On the ramified Siegel series

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Abstract

The ramified Siegel series is an essential factor in the Fourier coefficient of the Siegel-Eisenstein series. There are already many results, but the explicit formula of the general condition is currently an open problem. In this article, by programming PARI/GP, we computationally obtain the matrix of intertwining operators when n is 1 to 8. We also calculate the recursion formula of the ramified Siegel series.

1 Introduction

1.1 Background of the Siegel series

1.1.1 Hilbert's 11th problem

On August 8, 1900, at the Paris conference of the International Congress of Mathematicians, German mathematician David Hilbert presented ten open problems. Later, in 1902, he published an article titled "Mathematical Problems" [11]. This article has 23 open problems, now famous for Hilbert's 23 problems.

The 11th problem of this article is about "Quadratic forms with any algebraic numerical coefficients." Hilbert's article says that this problem is concerned *to attach successfully the theory of quadratic forms with any number of variables and with any algebraic numerical coefficients.*

Quadratic forms have been studied for centuries. It is defined as when K is a field,

$$Q(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j,$$

where the coefficients a_{ij} are elements in K.

Several famous results had already been known until the 1900s, for example, Fermat's theorem on sums of two squares. Let p be an odd prime number. The statement of this theorem is

$$p \equiv 1 \pmod 4 \iff {}^{\exists} x, y \in \mathbb{Z},\ p = x^2 + y^2.$$

Fermat announced this theorem in 1640 without any proof. The first proof was given by Euler, which is based on infinite descent [1] [2].

Quadratic forms had been usually studied with the coefficients in $\mathbb{R}$ or $\mathbb{C}$, but Minkowski [24] solved the equivalence theorem of quadratic forms over $\mathbb{Q}$. Hilbert himself studied in

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1899 [10] where the coefficients in a non-real algebraic number fields with odd class numbers. He introduced the Hilbert symbol and showed the condition if $ax^2 + by^2 = 1$ has a solution in K.

In 1923 and 1924, Hasse published five papers about quadratic forms [5–9]. The First to the third one was about the quadratic forms over $\mathbb{Q}$, and the fourth and the fifth were about algebraic number fields. His result is famous for Hasse's local-global principle.

We need some definition. Let D be an integral domain, and $f(x) = {}^t xSx$, $x = {}^t(x_1, \cdots, x_m)$, ${}^tS = S \in M_m(D)$ be a quadratic form with m variants with coefficients in D. We say the quadratic from $g(y) = {}^t yTy$ associated to T is represented by f over D if there exists $X \in M_{m,n}(D)$ such that $T = {}^tXSX$. When $m = n$ and X is invertible, f and g is called equivalent. For $c \in D$, we say f represents c over D when there exists $a \in D\backslash\{0\}$ such that $f(a) = c$.

Let k be an algebraic number field of finite degree. For each place $\mathfrak{p}$ of k, we write the completion of k as $k_{\mathfrak{p}}$. Let Ω be the set of all places of k.

Then Hasse's principle is written as

Theorem 1.1. *Let f, g be two quadratic forms over algebraic number field k. $P_i(k)$ $(i = 1, 2, 3)$ are propositions with variable k defined as*

$P_1(k)$: f represents 0 over k.

$P_2(k)$: f and g are equivalent over k.

$P_3(k)$: g is represented by f over k.

Then for each i, $P_i(k)$ holds if and only if $P_i(k_{\mathfrak{p}})$ holds for all $\mathfrak{p} \in \Omega$.

1.1.2 Siegel's study and the local density

In Siegel's paper [29], he studied the localization principle of the number of how many matrices X represent g by f.

Let S and T be positive definite symmetric matrix over $\mathbb{Z}$, and we write their size as m and n. Let $A(S,T)$ be a number of matrix $X \in M_{m,n}(\mathbb{Z})$ such that ${}^tXSX = T$. We write $E(S)$ for $A(S,S)$. When S and S_1 are equivalent over $\mathbb{R}$ and $\mathbb{Q}_p$ for all prime p, we say they are in the same genus. Each genus is the sum of finitely many equivalent classes, hence we denote by $S_1, \cdots, S_h$ the representative of the equivalent classes contained in the same genus with S. Put

$$M(S,T) = \sum_{k=1}^{h} \frac{A(S_k,T)}{E(S_k)}, \quad M(S) = \sum_{k=1}^{h} E(S_k)^{-1}, \quad A_0(S,T) = \frac{M(S,T)}{M(S)}.$$

Let $q > 0$ be a natural number, and $A_q(S,T)$ be the number of the solution of the equation ${}^tXSX \equiv T \mod q$. We assume $q = p^a$ for some prime p. In this case, when a is sufficiently large, the number $q^{-mn+n(n+1)/2}A_q(S,T)$ is independent of a. We put this value $\alpha_p(S,T)$.

We identify $\mathrm{Sym}_n(\mathbb{R})$, the set of symmetric matrices over $\mathbb{R}$ of size n, and $\mathbb{R}^{n(n+1)/2}$, the Euclid space of degree $n(n+1)/2$. Let $B \subset \mathbb{R}^{n(n+1)/2}$ be a Jordan measurable open subset which has a point T. Define B_1 as

$$B_1 = \{X \in \mathbb{R}^{n(n+1)/2} \mid {}^tXSX \in B\}.$$

Since the limit of the value $\dfrac{\mathrm{vol}(B_1)}{\mathrm{vol}(B)}$ as the radius of B approaches to 0 exists, we write this limit $\alpha_\infty(S,T)$. Then the Siegel's theorem is

Theorem 1.2. *We have*

$$A_0(S,T) = \alpha_\infty(S,T) \prod_p \alpha_p(S,T).$$

This theorem says that some weighted average of the measures $\mathbb{Z}$-solutions of the equation ${}^t xSx = T$ over the representative of each genus of S is related to local density. In [30], he extended this result to the case when S and T are not definite, and in [31], to the quadratic form over an arbitrary algebraic number field.

Weil reinterpreted this theorem in terms of representation theory and extended this result to more classical groups. [38] His result is known as the Siegel-Weil formula, which says that a theta integral corresponds to the Eisenstein series when both are absolutely convergent.

1.1.3 Local density

We assume the field F is a non-dyadic, non-archimedean local field. Later the Haar measure on $\mathrm{Sym}_n(F)$ and $M_{mn}(F)$ is normalized as

$$\int_{\mathrm{Sym}_n(\mathfrak{o})} dY = 1, \quad \int_{M_{mn}(\mathfrak{o})} dX = 1.$$

For $S \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o}),\ T \in \mathcal{H}_n(\mathfrak{o})$ and an integer $e > 0$, we define $\mathcal{A}_e(S,T)$ as

$$\mathcal{A}_e(S,T) = \{X \in M_{mn}(\mathfrak{o}) \mid S[X] - T \in \pi^e \mathcal{H}_n(\mathfrak{o})\}.$$

The local density is defined as follows.

Theorem 1.3. *The following limit exists, which is known as the local density related to S and T.*

$$\alpha(S,T) := \lim_{e\to\infty} q^{en(n+1)/2} \mathrm{vol}(\mathcal{A}_e(S,T)).$$

Siegel [29] calculated when p does not divide $2\det S \det T$, and after this, some results are known.

- Ozeki [26], when S is unimodular, size of $S \geq 4$, size of $T = 2$, p is odd.
- Kitaoka [21], when S is unimodular, size of $S \geq 3$, size of $T = 2$, p is odd.
- Kitaoka [22], when S is unimodular, size of $T = 3$, p is odd.
- Katsurada [20], when S is unimodular, size of $T = 3$.

Katsurada [19] gave a complete formula when S is unimodular. Yang [39] calculated when the size of T is 2 and p is odd. Compared to Katsurada's article, by using the recursion formula of the local density, Yang did by calculating some character sum. This idea was also used in the article of Sato and Hironaka.

1.1.4 The result of Sato and Hironaka

Sato and Hironaka [12] gives an explicit formula of the local density $\alpha(S,T)$ in the case $p \neq 2$ and $F = \mathbb{Q}_p$. They give the complete representative of $\Gamma_0(p)\backslash\mathrm{Sym}_n(\mathbb{Q}_p)$, where $\Gamma_0(p)$ is the Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$:

$$\Gamma_0(p) = \{\gamma = (\gamma_{ij}) \in \mathrm{GL}_n(\mathbb{Z}_p) \mid \gamma_{ij} \equiv 0 \mod p \text{ if } i > j\}.$$

This representative is written as $\{S_{\sigma,e,\varepsilon} \mid (\sigma,e,\varepsilon) \in \Lambda_n\}$, where the set Λ_n is defined as

$$\Lambda_n = \left\{(\sigma,e,\varepsilon) \in \mathfrak{S}_n \times \mathbb{Z}^n \times \{1,\delta\}^n \mid \sigma^2 = 1,\ e_{\sigma(i)} = e_i\,(1 \leq i \leq n),\ \varepsilon_i = 1\,(\sigma(i) \neq i)\right\},$$

and the symmetric matrix $S_{\sigma,e,\varepsilon}$ is defined as

$$S_{\sigma,e,\varepsilon} = \left(\varepsilon_i p^{e_i} \delta_{i,\sigma(j)}\right)_{i,j}.$$

Compared with $\Gamma = \mathrm{GL}_n(\mathbb{Z}_p)$, this choice of $\Gamma_0(p)$ can give us the calculation of the Gauss sum

$$\mathcal{G}_\Gamma(Y,T) = \int_\Gamma \psi(-\mathrm{tr}(Y \cdot {}^t\gamma T \gamma))d\gamma.$$

Since the local density is written using this term, an explicit formula of the local density can be represented.

In particular, the local density is given as

$$\begin{aligned}
\alpha(S,T) = &\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma^2=1}} 2^{-c_1(\sigma)}(1-p^{-1})^{c_2(\sigma)}p^{-c_2(\sigma)} \sum_{I=I_0\cup\cdots\cup I_r} p^{-\tau(\{I_i\})-t(\sigma,\{I_i\})} \\
&\times \sum_{k=0}^{r+1} \frac{2^{c_1^{(k)}(\sigma)}(1-p^{-1})^{c_1^{(k)}(\sigma)}p^{-\sum_{l=k+1}^{r} n(l)}}{\prod_{l=k}^{r}(1-p^{-n(l)})} \\
&\times {\sum_{\{\nu\}_k}}' p^{\sum_{l=0}^{k-1}\nu_l(n(k)-n(l))} \prod_{l=0}^{k-1} \Xi_{l,\nu_0+\cdots+\nu_l}(\sigma;T,S),
\end{aligned}$$

where the notations are

$$\begin{aligned}
I &= \{1,2,\cdots,n\}, \\
c_1(\sigma) &= \#\{i \in I \mid \sigma(i) = i\}, \\
c_1^{(k)}(\sigma) &= \#\{i \in I_k \mid \sigma(i) = i\}, \\
c_2(\sigma) &= \frac{1}{2}\#\{i \in I \mid \sigma(i) \neq i\}, \\
\tau(\{I_i\}) &= \sum_{l=1}^{r} \#\{(i,j) \in I_l \times (I_0 \cup \cdots \cup I_{l-1}) \mid j < i\}, \\
t(\sigma,\{I_i\}) &= \sum_{l=0}^{r} \#\{(i,j) \in I_l \times I_l \mid i < j < \sigma(i),\ \sigma(j) < \sigma(i)\}, \\
n(l) &= \#I_l,
\end{aligned}$$

and the definition of $\Xi_{l,\lambda}(\sigma;T,S)$ is

$$\Xi_{l,\lambda}(\sigma;T,S) = q^{\rho_{l,\lambda}(\sigma;T,S)} \prod_{\substack{i\in I_l \\ \sigma(i)=i}} \zeta_{i,\lambda}(T,S).$$

$$\rho_{l,\lambda}(\sigma;T,S) = \frac{n_l}{2}\sum_{k=1}^{m}\min\{\alpha_k+\lambda,0\} + \frac{1}{2}\sum_{i\in I_l}\sum_{k=1}^{n}\min\{e_k+e_{\sigma,i,k}+\lambda,0\}.$$

$$e_{\sigma,i,k} = \begin{cases} 0 & (k\le i,\ k\le\sigma(i)), \\ 1 & (\sigma(i)<k\le i \ \text{ or } \ i<k\le\sigma(i)), \\ 2 & (i<k,\ \sigma(i)<k). \end{cases}$$

$$\zeta_{i,\lambda}(T,S) = 2\prod_{k\in A(\lambda)}\chi(-u_k)\prod_{k\in B_i(\lambda)}\chi(v_k)$$

$$\times \begin{cases} 0 & (\beta_i+\lambda\ge 0, \quad \#A(\lambda)+\#B_i(\lambda)\not\equiv 0 \mod 2), \\ (1-q^{-1})\chi(-1)^{\frac{\#A(\lambda)+\#B_i(\lambda)}{2}} & (\beta_i+\lambda\ge 0, \quad \#A(\lambda)+\#B_i(\lambda)\equiv 0 \mod 2), \\ \chi(v_i)\chi(-1)^{\frac{\#A(\lambda)+\#B_i(\lambda)+1}{2}} & (\beta_i+\lambda=-1, \quad \#A(\lambda)+\#B_i(\lambda)\not\equiv 0 \mod 2), \\ -q^{-\frac{1}{2}}\chi(-1)^{\frac{\#A(\lambda)+\#B_i(\lambda)}{2}} & (\beta_i+\lambda=-1, \quad \#A(\lambda)+\#B_i(\lambda)\equiv 0 \mod 2). \end{cases}$$

$$A(\lambda) = \{k \mid 1\le k\le m,\ \alpha_k+\lambda<0,\ \alpha_k\not\equiv\lambda \mod 2\},$$
$$B_i(\lambda) = \{1\le k\le i-1,\ \beta_i+\lambda<0,\ \beta_k\not\equiv\lambda \mod 2\}$$
$$\cup\{k \mid i+1\le k\le n,\ \beta_k+\lambda+2<0,\ \beta_k\not\equiv\lambda \mod 2\}.$$

Here we assume that the matrix S and T is diagonal matrix such that

$$S = \mathrm{diag}(u_1p^{\alpha_1},\cdots,u_mp^{\alpha_m}),$$
$$T = \mathrm{diag}(v_1p^{\beta_1},\cdots,v_np^{\beta_n}).$$

The summation with respect to $I = I_0\cup\cdots\cup I_r$ is taken over all partitions of I into σ-invariant disjoint subsets, and the summation with respect to $\{\nu\}_k$ for $k\ge 1$ is taken over the finite set

$$\{(\nu_0,\nu_1,\cdots,\nu_{k-1})\in\mathbb{Z}\times\mathbb{N}^{k-1} \mid -b_l(\sigma,T)\le\nu_0+\nu_1+\cdots+\nu_l\le -1\ (0\le l\le k-1)\}.$$

where the value $b_l(\sigma,T)$ is defined as

$$b_l(\sigma,T) = \min\left\{\{\beta_i \mid i\in I_l,\sigma(i)>i\}\cup\{\beta_i+1 \mid i\in I_l,\sigma(i)\le i\}\right\}.$$

Thus, the local density is given by finite sums and finite products. We will quote their result to show the main theorem.

