. \tag{4.169}$$

Now observe that the p.g.fl. for the satellite processes satisfies

$$G_c[z \mid t] = G_c[z_t \mid 0] = G_c[z(t + \cdot) \mid 0], \tag{4.170}$$

so we get

$$\widehat{G}[z] = \exp\left\{ \int_0^{\infty} \mu(G_c[z(s + \cdot) \mid 0] - 1) \, \mathrm{d}s \right\}, \tag{4.171}$$

which is the p.g.fl. obtained by Hawkes–Oakes in [16] for the classical Hawkes process.

Finally, let us come back to the general case. We can relate the RHP with the stationary RHP in the limit through the following result.

Theorem 4.15. *Let N be an RHP and $\widehat{N}$ be the stationary version* (4.164). *Then, under assumptions **(A0)** and **(B0)**,*

$$N(\cdot + t) \xrightarrow[t \to \infty]{d} \widehat{N}(\cdot), \tag{4.172}$$

where $\mathcal{N}_{\mathbb{R}}^{\sharp}$ is equipped with the topology of vague convergence.

The proof of this Theorem follows from the convergence of the renewal process to its stationary version (see for example [1, Sec. VI. Example 2a]),

$$N_R(\cdot + t) \xrightarrow[t \to \infty]{d} \widehat{N}_R(\cdot). \tag{4.173}$$

Heuristically, we could say that

$$N(\cdot + t) = \int_{\mathbb{R}} N_c(\cdot + t \mid y) N_R(\mathrm{d}y) \tag{4.174}$$

$$\stackrel{\mathrm{d}}{=} \int_{\mathbb{R}} N_c(\cdot \mid y - t) N_R(\mathrm{d}y) \tag{4.175}$$

$$= \int_{\mathbb{R}} N_c(\cdot \mid y) N_R(\mathrm{d}y + t) \tag{4.176}$$

$$\xrightarrow[t \to \infty]{d} \int_{\mathbb{R}} N_c(\cdot \mid y) \widehat{N}_R(\mathrm{d}y). \tag{4.177}$$

This result can be formalized by the convergence of the p.g.fl. Let Ξ_c denote the class of continuous functions $z : \mathbb{R} \to (0,1]$ such that $1 - z$ vanishes outside a bounded set. Then, to prove (4.172) it is enough to show convergence of the p.g.fl. of N to that of $\widehat{N}$ for $z \in \Xi_c$ (c.f. [11, Proposition 11.1.VIII]), which we do as follows.

Proof. Let $G[z]$, $\widehat{G}[z]$ be the p.g.fls. of N, $\widehat{N}$ respectively, and $G_R[z]$, $\widehat{G}_R[z]$, the p.g.fls. for N_R and $\widehat{N}_R$ respectively. Notice that the p.g.fl. of $N(\cdot + t)$ is given by $G[z(\cdot - t)]$, then for $z \in \Xi_c$,

$$G[z(\cdot - t)] = G_R[G_c[z_{-t} \mid \cdot\,]] \tag{4.178}$$

$$= G_R[G_c[z \mid \cdot - t]]\,. \tag{4.179}$$

Note as well that $z \in \Xi_c$ implies

$$\widetilde{z}(\cdot) := G_c[z \mid \cdot] \in \Xi_c, \tag{4.180}$$

since for $t_0 \geq 0$ and $z \in \Xi$,

$$\lim_{t\to t_0} G_c[z \mid t] = \lim_{t\to t_0} \mathbb{E}\left[\exp \int_0^\infty \log z(s) N_c(\mathrm{d}s \mid t)\right] \tag{4.181}$$

$$= \lim_{t\to t_0} \mathbb{E}\left[\exp \int_0^\infty \log z(s+t) N_c(\mathrm{d}s \mid 0)\right], \tag{4.182}$$

and the integrals are nonpositive, so the exponential functions inside the expectation are dominated by the constant 1. Applying the Dominated Convergence Theorem once and using the continuity of the exponential function yields

$$\lim_{t\to t_0} G_c[z \mid t] = \mathbb{E}\left[\exp\left\{\lim_{t\to t_0} \int_0^\infty \log z(s+t) N_c(\mathrm{d}s \mid 0)\right\}\right], \tag{4.183}$$

and $N_c([0,\infty) \mid 0)$ is a.s. finite, thus applying the Dominated Convergence Theorem once more and using the continuity of the logarithm and z, yields

$$\lim_{t\to t_0} G_c[z \mid t] = \mathbb{E}\left[\exp \int_0^\infty \log z(s+t_0) N_c(\mathrm{d}s \mid 0)\right], \tag{4.184}$$

$$= \mathbb{E}\left[\exp \int_0^\infty \log z(s) N_c(\mathrm{d}s \mid t_0)\right] \tag{4.185}$$

$$= G_c[z \mid t_0]. \tag{4.186}$$

And from (4.173) we have

$$G_R[\widetilde{z}(\cdot - t)] \underset{t\to\infty}{\longrightarrow} \widehat{G}_R[\widetilde{z}] \quad \text{for } \widetilde{z} \in \Xi_c. \tag{4.187}$$

In summary

$$G[z(\cdot - t)] \underset{t\to\infty}{\longrightarrow} \widehat{G}[z] \quad \text{for all } z \in \Xi_c, \tag{4.188}$$

which proves (4.172). □

This last result implies that the p.g.fl. for the stationary RHP can be used as an approximation to the p.g.fl. of the general RHP as $t \to \infty$.

As we have seen throughout this section, many properties of the RHP depend on the properties of the corresponding renewal process. It is of particular interest for us the asymptotic behavior of the RHP. Because of this, the next Section is dedicated to finding necessary asymptotic results for renewal processes.

5 Decay rates for renewal processes

Since there is an imbedded renewal process in an RHP, as was shown in Section 4, it is only natural that some of the results on the asymptotic behavior of renewal processes will be of aid for establishing limit theorems for the RHP. The purpose of this section is twofold. First, we want to establish power law decay rates for the Key Renewal Theorem under the assumption of existence of moments. In particular, we are interested in the case of a spread out inter-arrival distribution. We obtain these results by means of a coupling argument as was done in the proof of the following Theorem (see for example [1, Theorem VII.2.10]).

Theorem 5.1 (Lund–Meyn–Tweedie [27]). *Assume that the distribution F is spread out with finite mean $m^{-1} := \int_0^\infty xF(\mathrm{d}x) < \infty$ and that for some $\eta > 0$, $\int_0^\infty e^{\eta x}F(\mathrm{d}x) < \infty$. Take $0 < \epsilon < \eta$. If the function z in (2.87) is measurable with $z(x) = O(e^{-\delta x})$ as $x \to \infty$ for some $\delta > \epsilon$, then*

$$\Phi * z(x) = m \int_0^\infty z(y)\mathrm{d}y + O(e^{-\epsilon x}) \quad \text{as } x \to \infty. \tag{5.1}$$

Secondly, we want to study the convergence in distribution of the recurrence times and compensator of a renewal process as elements of the càdlàg space $D([0,1])$. We do so by noting that these processes evolve in cycles between renewals and satisfy the *regenerative property* of Section 2.4.

We proceed then to state our results. The following Theorem improves the error term in Theorem 5.1.

Theorem 5.2. *Suppose that F satisfies* **(C0)** *and $\mathbb{E}[\tau_1^s] = \int_0^\infty x^s\, F(\mathrm{d}x) < \infty$ for $s \geq 2$. Let $z : [0,\infty) \to [0,\infty)$ be a measurable function that is integrable, bounded, and $z(x) = O(x^{-r})$ as $x \to \infty$ for $r > 1$. Then, for $q = s - 1$,*

$$\Phi * z(x) = m \int_0^\infty z(y)\mathrm{d}y + O\big(x^{\max\{1-r,-q\}}\big) \quad \text{as } x \to \infty. \tag{5.2}$$

Moreover, if $z(x) = o(x^{-r})$ as $x \to \infty$, then (5.2) holds with $o\big(x^{\max\{1-r,-q\}}\big)$ instead of $O\big(x^{\max\{1-r,-q\}}\big)$.

The next result establishes the speed of convergence in distribution for the compensator of the renewal process.

Theorem 5.3. *Let N be a zero-delayed renewal process satisfying* ***(B0)***. *For any $p > 0$, the convergence in distribution*

$$\left(\frac{1}{T^p}\left[\Lambda(Tv) - \Lambda(S_{N(Tv)-1})\right]\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} 0 \tag{5.3}$$

holds in the Skorokhod topology.

Finally, we have the following Theorem that establishes the speed of convergence for the recurrence times.

Theorem 5.4. *Let N be a renewal process satisfying $\mathbb{E}[\tau_1^s] = \int_0^\infty x^s\ F(\mathrm{d}x) < \infty$ for $s \geq 1$. Then, the convergence in distribution*

$$\left(\frac{1}{T^{1/s}}(A_{Tv}, B_{Tv})\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} (0,0) \tag{5.4}$$

holds in the Skorokhod topology.

5.1 Coupling for renewal processes

Throughout this section, we will suppose that **(C0)** holds and we will review some known consequences of this assumption. Given a finite signed measure μ on the measurable space $(\Omega, \mathcal{F})$, denote by $\|\mu\|_{\text{t.v.}}$ the total variation norm

$$\|\mu\|_{\text{t.v.}} := \sup_{B_1,B_2\in\mathcal{F}} \left(\mu(B_1) - \mu(B_2)\right). \tag{5.5}$$

Note that for measures, the total variation reduces to $\|\mu\|_{\text{t.v.}} = \mu(\Omega)$.

Consider two stochastic processes $\{X'_t\}_{t\geq 0}$, $\{X''_t\}_{t\geq 0}$, with the same state space and defined a priori on different probability spaces. By a *coupling* of X', X'', we mean a pair $\left(\widetilde{X}', \widetilde{X}''\right)$ and an associated random time $\mathcal{T}$ (*coupling time*), defined on a common probability space with

$$\widetilde{X}' \stackrel{\text{d}}{=} X', \quad \widetilde{X}'' \stackrel{\text{d}}{=} X'', \tag{5.6}$$

and such that

$$\widetilde{X}'_t = \widetilde{X}''_t, \quad \text{for all } t \geq \mathcal{T}. \tag{5.7}$$

We want to make use of the *coupling inequality.* Let $\{X'_t\}_{t\geq 0}$, $\{X''_t\}_{t\geq 0}$ be stochastic processes and $\mathcal{T} \leq \infty$ be a random time defined on a common probability space such that $X'_t = X''_t$ for all $t \geq \mathcal{T}$. Then,

$$\|\mathbb{P}(\theta_t X' \in \cdot) - \mathbb{P}(\theta_t X'' \in \cdot)\|_{t.v.} \leq 2\mathbb{P}(\mathcal{T} > t), \tag{5.8}$$

where θ_t stands for the shift operator given for any $t \geq 0$ as

$$\{\theta_t X\}_s = X_{t+s}, \tag{5.9}$$

for any stochastic process X.

For two arbitrary probability distributions λ and μ, we define a measure $\lambda \wedge \mu$ as

$$\lambda \wedge \mu(\cdot) := \int_{\cdot} (f \wedge g) \mathrm{d}(\lambda + \mu), \tag{5.10}$$

where f and g are the densities of μ and ν with respect to $\mu + \nu$ given by the Radon–Nikodym derivatives

$$f = \frac{\mathrm{d}\lambda}{\mathrm{d}(\lambda + \mu)}, \quad g = \frac{\mathrm{d}\mu}{\mathrm{d}(\lambda + \mu)}. \tag{5.11}$$

Then we have

$$\lambda \wedge \mu(\cdot) = \int_{\cdot} (\widetilde{f} \wedge \widetilde{g}) \mathrm{d}\nu, \tag{5.12}$$

if $\lambda(\cdot) = \int_{\cdot} \widetilde{f} \mathrm{d}\nu$ and $\mu(\cdot) = \int_{\cdot} \widetilde{g} \mathrm{d}\nu$ for an arbitrary dominating measure ν. Notice that

$$\int (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu = 1 - 1 = 0, \tag{5.13}$$

implies

$$\int_{\{\widetilde{f} > \widetilde{g}\}} (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu = -\int_{\{\widetilde{f} \leq \widetilde{g}\}} (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu. \tag{5.14}$$

As a consequence,

$$\begin{aligned}
\|\lambda - \mu\|_{\mathrm{t.v.}} &= \int \left| \widetilde{f} - \widetilde{g} \right| \mathrm{d}\nu && (5.15)\\
&= \int_{\{\widetilde{f} > \widetilde{g}\}} (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu - \int_{\{\widetilde{f} \leq \widetilde{g}\}} (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu && (5.16)\\
&= 2 \int_{\{\widetilde{f} > \widetilde{g}\}} (\widetilde{f} - \widetilde{g}) \mathrm{d}\nu && (5.17)\\
&= 2 \int (\widetilde{f} - \widetilde{f} \wedge \widetilde{g}) \mathrm{d}\nu && (5.18)\\
&= 2 (1 - \|\lambda \wedge \mu\|_{\mathrm{t.v.}}). && (5.19)
\end{aligned}$$

The following Lemma is sometimes referred to as *maximal coupling.*

Lemma 5.5. *Given two probability distributions F, G on a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, there exist random variables X, Y defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that X has distribution F, Y has distribution G, and*

$$\|F - G\|_{\mathrm{t.v.}} = 2\mathbb{P}(X \neq Y). \tag{5.20}$$

Proof. Write $\delta := \|F \wedge G\|_{\text{t.v.}}$ and $H = \delta^{-1}(F \wedge G)$ and define the distributions

$$F' = \frac{F - \delta H}{1 - \delta}, \quad G' = \frac{G - \delta H}{1 - \delta}. \tag{5.21}$$

It is clear that

$$F = \delta H + (1 - \delta)F', \quad G = \delta H + (1 - \delta)G'. \tag{5.22}$$

Now we take independent random variables ξ, X', Y' and Z defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that X' has distribution F', Y' has distribution G', Z has distribution H, and $\xi \sim \text{Bernoulli}(\delta)$. Define

$$X := \xi Z + (1 - \xi)X', \text{ and } Y := \xi Z + (1 - \xi)Y', \tag{5.23}$$

and notice that X has distribution F, Y has distribution G, and

$$\mathbb{P}(X \neq Y) = \mathbb{P}(\xi = 0) = 1 - \delta = \frac{1}{2}\|F - G\|_{t.v.}, \tag{5.24}$$

by (5.19). ☐

5.2 Construction of the coupling

We follow the construction of the coupling presented in [36]. First, we have the following Lemma for the forward recurrence time B_t that has been introduced in Definition 2.16.

Lemma 5.6. *For a zero-delayed spread out renewal process, there exist positive constants b and d such that the distributions of the B_t with $t \geq d$ have a common uniform component on $(0, b)$. That is, for some $\delta \in (0, 1)$ and all $t \geq d$,*

$$\mathbb{P}(u < B_t \leq b) \geq \delta \frac{v - u}{b}, \quad 0 < u < v < b. \tag{5.25}$$

We give a more detailed version of the proof found in [1, Lemma VII.2.8].

Proof. From Lemma 2.21 we have that for some $n_0 \geq 1$, F^{*n_0} has a uniform component on an interval, so there exist constants $0 < p < q$ and $\eta > 0$ s.t. $F^{*n_0}(v) - F^{*n_0}(u) \geq \eta(v - u)$ for $0 < p < u < v < q$. Let

$$b = \frac{(q - p)}{2}, \qquad a = \frac{(p + q)}{2}. \tag{5.26}$$

When $p < z < a$, $0 < u < v < b$, we have $(u + z, v + z) \subset (p, q)$, and hence from Lemma 2.24 we have

$$\mathbb{P}(u < B_t \leq b) = \int_0^t F((t + u - s, t + b - s])\, \Phi(\mathrm{d}s). \tag{5.27}$$

Note that

$$\Phi(\mathrm{d}s) = \sum_{n\geq 0} F^{*n}(\mathrm{d}s) \geq \sum_{n\geq n_0-1} F^{*n}(\mathrm{d}s) = F^{*(n_0-1)} * \Phi(\mathrm{d}s), \tag{5.28}$$

from which we can say that

$$\begin{aligned}
&\mathbb{P}(u < B_t \leq b) && (5.29)\\
&= \int_0^\infty 1_{[0,t]}(s_1) F((t+u-s_1, t+b-s_1])\,\Phi(\mathrm{d}s_1) && (5.30)\\
&= \int_0^\infty \int_0^\infty 1_{(t+u,t+b]}(s_1+s_2) 1_{[0,t]}(s_1) F(\mathrm{d}s_2)\Phi(\mathrm{d}s_1) && (5.31)\\
&\geq \int_0^\infty \int_0^\infty 1_{(t+u,t+v]}(s_1+s_2) 1_{[0,t]}(s_1) F(\mathrm{d}s_2)(F^{*(n_0-1)} * \Phi)(\mathrm{d}s_1) && (5.32)\\
&= \int_0^\infty \int_0^\infty \int_0^\infty 1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{[0,t]}(s_3+s_4) F(\mathrm{d}s_2) F^{*(n_0-1)}(\mathrm{d}s_3)\Phi(\mathrm{d}s_4). && (5.33)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{(t-p,t-a]}(s_4) && (5.34)\\
\geq &1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{[0,t-a]}(s_4) 1_{[0,v+p]}(s_3), && (5.35)\\
\geq &1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{[0,t]}(s_3+s_4), && (5.36)
\end{aligned}$$

hence

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_0^\infty 1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{[0,t]}(s_3+s_4) F(\mathrm{d}s_2) F^{*(n_0-1)}(\mathrm{d}s_3)\Phi(\mathrm{d}s_4) && (5.37)\\
&\geq \int_0^\infty \int_0^\infty \int_0^\infty 1_{(t+u,t+v]}(s_2+s_3+s_4) 1_{(t-p,t-a]}(s_4) F(\mathrm{d}s_2) F^{*(n_0-1)}(\mathrm{d}s_3)\Phi(\mathrm{d}s_4) && (5.38)\\
&= \int_0^\infty \int_0^\infty 1_{(t+u,t+v]}(s_4+s_5) 1_{(t-p,t-a]}(s_4) F^{*n_0}(\mathrm{d}s_5)\Phi(\mathrm{d}s_4) && (5.39)\\
&= \int_{t-p}^{t-a} F^{*n_0}((t+u-s_4, t+v-s_4])\Phi(\mathrm{d}s_4). && (5.40)
\end{aligned}$$

Using (5.27) and (5.40), we obtain the inequality

$$\begin{aligned}
\mathbb{P}(u < B_t \leq b) &\geq \int_{t-p}^{t-a} F^{*n_0}((t+u-s, t+v-s])\Phi(\mathrm{d}s) && (5.41)\\
&\geq \int_p^a F^{*n_0}((u+z, v+z])\,\Phi(t-\mathrm{d}z) && (5.42)\\
&\geq \int_p^a \eta(v+z-u-z)\,\Phi(t-\mathrm{d}z) && (5.43)\\
&= \eta(v-u)[\Phi(t-a)-\Phi(t-p)]\,. && (5.44)
\end{aligned}$$

Using Blackwell's renewal theorem, we have

$$\Phi(t-a) - \Phi(t-p) \underset{t\to\infty}{\longrightarrow} m(a-p) = mb. \tag{5.45}$$

So, for $0 < \delta < \min\{1, m\eta b^2\}$, there exists $d > 0$ such that for all $t \geq d$,

$$\Phi(t-a) - \Phi(t-p) \geq \frac{\delta}{b\eta}, \tag{5.46}$$

thus, for all $t \geq d$,

$$\mathbb{P}(u < B_t \leq b) \geq \delta \frac{v-u}{b}. \tag{5.47}$$

The proof is complete. $\square$

We proceed with the construction of the coupling for the forward recurrence time of the renewal process.

Lemma 5.7 (Chapter 10, Theorem 6.2 from [36])**.** *Let $\{S_n\}_{n\geq 0}$ be a zero-delayed renewal process satisfying **(C0)**, and $B = \{B_t\}_{t\geq 0}$ its forward recurrence time. Let $\{\widehat{S}_n\}_{n\geq 0}$ be the stationary version of the renewal process whose forward recurrence time $\widehat{B} = \{\widehat{B}_t\}_{t\geq 0}$ has stationary distribution Π. Then, the underlying probability space can be extended to support a coupling $(S', \widehat{S}')$ of S and $\widehat{S}$, and a geometric random variable σ such that the coupling event occurs in a σ number of trials.*

Proof. Let the positive constants b, d and δ be as in Lemma 5.6 so that for all $t \geq d$, the distribution of B_t has a common component δU where U has uniform distribution μ on $(0, b)$. Define a Markov process $(\eta_k, \widehat{\eta}_k)_{k\geq 0}$ in $[0,\infty) \times [0,\infty)$ in the following way:

$$(\eta_0, \widehat{\eta}_0) := (S_0, \widehat{S}_0), \text{ and } (\eta_k, \widehat{\eta}_k) := (S_{N_{L_k -}}, \widehat{S}_{\widehat{N}_{L_k -}}),\ k \geq 1, \tag{5.48}$$

where

$$L_k := \eta_k \vee \widehat{\eta}_k + d. \tag{5.49}$$

Then, since B is a time-homogeneous strong Markov process, we can see that conditionally on $(\eta_k, \widehat{\eta}_k) = (s, \widehat{s})$ the random variables

$$\beta_{k+1} := \eta_{k+1} - L_k, \quad \widehat{\beta}_{k+1} := \widehat{\eta}_{k+1} - L_k, \quad k \geq 0, \tag{5.50}$$

satisfy

$$\mathbb{P}(\beta_{k+1} \in A, \widehat{\beta}_{k+1} \in \widehat{A} \mid (\eta_k, \widehat{\eta}_k) = (s, \widehat{s})) = \mathbb{P}(B_{((\widehat{s}-s)_+ + d)-} \in A)\mathbb{P}(B_{((s-\widehat{s})_+ + d)-} \in \widehat{A}). \tag{5.51}$$

Thus, from Lemma 5.6,

$$\mathbb{P}((\beta_{k+1}, \widehat{\beta}_{k+1}) \in \cdot \mid (\eta_k, \widehat{\eta}_k)) \geq \delta^2 \mu \otimes \mu,\ k \geq 0. \tag{5.52}$$

Using [36, Chapter 3, Corollary 5.1], we can extend the underlying probability space to support 0-1 random variables $I_0, I_1, \ldots$ such that for $k \geq 0$,

$$(S, \widehat{S}, I_0, I_1, \ldots, I_{k-1}) \perp\!\!\!\perp I_k, \quad \text{given } ((\eta_k, \widehat{\eta}_k), (\eta_{k+1}, \widehat{\eta}_{k+1})), \tag{5.53}$$

and

$$\mathbb{P}(I_k = 1 \mid \eta_k, \widehat{\eta}_k) = \delta^2, \tag{5.54}$$
$$\mathbb{P}((\beta_{k+1}, \widehat{\beta}_{k+1}) \in \cdot \mid (\eta_k, \widehat{\eta}_k) = (s, \widehat{s}), I_k = 1) = \mu \otimes \mu. \tag{5.55}$$

Fix an $m \geq 1$. Because of (5.53) and the fact that $((\eta_i, \widehat{\eta}_i), (\eta_{i+1}, \widehat{\eta}_{i+1}))$ is a measurable mapping of $(\eta_k, \widehat{\eta}_k)_{k=0}^m$, we have that for $0 \leq i < m$

$$(S, \widehat{S}, I_0, I_1, \dots, I_{i-1}) \perp\!\!\!\perp I_i \quad \text{given } (\eta_k, \widehat{\eta}_k)_{k=0}^m. \tag{5.56}$$

In particular, for the m-tail of the Markov sequence it holds that

$$(\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \perp\!\!\!\perp I_i \quad \text{given } \left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{i-1}\right), \tag{5.57}$$

thus,

$$(\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \perp\!\!\!\perp I_{m-1} \quad \text{given } \left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{m-2}\right), \tag{5.58}$$
$$(\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \perp\!\!\!\perp I_{m-2} \quad \text{given } \left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{m-3}\right), \tag{5.59}$$
$$\vdots \tag{5.60}$$
$$(\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \perp\!\!\!\perp I_0 \quad \text{given } (\eta_k, \widehat{\eta}_k)_{k=0}^m, \tag{5.61}$$

which combined with the Markov property of $(\eta_k, \widehat{\eta}_k)_{k\geq 1}$ we can use to deduce that

$$(\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \perp\!\!\!\perp \left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{m-1}\right) \quad \text{given } (\eta_m, \widehat{\eta}_m). \tag{5.62}$$

Using (5.53) on I_m we obtain that

$$\left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{m-1}\right) \perp\!\!\!\perp I_m \quad \text{given } \left((\eta_m, \widehat{\eta}_m), (\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1}\right), \tag{5.63}$$

which in addition to (5.62) gives

$$\left((\eta_k, \widehat{\eta}_k)_{k=0}^m, I_0, \dots, I_{m-1}\right) \perp\!\!\!\perp \left(I_m, (\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1}\right) \quad \text{given } (\eta_m, \widehat{\eta}_m). \tag{5.64}$$

Once again, by using (5.53) repeatedly we note that for each $n \geq 1$

$$(\eta_k, \widehat{\eta}_k, I_k)_{k=0}^m \perp\!\!\!\perp I_{m+n} \quad \text{given } \left((\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1}, I_{m+1}, \dots, I_{m+n-1}\right), \tag{5.65}$$
$$(\eta_k, \widehat{\eta}_k, I_k)_{k=0}^m \perp\!\!\!\perp I_{m+n-1} \quad \text{given } \left((\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1}, I_{m+1}, \dots, I_{m+n-2}\right), \tag{5.66}$$
$$\vdots \tag{5.67}$$
$$(\eta_k, \widehat{\eta}_k, I_k)_{k=0}^m \perp\!\!\!\perp I_{m+1} \quad \text{given } (\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1}. \tag{5.68}$$

We note from (5.64) that

$$(\eta_k, \widehat{\eta}_k, I_k)_{k=0}^m \perp\!\!\!\perp (\eta_{m+k}, \widehat{\eta}_{m+k})_{k\geq 1} \quad \text{given } (\eta_m, \widehat{\eta}_m, I_m). \tag{5.69}$$

We can then conclude that

$$(\eta_k, \widehat{\eta}_k, I_k)_{k=0}^m \perp\!\!\!\perp (\eta_{m+k}, \widehat{\eta}_{m+k}, I_{m+k})_{k\geq 1} \quad \text{given } (\eta_m, \widehat{\eta}_m, I_m), \tag{5.70}$$

hence, $(\eta_k, \widehat{\eta}_k, I_k)_{k\geq 0}$ is a Markov process. Furthermore, from (5.54) and (5.64) we know that the random variables $I_1, I_2, \dots,$ are i.i.d. and $\mathbb{P}(I_1 = 1) = \delta^2$. Thus,

$$\sigma := \inf\{k \geq 0 : I_k = 1\}\,, \tag{5.71}$$

is geometrically distributed with $\mathbb{P}(\sigma = m) = (1-\delta^2)^m\delta^2$. Now, recall that $\eta_0 = 0$ (which implies $\eta_0 \vee \widehat{\eta}_0 = \widehat{\eta}_0$) and define

$$\mathcal{T} := L_\sigma + U = T_\sigma, \tag{5.72}$$

where for $n \geq 0$,

$$T_n := \widehat{\eta}_0 + d + \sum_{i=1}^{n}(\beta_i \vee \widehat{\beta}_i + d) + U. \tag{5.73}$$

Let us define as well

$$K := N_{L_\sigma -},\ \widehat{K} := \widehat{N}_{L_\sigma -}. \tag{5.74}$$

Since σ is a stopping time with respect to the Markov process $(\eta_k, \widehat{\eta}_k, I_k)_{k\geq 0}$, we obtain

$$\mathbb{P}\Big((\beta_{\sigma+1}, \widehat{\beta}_{\sigma+1}) \in \cdot \mid (\eta_\sigma, \widehat{\eta}_\sigma) = (s, \widehat{s})\Big) \tag{5.75}$$

$$= \mathbb{P}\Big((\beta_1, \widehat{\beta}_1) \in \cdot \mid (\eta_0, \widehat{\eta}_0) = (s, \widehat{s}), I_0 = 1\Big) = \mu \otimes \mu. \tag{5.76}$$

In other words, $\beta_{\sigma+1}$ and $\widehat{\beta}_{\sigma+1}$ are i.i.d. with distribution μ and $(\beta_{\sigma+1}, \widehat{\beta}_{\sigma+1})$ is independent of $(\eta_\sigma, \widehat{\eta}_\sigma)$ and thus of L_σ. From which we get that

$$S_K = \eta_{\sigma+1} = L_\sigma + \beta_{\sigma+1} \overset{\mathrm{d}}{=} L_\sigma + \widehat{\beta}_{\sigma+1} = \widehat{\eta}_{\sigma+1} = \widehat{S}_K. \tag{5.77}$$