1.1.5 The Eisenstein series for the modular group

The Eisenstein series is an essential example of the modular form.

Let $\mathcal{H}$ be the upper half-plane, which is $\mathcal{H} = \{z\in\mathbb{C} \mid \mathrm{Im}z>0\}$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane $\mathcal{H}$ as

$$\gamma(\tau) = \frac{a\tau+b}{c\tau+d}, \tag{1}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and $\tau\in\mathcal{H}$.

Definition 1.1. A modular form of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfies

- $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ for γ, τ is the same as (1),
- f is bounded when $z \to i\infty$.

When $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the condition is $f(\tau + 1) = f(\tau)$. It implies that the function f has the period one and the Fourier expansion.

Let $k \geq 2$ be an integer. We define the Eisenstein series $G_{2k}(\tau)$ as

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m + n\tau)^{2k}}.$$

The Eisenstein series satisfies the following Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \exp(2\pi i n\tau),$$

where $\zeta(z)$ is the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, and the division function $\sigma_k(n)$ is defined as $\sigma_k(n) = \sum_{d|n} d^k$.

1.1.6 The Siegel Eisenstein Series and the Siegel Series

Now, we consider the Siegel-Eisenstein series. In this section, Maass' lecture note [23] is fundamental.

Let $k \geq 2$, $l, n \geq 1$ be integers, and ψ be a Dirichlet character modulo l. The Siegel-Eisenstein series $E^n_{k,l,\psi}(Z)$ is defined as

$$E^n_{k,l,\psi}(Z) := \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n_\infty \setminus \Gamma^n_0(l)} \psi(\det D) \det(CZ + D)^{-k},$$

where the set Γ^n_∞ and $\Gamma^n_0(l)$ are subgroups of $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$ defined by

$$\Gamma^n_\infty := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \;\middle|\; C = 0 \right\},$$
$$\Gamma^n_0(l) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \;\middle|\; C \equiv 0 \mod l \right\},$$

respectively, and $Z \in \mathbb{H}^n := \{Z \in M_n(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\}$.

The Fourier expansion of the Siegel-Eisenstein series can be written as

$$E^n_{k,l,\psi}(Z) = \sum_{A\in S^*_n, A\geq 0} c(A) \exp(2\pi i \,\mathrm{Tr}(AZ)),$$

where

$$S_n^* := \{A = (a_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid a_{ii} \in \mathbb{Z},\ a_{ij} \in \mathbb{Z}/2\ (i \neq j)\},$$

and $c(A)$ is a constant depending only on the matrix A. The symbol $A \geq 0$ means that the matrix A is positive definite. More precisely, the Fourier coefficient $c(A)$ is calculated as

$$c(A) = \frac{2^{-\frac{n(n-1)}{2}}(-2\pi i)^{nk}}{\pi^{\frac{n(n-1)}{4}} \prod_{j=0}^{m-1} \Gamma(s - j/2)} (\det A)^{k-\frac{n+1}{2}} \prod_{p:\ \text{prime}} b_n^p(A, s),$$

when $A \geq 0$ and the factor $b_n^p(A, s)$ is called the ramified Siegel series when p is divided by l. Here, we note $\Gamma(s)$ the Gamma function.

1.1.7 Siegel series

Let F be a non-Archimedean, non-dyadic local field. Let $\mathfrak{o} = \mathfrak{o}_F$ denote the integral ring of the local field F. The matrix $B = (b_{ij}) \in \mathrm{Sym}_n(F)$ is called the half-integral matrix when

(1) $b_{ii} \in \mathfrak{o}$ $(1 \leq i \leq n)$.

(2) $2b_{ij} \in \mathfrak{o}$ $(1 \leq i < j \leq n)$.

$\mathcal{H}_n(\mathfrak{o})$ denotes the set of all half-integral matrices of size n, and we define $\mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o}) = \{B \in \mathcal{H}_n(\mathfrak{o}) \mid \det B \neq 0\}$.

For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$, we put $D_B = (-4)^{[n/2]} \det B$. We define the Kronecker invariant ξ_B and the Clifford invariant η_B as follows.

$$\xi_B = \xi(B) = \begin{cases} 1 & \text{if } D_B \in F^{\times 2}, \\ -1 & \text{if the extension } F(\sqrt{D_B})/F \text{ is unramified,} \\ 0 & \text{if the extension } F(\sqrt{D_B})/F \text{ is ramified.} \end{cases}$$

$$\eta_B = \eta(B) = \begin{cases} 1 & \text{if } B \text{ is split on } F, \\ 0 & \text{otherwise.} \end{cases}$$

For $B \in \mathrm{Sym}_n(F)$, there exists a matrix $X \in \mathrm{GL}_n(F)$ such that

$$B[X] = {}^tXBX = \mathrm{diag}(b_1, b_2, \cdots, b_n), \quad b_i \in F^\times.$$

When $B, B' \in \mathcal{H}_n(\mathfrak{o})$, they are called $\mathrm{GL}_n(\mathfrak{o})$-equivalent if there exists a matrix $X \in \mathrm{GL}_n(\mathfrak{o})$ such that $B = B'[X]$. $\{B\}$ denotes the equivalent class of the matrix $B \in \mathcal{H}_n(\mathfrak{o})$.

We write $\langle\ ,\ \rangle = \langle\ ,\ \rangle_F$ for the Hilbert symbol of the field F. The Hasse-Minkowski invariant of the matrix $B \in \mathrm{Sym}_n(F)$ is defined as

$$\varepsilon_B = \prod_{1 \leq i < j \leq n} \langle b_i, b_j \rangle,$$

where b_i is taken as above. This definition does not depend on the choice of the b_i. The Hasse-Minkowski invariant and the Clifford invariant satisfy the following relations

$$\eta_B = \begin{cases} \langle -1, -1 \rangle^{m(m+1)/2} \langle (-1)^m, \det B \rangle \varepsilon_B & \text{if } n = 2m+1, \\ \langle -1, -1 \rangle^{m(m-1)/2} \langle (-1)^{m+1}, \det B \rangle \varepsilon_B & \text{if } n = 2m. \end{cases}$$

The diagonal matrix $\mathrm{diag}(b_1, b_2, \cdots, b_n)$ is called the Jordan diagonal matrix when $0 \leq \mathrm{ord}(b_1) \leq \cdots \leq \mathrm{ord}(b_n)$. It is known that all matrix $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ is $\mathrm{GL}_n(\mathfrak{o})$-equivalent to a unique Jordan diagonal matrix. We call $GK(B) := (\mathrm{ord}(b_1), \cdots, \mathrm{ord}(b_n))$ the Gross-Keating invariant of the matrix B.

For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$, we define the polynomial $\gamma(B, X)$ as

$$\gamma(B,X) = \begin{cases} \dfrac{1-X}{1-q^{\frac{n}{2}}\xi_B X} \displaystyle\prod_{i=1}^{\frac{n}{2}} (1-q^{2i}X^2) & \text{if } n \text{ is even,} \\ (1-X) \displaystyle\prod_{i=1}^{\frac{n-1}{2}} (1-q^{2i}X^2) & \text{if } n \text{ is odd.} \end{cases}$$

The additive character $\psi : F \to \mathbb{C}$ is taken as order 0, which is

$$\{x \in F \mid \psi(xy) = 1, \ {}^{\forall} y \in \mathfrak{o}\} = \mathfrak{o}.$$

The (unramified) Siegel series $b(B, s)$ is defined as

$$b(B,s) = \int_{\mathrm{Sym}_n(F)} \psi(\mathrm{tr}(BX))[X\mathfrak{o}^n + \mathfrak{o}^n : \mathfrak{o}^n]^{-s} dX,$$

which is absolutely convergence when $\mathrm{Re}s \gg 0$. This definition is not independent of the choice of the additive character ψ.

The following theorem is well-known.

Theorem 1.4. *There is a polynomial $F(B, X) \in \mathbb{Z}(X)$ which satisfies*

$$b(B,s) = \gamma(B, q^{-s}) F(B, q^{-s}).$$

Let $\mathfrak{D}_B$ be a discriminant ideal of the field extension $F(\sqrt{D_B})/F$ and put $\mathfrak{e}_B$ as

$$\mathfrak{e}_B := \begin{cases} \mathrm{ord}(D_B) - \mathrm{ord}(\mathfrak{D}_B) & n \text{ is even,} \\ \mathrm{ord}(D_B) & n \text{ is odd.} \end{cases}$$

We put $\widetilde{F}(B, X) := X^{-\mathfrak{e}_B/2} F(B, q^{-(n+1)/2} X)$. From the definition of the polynomial F, we have

$$\widetilde{F}(B,X) \in \begin{cases} \mathbb{Q}[q^{1/2}][X, X^{-1}] & n \text{ is even,} \\ \mathbb{Q}[X^{1/2}, X^{-1/2}] & n \text{ is odd.} \end{cases}$$

The following functional equation is well known.

Theorem 1.5. *The functional equation of the polynomial $\widetilde{F}$ is as follows.*

$$\widetilde{F}(B, X^{-1}) = \begin{cases} \widetilde{F}(B,X) & n \textit{ is even,} \\ \eta_B \widetilde{F}(B,X) & n \textit{ is odd.} \end{cases}$$

Shimura [28], [27] gave the formula that the Siegel series is related to local density. In particular, when we put $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $S = H_k = H \perp H \perp \cdots \perp H$ (k times), then it follows

$$\alpha(H_k, B) = b(B, k) \quad (k \geq n).$$

On the Fourier expansion of the Siegel Eisenstein series, when $l = 1$, the explicit formula of the full-modular Siegel series is known by Katsurada [19]. He gives this formula in this article by calculating the local density $\alpha(S, T)$ where S is unimodular.

This result is closely related to the main theorem of this article, i.e., our result is the extension of Katsurada's result for the ramified character case. We now write down Katsurada's results.

We assume the matrix B is diagonal. Let $e, \tilde{e}$ be integers, and let ξ be a real number. The rational functions $C(e, \tilde{e}, \xi; Y, X)$ and $D(e, \tilde{e}, \xi; Y, X)$ in $Y^{\frac{1}{2}}$ and $X^{\frac{1}{2}}$ are defined as

$$\begin{aligned} C(e, \tilde{e}, \xi; Y, X) &= \frac{Y^{\tilde{e}/2} X^{-(e-\tilde{e})/2-1}(1 - \xi Y^{-1} X)}{X^{-1} - X}, \\ D(e, \tilde{e}, \xi; Y, X) &= \frac{Y^{\tilde{e}/2} X^{-(e-\tilde{e})/2}}{1 - \xi X}. \end{aligned}$$

For a positive integer i, we define the rational function $C_i(e, \tilde{e}, \xi; Y, X)$ as

$$C_i(e, \tilde{e}, \xi; Y, X) = \begin{cases} C(e, \tilde{e}, \xi; Y, X) & \text{if } i \text{ is even,} \\ D(e, \tilde{e}, \xi; Y, X) & \text{if } i \text{ is odd.} \end{cases}$$

Definition 1.2. Let $\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ be a sequence of integers. For an integer i which satisfies $1 \leq i \leq n$, we define $\mathfrak{e}_i = \mathfrak{e}_i(\underline{a})$ as

$$\mathfrak{e}_i = \begin{cases} a_1 + a_2 + \cdots + a_i - 1 & \text{if } i \text{ is even and } \sum_{k=1}^{n} a_k \text{ is odd,} \\ a_1 + a_2 + \cdots + a_i & \text{otherwise.} \end{cases}$$

We define ξ_B as

$$\xi_B = \begin{cases} 0 & \text{if } \sum_{k=1}^{n} e_k \text{ is even,} \\ \chi(D_B) & \text{if } \sum_{k=1}^{n} e_k \text{ is odd.} \end{cases}$$

Katsurada's result for the recursion formula of the unramified character can be written as follows.