And due to $L_\sigma + \beta_{\sigma+1} \overset{\mathrm{d}}{=} L_\sigma + U$, the processes S and $\widehat{S}$ have a common renewal at $\mathcal{T}$. Taking $S' = S$ and $\widehat{S}'$ with the same renewals as $\widehat{S}$ before $\mathcal{T}$ and with the same renewals as S after $\mathcal{T}$ gives the desired coupling. □

Remark. Due to (5.54) and the fact that $(\eta_k, \widehat{\eta}_k, I_k)_{k\geq 0}$ is a Markov process, we have for $i \in \{0, 1\}$ and $k \geq 1$ that

$$\mathbb{P}((\eta_0, \widehat{\eta}_0) \in \cdot, I_0 = i)) = \mathbb{P}(I_0 = i)\, \mathbb{P}((\eta_0, \widehat{\eta}_0) \in \cdot)\,, \tag{5.78}$$

and

$$\mathbb{P}((\eta_k, \widehat{\eta}_k) \in \cdot, I_k = i \mid (\eta_{k-1}, \widehat{\eta}_{k-1}) = (s, \widehat{s}), I_k = 0) \tag{5.79}$$

$$= \mathbb{P}(I_k = i)\, \mathbb{P}((\eta_k, \widehat{\eta}_k) \in \cdot \mid (\eta_{k-1}, \widehat{\eta}_{k-1}) = (s, \widehat{s}))\,. \tag{5.80}$$

Since the event

$$\{\sigma = m\} = \{I_0 = 0, I_1 = 0, \dots, I_{m-1} = 0, I_m = 1\}\,, \tag{5.81}$$

we have that for any Borel sets $D_0, D_1, \ldots, D_{m-1}$,

$$\mathbb{P}((\eta_0, \widehat{\eta}_0) \in D_0, \ldots, (\eta_{m-1}, \widehat{\eta}_{m-1}) \in D_{m-1}, \sigma = m) \tag{5.82}$$

$$= \int_{D_0} \cdots \int_{D_{m-1}} \mathbb{P}((\eta_{m-1}, \widehat{\eta}_{m-1}) \in \mathrm{d}u_{m-1}, I_{m-1} = 0 \mid (\eta_{m-2}, \widehat{\eta}_{m-2}) = u_{m-2}, I_{m-2} = 0) \cdots \tag{5.83}$$

$$\cdots \mathbb{P}((\eta_1, \widehat{\eta}_1) \in \mathrm{d}u_1, I_1 = 0 \mid (\eta_0, \widehat{\eta}_0) = u_0, I_0 = 0)\, \mathbb{P}((\eta_0, \widehat{\eta}_0) \in \mathrm{d}u_0, I_0 = 0) \tag{5.84}$$

$$= \mathbb{P}(I_m = 1)\mathbb{P}(I_{m-1} = 0) \cdots \mathbb{P}(I_0 = 0)\mathbb{P}((\eta_0, \widehat{\eta}_0) \in D_0, \ldots, (\eta_{m-1}, \widehat{\eta}_{m-1}) \in D_{m-1})\,, \tag{5.85}$$

hence,

$$\mathbb{P}((\eta_0, \widehat{\eta}_0) \in \cdot, \ldots, (\eta_{m-1}, \widehat{\eta}_{m-1}) \in \cdot, \sigma = m) \tag{5.86}$$

$$= \mathbb{P}(\sigma = m)\mathbb{P}((\eta_0, \widehat{\eta}_0) \in \cdot, \ldots, (\eta_{m-1}, \widehat{\eta}_{m-1}) \in \cdot)\,. \tag{5.87}$$

5.3 Decay rates for the Key Renewal Theorem

We work in the same context as the previous section. For the coupling time $\mathcal{T}$ constructed in Lemma 5.7 we have the following result that we prove along the lines of [26].

Lemma 5.8. *Assume $\mathbb{E}[\tau_1^s] < \infty$ for $s \geq 2$. Then, $\mathbb{E}[\mathcal{T}^q] < \infty$ for $q = s - 1$.*

Proof. Let us find some estimates for $\mathbb{E}[B_t^q]$. Consider first

$$\mathbb{E}[B_t] = \mathbb{E}\big[S_{N(t)}\big] - t \tag{5.88}$$

$$= m^{-1}\mathbb{E}[N(t)] - t \tag{5.89}$$

$$= m^{-1}\Phi(t) - t. \tag{5.90}$$

Since $\Phi(t)/t \underset{t\to\infty}{\longrightarrow} m$, for any $\epsilon_0 > 0$ we can find a $t_0 > 0$ such that

$$\Phi(t) < (\epsilon_0 + m)t, \quad \text{for all } t \geq t_0, \tag{5.91}$$

$$\Phi(t) \leq \Phi(t_0) + (\epsilon_0 + m)t, \quad \text{for all } t \geq 0. \tag{5.92}$$

Hence, for any arbitrary $\rho > 0$, there exists a constant a_ρ such that

$$\mathbb{E}[B_t] \leq a_\rho + \rho t, \quad \text{for all } t \geq 0. \tag{5.93}$$

For $\mathbb{E}[B_t^q]$, we can use Lemma 2.24 to find

$$\mathbb{E}[B_t^q] = \int_0^t \int_0^\infty x^q F(t - u + \mathrm{d}x)\, \Phi(\mathrm{d}u) \tag{5.94}$$

$$\leq \int_0^t \int_{t-u}^\infty x^q F(\mathrm{d}x)\, \Phi(\mathrm{d}u) \tag{5.95}$$

$$\leq \left(\int_0^\infty x^q F(\mathrm{d}x)\right) \Phi(t), \tag{5.96}$$

where $\int_0^\infty x^q F(\mathrm{d}x) < \infty$. Then, using (5.92) with $\epsilon_0 = 1$, we find positive constants a_1 and b_1 independent of k such that

$$\mathbb{E}[B_t^q] \le a_1 + b_1 t, \quad t \ge 0 \tag{5.97}$$

and by Fatou's Lemma also

$$\mathbb{E}[B_{t-}^q] \le \liminf_{n\to\infty} \mathbb{E}\left[B_{t-1/n}^q\right] \le a_1 + b_1 t, \quad t \ge 0. \tag{5.98}$$

Since there exist constants a_2 and b_2 such that $(d+x)^q \le a_2 + b_2 x^q$, for all $x \ge 0$, we have

$$\mathbb{E}\left[(d + \beta_k \vee \widehat{\beta}_k)^q\right] \le a_0 + b_0 \mathbb{E}\left[(\beta_k \vee \widehat{\beta}_k)^q\right]. \tag{5.99}$$

From (5.51) we have

$$\mathbb{E}\left[\left(\beta_k \vee \widehat{\beta}_k\right)^q \middle| (\eta_{k-1}, \widehat{\eta}_{k-1}) = (y, \widehat{y})\right] \tag{5.100}$$

$$\le \mathbb{E}\left[\beta_k^q + \widehat{\beta}_k^q \middle| (\eta_{k-1}, \widehat{\eta}_{k-1}) = (y, \widehat{y})\right] \tag{5.101}$$

$$= \mathbb{E}\left[B^q_{((\widehat{y}-y)_+ + d)-}\right] + \mathbb{E}\left[B^q_{((y-\widehat{y})_+ + d)-}\right]. \tag{5.102}$$

Then, from (5.98), we obtain

$$\mathbb{E}\left[(\beta_k \vee \widehat{\beta}_k)^q \mid \eta_{k-1}, \widehat{\eta}_{k-1}\right] \le 2a_1 + 2b_1 d + b_1 \left|\eta_{k-1} - \widehat{\eta}_{k-1}\right| \tag{5.103}$$

$$= 2a_1 + 2b_1 d + b_1 \left|\beta_{k-1} - \widehat{\beta}_{k-1}\right| \tag{5.104}$$

$$\le 2a_1 + 2b_1 d + b_1(\beta_{k-1} \vee \widehat{\beta}_{k-1}). \tag{5.105}$$

We define $a_3 := 2(a_1 + b_1 d)$ and $b_3 := b_1$ to write

$$\mathbb{E}\left[(\beta_k \vee \widehat{\beta}_k)^q \mid \eta_{k-1}, \widehat{\eta}_{k-1}\right] \le a_3 + b_3(\beta_{k-1} \vee \widehat{\beta}_{k-1}). \tag{5.106}$$

Now we shift our attention to $\mathbb{E}\left[\beta_k \vee \widehat{\beta}_k\right]$ for $k \ge 1$. We follow the same steps as in (5.102) (but with $q = 1$), and use (5.93) with $\rho = 1/2$ to find a constant a_ρ such that

$$\mathbb{E}\left[\beta_k \vee \widehat{\beta}_k \mid \eta_{k-1}, \widehat{\eta}_{k-1}\right] \le 2a_\rho + d + \frac{1}{2}\left(\beta_{k-1} \vee \widehat{\beta}_{k-1}\right). \tag{5.107}$$

By taking conditional expectation given $(\eta_{k-2}, \widehat{\eta}_{k-2}), (\eta_{k-3}, \widehat{\eta}_{k-3}), \ldots$, recursively and using (5.93) with $\rho = 1/2$ we ultimately obtain

$$\mathbb{E}\left[\beta_k \vee \widehat{\beta}_k \mid \eta_0, \widehat{\eta}_0\right] \le a_\rho(2 + 1 + \cdots + 2^{2-k}) + d(1 + 2^{-1} + \cdots + 2^{1-k}) + 2^{-k}(\eta_0 \vee \widehat{\eta}_0), \tag{5.108}$$

which can be simplified by recalling that $\eta_0 = 0$ and by noting that for any $k \geq 1$: $2 + 1 + \cdots + 2^{2-k} < 4$, $1 + 2^{-1} + \cdots + 2^{1-k} < 2$, and $2^{-k} < 1$. In summary,

$$\mathbb{E}\Big[\beta_k \vee \widehat{\beta}_k \mid \eta_0, \widehat{\eta}_0\Big] \leq 4a_\rho + 2d + \widehat{\eta}_0 = a_4 + \widehat{\eta}_0, \tag{5.109}$$

where $a_4 := 4a_\rho + 2d$. Now note that since $\widehat{\eta}_0$ is taken from the stationary distribution Π and $0 < q \leq s - 1$, then

$$\mathbb{E}[\widehat{\eta}_0^q] = \int_0^\infty x^q m\overline{F}(x)\mathrm{d}x \tag{5.110}$$

$$= \frac{m}{q+1} \int_0^\infty y^{q+1} F(\mathrm{d}y) < \infty. \tag{5.111}$$

There are two notable cases. Let us focus first on the case $q = s - 1 \geq 1$. In this case $c_0 := \mathbb{E}[(d + \widehat{\eta}_0)^q] < \infty$. We take the expectation on both sides of (5.109) and (5.99), which together with (5.106) yields

$$\mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k)^q\Big] \leq a_2 + b_2\Big(a_3 + b_3\mathbb{E}\Big[(\beta_k \vee \widehat{\beta}_k)^q\Big]\Big) \tag{5.112}$$

$$\leq a_2 + b_2\{a_3 + b_3(a_4 + c_0)\}\,. \tag{5.113}$$

We take $a_q := a_2 + b_2\{a_3 + b_3(a_4 + c_0)\}$. Because $q \geq 1$, we can apply Minkowski's inequality to obtain that

$$\mathbb{E}[T_n^q]^{1/q} \leq \mathbb{E}[(d + \widehat{\eta}_0)^q]^{1/q} + \sum_{k=1}^n \mathbb{E}\Big[\Big(d + \beta_k \vee \widehat{\beta}_k\Big)^q\Big]^{1/q} + (b^q)^{1/q} \tag{5.114}$$

$$\leq c_0^{1/q} + n a_q^{1/q} + b. \tag{5.115}$$

Since $\{\sigma = m\}$ is independent from T_n by (5.87) and (5.73), and since σ has geometric distribution, we have

$$\mathbb{E}[\mathcal{T}^q] = \mathbb{E}[T_\sigma^q] \tag{5.116}$$

$$= \sum_{n=0}^\infty \mathbb{E}[T_n^q]\,\mathbb{P}(\sigma = n) \tag{5.117}$$

$$\leq \sum_{n=0}^\infty (c_0^{1/q} + n a_q^{1/q} + b)^q \delta^2 (1 - \delta^2)^n < \infty, \tag{5.118}$$

because $(1 - \delta^2)^n$ vanishes exponentially.

For the case where $q = s - 1 < 1$ we use the subadditivity of the function $x \mapsto x^q$, $x \geq 0$, to get

$$\mathbb{E}[\mathcal{T}^q] = \mathbb{E}\left[\left(\sum_{k=0}^n (d + \beta_k \vee \widehat{\beta}_k) + U\right)^q ; \sigma = n\right] \tag{5.119}$$

$$\leq \mathbb{E}[(d + \widehat{\eta}_0)^q] + \sum_{k=1}^\infty \mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k)^q 1_{\{\sigma \geq k\}}\Big] + b^q \tag{5.120}$$

$$\leq c_0 + \sum_{k=1}^\infty \mathbb{E}\Big[\mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k)^q 1_{\{\sigma \geq k\}} \mid \eta_0, \widehat{\eta}_0\Big]\Big] + b^q. \tag{5.121}$$

Since $0 < q < 1$, we can apply Hölder's inequality with $p > 1$ such that $q + \frac{1}{p} = 1$ to obtain

$$\mathbb{E}\Big[\mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k)^q 1_{\{\sigma \geq k\}} \mid \eta_0, \widehat{\eta}_0\Big]\Big] \leq \mathbb{E}\Big[\mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k) \mid \eta_0, \widehat{\eta}_0\Big]^q \mathbb{E}\big[1_{\{\sigma \geq k\}} \mid \eta_0, \widehat{\eta}_0\big]^{1/p}\Big] \tag{5.122}$$

$$\leq \mathbb{E}\Big[d^q + \mathbb{E}\Big[\beta_k \vee \widehat{\beta}_k \mid \eta_0, \widehat{\eta}_0\Big]^q\Big] \,\mathbb{P}(\sigma \geq k)^{1/p}. \tag{5.123}$$

From (5.109) we know that

$$\mathbb{E}\Big[\beta_k \vee \widehat{\beta}_k \mid \eta_0, \widehat{\eta}_0\Big]^q \leq a_4^q + \widehat{\eta}_0^q, \tag{5.124}$$

hence,

$$\mathbb{E}\Big[\mathbb{E}\Big[(d + \beta_k \vee \widehat{\beta}_k)^q 1_{\{\sigma \geq k\}} \mid \eta_0, \widehat{\eta}_0\Big]\Big] \leq (d^q + a_4^q + \mathbb{E}[\widehat{\eta}_0^q])\,\mathbb{P}(\sigma \geq k)^{1/p} \tag{5.125}$$

$$= c_q \mathbb{P}(\sigma \geq k)^{1/p}, \tag{5.126}$$

where $c_q := d^q + a_4^q + \mathbb{E}[\widehat{\eta}_0^q] < \infty$. The random variable σ is geometrically distributed so $\mathbb{P}(\sigma \geq k) = (1 - \delta^2)^k$. From the above, we have that

$$\mathbb{E}[\mathcal{T}^q] \leq c_0 + b^q + \sum_{k=1}^{\infty} c_q (1 - \delta^2)^{k/p} < \infty, \tag{5.127}$$

since $(1 - \delta^2)^{k/p}$ decreases exponentially in k. This ultimately means that $\mathbb{E}[\mathcal{T}^q] < \infty$ for $0 < q \leq s - 1$, which concludes the proof. □

Before proceeding, let us make the following observation. Let $h : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $\lim_{t\to\infty} h(t) = \infty$ and suppose that $\mathbb{E}[h(\mathcal{T})] < \infty$ for the coupling time $\mathcal{T}$, then, by the Dominated Convergence Theorem,

$$h(t)\mathbb{P}(\mathcal{T} > t) \leq \mathbb{E}\big[h(\mathcal{T}) 1_{\{\mathcal{T} > t\}}\big] \underset{t\to\infty}{\longrightarrow} 0 \tag{5.128}$$

which means that

$$\mathbb{P}(\mathcal{T} > t) = o\left(\frac{1}{h(t)}\right) \quad \text{as } t \to \infty. \tag{5.129}$$

We use the previous remark to prove the following Lemma.

Lemma 5.9. *Let $\{S_n\}_{n\geq 0}$ be a zero-delayed renewal process that satisfies $\mathbb{E}[\tau_1^s] < \infty$ for $s \geq 2$. Then, for $q = s - 1$,*

$$\|\mathbb{P}(B_t \in \cdot) - \Pi(\cdot)\|_{\text{t.v.}} = o\big(t^{-q}\big) \ \text{ as } t \to \infty, \tag{5.130}$$

where Π is the stationary distribution given in (2.83)*, and*

$$\Phi_2([x, \infty)) = o\big(x^{-q}\big) \ \text{ as } x \to \infty, \quad \varphi_1(x) = m + o\big(x^{-q}\big) \ \text{ as } x \to \infty. \tag{5.131}$$

Proof. Let $q = s - 1$. Notice that the function $h : [0, \infty) \to [0, \infty)$ given by $h(t) = t^q$ is non-decreasing and $\mathbb{E}[\mathcal{T}^q] < \infty$, by Lemma 5.8. By definition of Π and $\mathcal{T}$ we have

$$\|\mathbb{P}(B_t \in \cdot) - \Pi(\cdot)\|_{t.v.} = \left\|\mathbb{P}(B_t \in \cdot) - \mathbb{P}(\widehat{B}_t \in \cdot)\right\|_{t.v.} \leq 2\mathbb{P}(\mathcal{T} > t)\,, \tag{5.132}$$

and thus, from (5.129),

$$\lim_{t\to\infty} t^q \|\mathbb{P}(B_t \in \cdot) - \Pi(\cdot)\|_{t.v.} \leq \lim_{t\to\infty} 2t^q\mathbb{P}(\mathcal{T} > t) = 0, \tag{5.133}$$

which implies

$$\|\mathbb{P}(B_t \in \cdot) - \Pi(\cdot)\|_{t.v.} = o\big(t^{-q}\big) \quad \text{as } t \to \infty. \tag{5.134}$$

Now consider Stone's decomposition $\Phi = \Phi_1 + \Phi_2$ obtained from the uniform component G_0 with density $g_0(x) = \frac{\|G_0\|_{\text{t.v.}}}{b} 1_{\{a \leq x < a+b\}}$, $x \geq 0$. Since $H = F^{*n_0} - G_0 < F$, from the finiteness of $\mathbb{E}[\tau_1^s]$, we get that

$$\int_0^\infty x^s H(\mathrm{d}x) < \infty, \tag{5.135}$$

and hence,

$$\int_0^\infty x^s \Phi_2(\mathrm{d}x) < \infty, \tag{5.136}$$

and

$$\limsup_{x\to\infty} x^q \int_{[x,\infty)} \Phi_2(\mathrm{d}t) \leq \lim_{x\to\infty} \int_{[x,\infty)} t^q \Phi_2(\mathrm{d}t) = 0, \tag{5.137}$$

from which we get that

$$\Phi_2([x, \infty)) = o\big(x^{-q}\big) \text{ as } x \to \infty. \tag{5.138}$$

From [1, Sec. V. Theorem 2.4 (iii)] we have that for a general renewal process

$$\Phi((x - a - b, x - a]) \tag{5.139}$$
$$=\mathbb{E}\big[N_{(x-a-b)+b} - N_{x-a-b}\big] \tag{5.140}$$
$$= \int_0^b \Phi(b - \xi)\mathbb{P}(B_{x-a-b} \in \mathrm{d}\xi) \tag{5.141}$$
$$= \int_0^b \Phi(b - \xi)\Pi(\mathrm{d}\xi) + \int_0^b \Phi(b - \xi)[\mathbb{P}(B_{x-a-b} \in \mathrm{d}\xi) - \Pi(\mathrm{d}\xi)]\,. \tag{5.142}$$

Since

$$\left|\int_0^b \Phi(b - \xi)[\mathbb{P}(B_{x-a-b} \in \mathrm{d}\xi) - \Pi(\mathrm{d}\xi)]\right| \leq \Phi(b)\, \|\mathbb{P}(B_t \in \cdot) - \Pi(\cdot)\|_{t.v.}\,, \tag{5.143}$$

we can use (5.134) and Lemma 2.17 to obtain,

$$\int_0^b \Phi(b-\xi)\Pi(\mathrm{d}\xi) + \int_0^b \Phi(b-\xi)[\mathbb{P}(B_{x-a-b}\in \mathrm{d}\xi) - \Pi(\mathrm{d}\xi)] \tag{5.144}$$

$$= mb + o\big((x-a-b)^{-q}\big) \tag{5.145}$$

$$= mb + o\big(x^{-q}\big)\,. \tag{5.146}$$

In other words, $\Phi((x-a-b, x-a]) = mb + o(x^{-q})$ as $x \to \infty$. From Stone's decomposition the density φ_1 is given as

$$\varphi_1(x) = \int_0^x \Phi * g_0(x-y)\Phi_0^{(2)}(\mathrm{d}y), \tag{5.147}$$

where from the previous reasoning

$$\Phi * g_0(x) - m\,\|G_0\|_{\text{t.v.}} = o(x^{-q}) \tag{5.148}$$

Notice that

$$\left|\int_0^{x/2} \left(\Phi * g_0(x-y) - m\,\|G_0\|_{\text{t.v.}}\right)\Phi_0^{(2)}(\mathrm{d}y)\right| \tag{5.149}$$

$$\leq \sup_{y'\leq x/2} \left|\Phi * g_0(x-y') - m\,\|G_0\|_{\text{t.v.}}\right| \left\|\Phi_0^{(2)}\right\|_{\text{t.v.}} = o(x^{-q}), \tag{5.150}$$

and

$$\left|\int_{x/2}^{x} \left(\Phi * g_0(x-y) - m\,\|G_0\|_{\text{t.v.}}\right)\Phi_0^{(2)}(\mathrm{d}y)\right| \tag{5.151}$$

$$\leq \left(\|\Phi * g_0\|_\infty + m\,\|G_0\|_{\text{t.v.}}\right)\Phi_0^{(2)}((x/2,\infty)) = o(x^{-q}), \tag{5.152}$$

from which we conclude that

$$\varphi_1(x) = m\,\|G_0\|_{\text{t.v.}} \left\|\Phi_0^{(2)}\right\|_{\text{t.v.}} + o(x^{-q}) \tag{5.153}$$

$$= m + o(x^{-q}), \tag{5.154}$$

by (2.117). □

We proceed to the proof of Theorem 5.2.

Proof of Theorem 5.2. We prove the case where $z(x) = o(x^{-r})$ as $x \to \infty$ and the case for $z(x) = O(x^{-r})$ as $x \to \infty$ follows similarly. Assume that $0 \leq q \leq s-1$. We compute the convolution explicitly using Stone's decomposition.

$$\Phi * z(x) = \int_0^x z(x-y)\Phi_2(\mathrm{d}y) + \int_0^x z(x-y)\varphi_1(y)\mathrm{d}y. \tag{5.155}$$

The first integral can be split as

$$\int_0^x z(x-y)\Phi_2(\mathrm{d}y) = \int_0^{x/2} z(x-y)\Phi_2(\mathrm{d}y) + \int_{x/2}^x z(x-y)\Phi_2(\mathrm{d}y), \tag{5.156}$$

where

$$\left|\int_0^{x/2} z(x-y)\Phi_2(\mathrm{d}y)\right| \leq \sup_{y'\geq x/2} |z(y')| \, \|\Phi_2\|_{\mathrm{t.v.}} = o(x^{-r}) \tag{5.157}$$

$$\left|\int_{x/2}^x z(x-y)\Phi_2(\mathrm{d}y)\right| \leq \|z\|_\infty \, \Phi_2((x/2,\infty)) = o(x^{-q}). \tag{5.158}$$

The second integral can be written as

$$\int_0^x z(y)\varphi_1(x-y)\mathrm{d}y = \int_0^{x/2} z(y)\varphi_1(x-y)\mathrm{d}y + \int_{x/2}^x z(y)\varphi_1(x-y)\mathrm{d}y. \tag{5.159}$$

From Lemma 5.9, we have $\widetilde{\varphi}_1(x) := \varphi_1(x) - m = o(x^{-q})$ as $x \to \infty$. This entails,

$$\left|\int_0^{x/2} z(y)\widetilde{\varphi}_1(x-y)\mathrm{d}y\right| \leq \int_0^{x/2} z(y)\,|\widetilde{\varphi}_1(x-y)|\,\mathrm{d}y \tag{5.160}$$

$$\leq \sup_{y'\geq x/2} |\widetilde{\varphi}_1(y')| \int_0^\infty z(y)\mathrm{d}y \tag{5.161}$$

$$= o(x^{-q}) \tag{5.162}$$

On the other hand,

$$\left|\int_{x/2}^x z(y)\widetilde{\varphi}_1(x-y)\mathrm{d}y\right| \leq \int_{x/2}^x z(y)\,|\widetilde{\varphi}_1(x-y)|\,\mathrm{d}y \tag{5.163}$$

$$\leq \|\widetilde{\varphi}_1\|_\infty \int_{x/2}^x z(y)\mathrm{d}y, \tag{5.164}$$

Let $\epsilon > 0$ be arbitrary. Since $z(x) = o(x^{-r})$ as $x \to \infty$, there exists $x_0 > 0$ such that $x^r\,|z(x)| < \epsilon$ for all $x > x_0$. Then, for all $x > 2x_0$,

$$\int_{x/2}^x |z(y)|\,\mathrm{d}y < \epsilon \int_{x/2}^x y^{-r}\mathrm{d}y \tag{5.165}$$

$$< \epsilon \frac{(x/2)^{1-r}}{r-1}, \tag{5.166}$$

from which we obtain that

$$\int_{x/2}^x |z(y)|\,\mathrm{d}y = o(x^{1-r}). \tag{5.167}$$

This means that,

$$\int_0^{x/2} z(y)\varphi_1(x-y)\mathrm{d}y = m\int_0^{x/2} z(y)\mathrm{d}y + o(x^{-q}), \tag{5.168}$$

$$\int_{x/2}^{x} z(y)\varphi_1(x-y)\mathrm{d}y = m\int_{x/2}^{x} z(y)\mathrm{d}y + o(x^{1-r}). \tag{5.169}$$

Ultimately, we get,

$$\Phi * z(x) = m\int_0^{\infty} z(y)\mathrm{d}y + o(x^{-q}) + o(x^{-r}) + o(x^{1-r}) + o(x^{-q}) \tag{5.170}$$

$$= m\int_0^{\infty} z(y)\mathrm{d}y + o(x^{\max\{1-r,-q\}}), \tag{5.171}$$

which is the desired result. □

5.4 Some Regenerative Processes

In this section we exhibit that the processes of interest satisfy the regenerative property. Let us begin with the recurrence times. Since the renewals are stopping times for a renewal process, from the strong Markov property it is obvious that the forward and backward recurrence times from Definition 2.16 are regenerative. We can show as well that the compensator of a zero-delayed renewal process that satisfies assumption **(B0)** can be written in terms of a regenerative process. Let $\mu : [0,\infty) \to [0,\infty)$ be the nonnegative measurable function given by

$$\mu(x) = \frac{f(x)}{1 - \int_0^x f(y)\mathrm{d}y}, \quad x \geq 0. \tag{5.172}$$

From [7, Sec. II Theorem T7], we have that the process $\{M(t)\}_{t\geq 0}$ given as

$$M(t) = N(t) - \int_0^t \mu(u - S_{N(u)})\mathrm{d}u, \quad t \geq 0, \tag{5.173}$$

is an $(\mathcal{F}_t^N)$-martingale, with *compensator* Λ:

$$\Lambda(t) := \int_0^t \mu(u - S_{N(u)})\mathrm{d}u, \quad t \geq 0. \tag{5.174}$$

Notice that Λ can be rewritten as

$$\Lambda(t) = \sum_{i=1}^{N(t)-1} \xi_i + \int_{S_{N(t)-1}}^{t} \mu(u - S_{N(t)-1})\mathrm{d}u \tag{5.175}$$

$$:= \sum_{i=1}^{N(t)-1} \int_{S_{i-1}}^{S_i} \mu(u - S_{i-1})\mathrm{d}u + \int_{S_{N(t)-1}}^{t} \mu(u - S_{N(t)-1})\mathrm{d}u, \tag{5.176}$$

where the random variables ξ_i are independent and identically distributed with

$$\xi_i \stackrel{\mathrm{d}}{=} \int_0^{\tau_1} \mu(u)\mathrm{d}u \quad i = 1, 2 \dots \tag{5.177}$$

Observe that the process

$$\int_{S_{N(t)-1}}^{t} \mu(u - S_{N(t)-1})\mathrm{d}u \tag{5.178}$$

is regenerative. In effect, let $n \geq 0$ be arbitrary, then from the strong Markov property of the backward recurrence time A_t we have

$$\int_{S_{N(t+S_n)-1}}^{t+S_n} \mu(u - S_{N(t+S_n)-1})\mathrm{d}u = \int_0^{t+S_n-S_{N(t+S_n)-1}} \mu(v)\mathrm{d}v \tag{5.179}$$

$$= \int_0^{A_{t+S_n}} \mu(v)\mathrm{d}v \tag{5.180}$$

$$\stackrel{\mathrm{d}}{=} \int_0^{A_t} \mu(v)\mathrm{d}v \tag{5.181}$$

and since this is a measurable function whose argument is the regenerative process A_t, from [1, Proposition VI.1.1], we can conclude that this process is itself regenerative.