Theorem 1.6. *We put $X = q^{-s}$ and $Y = q^{\frac{1}{2}}$. The function $\widetilde{F}(B, X)$ satisfies the following recursion formula.*

$$\begin{aligned} \widetilde{F}(B, X) = {} & C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X) \widetilde{F}(B^{(n-1)}, YX) \\ & + \zeta_i C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X^{-1}) \widetilde{F}(B^{(n-1)}, YX^{-1}), \end{aligned}$$

where ζ_i and ξ are defined as

$$\zeta_i = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \eta_B & \text{if } n \text{ is odd,} \end{cases} \qquad \xi = \begin{cases} \xi_B & \text{if } n \text{ is even,} \\ \xi_{B^{(n-1)}} & \text{if } n \text{ is odd.} \end{cases}$$

Later, several results of the Fourier expansion of the Siegel-Eisenstein series became known.

- Mizuno [25], when $n = 2$, l: square-free odd, ψ: primitive.
- Takemori [33], when $n = 2$, l: any integer, ψ: primitive.
- Takemori [34], when n: arbitrary, l: odd, $\psi = \prod \psi_p$: primitive, $\psi_p \neq \chi_p$.

Gunji [3] gives the case where n is=3, p is an odd prime, and ψ is primitive. We note that when $\psi_p \neq \chi_p$, the result is given by Takemori [34], so he calculated when $\psi_p = \chi_p$.

Also, he [4] calculated when n is arbitrary, p is an odd prime, and ψ is primitive. In this article, he used the theory of genus theta series. Sato and Hironaka's result gives the Fourier coefficients of the genus theta series so that he can give the formula. In our previous paper [36], we use Sato and Hironaka's result directly and locally to the Siegel series. We extend their result to general non-dyadic, non-archimedean local field cases, introducing the Weil constant and explicitly writing down the formula.

Ikeda and Katsurada's paper [17], [18] shows that the explicit formula given by Katsurada is written by using the extended Gross-Keating invariant of the matrix.

1.1.8 Functional equation of the Siegel series

Sweet gives the precise formula of the functional equation of the Whittaker functional [32]. Before his article, Igusa [13, 14] investigated the functional equations of p-adic zeta integrals concerning some prehomogeneous vector spaces. Sweet calculated when $X = \mathrm{Sym}_n(k)$ where k is a non-archimedean field of characteristic 0. In this case, the calculation of the zeta integrals means to that of the Siegel series.

Ikeda [15] reformulated Sweet's result more explicitly. He showed the functional equation of the Whittaker functional over the non-archimedean local field F. He used the Weil constant $\alpha_\psi(a)$ which satisfies

$$\int_F \phi(x)\psi(ax^2)dx = \alpha_\psi(a)|2a|^{-\frac{1}{2}} \int_F \hat{\phi}(x)\psi\left(-\frac{x^2}{4a}\right)dx,$$

where $\phi \in \mathcal{S}(F)$ is a Schwartz function and $\hat{\phi}$ is a Fourier transform of ϕ with self-dual Haar measure. Weil first introduced this constant [37].

Let $M_{w_n}^{(s)}$ and $\mathrm{Wh}_B(s)$ be as following integration: **

$$M_{w_n}^{(s)} f(g) = \int_{\mathrm{Sym}_n(F)} f\left(w_n \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} g\right) dX,$$

$$\mathrm{Wh}_B(s)f = \int_{\mathrm{Sym}_n(F)} f\left(w_n \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right) \psi(-\mathrm{tr}(BX))dX.$$

w_n denotes the matrix $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Then Ikeda's article says that there exists a functional equation

$$\mathrm{Wh}_B(-s) \circ M_{w_n}^{(s)} = \omega^{-1}(\det B)|\det B|^{-s} c_B(\omega, s)\mathrm{Wh}_B(s)$$

and the term $c_B(\omega, s)$ is calculated as

Theorem 1.7 ([15] Theorem 2.1). *If $n = 2m+1$, then we have*

$$c_Q(\omega, s) = \varepsilon'(s-m, \omega)^{-1} \prod_{r=1}^{m} \varepsilon'(2s-2m-1+2r, \omega^2)^{-1} \times |2|^{-2ms+\frac{m(2m+1)}{2}} \omega^{-m}(4)\eta_Q.$$

If $n = 2m$, then we have

$$c_Q(\omega, s) = \varepsilon'(s - m + \frac{1}{2}, \omega)^{-1} \prod_{r=1}^{m} \varepsilon'(2s - 2m + 2r, \omega^2)^{-1}$$
$$\times |2|^{-2ms + \frac{m(2m-1)}{2}} \omega^{-m}(4) \frac{\alpha(D_Q)}{\alpha(1)} \varepsilon'(s + \frac{1}{2}, \omega\chi_{D_Q}).$$

Here we note that $\varepsilon'(s, \omega)$ is defined as

$$\varepsilon'(s, \omega) = \varepsilon'(s, \omega, \psi) = \varepsilon(s, \omega, \psi) \frac{L(1 - s, \omega^{-1})}{L(s, \omega)},$$

$\alpha(a) = \alpha_\psi(a)$ is Weil constant and η_Q is the Clifford invariant of the matrix Q.

1.2 Main theorems on this article

1.2.1 Notations and definitions

We need some notations. Let $G = \mathrm{Sp}_n(F)$ be the symplectic group of rank n over a non-dyadic, non-archimedean local field F. Put $\mathfrak{o}$ and $\mathfrak{p}$ be the ring of integers of F and the maximal ideal of $\mathfrak{o}$, respectively. We fix a prime element π. We write $K = \mathrm{Sp}_n(\mathfrak{o})$. The set Γ is defined as

$$\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; C \equiv 0 \mod \mathfrak{p} \right\}.$$

We can choose $\{w_i\}_{0 \le i \le n}$, a complete set of representatives of the double coset $P \backslash G / \Gamma$, by

$$w_i = \left(\begin{array}{cc|cc} 1_{n-i} & & & \\ & & & -1_i \\ \hline & & 1_{n-i} & \\ & 1_i & & \end{array} \right).$$

Let ψ be an additive character of F of order 0 and χ be a ramified nontrivial character of $F^\times$ satisfying $\chi^2 = 1$. We extend the character χ on Γ as

$$\chi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \chi(\det D).$$

We will denote by $I_n(\chi, s) = \mathrm{Ind}_P^G(\chi \circ |\det|^s)$ the space of the induced representation; the space of C^∞-functions on G satisfying

$$f\left(\begin{pmatrix} A & * \\ 0 & {}^tA^{-1} \end{pmatrix} g \right) = \chi(\det A) |\det A|^{s + \frac{n+1}{2}} f(g),$$

and we also define $I_n(\omega, s)^{\Gamma, \chi}$ as

$$I_n(\chi, s)^{\Gamma, \chi} = \{ f \in I(\omega, s) \mid f(gk) = \chi(k) f(g) \text{ for all } k \in \Gamma \}.$$

For $0 \le t \le n$, let $f_t \in I_n\left(\chi, s - \frac{n+1}{2}\right)^{\Gamma, \chi}$ which satisfies $f_t(\omega_i) = \delta_{ti}$. This function f_t is uniquely determined.

We assume the matrix B is the diagonal matrix

$$B = \mathrm{diag}(\alpha_1\pi^{e_1}, \cdots, \alpha_n\pi^{e_n}), \quad \alpha_i \in \mathfrak{o}^\times,\ 0 \le e_1 \le \cdots \le e_n.$$

The ramified Siegel series $S_t(B,s)^\chi$ is defined as, when $0 \le t \le n$,

$$S_t(B,s)^\chi = \int_{\mathrm{Sym}_n(F)} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}\right)\psi(-\mathrm{tr}(BX))dX.$$

Let $\beta_0 = \alpha_\psi(\pi)\chi(\pi)q^{-\frac{1}{2}}$, and $D_B = (-4)^{[\frac{n}{2}]}\det B$. We write the function $f_\beta(s) = f_{\beta,n}(s)$ for $f_\beta(s) = \sum_{t=0}^{n}\beta^t f_t(s)$. (Here the function f_t is taken in the space $I_n(\chi,s)^{\Gamma,\chi}$.) We also define $F_B(s)$, $\tilde{F}_B(s)$ as

$$F_B(s) = \begin{cases} \dfrac{\mathrm{Wh}_B(s)f_\beta^{(s)}}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})} & \text{if } n \text{ is odd,} \\ \dfrac{\mathrm{Wh}_B(s)f_\beta^{(s)}}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} & \text{if } n \text{ is even and } \sum_{k=1}^n e_k \text{ is even,} \\ \dfrac{(1-\chi(-D_B)q^{-s-\frac{1}{2}})\mathrm{Wh}_B(s)f_\beta^{(s)}}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} & \text{if } n \text{ is even and } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

$$\tilde{F}(s) = \begin{cases} q^{\frac{s}{2}(\sum_{k=1}^n e_k+1)}F(s) & \text{if } n \text{ is odd,} \\ q^{\frac{s}{2}\sum_{k=1}^n e_k}F(s) & \text{if } n \text{ is even and } \sum_{k=1}^n e_k \text{ is even,} \\ q^{\frac{s}{2}(\sum_{k=1}^n e_k+1)}F(s) & \text{if } n \text{ is even and } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

1.2.2 Main theorems

The first main theorem of this article is the explicit formula of the ramified Siegel series. The statement is written as follows.

Theorem 1.8. *The Siegel series $S_t(B,s)^\chi$ associated with the function f_t $(0 \le t \le n)$ is as follows;*

$$S_t(B,s)^\chi = \alpha_\psi(\pi)^{n-t}\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma^2=1}}(1-q^{-1})^{c_2(\sigma)}q^{-c_2(\sigma)}\sum_{\substack{I=I_0\cup\cdots\cup I_r \\ n^{(k)}=t}} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}\frac{(1-q^{-1})^{\sum_{l=k}^r c_1^{(l)}(\sigma)}q^{n(k)}}{\prod_{l=k}^r(q^{n(l)}-1)}$$
$$\times\sum_{\{\nu\}_k^t}\prod_{l=0}^{k-1}\chi(-1)^{\nu_l(n^{(l)}-n^{(k)})}q^{\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))+\tilde{\rho}_{l,\nu_0+\cdots+\nu_l}(\sigma;B)}\prod_{\substack{i\in I_l \\ \sigma(i)=i}}\xi_{i,\nu_0+\cdots+\nu_l}(B)_\chi.$$

Here the summation with respect to $\{\nu\}_k^t$ for $k \ge 1$ is taken over the finite set

$$\left\{(\nu_0,\nu_1,\cdots,\nu_{k-1}) \in \mathbb{Z}\times\mathbb{Z}_{>0}^{k-1} \;\middle|\; -b_l(\sigma,B) \le \nu_0+\nu_1+\cdots+\nu_l \le -1\,(0 \le l \le k-1),\ n^{(k)} = t\right\}.$$

The second main theorem of this article is the recursion formula of the ramified Siegel series. It is written as follows. The rational function $C_i(e,\tilde{e},\xi;Y,X)$ is as above. However, the definition of $\mathfrak{e}_i$ and ξ_B differs from the unramified case.

Definition 1.3. Let $\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ be a sequence of integers. For an integer i which satisfies $1 \le i \le n$, we define $\mathfrak{e}_i = \mathfrak{e}_i(\underline{a})$ as

$$\mathfrak{e}_i = \begin{cases} a_1 + a_2 + \cdots + a_i & \text{if } i \text{ is even and } \sum_{k=1}^n a_k \text{ is even,} \\ a_1 + a_2 + \cdots + a_i + 1 & \text{otherwise.} \end{cases}$$

We define ξ_B as

$$\xi_B = \begin{cases} 0 & \text{if } \sum_{k=1}^n e_k \text{ is even,} \\ \chi(-D_B) & \text{if } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

Then it follows that

Theorem 1.9. *We put $X = q^{-s}$ and $Y = q^{\frac{1}{2}}$ and write $\tilde{F}_B(q^{-s}) = \tilde{F}_B(s)$. The function $\tilde{F}_B(X)$ satisfies the following recursion formula.*

$$\begin{aligned} \tilde{F}_B(X) = & \beta_0 C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X)\tilde{F}_{B^{(n-1)}}(YX) \\ & + \beta_0 \zeta_i C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X^{-1})\tilde{F}_{B^{(n-1)}}(YX^{-1}), \end{aligned}$$

where ζ_i and ξ are defined as

$$\zeta_i = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \eta_B \chi(-D_B) & \text{if } n \text{ is odd,} \end{cases} \quad \xi = \begin{cases} \xi_B & \text{if } n \text{ is even,} \\ \xi_{B^{(n-1)}} & \text{if } n \text{ is odd.} \end{cases}$$

In Section 2, we recall some fundamental notations and results. More precisely, Section 2.1 is the induced representation. Section 2.2 is the result of Sato and Hironaka, and Section 2.3 is the Weil constant.

Section 3 is the first main theorem about the explicit formula for the ramified Siegel series $S_t(B, s)^\chi$. The content of Sections 2 and 3 is the same as our previous paper, which is under publishing now.

Section 4 is about calculating the matrix of the intertwining operator $M_{w_n}^{(s)}$. In Section 4.1, we will show why β_0^i is needed from the point of view of the Weil representation on the finite field. In Section 4.2, we recall how $E_n^{(s),\chi}$ is calculated when $n = 2$ and, in Section 4.3, we write a programming code in PARI/GP to calculate when $n \ge 3$. In Section 4.4, we show the eigenvectors and eigenvalues of the matrix when n is 1 to 8.

Section 5 shows the functional equation of the ramified Siegel series.

In Section 6, we prove the recursion formula of the ramified Siegel series. In Section 6.1, we calculate the term where e_n shows in the explicit formula of the ramified Siegel series. In Section 6.2, we show that the Clifford invariant η_B is written explicitly with $\chi(\alpha_i)$ and $\chi(-1)$, and the term of Gauss sum $\xi_{n,\nu_0}(B)_\chi$ is written concisely. In Section 6.3, we state the second main theorem and prove it in Section 6.4.

The author would like to thank Tamotsu Ikeda of Kyoto University for his constant support.

2 Basic notations

2.1 Induced representations

Let F be a non-archimedean local field of characteristic 0. We denote by $\mathfrak{o} = \mathfrak{o}_F$ the ring of integers of F. We write $\mathfrak{p} = \mathfrak{p}_F$, $\mathfrak{k} = \mathfrak{k}_F$ for the maximal ideal and the reside field of $\mathfrak{o}$, respectively. We fix a prime element $\pi \in \mathfrak{p}$ satisfying $\mathfrak{p} = \pi\mathfrak{o}$. Let q denote the cardinality of $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$, and we assume q is odd.

The set of symmetric matrices of degree n over F is defined by

$$\mathrm{Sym}_n(F) := \{X = (x_{ij}) \in M_n(F) \mid x_{ij} = x_{ji} \text{ for all } 1 \le i, j \le n\},$$

and we denote by $S_n(F)$ the subset of non-degenerate symmetric matrices.

The symplectic group of degree n over F is defined by

$$G = \mathrm{Sp}_n(F) := \{M \in \mathrm{GL}_{2n}(F) \mid {}^tM w_n M = w_n\},$$

where we write ${}^tM$ for the transpose of a matrix M, and w_n for $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. The group G is now considered an algebraic group over F. We write $K = \mathrm{Sp}_n(\mathfrak{o})$ as a maximal compact subgroup of G. Also, we define

$$P = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(F) \;\middle|\; C = 0 \right\}$$

by the Siegel parabolic subgroup of G. The decomposition $G = PK$ is well-known and called the Iwasawa decomposition. Set

$$\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K \;\middle|\; C \equiv 0 \bmod \mathfrak{p} \right\}.$$

Let ω be a character of $F^\times$ satisfying $\omega^2 = 1$. Later, we only consider $\omega = \mathbf{1}$ the trivial character or $\omega = \chi$ a ramified nontrivial character. We define a character ω^Γ on Γ with

$$\omega^\Gamma\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \omega(\det D)$$

and we write ω for ω^Γ (by agree of notation).

We will write $I_n(\omega, s) = \mathrm{Ind}_P^G(\omega \circ |\det|^s)$ for the space of the induced representation; the space of C^∞-functions on G satisfying

$$f\left(\begin{pmatrix} A & * \\ 0 & {}^tA^{-1} \end{pmatrix} g \right) = \omega(\det A) |\det A|^{s + \frac{n+1}{2}} f(g)$$

and we also define $I_n(\omega, s)^{\Gamma,\omega}$ as

$$I_n(\omega, s)^{\Gamma,\omega} = \{f \in I(\omega, s) \mid f(gk) = \omega(k) f(g) \text{ for all } k \in \Gamma\}.$$

The double coset $P\backslash G/\Gamma = (P \cap K)\backslash K/\Gamma \simeq \Gamma\backslash K/\Gamma$ is a finite set, and one can choose a complete set of representatives $\{w_i\}_{0 \le i \le n}$ by

$$w_i = \left(\begin{array}{cc|cc} 1_{n-i} & & & \\ & & & -1_i \\ \hline & & 1_{n-i} & \\ & 1_i & & \end{array} \right).$$

It is known that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K$ is an element of $\Gamma w_i \Gamma$ if and only if the matrix C has rank i when it is considered as $\bmod\, \mathfrak{p}$.

Each $f \in I_n(\omega, s)^{\Gamma,\omega}$ is determined by its value on K, and also by $\{f(w_i)\}_{0 \le i \le n}$. Now we take the following result;

Proposition 2.1. *Define $f_i \in I(\omega, s)^{\Gamma,\omega}$ satisfying $f_i(\omega_j) = \delta_{ij}$. Then*

$$I_n(\omega, s)^{\Gamma,\omega} = \bigoplus_{i=0}^{n} \mathbb{C} f_i.$$

2.2 The result of Sato and Hironaka

For the proofs of the following results when $F = \mathbb{Q}_p$, we refer the reader to [12]. This proof remains valid when F is an arbitrary non-archimedean non-dyadic local field, considering the structure of a non-archimedean local field.

We put

$$S_n(F) = \{X \in \mathrm{Sym}_n(F) \mid \det X \neq 0\}$$

and we define

$$\Gamma_0 = \{\gamma = (\gamma_{ij}) \in \mathrm{GL}_n(\mathfrak{o}) \mid \gamma_{ij} \in \mathfrak{p}\ (i > j)\},$$

which acts on $S_n(F)$ by $Y \mapsto \gamma \cdot Y = \gamma Y {}^t\gamma$. We fix a non-square unit $\delta \in \mathfrak{o}^\times \backslash (\mathfrak{o}^\times)^2$. The following theorem determines the orbits of this action. Put $I = \{1, 2, \cdots, n\}$ and consider the standard action of $\mathfrak{S}_n$ on I.

Theorem 2.1 ([12] Theorem 2.1)**.** *Let Λ_n be the collection of $(\sigma, e, \varepsilon) \in \mathfrak{S}_n \times \mathbb{Z}^n \times \{1, \delta\}^n$ satisfying*

$$\sigma^2 = 1,\ e_{\sigma(i)} = e_i\ (i \in I),\ \varepsilon_i = 1\,(i \in I, \sigma(i) \neq i).$$

For a $(\sigma, e, \varepsilon) \in \Lambda_n$, we define a symmetric matrix $S_{\sigma,e,\varepsilon}$ by

$$S_{\sigma,e,\varepsilon} = (s_{ij}),\ s_{ij} = \varepsilon_i \pi^{e_i} \delta_{i,\sigma(j)},$$

where $\delta_{i,\sigma(j)}$ is the Kronecker delta. Then the set $\{S_{\sigma,e,\varepsilon} \mid (\sigma, e, \varepsilon) \in \Lambda_n\}$ gives the complete set of representatives of Γ_0-equivalence classes in $S_n(F)$.

Before writing the second theorem, we need some preparation.

For $Y \in S_n(F)$, we define

$$\alpha(\Gamma_0; Y) = \lim_{l \to \infty} q^{-ln(n-1)/2} N_l(\Gamma_0; Y),$$

where

$$N_l(\Gamma_0; Y) = \#\{\gamma \in \Gamma_0 \bmod \mathfrak{p}^l \mid \gamma Y {}^t\gamma \equiv Y \bmod \mathfrak{p}^l\}.$$

We normalize the Haar measures $d\gamma$ and dY on $M_n(F)$ and $\mathrm{Sym}_n(F)$, respectively, by

$$\int_{M_n(\mathfrak{o})} d\gamma = 1, \quad \int_{\mathrm{Sym}_n(\mathfrak{o})} dY = 1.$$

Theorem 2.2 ([12] Proposition 1.2)**.** *Let $Y_0 \in S_n(F)$ then the following integral formula holds for any continuous function f on $\Gamma_0 \cdot Y_0$:*

$$\int_{\Gamma_0 \cdot Y_0} f(Y) dY = \alpha(\Gamma_0; Y_0)^{-1} \int_{\Gamma_0} f(\gamma Y_0 {}^t\gamma) d\gamma.$$

Now we introduce some notations.

For a $(\sigma, e, \varepsilon) \in \Lambda_n$, let

$$\{\lambda_0, \lambda_1, \cdots, \lambda_r\},\ \lambda_0 < \lambda_1 < \cdots < \lambda_r$$

be the set of integers λ such that $\lambda = e_i$ for some $i \in I$. We put

$$I_i = \{j \in I \mid e_j = \lambda_i\},\ 0 \leq i \leq r.$$

Then $I_0, \cdots, I_r$ are σ-stable subsets of I and $I = I_0 \cup I_1 \cup \cdots \cup I_r$ (disjoint union). We also put

$$I^{(i)} = I_i \cup I_{i+1} \cup \cdots \cup I_r, \quad 0 \leq i \leq r.$$

We set

$$n_i = \#(I_i),\ n^{(i)} = \#(I^{(i)}) = n_i + \cdots + n_r,\ n(i) = \frac{n^{(i)}(n^{(i)}+1)}{2}.$$

Put

$$\nu_i = \lambda_i - \lambda_{i-1} (1 \leq i \leq r), \quad \nu_0 = \lambda_0.$$

Then $\nu_0 \in \mathbb{Z}$ and $\nu_1, \cdots, \nu_r \in \mathbb{Z}_{>0}$.

Theorem 2.3 ([12] Theorem 2.2)**.** *Put*

$$\begin{aligned}
c_1(\sigma) &= \#\{i \in I \mid \sigma(i) = i\}, \\
c_2(\sigma) &= \frac{1}{2}\#\{i \in I \mid \sigma(i) \neq i\}, \\
t(\sigma, \{I_i\}) &= \sum_{l=0}^{r} \#\{(i,j) \in I_l \times I_l \mid i < j < \sigma(i),\ \sigma(j) < \sigma(i)\}, \\
\tau(\{I_i\}) &= \sum_{l=1}^{r} \#\{(i,j) \in I_l \times (I_0 \cup \cdots \cup I_{l-1}) \mid j < i\}.
\end{aligned}$$

Then we have

$$\alpha(\Gamma_0; S_{\sigma,e,\varepsilon}) = 2^{c_1(\sigma)}(1-q^{-1})^{c_2(\sigma)} q^{c(\sigma,e,\varepsilon)},$$

where

$$c(\sigma, e, \varepsilon) = -\frac{n(n-1)}{2} + \tau(\{I_i\}) + t(\sigma, \{I_i\}) + c_2(\sigma) + \sum_{l=0}^{r} \nu_l n(l).$$

For $a \in F$, we put

$$I(a) = \int_{\mathfrak{o}} \psi(ax^2)dx,\ I^*(a) = \int_{\mathfrak{o}^\times} \psi(ax^2)dx = I(a) - \frac{1}{q} I(a\pi^2).$$

For $T, Y \in S_n(F)$, put

$$\mathscr{G}_{\Gamma_0}(Y, T) = \int_{\Gamma_0} \psi(-\mathrm{tr}(Y \cdot T[\gamma]))d\gamma.$$

where we write $T[\gamma] = {}^t\gamma T \gamma$. Let $T = \mathrm{diag}(v_1\pi^{\beta_1}, v_2\pi^{\beta_2}, \cdots, v_n\pi^{\beta_n})$ $(v_i \in \mathfrak{o}^\times,\ \beta_i \in \mathbb{Z})$.

Theorem 2.4 ([12] Proposition 3.3). *For $(\sigma, e, \varepsilon) \in \Lambda_n$, the character sum $\mathscr{G}_{\Gamma_0}(S_{\sigma,e,\varepsilon}, T)$ vanishes unless*

$$e_i \geq \begin{cases} -\beta_i - 1 & \text{if } \sigma)i) \geq i, \\ -\beta_i & \text{if } \sigma(i) > i \end{cases}$$

for any $i \in I$. When the condition above is satisfied, we have

$$\begin{aligned} \mathscr{G}_{\Gamma_0}(S_{\sigma,e,\varepsilon}, T) =& (1-q^{-1})^{2c_2(\sigma)} q^{-\frac{n(n-1)}{2}+d(\sigma,e,\beta)} \\ & \times \prod_{\substack{i=1 \\ \sigma(i)=i}}^{n} \left\{ I^*(-\varepsilon_i v_i \pi^{e_i+\beta_i}) \prod_{k=1}^{i-1} I(-\varepsilon_i v_k \pi^{e_i+\beta_k}) \prod_{k=i+1}^{n} I(-\varepsilon_i v_k \pi^{e_i+\beta_k+2}) \right\}, \end{aligned}$$

where

$$d(\sigma, e, \beta) = \sum_{\substack{i=1 \\ \sigma(i)>i}}^{n} \left\{ \sum_{k=1}^{i-1} \min\{e_i + \beta_k, 0\} + \sum_{k=i+1}^{\sigma(i)-1} \min\{e_i + \beta_k + 1, 0\} + \sum_{k=\sigma(i)+1}^{n} \min\{e_i + \beta_k + 2, 0\} \right\}.$$

2.3 Weil constant

We recall the definition of the Weil constant as in [15].

For each Schwartz function $\phi \in \mathcal{S}(F)$, the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = \int_F \phi(y)\psi(xy)dy.$$

Note that the Haar measure dy satisfying $\int_{\mathfrak{o}} dy = 1$ is the self-dual Haar measure for the Fourier transform $\phi \mapsto \hat{\phi}$.

Definition 2.1. Let ψ be an additive character over F and let $a \in F^\times$. The Weil constant $\alpha_\psi(a)$ is a complex number satisfying

$$\int_F \phi(x)\psi(ax^2)dx = \alpha_\psi(a)|2a|^{-\frac{1}{2}} \int_F \hat{\phi}(x)\psi\left(-\frac{x^2}{4a}\right)dx \tag{1}$$

for any $\phi \in \mathcal{S}(F)$.

The following lemmas are fundamental. Let $\langle *, * \rangle$ denote the Hilbert symbol of index two on F.

Lemma 2.1. *For any $a, b \in F^\times$,*

$$\frac{\alpha_\psi(a)\alpha_\psi(b)}{\alpha_\psi(ab)\alpha_\psi(1)} = \langle a, b \rangle.$$

Here we note that, for $a, b \in F^\times$, $\alpha_\psi(ab^2) = \alpha_\psi(a)$.