5.5 Convergence of processes

Now we can proceed with the proof of our Theorems 5.3 and 5.4. We work in the context of regenerative processes introduced in the previous section.

Proof of Theorem 5.3. As we noted above, for all $i = 1, 2, \dots,$

$$\xi_i := \int_{S_{i-1}}^{S_i} \mu(s - S_{i-1})\mathrm{d}s \stackrel{\mathrm{d}}{=} \int_0^{\tau_1} \mu(s)\mathrm{d}s. \tag{5.182}$$

We now look for the distribution of the latter. Let $x > 0$, we have

$$\mu(x) = \frac{f(x)}{1 - F(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} \log(1 - F(x)), \tag{5.183}$$

or in other words,

$$\exp\left(-\int_0^x \mu(s)\mathrm{d}s\right) = 1 - F(x), \tag{5.184}$$

which implies that

$$G(x) := \mathbb{P}\left(\int_0^{\tau_1} \mu(s)\mathrm{d}s \leq x\right) \tag{5.185}$$

$$= \mathbb{P}\left(\exp\left(-\int_0^{\tau_1} \mu(s)\mathrm{d}s\right) \geq e^{-x}\right) \tag{5.186}$$

$$= \mathbb{P}\big(F(\tau_1) \leq 1 - e^{-x}\big) = 1 - e^{-x}, \tag{5.187}$$

where in the last equality we used that if $F(x) = \mathbb{P}(\tau_1 \le x)$, then the random variable $F(\tau_1) \sim \text{Uniform}[0,1]$. This means that

$$\int_0^{\tau_1} \mu(s)\mathrm{d}s \sim \exp(1). \tag{5.188}$$

We have for any $p > 0$ that

$$\sup_{v\in[0,1]} \frac{1}{T^p}\big[\Lambda(Tv) - \Lambda(S_{N(Tv)-1})\big] = \sup_{v\in[0,1]} \frac{1}{T^p}\int_{S_{N(Tv)-1}}^{Tv} \mu(s - S_{N(Tv)-1})\mathrm{d}s \tag{5.189}$$

$$\le \frac{1}{T^p} \max_{k\le N(T)} \xi_k, \tag{5.190}$$

and since G does not have finite support, from Theorem 2.28, we know that $F_T(x) := \mathbb{P}\left(\max_{k\le N(T)} \xi_k \le x\right)$ can be uniformly approximated by $G(x)^{mT}$, in the sense that

$$\lim_{T\to\infty} \sup_{x\ge 0} \left|F_T(x) - G(x)^{mT}\right| = 0, \tag{5.191}$$

where $G(x) = \mathbb{P}\big(\int_0^{\tau_1} \mu(s)\mathrm{d}s \le x\big) = 1 - e^{-x}$. If we take $x = \epsilon T^p$ for $\epsilon > 0$, we have,

$$G(\epsilon T^p)^{mT} = \big(1 - e^{-\epsilon T^p}\big)^{mT} = \exp\big\{mT\log\big(1 - e^{-\epsilon T^p}\big)\big\} \tag{5.192}$$

$$= \exp\big\{mT\big(e^{-\epsilon T^p}(1 + o(1))\big)\big\} \underset{T\to\infty}{\longrightarrow} 1, \tag{5.193}$$

which proves that for any $p > 0$

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{1}{T^p}\max_{k\le N(T)} \xi_k \le \epsilon\right) = 1 \tag{5.194}$$

and hence,

$$\left(\frac{1}{T^p}\int_{S_{N(Tv)-1}}^{Tv} \mu(s - S_{N_R(Tv)-1})\mathrm{d}s\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} 0. \tag{5.195}$$

The proof is complete. □

We proceed similarly in the case of the recurrence times. As we will see, the dominating process will be the maximum process of the inter-arrival times.

Proof of Theorem 5.4. Let $p \le s$. Notice that

$$\sup_{v\in[0,1]} \frac{1}{T^{1/p}}\big(Tv - S_{N(Tv)-1}\big) \le \frac{1}{T^{1/p}} \max_{k\le N(T)} \tau_k, \tag{5.196}$$

$$\sup_{v\in[0,1]} \frac{1}{T^{1/p}}\big(S_{N(Tv)} - Tv\big) \le \frac{1}{T^{1/p}} \max_{k\le N(T)} \tau_k. \tag{5.197}$$

It is clear that if $F(x) := \mathbb{P}(\tau_1 \le x)$ has finite support, the result is trivial. Thus, we assume that F does not have a finite support. In this case we can approximate the distribution of the maximum by

$$F(x)^{mT}, \tag{5.198}$$

and by taking $x = \epsilon T^{1/p}$ we can compute,

$$\begin{aligned} F(\epsilon T^{1/p})^{mT} &= \exp\big\{mT \log\big(F(\epsilon T^{1/p})\big)\big\} && (5.199)\\ &= \exp\big\{mT \log\big(1 - \overline{F}(\epsilon T^{1/p})\big)\big\} && (5.200)\\ &= \exp\big\{mT\big(\overline{F}(\epsilon T^{1/p})(1 + o(1))\big)\big\}. && (5.201)\end{aligned}$$

Since $\mathbb{E}[\tau_1^s] < \infty$, it follows by the Dominated Convergence Theorem that

$$x^p \overline{F}(x) = x^p \mathbb{P}(\tau_1 > x) \le \mathbb{E}[\tau_1^p \; ; \; \tau_1 > x] \underset{x\to\infty}{\longrightarrow} 0. \tag{5.202}$$

The previous reasoning implies that

$$T\overline{F}(\epsilon T^{1/p}) \underset{T\to\infty}{\longrightarrow} 0, \tag{5.203}$$

$$\exp\big\{mT\big(\overline{F}(\epsilon T^{1/p})(1 + o(1))\big)\big\} \underset{T\to\infty}{\longrightarrow} 1, \tag{5.204}$$

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{1}{T^{1/p}} \max_{k\le N(T)} \tau_k \le \epsilon\right) = 1, \tag{5.205}$$

from which we can conclude that

$$\left(\frac{1}{T^{1/p}}\big(Tv - S_{N(Tv)-1}\big), \frac{1}{T^{1/p}}\big(S_{N(Tv)} - Tv\big)\right)_{v\in[0,1]} \overset{d}{\underset{T\to\infty}{\longrightarrow}} (0,0), \tag{5.206}$$

which completes the proof. □

With this, we conclude the exposition of our convergence rates results for renewal processes. Nevertheless, each one of them will see an application in the derivation of the limit theorems of the following Section.

6 Limit theorems for renewal Hawkes processes

The purpose of this Section is to establish limit theorems for the RHP, namely, a law of large numbers and a central limit theorem, as Bacry–Delattre–Hoffmann–Muzy [5] did for the classical (multivariate) Hawkes process via a martingale approach. In the case of one dimension, if $\alpha := \int_0^\infty h(t)\mathrm{d}t < 1$, a law of large numbers (LLN) is given as

Theorem 6.1 (Bacry–Delattre–Hoffmann–Muzy [5, Theorem 1]). *We have $N(t) \in L^2(\mathbb{P})$ for all $t \ge 0$ and the convergence*

$$\sup_{v\in[0,1]} \left| T^{-1}N(Tv) - v\frac{\mu}{1-\alpha}\right| \underset{T\to\infty}{\longrightarrow} 0 \tag{6.1}$$

holds a.s. and in $L^2(\mathbb{P})$.

A central limit theorem (CLT) was also proved, namely,

Theorem 6.2 (Bacry–Delattre–Hoffmann–Muzy [5, Theorem 2]). *The convergence*

$$\left(\frac{1}{\sqrt{T}}(N_{Tv} - \mathbb{E}[N_{Tv}])\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} \left(\sqrt{\frac{\mu}{(1-\alpha)^3}}W_v\right)_{v\in[0,1]} \tag{6.2}$$

holds in the Skorokhod topology, where $(W_v)_{v\in[0,1]}$ *is a standard Brownian motion.*

6.1 Limit theorems for the renewal Hawkes process

We extend these results to the RHP. First, we have a law of large numbers, in which we show that the mean number of arrivals can be consistently estimated as follows.

Theorem 6.3. *Assume* **(A0, A1)** *and* **(B0)**. *Then,*

$$\sup_{v\in[0,1]} \left| T^{-1}N(Tv) - v\frac{m}{1-\alpha} \right| \xrightarrow[T\to\infty]{a.s.} 0. \tag{6.3}$$

The central limit theorem for the RHP takes the form,

Theorem 6.4. *Under assumptions* **(A0, A1)** *and* **(B0)**, *if* $\int_0^\infty x^2 F(\mathrm{d}x) < \infty$, *the convergence in distribution*

$$\left(\frac{1}{\sqrt{T}}(N(Tv) - \mathbb{E}[N(Tv)])\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} (\sigma W(v))_{v\in[0,1]}, \tag{6.4}$$

holds in the Skorokhod topology, where $(W(v))_{v\in[0,1]}$ *is a standard Brownian motion and*

$$\sigma = \frac{\sqrt{\sigma_M^2 + \sigma_R^2}}{(1-\alpha)}, \quad \sigma_M^2 = \frac{m}{1-\alpha}, \quad \frac{\sigma_R^2}{m} = 3 + m^2\,\mathrm{Var}[\tau] - 2m\mathbb{E}\left[\tau\int_0^\tau \mu(s)\mathrm{d}s\right], \tag{6.5}$$

and τ *is a random variable such that for* $x \geq 0$, $\mathbb{P}(\tau \leq x) = F(x)$.

And finally, we have the following result of asymptotic normality:

Corollary 6.5. *Under assumptions* **(A0, A1)** *and* **(B0)**, *if* $\int_0^\infty x^3 F(\mathrm{d}x) < \infty$ *and* $\int_0^\infty x^r h(x)\mathrm{d}x < \infty$ *for some* $r > 1$, *the convergence in distribution*

$$\left(\frac{1}{\sqrt{T}}N(Tv) - v\frac{m}{1-\alpha}\sqrt{T}\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} (\sigma W(v))_{v\in[0,1]}, \tag{6.6}$$

holds in the Skorokhod topology, where σ *is the same as in Theorem 6.4 and* $(W(v))_{v\in[0,1]}$ *is a standard Brownian motion.*

We make use of the decay rate results found in Section 5 for the proof of our limit theorems. In the following section, we find renewal type equations whose solutions yield useful identities for our study.

6.2 Results for the mean number of arrivals

As mentioned in Section 4, the arrivals in the RHP are of two types: those coming from a renewal process and those coming from the self-exciting part of the process. For the proofs of our results, it is often convenient to work with the *imbedded renewal process* separately. To make this distinction denote $N_R(\cdot) := \{T_i : D_i = 0\} = \{0 = S_0, S_1, S_2, \dots\}$ with $S_0 < S_1 < S_2 < \dots$, and consider the counting process $N_R(t) = \min\{i : S_i \leq t\}$, $t \geq 0$. Then, it is clear that

$$\mu(t - T_{I(t)}) = \mu(t - S_{N_R(t)-1}), \quad t \geq 0. \tag{6.7}$$

Furthermore, N_R is a renewal process with inter-arrivals $\tau_i := S_i - S_{i-1}$ which have distribution F, and the *renewal function*

$$\Phi(t) := \mathbb{E}[N_R(t)] = \sum_{n\geq 0} F^{*n}(t), \quad t \geq 0, \tag{6.8}$$

where

$$F^{*0}(x) = \delta_0(x), \quad F^{*(n+1)}(x) = \int_0^x F^{*n}(x-y)F(\mathrm{d}y), \quad x \geq 0,\ n \geq 0, \tag{6.9}$$

as shown in the following Lemma.

Lemma 6.6. *Let $\tau_1, \tau_2, \dots$ be i.i.d. random variables with distribution F that satisfies assumption* **(B0)***, and define the partial sums $S_n := \tau_1 + \cdots + \tau_n$, $n \geq 1$. Then the counting process $N_R(t) = \sum_{i\geq 0} 1_{\{S_i \leq t\}}$, $t \geq 0$, admits the $(\mathcal{F}_t)$-intensity $\mu(t - T_{I(t)})$. Moreover*

$$\mathbb{E}\left[\int_0^t \mu\big(s - T_{I(s)}\big)\,\mathrm{d}s\right] = \Phi(t). \tag{6.10}$$

Proof. From Theorem 2.14 we know that the intensity for the process $N_R(\cdot)$ is given by

$$\lambda_R(t) = \sum_{n\geq 0} \frac{f^{(n+1)}(t - S_n)}{1 - \int_0^{t-S_n} f^{(n+1)}(x)\mathrm{d}x} 1_{\{S_n \leq t < S_{n+1}\}}, \tag{6.11}$$

where for a Borel set A, we have

$$F^{(n+1)}(A) := \mathbb{P}[S_{n+1} \in A \mid \mathcal{F}_{S_n}] = \int_A f^{(n+1)}(x)\mathrm{d}x. \tag{6.12}$$

In the case of $N_R(\cdot)$, we have that $F^{(n+1)} = F$ and $f^{(n+1)} = f$ for all $n \geq 0$. We can the substitute this in (6.11) to get

$$\begin{aligned} \lambda_R(t) =& \sum_{n\geq 0} \frac{f(t - S_n)}{1 - F(t - S_n)} 1_{\{S_n \leq t < S_{n+1}\}} && (6.13)\\ =& \sum_{n\geq 0} \mu(t - S_n)\, 1_{\{S_n \leq t < S_{n+1}\}} && (6.14)\\ =& \sum_{n\geq 0} \mu\big(t - S_{N_R(t)-1}\big)\, 1_{\{S_n \leq t < S_{n+1}\}} && (6.15)\\ =& \mu\big(t - S_{N_R(t)-1}\big)\,. && (6.16) \end{aligned}$$

Now, since the process $C(s) = 1_{(0,t]}(s)$, $s \geq 0$ is $(\mathcal{F}_t)$-predictable, we can compute

$$\mathbb{E}\left[\int_0^t \mu(s - S_{N_R(s)-1})\mathrm{d}s\right] = \mathbb{E}\left[\int_0^t N_R(\mathrm{d}s)\right] = \mathbb{E}[N_R(t)] = \Phi(t). \tag{6.17}$$

Finally, from the observation (6.7), the proof follows. □

As stated in Theorem 2.22, if the imbedded renewal process has an inter-arrival distribution that satisfies **(B0)**, then the induced renewal measure $\Phi(\mathrm{d}t)$ can be decomposed using Stone's decomposition (2.121) as

$$\Phi = \Phi_1 + \Phi_2, \tag{6.18}$$

where $\Phi_2([0,\infty)) < \infty$ and Φ_1 is absolutely continuous with bounded density φ_1 that satisfies

$$\varphi_1(t) \underset{t\to\infty}{\longrightarrow} m. \tag{6.19}$$

In order to study the arrivals related to the self-exciting part of the process, we define the following function:

$$\psi(t) = \sum_{n\geq 1} h^{*n}(t), \quad t \geq 0, \tag{6.20}$$

where h^{*n} denotes the n-fold convolution of h. Then, we can state the following Lemma.