Lemma 2.2. *For $a \in F^\times$, we write $a = \varepsilon\pi^n (\varepsilon \in \mathfrak{o}^\times, n \in \mathbb{Z})$. Then we have*

$$\alpha_\psi(\varepsilon\pi^n) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ \chi(\varepsilon)\alpha_\psi(\pi) & \text{if } n \text{ is odd}. \end{cases}$$

Proof. It is sufficient to show the following equations.

$$\alpha_\psi(\delta) = \alpha_\psi(1) = 1. \tag{2}$$
$$\alpha_\psi(\delta\pi) = -\alpha_\psi(\pi). \tag{3}$$

(2) is trivial. To prove (3), we use Lemma 2.1. We have

$$\frac{\alpha_\psi(\delta)\alpha_\psi(\pi)}{\alpha_\psi(\delta\pi)\alpha_\psi(1)} = \langle \delta, \pi \rangle.$$

Because the fact that $\alpha_\psi(\delta) = \alpha_\psi(1) = 1$ and $\langle \delta, \pi \rangle = -1$, the lemma is proved. □

3 The explicit formula of the ramified Siegel series

3.1 Calculation of the Siegel series associated to $\varphi = f_t$

This section considers the case where $\varphi = f_t$ $(0 \le t \le n)$.

Let $B \in \mathrm{Sym}_n(F)$. Recall that we define the Siegel series as

$$\begin{aligned} S_t(B,s)^\chi &= \int_{\mathrm{Sym}_n(F)} f_t\left(w_n \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right) \psi(-\mathrm{tr}(BX))dX \\ &= \int_{\mathrm{Sym}_n(F)} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}\right) \psi(-\mathrm{tr}(BX))dX. \end{aligned}$$

Note that the function f_t is taken in the space $I_n\left(\omega, s - \frac{n+1}{2}\right)^{\Gamma,\omega}$. We can certainly assume that X is an invertible matrix since the measure of $\mathrm{Sym}_n(F) \backslash S_n(F)$ is 0. We need the Iwasawa decomposition of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}$. The following lemma is well-known and called the Jordan splitting;

Lemma 3.1. *For $X \in \mathrm{Sym}_n(\mathfrak{o})$, there are $U \in \mathrm{GL}_n(\mathfrak{o})$ and diagonal matrix Y such that $X = {}^tUYU$. Moreover, when we write $Y = \mathrm{diag}(\alpha_1\pi^{v_1}, \cdots, \alpha_n\pi^{v_n})$ $(\alpha_i \in \mathfrak{o}^\times, v_i \ge 0)$ then for each $m \ge 0$, the value*

$$\#\{i \mid v_i = m\} \quad \textit{and} \quad \prod_{v_i=m} \alpha_i$$

are uniquely determined by the matrix X.

When we write $X = {}^tUYU$ where $U \in \mathrm{GL}_n(\mathfrak{o})$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & {}^tUYU \end{pmatrix} = \begin{pmatrix} U^{-1} & \\ & {}^tU \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & Y \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \\ & U \end{pmatrix}.$$

Then

$$\begin{aligned} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}\right) &= f_t\left(\begin{pmatrix} U^{-1} & \\ & {}^tU \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & Y \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & \\ & U \end{pmatrix}\right) \\ &= \omega(\det U^{-1} {}^tU^{-1}) \left|\det U^{-1} {}^tU^{-1}\right|^{-s} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & Y \end{pmatrix}\right) \\ &= f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & Y \end{pmatrix}\right) \end{aligned}$$

since $\omega^2 = 1$ and $\det U \in \mathfrak{o}^\times$.

We write $Y = \begin{pmatrix} Y' & \\ & Y'' \end{pmatrix}$ where Y' and Y'' are diagonal matrices of degree r and $n-r$ $(0 \leq r \leq n)$, respectively. We assume that $Y' \in M_r(\mathfrak{o})$ and $Y''^{-1} \in \mathfrak{p}M_{n-r}(\mathfrak{o})$. Since

$$\left(\begin{array}{cc|cc} 0 & & -1 & \\ & 0 & & -1 \\ \hline 1 & & Y' & \\ & 1 & & Y'' \end{array}\right) = \left(\begin{array}{cc|cc} 0 & & -1 & \\ & 0 & & -1 \\ \hline 1 & & 0 & \\ & 1 & & Y'' \end{array}\right) \left(\begin{array}{cc|cc} 1 & & Y' & \\ & 1 & & 0 \\ \hline 0 & & 1 & \\ & 0 & & 1 \end{array}\right),$$

we may assume that $Y' = 0$. Now we consider an Iwasawa decomposition;

$$\left(\begin{array}{cc|cc} 0 & & -1 & \\ & 0 & & -1 \\ \hline 1 & & 0 & \\ & 1 & & Y'' \end{array}\right) = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & -Y'' \\ \hline 0 & & 1 & \\ & 0 & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & & 0 & \\ & Y''^{-1} & & 0 \\ \hline 0 & & 1 & \\ & 0 & & Y'' \end{array}\right) \left(\begin{array}{cc|cc} 0 & & -1 & \\ & 1 & & 0 \\ \hline 1 & & 0 & \\ & Y''^{-1} & & 1 \end{array}\right).$$

We note that ${}^tY'' = Y''$, since Y'' is diagonal. Therefore, in order to $f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & Y \end{pmatrix}\right) \neq 0$, we assume the rank of the matrix $\begin{pmatrix} 1 & \\ & Y''^{-1} \end{pmatrix}$ as mod $\mathfrak{p}$ $\left(= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right)$ is t. We define $S_{n,t}(F)$ the subset of $S_n(F)$ by

$$S_{n,t}(F) = \{X \in S_n(F) \mid (*) : \#\{i \mid e_i < 0\} = n - t\},$$

where the notation e_i is the same as in the Lemma 3.1. From the discussion above, we have

$$\begin{aligned} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}\right) &= f_t\left(\left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & -Y'' \\ \hline 0 & & 1 & \\ & 0 & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & & 0 & \\ & Y''^{-1} & & 0 \\ \hline 0 & & 1 & \\ & 0 & & Y'' \end{array}\right) \left(\begin{array}{cc|cc} 0 & & -1 & \\ & 1 & & 0 \\ \hline 1 & & 0 & \\ & Y''^{-1} & & 1 \end{array}\right)\right) \\ &= \omega(\det Y''^{-1})|\det Y''^{-1}|^s f_t\left(\left(\begin{array}{cc|cc} 0 & & -1 & \\ & 1 & & 0 \\ \hline 1 & & 0 & \\ & Y''^{-1} & & 1 \end{array}\right)\right) \\ &= \omega(\det Y'')|\det Y''|^{-s}, \end{aligned}$$

hence the ramified Siegel series can be written as

$$\begin{aligned} S_t(B,s)^\omega &= \int_{S_n(F)^t} f_t\left(\begin{pmatrix} 0 & -1 \\ 1 & X \end{pmatrix}\right) \psi(-\mathrm{tr}(BX))dX \\ &= \int_{S_n(F)^t} \omega\left(\det Y''\right) \left|\det Y''\right|^{-s} \psi(-\mathrm{tr}(BX))dX. \end{aligned}$$

3.2 Calculation of the integral

Now, we calculate the integral

$$S_t(B,s)^\omega = \int_{S_n(F)^t} \omega\left(\det Y''\right) \left|\det Y''\right|^{-s} \psi(-\mathrm{tr}(BX))dX, \quad 0 \leq t \leq n$$

by using the above theorems. Later we put $B = \mathrm{diag}(\alpha_1\pi^{e_1}, \cdots, \alpha_n\pi^{e_n})$, where $\alpha_i \in \mathfrak{o}^\times$, and $0 \le e_1 \le e_2 \le \cdots \le e_n$. From lemma 3.1, we assume B as this type without loss of generality.

First, we divide the domain of integration by Γ_0 orbits. Each representatives $S_{\sigma,h,\varepsilon}$ is in the element of $S_n(F)^t$ if and only if

$$(*) : \#\{i \mid h_i < 0\} = n - t.$$

Therefore

$$S_t(B,s)^\omega = \sum_{\sigma,h,\varepsilon(*)} \int_{\Gamma_0 S_{\sigma,h,\varepsilon}} \omega(\det Y'')|\det Y''|^{-s}\psi(-\mathrm{tr}(BX))dX.$$

Lemma 3.2. *The term $\omega(\det Y'')|\det Y''|^{-s}$ depends only on its Γ_0-orbit.*

Proof. Let X' be a matrix with the same equivalence class as X. Here we write $X' = \gamma X {}^t\gamma$ for some $\gamma \in \Gamma_0$. We note that $\Gamma_0 \subset \mathrm{GL}_n(\mathfrak{o})$. Therefore the two matrices X and X' are $\mathrm{GL}_n(\mathfrak{o})$-equivalent.

From Lemma 3.1, since $\prod_{v_i<0} \alpha_i = \det Y''$ is uniquely determined by its $\mathrm{GL}_n(\mathfrak{o})$-equivalence class, we can show the statement of the lemma. □

Let $S_{\sigma,h,\varepsilon}$ satisfy the condition $(*)$. Let

$$\{\mu_1, \mu_2, \cdots, \mu_{n-t}\},\ \mu_1 < \mu_2 < \cdots < \mu_{n-t}$$

be the set of integers $\mu : 1 \le \mu \le n$ such that $h_\mu < 0$. We define a square matrix $S^{(t)}_{\sigma,h,\varepsilon}$ of size $n-t$ defined by

$$\left(S^{(t)}_{\sigma,h,\varepsilon}\right)_{ij} = \left(S_{\sigma,h,\varepsilon}\right)_{\mu_i,\mu_j}.$$

Then we note that Y'' and $S^{(t)}_{\sigma,h,\varepsilon}$ are $\mathrm{GL}_{n-t}(\mathfrak{o})$-equivalent.

We denote $c_{\sigma,h,\varepsilon,\omega}(s) = \omega(\det S^{(t)}_{\sigma,h,\varepsilon})|\det S^{(t)}_{\sigma,h,\varepsilon}|^{-s}$. Using the previous lemma,

$$S_t(B,s)^\omega = \sum_{\sigma,h,\varepsilon(*)} c_{\sigma,h,\varepsilon,\omega}(s) \int_{\Gamma_0 S_{\sigma,h,\varepsilon}} \psi(-\mathrm{tr}(BX))dX.$$

Now using Theorem 2.2 when $f(X) = \psi(-\mathrm{tr}(BX))$, we can deduce

$$\begin{aligned}\int_{\Gamma_0 S_{\sigma,h,\varepsilon}} \psi(-\mathrm{tr}(BX))dX &= \frac{1}{\alpha(\Gamma_0; S_{\sigma,h,\varepsilon})}\int_{\Gamma_0} \psi(-\mathrm{tr}(B\gamma S_{\sigma,h,\varepsilon}{}^t\gamma))d\gamma \\ &= \frac{1}{\alpha(\Gamma_0; S_{\sigma,h,\varepsilon})}\int_{\Gamma_0} \psi(-\mathrm{tr}(B\cdot S_{\sigma,h,\varepsilon}[{}^t\gamma]))d\gamma \\ &= \frac{\mathscr{G}_{\Gamma_0}(B, S_{\sigma,h,\varepsilon})}{\alpha(\Gamma_0; S_{\sigma,h,\varepsilon})}.\end{aligned}$$

We can write the main theorem of this article:

Theorem 3.1. *Let the notation be as above. Then we have*

$$S_t(B,s)^\omega = \sum_{\sigma,h,\varepsilon(*)} c_{\sigma,h,\varepsilon,\omega}(s)\frac{\mathscr{G}_{\Gamma_0}(B, S_{\sigma,h,\varepsilon})}{\alpha(\Gamma_0; S_{\sigma,h,\varepsilon})}.$$

Here we note that the sum for σ, h, ε is finite.

3.3 An explicit formula for the Siegel series

In this section, we give an explicit formula for

$$S_t(B,s)^{\omega} = \sum_{\substack{\sigma,h,\varepsilon \\ e_i \le -1}} c_{\sigma,h,\varepsilon,\omega}(s) \frac{\mathscr{G}_{\Gamma}(B, S_{\sigma,h,\varepsilon})}{\alpha(\Gamma; S_{\sigma,h,\varepsilon})}$$

with the notation in the previous section. At first, we assume that $\omega = \chi$.

We put

$$e_{\sigma,i,k} = \begin{cases} 0 & \text{if } k \le i,\ k \le \sigma(i) \\ 1 & \text{if } \sigma(i) < k \le i \text{ or } i < k \le \sigma(i) \\ 2 & \text{if } i < k,\ \sigma(i) < k, \end{cases}$$

$$\begin{aligned} b_l(\sigma, B) &= \min\{\{e_i \mid i \in I_l, \sigma(i) > i\} \cup \{e_i + 1 \mid i \in I_l, \sigma(i) \le i\}\}, \\ B_i(\lambda) &= \{k \mid 1 \le k \le i-1,\ e_k + \lambda < 0,\ e_k \not\equiv \lambda \bmod 2\} \\ &\quad \cup \{k \mid i+1 \le k \le n,\ \beta_k + \lambda + 2 < 0,\ \beta_k \not\equiv \lambda \bmod 2\}, \\ \tilde{\rho}_{l,\lambda}(\sigma; B) &= \frac{1}{2} \sum_{i \in I_l} \sum_{k=1}^{n} \min\{e_k + e_{\sigma,i,k} + \lambda,\ 0\}, \end{aligned}$$

$$\xi_{i,\lambda}(B)_{\chi} = \prod_{k \in B_i(\lambda)} \chi(v_k) \times \begin{cases} 0 & e_i + \lambda \ge 0,\ \#B_i(\lambda) : \text{even} \\ (1-q^{-1})\chi(-1)^{[\#B_i(\lambda)/2]+1} & e_i + \lambda \ge 0,\ \#B_i(\lambda) : \text{odd} \\ \chi(v_i)\chi(-1)^{[\#B_i(\lambda)/2]+1} & e_i + \lambda = -1,\ \#B_i(\lambda) : \text{even} \\ -q^{-1/2}\chi(-1)^{[\#B_i(\lambda)/2]+1} & e_i + \lambda = -1,\ \#B_i(\lambda) : \text{odd}. \end{cases}$$

Now we have the following theorem.