Lemma 6.7. *Assume* **(A0)** *and* **(B0)**. *For any $t \geq 0$, the mean number of events $\mathbb{E}[N(t)]$ is given as,*

$$\mathbb{E}[N(t)] = \Phi(t) + \int_0^t \psi(t-s)\Phi(s)\mathrm{d}s. \tag{6.21}$$

Proof. Let $t \geq 0$. Since the process $C(s) = 1_{(0,t]}(s)$, $s \geq 0$ is $(\mathcal{F}_t)$-predictable, from the property (2.17) of the intensity and Lemma 6.7, we have

$$\mathbb{E}[N(t)] = \mathbb{E}\left[\int_0^t \mu(s - T_{I(s)})\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_0^s h(s-u)N(\mathrm{d}u)\mathrm{d}s\right] \tag{6.22}$$

$$= \Phi(t) + \mathbb{E}\left[\int_0^t h(t-s)N(s)\mathrm{d}s\right] \tag{6.23}$$

$$= \Phi(t) + \int_0^t h(t-s)\mathbb{E}[N(s)]\,\mathrm{d}s. \tag{6.24}$$

This is a renewal type integral equation for $\mathbb{E}[N(t)]$. Since the renewal function is always finite, Φ is bounded on finite intervals, and the integral equation has a unique solution bounded on finite intervals given by

$$\mathbb{E}[N(t)] = \Phi(t) + \int_0^t \psi(t-s)\Phi(s)\mathrm{d}s, \quad t \geq 0, \tag{6.25}$$

and this concludes the proof. □

In the following, it will be useful to write the process

$$X(t) := N(t) - \mathbb{E}[N(t)], \quad t \geq 0, \tag{6.26}$$

as a linear functional of the characteristic martingale, as it is done below.

Lemma 6.8. *Assume* **(A0)** *and* **(B0)**. *Set*

$$A(t) := M(t) + \int_0^t \mu\big(s - T_{I(s)}\big)\, \mathrm{d}s - \Phi(t). \tag{6.27}$$

Then, for all $t \geq 0$*, the process* $(X_t)_{t\geq 0}$ *satisfies,*

$$X(t) = A(t) + \int_0^t \psi(t-s)A(s)\mathrm{d}s. \tag{6.28}$$

Proof. We have

$$X(t) = M(t) + \int_0^t \lambda(s)\mathrm{d}s - \mathbb{E}[N(t)]. \tag{6.29}$$

From the proof of Lemma 6.7 we have

$$X(t) = M(t) + \int_0^t \lambda(s)\mathrm{d}s - \Phi(t) - \int_0^t h(t-s)\mathbb{E}[N(s)]\,\mathrm{d}s \tag{6.30}$$

$$= M(t) + \int_0^t \mu(s - T_{I(s)})\mathrm{d}s + \int_0^t \int_0^s h(s-u)N(\mathrm{d}u)\mathrm{d}s - \Phi(t) - \int_0^t h(t-s)\mathbb{E}[N(s)]\,\mathrm{d}s \tag{6.31}$$

$$= M(t) + \int_0^t \mu(s - T_{I(s)})\mathrm{d}s + \int_0^t h(t-s)N(s)\mathrm{d}s - \Phi(t) - \int_0^t h(t-s)\mathbb{E}[N(s)]\,\mathrm{d}s \tag{6.32}$$

$$= M(t) + \int_0^t \mu(s - T_{I(s)})\mathrm{d}s - \Phi(t) + \int_0^t h(t-s)X(s)\mathrm{d}s. \tag{6.33}$$

$$= A(t) + \int_0^t h(t-s)X(s)\mathrm{d}s. \tag{6.34}$$

Since the function μ is locally integrable, then the process $(A_t)_{t\geq 0}$ is a.s. bounded on finite intervals, therefore we have a solution for $X(t)$ given by

$$X(t) = A(t) + \int_0^t \psi(t-s)A(s)\mathrm{d}s, \quad t \geq 0. \tag{6.35}$$

The proof is complete. □

6.3 Law of large numbers

In preparation for the proof of Theorem 6.3, we need the following Lemmata.

Lemma 6.9. *Assume* **(A0, A1)** *and* **(B0)**. *Then,*

$$\sup_{v\in[0,1]}\left|T^{-1}\mathbb{E}[N(Tv)]-v\frac{m}{1-\alpha}\right| \underset{T\to\infty}{\longrightarrow} 0. \tag{6.36}$$

If additionally we assume $\int_0^\infty x^r h(x)\mathrm{d}x<\infty$ *for* $r>1$ *and* $\int_0^\infty x^3 F(\mathrm{d}x)<\infty$, *we have for* $0\le p<1$ *that*

$$T^p\sup_{v\in[0,1]}\left|T^{-1}\mathbb{E}[N(Tv)]-v\frac{m}{1-\alpha}\right| \underset{T\to\infty}{\longrightarrow} 0. \tag{6.37}$$

Proof. We define the function

$$G(t):=\int_0^t \psi(t-s)\Phi(s)\mathrm{d}s,\quad t\ge 0. \tag{6.38}$$

By changing the order of integration and setting $\Psi(t)=\int_0^t\psi(s)\mathrm{d}s$ (note that $\Phi(t)=\int_0^t\Phi(\mathrm{d}s)$) we can rewrite G as

$$G(t)=\int_0^t \Psi(t-s)\Phi(\mathrm{d}s),\quad t\ge 0. \tag{6.39}$$

From the relation $(f*g)'=f'*g$, we can deduce

$$G'(t)=\int_0^t \psi(t-s)\Phi(\mathrm{d}s),\quad t\ge 0. \tag{6.40}$$

Let us analyze the asymptotics of $\psi=\sum_{n\ge1}h^{*n}$. For this, note that we can write ψ as the solution to the renewal equation,

$$\psi=h+H*\psi, \tag{6.41}$$

where $H:=\int_0^t h(u)\mathrm{d}u$. Equation (6.41) is a defective renewal equation (i.e. $H(\infty)=\lim_{t\to\infty}H(t)=\alpha<1$) with solution $\psi=\sum_{n\ge1}h^{*n}=h*(1+\psi)$. In the case of a defective renewal equation, it holds (c.f. V.7.4 in [1]),

$$\psi(t)\underset{t\to\infty}{\longrightarrow}\frac{h(\infty)}{1-H(\infty)}=\frac{0}{1-\alpha}=0. \tag{6.42}$$

Furthermore, ψ is uniformly continuous since it is the convolution of the integrable function h with the bounded function $1+\psi$. Boundedness of ψ can be seen from

$$h^{*n}*h=\int_0^t h^{*n}(t-s)h(s)\mathrm{d}s\le\alpha\left\|h^{*n}\right\|_\infty, \tag{6.43}$$

$$\left\|h^{*n}\right\|_\infty\le\alpha^{n-1}\left\|h\right\|_\infty, \tag{6.44}$$

$$\left\|\psi\right\|_\infty\le\frac{\left\|h\right\|_\infty}{1-\alpha}. \tag{6.45}$$

Then from (6.42), (6.45), and the Key renewal Theorem for spread-out distributions (c.f. Corollary VII.1.3 in [1]),

$$G'(\infty) := \lim_{t\to\infty} G'(t) = m\int_0^\infty \psi(s)\mathrm{d}s \tag{6.46}$$

$$= m\sum_{n\geq 1}\int_0^\infty h^{*n}(s)\mathrm{d}s = m\sum_{n\geq 1}\alpha^n = \frac{m\alpha}{1-\alpha}. \tag{6.47}$$

From Lemma 6.7, (2.121), and the fact that $v\frac{m}{1-\alpha} = v\big(m + \frac{m\alpha}{1-\alpha}\big)$, we have

$$T^p\left(T^{-1}\mathbb{E}[N(Tv)] - v\frac{m}{1-\alpha}\right) \tag{6.48}$$

$$= T^p\left[\left(vm - \frac{\Phi(Tv)}{T}\right) + \left(v\frac{m\alpha}{1-\alpha} - \frac{\int_0^{Tv}\psi(Tv-s)\Phi(s)\mathrm{d}s}{T}\right)\right] \tag{6.49}$$

$$= T^p\left[\left(vm - \frac{\Phi_2(Tv) + \int_0^{Tv}\varphi_1(s)\mathrm{d}s}{T}\right) + \left(vG'(\infty) - \frac{\int_0^{Tv}G'(s)\mathrm{d}s}{T}\right)\right] \tag{6.50}$$

$$= \left[\left(\frac{\int_0^{Tv} m - \varphi_1(s)\mathrm{d}s}{T^{1-p}}\right) + \left(\frac{\int_0^{Tv} G'(\infty) - G'(s)\mathrm{d}s}{T^{1-p}}\right) - \frac{\Phi_2(Tv)}{T^{1-p}}\right]. \tag{6.51}$$

Now we notice that

$$T^p \sup_{v\in[0,1]}\left|\left(T^{-1}\mathbb{E}[N(Tv)] - v\frac{m}{1-\alpha}\right)\right| \tag{6.52}$$

$$\leq \sup_{v\in[0,1]}\left|\frac{\int_0^{Tv} m - \varphi_1(s)\mathrm{d}s}{T^{1-p}}\right| + \sup_{v\in[0,1]}\left|\frac{\int_0^{Tv} G'(\infty) - G'(s)\mathrm{d}s}{T^{1-p}}\right| + \frac{\Phi_2(Tv)}{T^{1-p}} \tag{6.53}$$

$$\leq \frac{\int_0^T |m - \varphi_1(s)|\,\mathrm{d}s}{T^{1-p}} + \frac{\int_0^T |G'(\infty) - G'(s)|\,\mathrm{d}s}{T^{1-p}} + \frac{\Phi_2([0,\infty))}{T^{1-p}}. \tag{6.54}$$

If we assume **(B0)** and take $p = 0$, the result (6.36) follows immediately from the finiteness of the measure Φ_2 and the convergence of $G'(t) \underset{t\to\infty}{\longrightarrow} G'(\infty)$ and $\varphi_1(t) \underset{t\to\infty}{\longrightarrow} m$.

For $p > 0$ we want to study the rates of convergence of φ_1 and G'. From Lemma 5.9 we have that $\varphi_1(s) = m + o(s^{-2})$ as $s \to \infty$. Hence,

$$\frac{\int_0^T |m - \varphi_1(s)|\,\mathrm{d}s}{T^{1-p}} = \frac{O(\log T)}{T^{1-p}} \underset{T\to\infty}{\longrightarrow} 0. \tag{6.55}$$

For the second term, we have that

$$\int_0^\infty x^r\psi(x)\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty x^r h^{*n}(x)\mathrm{d}x \tag{6.56}$$

$$= \sum_{n=1}^\infty \int_0^\infty \cdots \int_0^\infty (x_1+\cdots+x_n)^r h(x_1)\cdots h(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{6.57}$$

$$\le \sum_{n=1}^\infty \int_0^\infty \cdots \int_0^\infty n^{r-1}(x_1^r+\cdots+x_n^r) h(x_1)\cdots h(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{6.58}$$

$$= \sum_{n=1}^\infty n^r\alpha^{n-1}\int_0^\infty x^r h(x)\mathrm{d}x < \infty. \tag{6.59}$$

It suffices to show that

$$\int_1^T |G'(\infty)-G'(s)|\,\mathrm{d}s = O(\log T). \tag{6.60}$$

We can proceed along the lines of the proof of Theorem 5.2 from which we have the bound

$$|G'(\infty)-G'(x)| \le C\int_{x/2}^\infty \psi(y)\mathrm{d}y + \int_{x/2}^x |\varphi_1(y)-m|\,\mathrm{d}y + \int_0^x \psi(x-y)\Phi_2(\mathrm{d}y), \tag{6.61}$$

and the constant C can be taken as $m+\|\varphi_1-m\|_\infty<\infty$. Looking at the first integral on the RHS, we notice that

$$\int_1^\infty\int_{x/2}^\infty \psi(y)\mathrm{d}y\mathrm{d}x \le \int_1^\infty\int_{x/2}^\infty \frac{(2y)^r}{x^r}\psi(y)\mathrm{d}y\mathrm{d}x \tag{6.62}$$

$$\le \int_1^\infty x^{-r}\mathrm{d}x\int_{x/2}^\infty (2y)^r\psi(y)\mathrm{d}y \tag{6.63}$$

$$\le \int_1^\infty x^{-r}\mathrm{d}x\int_0^\infty (2y)^r\psi(y)\mathrm{d}y < \infty. \tag{6.64}$$

For the second term, we see that the integral is at most of logarithmic order. Indeed,

$$\int_1^T\int_{x/2}^x |\varphi_1(y)-m|\,\mathrm{d}y\mathrm{d}x \le \int_{1/2}^T\int_y^{2y} |\varphi_1(y)-m|\,\mathrm{d}x\mathrm{d}y \tag{6.65}$$

$$= \int_{1/2}^T y\,|\varphi_1(y)-m|\,\mathrm{d}y \tag{6.66}$$

$$= O(\log T). \tag{6.67}$$

On the other hand,

$$\int_0^\infty\int_0^x \psi(x-y)\Phi_2(\mathrm{d}y)\mathrm{d}x = \int_0^\infty \psi(u)\mathrm{d}u\int_0^\infty \Phi_2(\mathrm{d}y) < \infty. \tag{6.68}$$

This concludes the proof. □

The next result treats the asymptotic behavior of the renewal process part.