Theorem 3.2. *The Siegel series $S_t(B,s)^{\chi}$ associated with the function f_t $(0 \le t \le n)$ is as follows;*

$$S_t(B,s)^{\chi} = \alpha_{\psi}(\pi)^{n-t} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^2 = 1}} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{\substack{I = I_0 \cup \cdots \cup I_r \\ n^{(k)} = t}} q^{-\tau(\{I_i\}) - t(\sigma, \{I_i\})} \frac{(1-q^{-1})^{\sum_{l=k}^{r} c_1^{(l)}(\sigma)} q^{n(k)}}{\prod_{l=k}^{r} (q^{n(l)} - 1)}$$

$$\times \sum_{\{\nu\}_k^t} \prod_{l=0}^{k-1} \chi(-1)^{\nu_l (n^{(l)} - n^{(k)})} q^{\nu_l((sn^{(l)} - n(l)) - (sn^{(k)} - n(k))) + \tilde{\rho}_{l,\nu_0 + \cdots + \nu_l}(\sigma; B)} \prod_{\substack{i \in I_l \\ \sigma(i) = i}} \xi_{i,\nu_0 + \cdots + \nu_l}(B)_{\chi}.$$

Here the summation with respect to $\{\nu\}_k^t$ for $k \ge 1$ is taken over the finite set

$$\left\{(\nu_0, \nu_1, \cdots, \nu_{k-1}) \in \mathbb{Z} \times \mathbb{Z}_{>0}^{k-1} \;\middle|\; -b_l(\sigma, B) \le \nu_0 + \nu_1 + \cdots + \nu_l \le -1 \, (0 \le l \le k-1),\ n^{(k)} = t\right\}$$

and we put

$$c_1^{(k)}(\sigma) = \#\{i \in I_k \mid \sigma(i) = i\}.$$

Proof. From Theorem 3.1, we consider

$$S_0(B,s)^{\chi} = \sum_{\substack{\sigma,h,\varepsilon \\ e_i \le -1}} c_{\sigma,h,\varepsilon,\chi}(s) \frac{\mathscr{G}_{\Gamma_0}(B, S_{\sigma,h,\varepsilon})}{\alpha(\Gamma_0; S_{\sigma,h,\varepsilon})},$$

where

$$\begin{aligned} c_{\sigma,h,\varepsilon,\chi}(s) &= \chi(\det S^{(t)}_{\sigma,h,\varepsilon})|\det S^{(t)}_{\sigma,h,\varepsilon}|^{-s} \\ &= \chi\left((-1)^{c_2^t(\sigma)}\prod_{h_i<0}\varepsilon_i\pi^{\sum_{h_j<0}h_j}\right)\left|(-1)^{c_2^t(\sigma)}\prod_{h_i<0}\varepsilon_i\pi^{\sum_{h_j<0}h_j}\right|^{-s} \\ &= \chi(-1)^{c_2^t(\sigma)}\chi(\pi)^{\sum_{h_j<0}h_j}\prod_{h_i<0}\chi(\varepsilon_i)q^{s\sum_{h_j<0}h_j} \end{aligned}$$

since $\chi(\pi)=\chi(-1)$. Here we note that χ satisfies $\chi(-\pi)=\langle\pi,-\pi\rangle=1$, and we put

$$c_2^t(\sigma)=\frac{1}{2}\#\{i\in I\mid \sigma(i)\neq i,\ h_i<0\}.$$

Therefore we have

$$\begin{aligned} S_t(B,s)^\chi &= \int_{S_n(F)^t} f_t\left(\begin{pmatrix}0 & -1\\ 1 & X\end{pmatrix}\right)\psi(-\mathrm{tr}(BX))dX \\ &= \sum_{\sigma,h,\varepsilon(*)}\int_{\Gamma_0 S_{\sigma,h,\varepsilon}}\chi\left(\det Y''^{-1}\right)\left|\det Y''^{-1}\right|^{-s}\psi(-\mathrm{tr}(BX))dX \\ &= \sum_{\sigma,h,\varepsilon(*)}\int_{\Gamma_0 S_{\sigma,h,\varepsilon}}\chi(-1)^{c_2^t(\sigma)}\chi(\pi)^{\sum_{h_j<0}h_j}\prod_{e_i<0}\chi(\varepsilon_i)q^{s\sum_{h_j<0}h_j}\frac{\mathscr{G}_{\Gamma_0}(B,S_{\sigma,h,\varepsilon})}{\alpha(\Gamma_0;S_{\sigma,h,\varepsilon})}dX. \end{aligned}$$

By using Theorem 2.3 and Theorem 2.4, we have

$$\begin{aligned} S_t(B,s)^\chi = &\sum_{\sigma,h,\varepsilon(*)}\chi(-1)^{c_2^t(\sigma)}\chi(-1)^{\sum_{h_j<0}h_j}\prod_{h_i<0}\chi(\varepsilon_i)q^{s\sum_{h_j<0}h_j}(1-q^{-1})^{2c_2(\sigma)}q^{-\frac{n(n-1)}{2}+d(\sigma,h,e)} \\ &\times\prod_{\substack{i=1\\ \sigma(i)=i}}^{n}\left\{I^*(-\varepsilon_i v_i\pi^{h_i+e_i})\prod_{k=1}^{i-1}I(-\varepsilon_i v_k\pi^{h_i+e_k})\prod_{k=i+1}^{n}I(-\varepsilon_i v_k\pi^{h_i+e_k+2})\right\} \\ &\times 2^{-c_1(\sigma)}(1-q^{-1})^{-c_2(\sigma)}q^{\frac{n(n-1)}{2}-\tau(\{I_i\})-t(\sigma,\{I_i\})-c_2(\sigma)-\sum_{l=0}^{r}\nu_l n(l)}, \end{aligned}$$

where the summation with respect to (σ,h,ε) is taken over all $(\sigma,h,\varepsilon)\in\Lambda_n$ satisfying the condition $(*)$.

We replace the sum with respect to (σ,h) with $(\sigma,I,\{\nu\})$. Then it follows

$$\begin{aligned} S_t(B,s)^\chi = &\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}}\chi(-1)^{c_2^t(\sigma)}2^{-c_1(\sigma)}(1-q^{-1})^{c_2(\sigma)}q^{-c_2(\sigma)}\sum_{I=I_0\cup\cdots\cup I_r}q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})} \\ &\times\sum_{\nu_0,\cdots,\nu_r(**)}\chi(-1)^{\sum_{h_j<0}h_j}q^{d(\sigma,h,e)+s\sum_{h_j<0}h_j-\sum_{l=0}^{r}\nu_l n(l)} \\ &\times\sum_{\varepsilon}\chi(\varepsilon_i)'\prod_{l=0}^{r}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\left\{I^*(-\varepsilon_i v_i\pi^{\lambda_l+e_i})\prod_{k=1}^{i-1}I(-\varepsilon_i v_k\pi^{\lambda_l+e_k})\prod_{k=i+1}^{n}I(-\varepsilon_i v_k\pi^{\lambda_l+e_k+2})\right\}, \end{aligned}$$

where $\lambda_l=\nu_0+\nu_1+\cdots+\nu_l$ $(l=0,1,\cdots,r)$, and the summation with respect to $\nu_0,\nu_1,\cdots,\nu_r$ is taken over all $(\nu_0,\nu_1,\cdots,\nu_r)$ satisfying

$$(**)\quad \nu_1,\cdots,\nu_r\geq 1,\ \ -b_l(\sigma,B)\leq\lambda_l\leq -1\quad (l=0,1,\cdots,r).$$

Here we also define $\chi(\varepsilon_i)'$ as

$$\chi(\varepsilon_i)' = \begin{cases} \chi(\varepsilon_i) & (e_i < 0) \\ 1 & (e_i \geq 0). \end{cases}$$

Put

$$Q_{l,\lambda_l}(\sigma; B) = \prod_{\substack{i \in I_l \\ \sigma(i)=i}} q_{i,\lambda_l}(\sigma; B),$$

where

$$q_{i,\lambda_l}(\sigma; B) = \sum_{\varepsilon=1,\delta} \left\{ \chi(\varepsilon) I^*(-\varepsilon v_i \pi^{e_i+\lambda_l}) \prod_{k=1}^{i-1} I(-\varepsilon v_k \pi^{e_k+\lambda_l}) \prod_{k=i+1}^{n} I(-\varepsilon v_k \pi^{e_k+\lambda_l+2}) \right\}.$$

We note that, when $\sigma(i) \neq i$, ε_i must be 1 and $\prod_i \chi(\varepsilon_i) = \prod_{\sigma(i)=i} \chi(\varepsilon_i)$.

We conclude

$$\begin{aligned} S_t(B, s)^\chi = & \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^2 = 1}} \chi(-1)^{c_2^t(\sigma)} 2^{-c_1(\sigma)} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{I = I_0 \cup \cdots \cup I_r} q^{-\tau(\{I_i\}) - t(\sigma, \{I_i\})} \\ & \times \sum_{\nu_0, \cdots, \nu_r (**)'} \chi(-1)^{\sum_{h_j<0} h_j} q^{d(\sigma,h,e) + s\sum_{h_j<0} h_j - \sum_{l=0}^r \nu_l n(l)} \prod_{l=0}^{r} \prod_{\substack{i \in I_l \\ \sigma(i)=i}} q'_{i,\lambda_l}(\sigma; B) \end{aligned} \tag{4}$$

where $\{\nu_i\}$ takes

$$(**)' \quad \nu_1, \cdots, \nu_r \geq 1, \ -b_l(\sigma, B) \leq \lambda_l, \quad \sum_{\substack{l \\ \nu_l < 0}} n(l) = n - t.$$

We define

$$q'_{i,\lambda_l}(\sigma; B) = \sum_{\varepsilon=1,\delta} \left\{ \chi(\varepsilon)' I^*(-\varepsilon v_i \pi^{e_i+\lambda_l}) \prod_{k=1}^{i-1} I(-\varepsilon v_k \pi^{e_k+\lambda_l}) \prod_{k=i+1}^{n} I(-\varepsilon v_k \pi^{e_k+\lambda_l+2}) \right\}$$

and since $i \in I_l$ we get

$$q'_{i,\lambda_l}(\sigma; B) = \begin{cases} q_{i,\lambda_l}(\sigma; B) & (\lambda_l < 0) \\ 2(1-q^{-1}) & (\lambda_l \geq 0). \end{cases}$$

We divide the summation with respect to $(\nu_0, \cdots, \nu_r)$ as follows:

$$\sum_{\nu_0, \cdots, \nu_r (**)'} = \sum_{k=0}^{r+1} \sum_{\substack{\nu_0 + \cdots + \nu_{k-1} < 0 \\ \nu_0 + \cdots + \nu_k \geq 0}}.$$

Note here that, if $\nu_0 + \cdots + \nu_k \geq 0$, then $\nu_0 + \cdots + \nu_l \geq 0$ for any $l \geq k$, since $\nu_1, \cdots, \nu_r \geq 1$. The term $\sum_{h_j<0} h_j$ is

$$\begin{aligned} \sum_{h_j<0} h_j &= n_0\nu_0 + n_1(\nu_0 + \nu_1) + \cdots + n_{k-1}(\nu_0 + \nu_1 + \cdots + \nu_{k-1}) \\ &= \sum_{l=0}^{k-1} \nu_l (n_l + \cdots + n_{k-1}) \\ &= \sum_{l=0}^{k-1} \nu_l (n^{(l)} - n^{(k)}). \end{aligned}$$

Thus we have

$$\sum_{\substack{\nu_0+\cdots+\nu_{k-1}<0\\ \nu_0+\cdots+\nu_k\geq 0}} \chi(-1)^{\sum_{h_j<0}h_j} q^{s\sum_{h_j<0}h_j-\sum_{l=0}^{r}\nu_l n(l)} \prod_{l=0}^{r}\prod_{\substack{i\in I_l\\ \sigma(i)=i}} q'_{i,\lambda_l}(\sigma;B)$$

$$\begin{aligned}
&= \sum_{\nu_0,\cdots,\nu_{k-1}} \chi(-1)^{\sum_{l=0}^{k-1}\nu_l(n^{(l)}-n^{(k)})} q^{\sum_{l=0}^{k-1}\nu_l(sn^{(l)}-sn^{(k)}-n(l))} \prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}} q'_{i,\lambda_l}(\sigma;B)\\
&\quad\times \sum_{\nu_k=-(\nu_0+\cdots+\nu_{k-1})}^{\infty}\ \sum_{\nu_{k+1}=1}^{\infty}\cdots\sum_{\nu_r=1}^{\infty} q^{-\sum_{l=k}^{r}\nu_l n(l)}\{2(1-q^{-1})\}^{\sum_{l=k}^{r}c_1^{(l)}(\sigma)}\\
&= \frac{\{2(1-q^{-1})\}^{\sum_{l=k}^{r}c_1^{(l)}(\sigma)}q^{n(k)}}{\prod_{l=k}^{r}(q^{n(l)}-1)}\\
&\quad\times \sum_{\nu_0,\cdots,\nu_{k-1}} \chi(-1)^{\sum_{l=0}^{k-1}\nu_l(n^{(l)}-n^{(k)})} q^{\sum_{l=0}^{k-1}\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))} \prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}} q'_{i,\lambda_l}(\sigma;B).
\end{aligned}\tag{5}$$

It is easy to check that, when $\lambda_l \geq 0$,

$$\prod_{\substack{i\in I_l\\ \sigma(i)=i}} q'_{i,\lambda_l}(\sigma;B) = \{2(1-q^{-1})\}^{c_1^{(l)}(\sigma)}$$

where we put

$$c_1^{(k)}(\sigma) = \#\{i\in I_l \mid \sigma(i)=i\}.$$

Lemma 3.3. *When we put*

$$r'_i = \frac{1}{2}\sum_{k=1}^{i-1}\min\{e_k+\lambda_l,0\} + \frac{1}{2}\sum_{k=i+1}^{n}\min\{e_k+\lambda_l+2,0\},$$

we have

$$q_{i,\lambda_l}(\sigma,B) = 2\alpha_\psi(\pi)q^{r'_i+\frac{1}{2}\min\{e_i+\lambda_l,0\}}\xi_{i,\lambda_l}(B)_\chi.$$

Proof. Now we use the fact that, for $v\in\mathfrak{o}^\times$ and $k\in\mathbb{Z}$,

$$I(v\pi^k) = \begin{cases} 1 & k\geq 0\\ \alpha_\psi(v\pi^k)q^{k/2} & k<0.\end{cases}$$

It becomes

$$q_{i,\lambda_l}(\sigma;B) = q^{r'_i}\sum_{\varepsilon=1,\delta}\left\{\chi(\varepsilon)I^*(-\varepsilon v_i\pi^{e_i+\lambda_l})\prod_{\substack{k=1\\ e_k+\lambda_l<0}}^{i-1}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l})\prod_{\substack{k=i+1\\ e_k+\lambda_l+2<0}}^{n}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l+2})\right\}.$$

We recall

$$I^*(v\pi^k) = I(v\pi^k)-\frac{1}{q}I(v\pi^{k+2}) = \begin{cases} 1-q^{-1} & (k\geq 0)\\ \alpha_\psi(v\pi^{-1})q^{-1/2}-q^{-1} & (k=-1)\\ 0 & (k\leq -2).\end{cases}$$

Hence

$$q_{i,\lambda_l}(\sigma;B)=q^{r_i'}\sum_{\varepsilon=1,\delta}\left\{\chi(\varepsilon)\prod_{\substack{k=1\\ e_k+\lambda_l<0}}^{i-1}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l})\prod_{\substack{k=i+1\\ e_k+\lambda_l+2<0}}^{n}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l+2})\right\}$$
$$\times\begin{cases}1-q^{-1} & (e_i+\lambda_l\geq 0)\\ \alpha_\psi(-\varepsilon v_i\pi^{-1})q^{-1/2}-q^{-1} & (e_i+\lambda_l=-1).\end{cases}$$

To calculate $q_{i,\lambda_l}(\sigma;B)$, we use following lemmas.

Lemma 3.4. *For $a_k\in F^\times(1\leq k\leq r)$, we define $A:=\{k\mid \operatorname{ord}a_k:\text{odd}\}$. In this situation,*

$$\sum_{\varepsilon=1,\delta}\chi(\varepsilon)\prod_{k=1}^{r}\alpha_\psi(\varepsilon a_k)=\begin{cases}2\prod_{k=1}^{r}\alpha_\psi(a_k) & \#A:\text{odd}\\ 0 & \#A:\text{even}.\end{cases}$$

Proof. From Lemma 2.2, we have $\alpha_\psi(\delta a)=(-1)^{\operatorname{ord}a}\alpha_\psi(a)$. Hence

$$\begin{aligned}\sum_{\varepsilon=1,\delta}\chi(\varepsilon)\prod_{k=1}^{r}\alpha_\psi(\varepsilon a_k)&=\prod_{k=1}^{r}\alpha_\psi(a_k)+\chi(\delta)\prod_{k=1}^{r}\alpha_\psi(\delta a_k)\\ &=\prod_{k=1}^{r}\alpha_\psi(a_k)-\prod_{k=1}^{r}(-1)^{\operatorname{ord}a_k}\alpha_\psi(a_k)\\ &=\left(1-\prod_{k=1}^{r}(-1)^{\operatorname{ord}a_k}\right)\prod_{k=1}^{r}\alpha_\psi(a_k).\end{aligned}$$

Since the term $1-\prod_{k=1}^{r}(-1)^{\operatorname{ord}a_k}$ is 2 and 0 when $\#A$ is odd and even, respectively, we can prove this lemma. □

Lemma 3.5. *Let $a,b\in\pi\mathfrak{o}$. Then $\alpha_\psi(a)\alpha_\psi(b)=\chi(-ab)$.*

Proof. From the Lemma 2.1,

$$\frac{\alpha_\psi(a)\alpha_\psi(b)}{\alpha_\psi(1)\alpha_\psi(ab)}=\langle a,b\rangle.$$

Now the denominator of the left-hand side is $\alpha_\psi(1)\alpha_\psi(ab)=1\cdot 1=1$ and the right-hand side of the Hilbert symbol is $\chi(-ab)$, we proved the lemma. □

First, we assume $e_i+\lambda_l\geq 0$. We note that the order of the set

$$\begin{aligned}B_i(\lambda_l)=&\{k\mid 1\leq k\leq i-1,\ e_k+\lambda_l<0,\ e_k\not\equiv\lambda_l \bmod 2\}\\ &\cup\{k\mid i+1\leq k\leq g,\ e_k+\lambda_l+2<0,\ e_k\not\equiv\lambda_l \bmod 2\}\end{aligned}$$

is odd. If we fix an element $k_0\in B_i(\lambda_l)$, then $\#B_i(\lambda_l)\backslash\{k_0\}$ is even and by Lemma 3.5, we give

$$
\begin{aligned}
q_{i,\lambda_l}(\sigma;B) &= (1-q^{-1})q^{r_i'}\sum_{\varepsilon=1,\delta}\left\{\chi(\varepsilon)\prod_{\substack{k=1\\ e_k+\lambda_l<0}}^{i-1}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l})\prod_{\substack{k=i+1\\ e_k+\lambda_l+2<0}}^{n}\alpha_\psi(-\varepsilon v_k\pi^{e_k+\lambda_l+2})\right\}\\
&= 2(1-q^{-1})q^{r_i'}\prod_{k\in B_i(\lambda_l)}\alpha_\psi(-v_k\pi^{e_k+\lambda_l})\\
&= 2(1-q^{-1})q^{r_i'}\prod_{k\in B_i(\lambda_l)\setminus\{k_0\}}\alpha_\psi(-v_k\pi^{e_k+\lambda_l})\cdot\alpha_\psi(-v_{k_0}\pi^{e_{k_0}+\lambda_l})\\
&= 2(1-q^{-1})q^{r_i'}\chi(-1)^{[\#B_i(\lambda_l)/2]}\prod_{k\in B_i(\lambda_l)\setminus\{k_0\}}\chi(v_k)\alpha_\psi(-v_{k_0}\pi^{e_{k_0}+\lambda_l})\\
&= 2(1-q^{-1})q^{r_i'}\chi(-1)^{[\#B_i(\lambda_l)/2]+1}\alpha_\psi(\pi)\prod_{k\in B_i(\lambda_l)}\chi(v_k).
\end{aligned}
$$

We use $\alpha_\psi(-v_{k_0}\pi^{e_{k_0}+\lambda_l})=\chi(-v_{k_0})\alpha_\psi(\pi)$, since $e_{k_0}+\lambda_l$ is an odd number.

When $e_i+\lambda_l=-1$, we can similarly calculate as

$$
\begin{aligned}
&q_{i,\lambda_l}(\sigma;B)\\
&=\begin{cases}2q^{r_i'-\frac{1}{2}}\chi(v_i)\chi(-1)^{[\#B_i(\lambda_l)/2]+1}\alpha_\psi(\pi)\prod_{k\in B_i(\lambda_l)}\chi(v_k) & \#B_i(\lambda_l):\text{even}\\ 2q^{r_i'-1}\chi(-1)^{[\#B_i(\lambda_l)/2]+1}\alpha_\psi(\pi)\prod_{k\in B_i(\lambda_l)}\chi(v_k) & \#B_i(\lambda_l):\text{odd}.\end{cases}
\end{aligned}
$$

Therefore,

$$q_{i,\lambda_l}(\sigma,B)=2\alpha_\psi(\pi)q^{r_i'+\frac{1}{2}\min\{e_i+\lambda_l,0\}}\xi_{i,\lambda_l}(B)_\chi$$

and the lemma holds. □

From (4), (5), and Lemma 4.1, It follows that

$$
\begin{aligned}
S_t(B,s)^\chi &= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}}\chi(-1)^{c_2^t(\sigma)}2^{-c_1(\sigma)}(1-q^{-1})^{c_2(\sigma)}q^{-c_2(\sigma)}\sum_{\substack{I=I_0\cup\cdots\cup I_r\\ n^{(k)}=t}}q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}\\
&\times\frac{\{2(1-q^{-1})\}^{\sum_{l=k}^{r}c_1^{(l)}(\sigma)}q^{n(k)}}{\prod_{l=k}^{r}(q^{n(l)}-1)}\\
&\times\sum_{\nu_0,\cdots,\nu_{k-1}}\chi(-1)^{\sum_{l=0}^{k-1}\nu_l(n^{(l)}-n^{(k)})}q^{d(\sigma,e,\beta)+\sum_{l=0}^{k-1}\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))}\\
&\times\prod_{l=0}^{k-1}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}2\alpha_\psi(\pi)q^{r_i'+\frac{1}{2}\min\{\beta_i+\lambda_l,0\}}\xi_{i,\lambda_l}(B)_\chi,
\end{aligned}
$$

hence we have

$$S_t(B,s)^\chi = \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{\substack{I=I_0\cup\cdots\cup I_r\\ n^{(k)}=t}} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})} \frac{(1-q^{-1})^{\sum_{l=k}^r c_1^{(l)}(\sigma)} q^{n(k)}}{\prod_{l=k}^r (q^{n(l)}-1)}$$

$$\times \sum_{\{\nu\}_k^t} \prod_{l=0}^{k-1} \chi(-1)^{\nu_l(n^{(l)}-n^{(k)})} q^{\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))+\tilde{\rho}_{l,\nu_0+\cdots+\nu_l}(\sigma;B)} \prod_{\substack{i\in I_l\\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_l}(B)_\chi$$

To show the main theorem, we recall

$$\begin{aligned} d(\sigma,h,e) &= \sum_{\substack{i=1\\ \sigma(i)>i}}^n \left\{ \sum_{k=1}^{i-1} \min\{h_i+e_k,0\} + \sum_{k=i+1}^{\sigma(i)-1} \min\{h_i+e_k+1,0\} \right. \\ &\qquad \left. + \sum_{k=\sigma(i)+1}^{n} \min\{h_i+e_k+2,0\} \right\} \\ &= \sum_{\substack{i=1\\ \sigma(i)>i}}^n \sum_{k\neq i,\sigma(i)} \min\{h_i+e_k+e_{\sigma,i,k},0\} \end{aligned}$$

and

$$\begin{aligned} & d(\sigma,h,e) + \sum_{l=0}^{k-1} \sum_{\substack{i\in I_l\\ \sigma(i)=i}} \left(r_i' + \frac{1}{2}\min\{e_i+\lambda_l,0\} \right) \\ &= \frac{1}{2}\sum_{l=0}^{k-1}\sum_{i\in I_l}\sum_{k=1}^{n} \min\{e_k+\lambda_l+e_{\sigma,i,k},0\} + \frac{1}{2}\sum_{l=k}^{r}\sum_{\substack{i\in I_l\\ \sigma(i)\neq i}}\sum_{k=1}^{n} \min\{e_k+\lambda_l+e_{\sigma,i,k},0\} \\ &= \sum_{l=0}^{k-1} \tilde{\rho}_{l,\lambda_l}(\sigma;B). \end{aligned}$$

□

4 The action of the intertwining operator

In this section, we first recall the calculation of the matrix of the intertwining operators when $n=2$. We use this method to compute the matrix $E_n^{(s),\chi}$ where $1\le n\le 8$ by using a computer algebra system, PARI/GP. Moreover, we write down the eigenvalue and eigenvector of the matrix where $1\le n\le 8$.

We define an intertwining operator $M_{w_n}^{(s)} : I_n(\omega,s)^{\Gamma,\omega} \to I_n(\omega,n+1-s)^{\Gamma,\omega}$ by

$$M_{w_n}^{(s)}(f) := \int_{X\in \mathrm{Sym}_n(F)} f\left(w_n \begin{pmatrix} 1 & X\\ 0 & 1\end{pmatrix} g\right) dX.$$

We can give a matrix representation of the linear map $M_{w_n}^{(s)}$ as

$$M_{w_n}^{(s)}\begin{bmatrix} f_0^{(s)} & f_1^{(s)} & \cdots & f_n^{(s)} \end{bmatrix} = \begin{bmatrix} f_0^{(n+1-s)} & f_1^{(n+1-s)} & \cdots & f_n^{(n+1-s)} \end{bmatrix} ((E_n^{(s),\omega})_{ij})$$

where $E_n^{(s),\omega} = ((E_n^{(s),\omega})_{ij})_{0\le i,j\le n}$ is a square matrix of size $n+1$.

We can write down as

$$M_{w_n}^{(s)} f_j^{(s)} = (E_n^{(s),\omega})_{0j} f_0^{(n+1-s)} + (E_n^{(s),\omega})_{1j} f_1^{(n+1-s)} + \cdots + (E_n^{(s),,\omega})_{nj} f_n^{(n+1-s)}.$$

The value of the above function at $g = w_i$ is

$$M_{w_n}^{(s)} f_j^{(s)}(w_i) = (E_n^{(s),\omega})_{ij},$$

therefore the value of intertwining operator at w_i is the entries of the matrix $E_n^{(s),\omega}$.