Lemma 6.10. *Under* **(B0)***, we have almost surely that*

$$T^{-1}\sup_{t\le T}\left|\int_0^t \mu(s-T_{I(s)})\mathrm{d}s-\Phi(t)\right|\underset{T\to\infty}{\longrightarrow}0. \tag{6.69}$$

Proof. We rewrite the integral term as

$$\int_0^t \mu(s-T_{I(s)})\mathrm{d}s=\int_0^t \mu(s-S_{N_R(s)-1})\mathrm{d}s \tag{6.70}$$

$$=\sum_{j=1}^{N_R(t)-1}\int_{S_{j-1}}^{S_j}\mu(s-S_{j-1})\mathrm{d}s+\int_{S_{N_R(t)-1}}^{t}\mu(s-S_{N_R(t)-1})\mathrm{d}s \tag{6.71}$$

$$=\sum_{j=1}^{N_R(t)-1}\xi_j+\int_{S_{N_R(t)-1}}^{t}\mu(s-S_{N_R(t)-1})\mathrm{d}s, \tag{6.72}$$

where the ξ_j are i.i.d. random variables. To compute their mean we use the property (2.17) of the intensity and the fact that the process $1_{(S_{j-1},S_j]}(t)$ is predictable. We have,

$$\mathbb{E}[\xi_j]=\mathbb{E}\left[\int_{S_{j-1}}^{S_j}\mu(s-S_{j-1})\mathrm{d}s\right] \tag{6.73}$$

$$=\mathbb{E}\left[\int_0^\infty 1_{(S_{j-1},S_j]}(s)\mu(s-S_{j-1})\mathrm{d}s\right] \tag{6.74}$$

$$=\mathbb{E}\left[\int_0^\infty 1_{(S_{j-1},S_j]}(s)N_R(\mathrm{d}s)\right] \tag{6.75}$$

$$=\mathbb{E}[N_R((S_{j-1},S_j])]=1, \tag{6.76}$$

for all $1\le j\le N_R(t)-1$. On the one hand, from the Law of Large Numbers for ξ_j we have

$$\frac{1}{n}\sum_{j=1}^{n}\xi_j\xrightarrow{a.s.}1, \tag{6.77}$$

while from the LLN for the inter-arrival times of $N_R(\cdot)$ we have

$$\frac{1}{n}S_n=\frac{1}{n}\sum_{j=1}^{n}\tau_j\xrightarrow{a.s.}\frac{1}{m}. \tag{6.78}$$

Combining these two facts we obtain that

$$\frac{1}{S_n}\int_0^{S_n}\mu(s-T_{I(s)})\mathrm{d}s=\frac{1}{S_n}\sum_{j=1}^{n}\int_{S_{j-1}}^{S_j}\mu(s-S_{j-1})\mathrm{d}s=\frac{1}{S_n}\sum_{j=1}^{n}\xi_j\xrightarrow{a.s.}m \tag{6.79}$$

Since for $t > 0$ we may take $n = n(t)$ such that $S_{n-1} < t \leq S_n$,

$$R_t := \frac{1}{t} \int_0^t \mu(s - T_{I(s)}) \mathrm{d}s \xrightarrow[t\to\infty]{a.s.} m, \tag{6.80}$$

because,

$$\frac{1}{S_{n-1} + \tau_n} \int_0^{S_{n-1}} \mu(s - T_{I(s)}) \mathrm{d}s \leq R_t \leq \frac{1}{S_n - \tau_{n-1}} \int_0^{S_n} \mu(s - T_{I(s)}) \mathrm{d}s, \tag{6.81}$$

and the limit on each side is equal to m a.s.

Furthermore, from the elementary renewal theorem, i.e.

$$\frac{\Phi(t)}{t} \xrightarrow[t\to\infty]{} m, \tag{6.82}$$

and the fact that $\Phi(t) \geq mt$ for all $t \geq 0$, we obtain the desired conclusion. □

We can proceed with the proof of Theorem 6.3.

Proof of Theorem 6.3. We use (6.36) of Lemma 6.9, then it suffices to prove that

$$T^{-1} \sup_{v\in[0,1]} |N(Tv) - \mathbb{E}[N(Tv)]| \xrightarrow[T\to\infty]{a.s.} 0. \tag{6.83}$$

Set

$$X(t) := N(t) - \mathbb{E}[N(t)], \quad t \geq 0. \tag{6.84}$$

From Lemma 6.8 we know that for $T \geq 0$ and $v \in [0, 1]$

$$X(Tv) = A(Tv) + \int_0^{Tv} \psi(Tv - s) A(s) \mathrm{d}s, \tag{6.85}$$

with $A(Tv) = M(Tv) + \int_0^{Tv} \mu\big(s - T_{I(s)}\big)\, \mathrm{d}s - \Phi(Tv)$. Then

$$\sup_{v\in[0,1]} |X(Tv)| \leq \sup_{t\leq T} |A(t)| + \sup_{t\leq T} \int_0^t \psi(t-s)\, |A(s)|\, \mathrm{d}s \tag{6.86}$$

$$\leq \sup_{t\leq T} |A(t)| \left(1 + \int_0^\infty \psi(s) \mathrm{d}s\right), \tag{6.87}$$

where we note that $\psi(\cdot)$ is integrable. Now, shifting attention to $\sup\limits_{t\leq T} |A(t)|$, we obtain the bound

$$\sup_{t\leq T} |A(t)| \leq \sup_{t\leq T} |M(t)| + \sup_{t\leq T} \left| \int_0^t \mu(s - T_{I(s)}) \mathrm{d}s - \Phi(t) \right|. \tag{6.88}$$

Consider the characteristic martingale $M(t)$, and define the martingale

$$Z(t) = \int_{(0,t]} \frac{1}{s+1} \mathrm{d}M_s. \tag{6.89}$$

We compute its quadratic variation

$$\begin{aligned}
[Z, Z]_t &= \sum_{0<u\leq t} (Z(u) - Z(u-))^2 && (6.90)\\
&= \sum_{0<u\leq t} \left(\int_{(0,u]} \frac{1}{s+1} \mathrm{d}M_s - \int_{(0,u)} \frac{1}{s+1} \mathrm{d}M_s \right)^2 && (6.91)\\
&= \sum_{0<u\leq t} \left(\frac{1}{u+1} (N(u) - N(u-)) \right)^2 && (6.92)\\
&= \int_{(0,t]} \frac{1}{(s+1)^2} N(\mathrm{d}s). && (6.93)
\end{aligned}$$

Now, by integration by parts, we obtain

$$\begin{aligned}
\int_0^t \frac{1}{(u+1)^2} N(\mathrm{d}u) - \frac{N(t)}{(t+1)^2} &= \int_0^t \left[\frac{1}{(u+1)^2} - \frac{1}{(t+1)^2} \right] N(\mathrm{d}u) && (6.94)\\
&= \int_0^t \int_u^t \frac{2}{(s+1)^3} \mathrm{d}s N(\mathrm{d}u) && (6.95)\\
&=2 \int_0^t \int_0^s \frac{N(\mathrm{d}u)}{(s+1)^3} \mathrm{d}s && (6.96)\\
&=2 \int_0^t \frac{N(s)}{(s+1)^3} \mathrm{d}s. && (6.97)
\end{aligned}$$

From the previous equalities, we have

$$\mathbb{E}\left[\int_0^t \frac{1}{(s+1)^2} N(\mathrm{d}s) \right] = 2\mathbb{E}\left[\int_0^t \frac{N(s)}{(s+1)^3} \mathrm{d}s \right] + \frac{\mathbb{E}[N(t)]}{(t+1)^2}. \tag{6.98}$$

We can analyze the second term on the RHS by using Lemma 6.7 and the increasingness of $\Phi(\cdot)$,

$$\frac{\mathbb{E}[N(t)]}{(t+1)^2} = \frac{\Phi(t)}{(t+1)^2} + \int_0^t \frac{\psi(t-s)}{(t+1)^2} \Phi(s) \mathrm{d}s \leq \frac{\Phi(t)}{(t+1)^2} \left\{ 1 + \int_0^t \psi(t-s) \mathrm{d}s \right\} \underset{t\to\infty}{\longrightarrow} 0. \tag{6.99}$$

Using the monotone convergence theorem and the elementary renewal theorem, we obtain,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \frac{1}{(s+1)^2} N(\mathrm{d}s) \right] = 2\mathbb{E}\left[\int_0^\infty \frac{N(s)}{(s+1)^3} \mathrm{d}s \right] && (6.100)\\
&=2 \int_0^\infty \frac{\mathbb{E}[N(s)]}{(s+1)^3} \mathrm{d}s = 2 \int_0^\infty \frac{\Phi(s)}{(s+1)^3} \mathrm{d}s + 2 \int_0^\infty \frac{\int_0^s \psi(s-u)\Phi(u)\mathrm{d}u}{(s+1)^3} \mathrm{d}s && (6.101)\\
&\leq 2 \int_0^\infty \frac{\Phi(s)}{(s+1)^3} \mathrm{d}s + 2 \int_0^\infty \frac{\Phi(s)}{(s+1)^3} \mathrm{d}s \int_0^\infty \psi(u)\mathrm{d}u < +\infty. && (6.102)
\end{aligned}$$

This tells us that $Z(\cdot)$ is a martingale bounded in $L^2(\mathbb{P})$, therefore, by the martingale convergence theorem $\lim_{t\to\infty} Z(t)$ exists and is finite a.s. Let us recall that $M(0) = 0$, and consider

$$\int_0^t Z(s)\mathrm{d}s = \int_0^t \int_0^s \frac{\mathrm{d}M_u}{u+1}\mathrm{d}s \tag{6.103}$$

$$= \int_0^t \frac{t-u}{u+1}\mathrm{d}M_u \tag{6.104}$$

$$=(t+1)\int_0^t \frac{\mathrm{d}M_u}{u+1} - \int_0^t \frac{u+1}{u+1}\mathrm{d}M_u \tag{6.105}$$

$$=(t+1)Z(t) - M(t). \tag{6.106}$$

Furthermore, from the finiteness of the limit of $Z(t)$ it holds that

$$\frac{1}{t+1}M(t) = Z(t) - \frac{1}{t+1}\int_0^t Z(s)\mathrm{d}s \xrightarrow[t\to\infty]{a.s.} 0. \tag{6.107}$$

Finally, we show that the convergence is uniform in $v \in [0,1]$. Let $0 < \epsilon < 1$. For $0 \le v < \epsilon$, we have,

$$\left|\frac{M(Tv)}{T}\right| = \left|\frac{M(Tv)}{Tv}v\right| \le \epsilon \sup_{0<t<\infty}\left|\frac{M(t)}{t}\right|. \tag{6.108}$$

Meanwhile, for $\epsilon \le v \le 1$,

$$\left|\frac{M(Tv)}{T}\right| \le \sup_{T\epsilon<t<\infty}\left|\frac{M(t)}{t}\right|. \tag{6.109}$$

Hence, we obtain,

$$\limsup_{T\to\infty}\left(\sup_{v\in[0,1]}\left|\frac{M(Tv)}{T}\right|\right) \le \sup_{0<t<\infty}\left|\frac{M(t)}{t}\right|\epsilon. \tag{6.110}$$

Since this was done for an arbitrary ϵ, we conclude the desired uniform convergence. □

Remark. Notice that we recover the result of Bacry–Delattre–Hoffman–Muzy [5] by taking the imbedded renewal process as an homogeneous Poisson process of constant intensity m.

6.4 Central limit theorem

Lemma 6.11. *Assume* **(A0, A1)**, **(B0)** *and* $\int_0^\infty x^2 F(\mathrm{d}x) < \infty$. *Set*

$$Q(t) := \int_0^t \mu\big(s - T_{I(s)}\big)\,\mathrm{d}s - \Phi(t), \tag{6.111}$$

and define for each $T > 0$,

$$Q^{(T)}(v) := \big(T^{-1/2} Q(Tv)\big)_{v\in[0,1]} \quad \textit{and} \quad M^{(T)} := \big(T^{-1/2} M(Tv)\big)_{v\in[0,1]}. \tag{6.112}$$

Then, the processes $\big(M^{(T)}, Q^{(T)}\big)$ converge jointly in distribution to $\Big(\sigma_M W, \sigma_R \widetilde{W}\Big)$, where $W = (W(v))_{v\in[0,1]}$ and $\widetilde{W} = (\widetilde{W}(v))_{v\in[0,1]}$ are independent standard Brownian motions and,

$$\sigma_M^2 = \frac{m}{1-\alpha}, \qquad \frac{\sigma_R^2}{m} = 3 + m^2 \operatorname{Var}[\tau] - 2m\mathbb{E}\left[\tau \int_0^\tau \mu(s)\mathrm{d}s\right]. \tag{6.113}$$

Proof. Since from Lemma 2.26 we can conclude that

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} (\Phi(Tv) - mTv) = 0, \tag{6.114}$$

then we can look instead of $Q^{(T)}$ at the convergence of

$$\widetilde{Q}^{(T)}(v) := \frac{1}{\sqrt{T}} \left[\int_0^{Tv} \mu(s - T_{I(s)})\mathrm{d}s - mTv\right], \tag{6.115}$$

as $T \to \infty$. Recall our definition for $\xi_j := \int_{S_{j-1}}^{S_j} \mu(s - S_{j-1})\mathrm{d}s$. We can rewrite,

$$\widetilde{Q}^{(T)}(v) = \frac{1}{\sqrt{T}} \left[\sum_{j=1}^{N_R(Tv)-1} \xi_j + \int_{S_{N_R(Tv)-1}}^{Tv} \mu(s - S_{N_R(Tv)-1})\mathrm{d}s - mTv\right]. \tag{6.116}$$

From Theorem 5.3 we know that,

$$\frac{1}{\sqrt{T}} \int_{S_{N_R(Tv)-1}}^{Tv} \mu(s - S_{N_R(Tv)-1})\mathrm{d}s \xrightarrow[T\to\infty]{d} 0, \tag{6.117}$$

hence, it is enough to study the convergence of

$$\frac{1}{\sqrt{T}} \left[\sum_{j=1}^{N_R(Tv)-1} \xi_j - mTv\right] = \frac{1}{\sqrt{T}} \left[\sum_{j=1}^{N_R(Tv)-1} (\xi_j - m\tau_j) - m\big(Tv - S_{N_R(Tv)-1}\big)\right]. \tag{6.118}$$

From Theorem 5.4 we also have,

$$\frac{1}{\sqrt{T}} \left(Tv - \sum_{j=1}^{N_R(Tv)-1} m\tau_j\right) = \frac{1}{\sqrt{T}} \big(Tv - S_{N_R(Tv)-1}\big) \xrightarrow[T\to\infty]{d} 0. \tag{6.119}$$

This means that $Q^{(T)}$ has the same limit in distribution as

$$M_Q^{(T)}(v) := \frac{1}{\sqrt{T}} \sum_{j=1}^{N_R(Tv)-1} (\xi_j - m\tau_j). \tag{6.120}$$

Notice that $\{\xi_j - m\tau_j\}_{j\geq 1}$ is a sequence of i.i.d. random variables with mean zero, as $\mathbb{E}[\xi_j - m\tau_j] = 1 - m\frac{1}{m} = 0$. Thus, $M_Q^{(T)}(v)$ is integrable, and clearly it is adapted. Now, take $0 < \delta < v$,

$$\mathbb{E}\left[M_Q^{(T)}(v)\middle|\mathcal{F}_{T\delta}\right] = \frac{1}{\sqrt{T}}\sum_{j=1}^{N_R(T\delta)-1}(\xi_j - m\tau_j) + \mathbb{E}\left[\frac{1}{\sqrt{T}}\sum_{j=N_R(T\delta)}^{N_R(Tv)}(\xi_j - m\tau_j)\middle|\mathcal{F}_{T\delta}\right] \tag{6.121}$$

$$= \frac{1}{\sqrt{T}}\sum_{j=1}^{N_R(T\delta)-1}(\xi_j - m\tau_j) = M_Q^{(T)}(\delta). \tag{6.122}$$

Thus, $M_Q^{(T)}(v)$ is a martingale. We have then reduced the analysis of the convergence of $M^{(T)} + Q^{(T)}$ to that of the martingale $M^{(T)} + M_Q^{(T)}$. Since the martingales have uniformly bounded jumps, by appealing to a suitable version of the martingale convergence theorem, c.f.(Theorem 14.14 in [20]), we can find the limit if we compute the quadratic variation. From Theorem 6.3 we get,

$$\left[M^{(T)}, M^{(T)}\right]_v = \sum_{s\leq v}\left[T^{-1/2}(N(Ts) - N(Ts-))\right]^2 \tag{6.123}$$

$$= T^{-1}N(Tv) \underset{T\to\infty}{\longrightarrow} \sigma_M^2 v. \tag{6.124}$$

For the cross-term, we have that only the common jumps of $M^{(T)}$ and $M_Q^{(T)}$ do not vanish,

$$\left[M^{(T)}, M_Q^{(T)}\right]_v = \frac{1}{T}\sum_{s\leq v}\left(M^{(T)}(s) - M^{(T)}(s-)\right)\left(M_Q^{(T)}(s) - M_Q^{(T)}(s-)\right) \tag{6.125}$$

$$= \frac{1}{T}\sum_{s\leq v}(N_R(Ts) - N_R(Ts-))\left(\xi_{N_R(Ts)} - m\tau_{N_R(Ts)}\right) \tag{6.126}$$

$$= \frac{1}{T}\sum_{s\leq v} 1\cdot\left(\xi_{N_R(Ts)} - m\tau_{N_R(Ts)}\right) \tag{6.127}$$

$$= \frac{1}{T}\sum_{j=1}^{N_R(Tv)-1}(\xi_j - m\tau_j) \underset{T\to\infty}{\longrightarrow} 0. \tag{6.128}$$

Finally,

$$\left[M_Q^{(T)}, M_Q^{(T)}\right]_v = \frac{1}{T}\sum_{s\leq v}\left(\xi_{N_R(Ts)} - m\tau_{N_R(Ts)}\right)^2 \tag{6.129}$$

$$= \frac{1}{T}\sum_{j=1}^{N_R(Tv)-1}(\xi_j - m\tau_j)^2 \underset{T\to\infty}{\longrightarrow} mv\,\mathrm{Var}[\xi - m\tau]\,. \tag{6.130}$$

Let us compute this variance:

$$\mathrm{Var}[\xi - m\tau] = \mathrm{Var}[\xi] + m^2\,\mathrm{Var}[\tau] - 2m\,\mathrm{Cov}[\xi, \tau]\,. \tag{6.131}$$

We need to find $\mathrm{Var}[\xi]$. First we compute

$$\mathbb{E}\left[\xi^2\right] = \mathbb{E}\left[\left(\int_0^{\tau_1} \mu(s)\,\mathrm{d}s\right)^2\right] = \int_0^\infty \left(\int_0^t \mu(s)\,\mathrm{d}s\right)^2 f(t)\mathrm{d}t. \tag{6.132}$$

From the definition of the hazard function in (3.3) we have

$$\left(\int_0^t \mu(s)\mathrm{d}s\right)^2 = \left(\log\left(\frac{f(t)}{\mu(t)}\right)\right)^2 = \log^2\left(\frac{f(t)}{\frac{f(t)}{1-\int_0^t f(s)\mathrm{d}s}}\right) \tag{6.133}$$

$$= \left(\log\left(1 - \int_0^t f(s)\mathrm{d}s\right)\right)^2. \tag{6.134}$$

Substituting this into (6.132) yields

$$\mathbb{E}\left[\left(\int_0^{\tau_1} \mu(s)\,\mathrm{d}s\right)^2\right] = \int_0^\infty \log^2\left(1 - \int_0^t f(s)\mathrm{d}s\right) f(t)\mathrm{d}t. \tag{6.135}$$

We make the change of variable $u = 1 - \int_0^t f(s)\mathrm{d}s$, so $\mathrm{d}u = -f(t)\mathrm{d}t$.

$$\int_0^\infty \left(\log\left(1 - \int_0^t f(s)\mathrm{d}s\right)\right)^2 f(t)\mathrm{d}t = \int_0^1 (\log(u))^2\mathrm{d}u = 2. \tag{6.136}$$

Thus,

$$\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - \mathbb{E}[\xi]^2 = 2 - 1 = 1, \tag{6.137}$$

and hence,

$$\mathrm{Var}[\xi - m\tau] = 1 + m^2\,\mathrm{Var}[\tau] - 2m(\mathbb{E}[\xi\tau] - \mathbb{E}[\xi]\,\mathbb{E}[\tau]) \tag{6.138}$$

$$= 1 + m^2\,\mathrm{Var}[\tau] - 2m\left(\mathbb{E}\left[\tau\int_0^\tau \mu(s)\mathrm{d}s\right] - \frac{1}{m}\right) \tag{6.139}$$

$$= 3 + m^2\,\mathrm{Var}[\tau] - 2\mathbb{E}\left[\tau\int_0^\tau \mu(s)\mathrm{d}s\right] = \frac{\sigma_R^2}{m}. \tag{6.140}$$

We therefore obtain the desired result. □

We can now present the proof of the central limit theorem.

Proof of Theorem 6.4. Let us write $\|f\|_\infty = \sup_{v\in[0,1]} |f(v)|$. For $\delta > 0$, we denote,

$$\omega_\delta(f) := \sup_{|u-u'|\le\delta} |f(u) - f(u')|\,. \tag{6.141}$$

From Lemma 6.11 we have $M^{(T)} \xrightarrow{d} \sigma_M W$, so by Skorokhod's Representation Theorem there exists an a.s. convergent coupling $\left(\widehat{M}^{(T)}, \widehat{W}\right)$, i.e.,

$$M^{(T)} \overset{\mathrm{d}}{=} \widehat{M}^{(T)} \xrightarrow{a.s.} \sigma_M \widehat{W} \overset{\mathrm{d}}{=} \sigma_M W. \tag{6.142}$$

Because $\widehat{W}$ is a continuous process, we have $\left\|\widehat{M}^{(T)} - \sigma_M \widehat{W}\right\|_\infty \xrightarrow{a.s.} 0$. Hence we obtain

$$\left\|M^{(T)}\right\|_\infty \xrightarrow{d} \left\|\sigma_M \widehat{W}\right\|_\infty, \tag{6.143}$$

$$\omega_\delta\big(M^{(T)}\big) \xrightarrow{d} \omega_\delta\left(\sigma_M \widehat{W}\right). \tag{6.144}$$

Set

$$X^{(T)}(v) := T^{-1/2}(N(Tv) - \mathbb{E}[N(Tv)]) \tag{6.145}$$

$$= T^{-1/2}\left\{M(Tv) + Q(Tv) + \int_0^{Tv} \psi(Tv - s)[M(s) + Q(s)]\,\mathrm{d}s\right\}. \tag{6.146}$$

From Lemma 6.11, it is enough to show that

$$\left\|X^{(T)} - \frac{1}{1-\alpha}M^{(T)} - \frac{1}{1-\alpha}Q^{(T)}\right\|_\infty \xrightarrow[T\to\infty]{p} 0. \tag{6.147}$$

Since $\frac{\alpha}{1-\alpha} = \int_0^\infty \psi(t)\mathrm{d}t$,

$$X^{(T)}(v) - \frac{1}{1-\alpha}M^{(T)}(v) - \frac{1}{1-\alpha}Q^{(T)}(v) = X_M^{(T)}(v) + X_Q^{(T)}(v), \tag{6.148}$$

where,

$$X_M^{(T)}(v) := \int_0^v T\psi(Tu)M^{(T)}(v-u)\mathrm{d}u - \left(\int_0^\infty \psi(t)\mathrm{d}t\right) M^{(T)}(v), \tag{6.149}$$

$$X_Q^{(T)}(v) := \int_0^v T\psi(Tu)Q^{(T)}(v-u)\mathrm{d}u - \left(\int_0^\infty \psi(t)\mathrm{d}t\right) Q^{(T)}(v). \tag{6.150}$$

We want to find a bound for

$$\left\|X_M^{(T)} + X_Q^{(T)}\right\|_\infty \le \left\|X_M^{(T)}\right\|_\infty + \left\|X_Q^{(T)}\right\|_\infty. \tag{6.151}$$

Let us consider $\left\|X_M^{(T)}\right\|_\infty$. Take $0 < \delta \le 1$ and note that

$$\sup_{v\in[0,\delta]} \left|X_M^{(T)}(v)\right| \le 2\left(\sup_{v\in[0,\delta]} \left|M^{(T)}(v)\right|\right) \int_0^\infty \psi(u)\mathrm{d}u \xrightarrow{d} 2\left(\sup_{v\in[0,\delta]} |\sigma_M W(v)|\right) \int_0^\infty \psi(u)\mathrm{d}u. \tag{6.152}$$

Note also that

$$\sup_{v\in[\delta,1]}\left|\int_\delta^v T\psi(Tu)M^{(T)}(v-u)\mathrm{d}u\right| \leq \left\|M^{(T)}\right\|_\infty \int_\delta^\infty T\psi(Tu)\mathrm{d}u \tag{6.153}$$

$$= \left\|M^{(T)}\right\|_\infty \int_{T\delta}^\infty \psi(u)\mathrm{d}u \xrightarrow[T\to\infty]{d} 0, \tag{6.154}$$

since $\left\|M^{(T)}\right\|_\infty \xrightarrow{d} \|\sigma_M W\|_\infty < \infty$. It is easily seen that,

$$\sup_{v\in[\delta,1]}\left|\int_0^\delta T\psi(Tu)\mathrm{d}u\, M^{(T)}(v) - \int_0^\infty \psi(u)\mathrm{d}u\, M^{(T)}(v)\right| \leq \left\|M^{(T)}\right\|_\infty \int_{T\delta}^\infty \psi(u)\mathrm{d}u \xrightarrow{d} 0, \tag{6.155}$$

and that,

$$\sup_{v\in[\delta,1]}\left|\int_0^\delta T\psi(Tu)M^{(T)}(v-u)\mathrm{d}u - \int_0^\delta T\psi(Tu)M^{(T)}(v)\mathrm{d}u\right| \tag{6.156}$$

$$\leq \omega_\delta(M^{(T)})\int_0^\infty \psi(u)\mathrm{d}u \xrightarrow[T\to\infty]{d} \omega_\delta(\sigma_M W)\int_0^\infty \psi(u)\mathrm{d}u. \tag{6.157}$$

In summary, if we denote

$$G_M(T,\delta) := 2\left\|M^{(T)}\right\|_\infty \int_{T\delta}^\infty \psi(u)\mathrm{d}u + \left(2\sup_{v\in[0,\delta]}\left|M^{(T)}(v)\right| + \omega_\delta(M^{(T)})\right)\int_0^\infty \psi(u)\mathrm{d}u, \tag{6.158}$$

$$G_M(\delta) := \left(2\sup_{v\in[0,\delta]}|\sigma_M W(v)| + \omega_\delta(\sigma_M W)\right)\int_0^\infty \psi(u)\mathrm{d}u, \tag{6.159}$$

then,

$$\left\|X_M^{(T)}\right\|_\infty \leq G_M(T,\delta) \xrightarrow[T\to\infty]{d} G_M(\delta) \xrightarrow[\delta\downarrow 0]{d} 0. \tag{6.160}$$

For $\epsilon > 0$, from the Portmanteau Theorem applied to the closed set $[\epsilon,\infty)$, we have

$$\limsup_{T\to\infty}\mathbb{P}\left(\left\|X_M^{(T)}\right\|_\infty \geq \epsilon\right) \leq \limsup_{T\to\infty}\mathbb{P}(G_M(T,\delta)\geq\epsilon) \leq \mathbb{P}(G_M(\delta)\geq\epsilon) \xrightarrow[\delta\downarrow 0]{} 0. \tag{6.161}$$

Since ϵ was taken arbitrarily, that means,

$$\left\|X_M^{(T)}\right\|_\infty \xrightarrow[T\to\infty]{p} 0. \tag{6.162}$$

Notice that the proof of $\left\|X_Q^{(T)}\right\|_\infty \xrightarrow[T\to\infty]{p} 0$ can be carried out in exactly the same way. Ultimately, the previous reasoning proves (6.147), and therefore the proof is complete. □

Remark. This result reduces to that of Bacry–Delattre–Hoffman–Muzy [5] since by taking the imbedded renewal process as an homogeneous Poisson process of constant intensity m, then $\sigma_R = 0$ and σ_M is that of Theorem 6.4.

Finally, we prove the asymptotic normality stated in Corollary 6.5.

Proof of Corollary 6.5. Taking Lemma 6.9 with $p = \frac{1}{2}$ yields

$$\sup_{v\in[0,1]} \left| \frac{1}{\sqrt{T}}\mathbb{E}[N(Tv)] - v\frac{m}{1-\alpha}\sqrt{T} \right| \underset{T\to\infty}{\longrightarrow} 0. \tag{6.163}$$

Furthermore, from Theorem 6.4 we know that

$$\left(\frac{1}{\sqrt{T}}N(Tv) - \frac{1}{\sqrt{T}}\mathbb{E}[N(Tv)] \right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} (\sigma W(v))_{v\in[0,1]}, \tag{6.164}$$

from which the result follows. □

This consolidates our treatment of the RHP and renewal processes. As for possible room for improvement there can be further refinements for the speed of convergence in the law of large numbers. We can as well look into convergence in L_1 and L_2 to completely extend the law of large numbers presented in Bacry–Delattre–Hoffman–Muzy. This could lead to the possibility of analyzing second order properties too, namely, the estimation of the covariance. We conclude this thesis here.

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