When $n = 1$, we give the representation matrix of the intertwining operator as

$$E_1^{(s),\chi} = \begin{pmatrix} 0 & 1 \\ \chi(-1)q^{-1} & 0 \end{pmatrix}.$$

In other words, the intertwining operator M satisfies the following two equations.

$$\begin{aligned} M_{w_1}^{(s)} f_0^{(s)} &= \chi(-1)q^{-1} f_1^{(2-s)}, \\ M_{w_1}^{(s)} f_1^{(s)} &= f_0^{(2-s)}. \end{aligned}$$

Here if we fix a complex number β that satisfies $\beta^2 = \chi(-1)q^{-1}$, the above two equations equal to

$$\begin{aligned} M_{w_1}^{(s)}(f_0^{(s)} + \beta f_1^{(s)}) &= \beta((f_0^{(2-s)} + \beta f_1^{(2-s)})), \\ M_{w_1}^{(s)}(f_0^{(s)} - \beta f_1^{(s)}) &= -\beta((f_0^{(2-s)} - \beta f_1^{(2-s)})). \end{aligned}$$

Hence the functions $f_0 \pm \beta f_1$ can be seen as the eigenvector of the intertwining operator, and the eigenvalue is $\pm\beta$.

When $n = 2$, the representation matrix is

$$E_2^{(s),\chi} = \begin{pmatrix} \chi(-1)q^{-1}(1-q^{-1})\frac{q^{-2s}}{1-q^{-2s}} & 0 & 1 \\ 0 & \chi(-1)q^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}} & 0 \\ q^{-3} & 0 & \chi(-1)q^{-1}(1-q^{-1})\frac{1}{1-q^{-2s}} \end{pmatrix},$$

and we can calculate the eigenvector as above. Each eigenvector and its eigenvalue are

$$\left(f_0 \pm \beta f_1 + \beta^2 f_2,\ \beta^2\frac{1-q^{-2s-1}}{1-q^{-2s}}\right), \quad \left(f_0 - \beta^2 q^{-1} f_2,\ -\beta^2 q^{-1}\frac{1-q^{-2s+1}}{1-q^{-2s}}\right).$$

We note that, for example when $n = 1$, the representation matrix $E_1^{(s),\chi}$ is written as

$$E_1^{(s),\chi} = \begin{pmatrix} 1 & 1 \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} \beta & \\ & -\beta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta & -\beta \end{pmatrix}^{-1}.$$

Therefore the information of eigenvectors and eigenvalues derive the matrix of the intertwining operator.

Remark 4.1. The definition of β depends on the definition of χ, which is a fixed one of two ramified characters. Since this situation is an inconvenience, we define a character $\chi = \chi_\pi$ and a complex number $\beta = \beta_0$ as follows:

$$\begin{aligned} \chi(x) = \chi_\pi(x) &:= \langle x, \pi \rangle, \\ \beta = \beta_0 &:= \alpha_\psi(\pi)\chi(\pi)q^{-\frac{1}{2}} = \varepsilon'(1,\chi,\psi). \end{aligned}$$

Here we note that $\chi(\pi) = \chi(-1)$, since $\chi(\pi) = \langle \pi, \pi \rangle = \alpha_\psi(\pi)^2 = \chi(-1)$.

4.1 The Weil representation on the finite field

Let notations as above. We write $\mathbb{F}_q$ as a finite field with q elements. $\mathcal{S}(\mathbb{F}_q^n)$ denotes the set of the Schwartz functions over $\mathbb{F}_q$. Let $\mathrm{Sp}_n = \mathrm{Sp}_n(\mathbb{F}_q)$ be the symplectic group over $\mathbb{F}_q$ of size n. The Weil representation ω_ψ is defined as

$$\omega_\psi : \mathrm{Sp}_n(\mathbb{F}_q) \curvearrowright \mathcal{S}(\mathbb{F}_q^n)$$

is defined as

$$\begin{aligned} \omega_\psi(m(A))\Phi(v) &= \chi(\det A)\Phi(vA), \quad A \in \mathrm{GL}_n(\mathbb{F}_q), \\ \omega_\psi(n(B))\Phi(v) &= \psi(vB^tv)\Phi(v), \quad B \in \mathrm{Sym}_n(\mathbb{F}_q), \\ \omega_\psi(w_n)\Phi(v) &= G_\psi^{-n}\mathcal{F}\Phi(-v). \end{aligned}$$

Here the unitary Fourier transform $\mathcal{F}$ is defined as

$$\mathcal{F}\Phi(v) = \sum_{y\in\mathbb{F}_q^n} \Phi(v)\psi(2y \cdot {}^tv)$$

and the Gauss sum G_ψ is defined as $G_\psi = \sum_{x\in\mathbb{F}_q} \psi(x^2)$. We define the matrices $m(A)$ and $n(B)$ by

$$\begin{aligned} m(A) &= \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad A \in \mathrm{GL}_n(\mathbb{F}_q), \\ n(B) &= \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}, \quad B \in \mathrm{Sym}_n(\mathbb{F}_q). \end{aligned}$$

Let $P \subset \mathrm{Sp}_n$ be a Siegel parabolic subgroup. We define the space of induced representation as

$$\mathrm{Ind}_P^{\mathrm{Sp}_n}\chi = \{f : \mathrm{Sp}_n \to \mathbb{C} \mid f(pg) = \chi(\det A)f(g),\ p = m(A)n(B) \in P\}.$$

For each Schwartz function $\phi \in \mathcal{S}(\mathbb{F}_q^n)$, the function $g \mapsto \omega_\psi(g)\phi(0)$ is an element of the space of induced representation, i.e.,

$$\begin{aligned} \widetilde{\omega_\psi} : \mathcal{S}(\mathbb{F}_q^n) &\to \mathrm{Ind}_P^{\mathrm{Sp}_n}\chi \\ \phi &\mapsto [g \mapsto \omega_\psi(g)\phi(0)]\,. \end{aligned}$$

Let ϕ_0 be the characteristic function of $\{0\}$ and let f be the image of the above map $\widetilde{\omega_\psi}$: in other words, $f(g) = (\omega_\psi(g)\phi_0)(0)$. This f plays an important role in $\mathrm{Ind}_P^{\mathrm{Sp}_n}\chi$, since it is in the set $(\mathrm{Im}\ \widetilde{\omega_\psi})^{P,\chi}$. Since $\omega_\psi^+ := \mathrm{Im}\ \widetilde{\omega_\psi}$ is the function of the even function, this is an irreducible representation. Also we note that since the Hecke ring $\mathcal{H}(P\backslash\mathrm{Sp}_n/P, \chi)$ is commutative, the dimension of $(\mathrm{Im}\ \widetilde{\omega_\psi})^{P,\chi}$ is at most 1.

We conclude that this f generates this P, χ-invariant subspace.

Let w_i be an element of the Weyl group

$$w_i = \left(\begin{array}{cc|cc} 1_{n-i} & & & \\ & & & -1_i \\ \hline & & 1_{n-i} & \\ & 1_i & & \end{array}\right),$$

and we want to know the value of $\omega_\psi(w_i)\phi_0$. We consider the inclusion $\mathrm{Sp}_i \hookrightarrow \mathrm{Sp}_n$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \hookrightarrow \left(\begin{array}{cc|cc} 1_{n-i} & & & \\ & A & & B \\ \hline & & 1_{n-i} & \\ & C & & D \end{array}\right).$$

From this inclusion, the space $\mathcal{S}(\mathbb{F}_q^n)$ is written as $\mathcal{S}(\mathbb{F}_q^{n-i}) \otimes \mathcal{S}(\mathbb{F}_q^i)$, and $\phi_0 = \phi_0^{(n-i)} \otimes \phi_0^{(i)}$. We also note that $\omega_\psi = \omega_\psi^{(n)} = \omega_\psi^{(n-i)} \otimes \omega_\psi^{(i)}$ and $\omega_\psi^{(n-i)}$ trivially acts on $\phi_0^{(n-i)}$, thus we have

$$\omega_\psi(w_i)\phi_0(0) = \omega_\psi^{(i)}(w_i)\phi_0^{(i)}(0) = G_\psi^{-i}$$

From the fact that the Gauss sum is equal to β_0^{-1}, we conclude that the function $f_\beta := f_0 + \beta_0 f_1 + \cdots + \beta_0^n f_n$ gives an eigenvector of the intertwining operator (i.e., an idempotent element of the Hecke ring).

4.2 The case where $n = 2$ and $\omega = \chi$

We recall the calculation of the matrix of order 2. We wrote this method in our previous paper.

Now, we define some notations which the same as [16].

Let α_i $(i = 1, 2)$ be simple roots of the Lie group $G = \mathrm{Sp}_2(F)$;

$$\alpha_1 = x_1 - x_2$$
$$\alpha_2 = 2x_2.$$

We denote by $\iota_\alpha : \mathrm{SL}_2(F) \to G$ the corresponding homomorphism of α. That is, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put

$$\iota_{\alpha_1}(A) = \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}, \quad \iota_{\alpha_2}(A) = \left(\begin{array}{cc|cc} 1 & & & \\ & a & & b \\ \hline & & 1 & \\ & c & & d \end{array}\right),$$

and we set

$$w_1^{B_2} = \iota_{\alpha_1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \quad w_2^{B_2} = \iota_{\alpha_2}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Let B_2 be the Borel subgroup of $\mathrm{Sp}_2(F)$, i.e.

$$B_2 = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline & & * & 0 \\ & & * & * \end{array}\right) \in \mathrm{Sp}_2(F) \right\}.$$

The space of the induced representation $\mathrm{Ind}_{B_2}^G(\chi|\cdot|^{t_1}, \chi|\cdot|^{t_2})$ is the space of C^∞ functions on G satisfying

$$f(bg) = \chi(a_1)|a_1|^{t_1+2}\chi(a_2)|a_2|^{t_2+1}f(g)$$

for any $b = \left(\begin{array}{cc|cc} a_1 & * & * & * \\ & a_2 & * & * \\ \hline & & a_1^{-1} & \\ & & * & a_2^{-1} \end{array}\right) \in B_2$. We consider that

$$I_2(\chi, s) \subset \mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{s-\frac{1}{2}}, \chi|\cdot|^{s+\frac{1}{2}}).$$

We define the intertwining operator

$$M_w = M(w, \chi) : \mathrm{Ind}_{B_2}^{G}(\chi) \to \mathrm{Ind}_{B_2}^{G}(\chi^w)$$

by

$$M_w f(h) = \int_{N_2 \cap w N_2^- w^{-1}} f(w^{-1} n h) dn$$

where N_2 and N_2^- denotes the unipotent radical of B_2 (resp. of the opposite parabolic subgroups of B_2).

Then for $f \in \mathrm{Ind}_{B_2}^{G}(\chi)$,

$$\iota_\alpha^*(M(w, \chi)f) = M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \iota_\alpha^* \chi\right)(\iota_\alpha^* f)$$

as a function on $\mathrm{SL}_2(F)$. We note the fact that when $l(w_1)+l(w_2) = l(w_1w_2)$, $M_{w_1}M_{w_2} = M_{w_1w_2}$.

According to the above notations,

$$w_1^{B_2} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad w_2^{B_2} = \begin{pmatrix} 1 & & & \\ & & & -1 \\ & & 1 & \\ & 1 & & \end{pmatrix},$$

and we put

$$w_3^{B_2} := w_1^{B_2} w_2^{B_2} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & -1 & & \\ & & 1 & \end{pmatrix}, \quad w_4^{B_2} := w_2^{B_2} w_1^{B_2} = \begin{pmatrix} & -1 & & \\ & & -1 & \\ & & & -1 \\ 1 & & & \end{pmatrix},$$

$$w_5^{B_2} := w_1^{B_2} w_2^{B_2} w_1^{B_2} = \begin{pmatrix} & & 1 & \\ & -1 & & \\ -1 & & & \\ & & & -1 \end{pmatrix}, \quad w_6^{B_2} := w_2^{B_2} w_1^{B_2} w_2^{B_2} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},$$

$$w_7^{B_2} := w_1^{B_2} w_2^{B_2} w_1^{B_2} w_2^{B_2} = w_2^{B_2} w_1^{B_2} w_2^{B_2} w_1^{B_2} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

The Weyl group of G is defined as $W := \{w_i^{B_2} \mid 1 \le i \le 7\} \cup \{w_0^{B_2} = 1_4\}$. Next we define the function $f_i^{B_2,\chi}(0 \le i \le 7)$ by satisfying its support has $B_2 w_i^{B_2} B_2$ and $f_i^{B_2,\chi}\left(w_i^{B_2}\right) = 1$. The function $f_i^{B_2,\mathbf{1}}$ are similarly defined.

Put $q^{-s} = X$, $\chi(-1) = \eta$, and $f(s) = \dfrac{1-q^{-1}}{1-q^{-s}}$. We define the square matrices $A(s)$ and $B(s)$ of size 8 by

$$\int_{\mathrm{Sym}_2(F)} f_i^{B_2,\chi}\left(w_2\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix} g\right) dX = \sum_{j=0}^{7} f_j^{B_2,\chi}(g)(A(s))_{ij}$$

$$\int_{\mathrm{Sym}_2(F)} f_i^{B_2,\mathbf{1}}\left(w_2\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix} g\right) dX = \sum_{j=0}^{7} f_j^{B_2,\mathbf{1}}(g)(B(s))_{ij}.$$

Because $\iota_\alpha^*(M(w,\chi)f) = M\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \iota_\alpha^*\chi\right)(\iota_\alpha^* f)$, we can easily calculate the entries of $A(s)$ and $B(s)$ by using the entries of $E_1^{(s),\chi}$ and $E_1^{(s),\mathbf{1}}$;

$$A(s) = \begin{pmatrix}
0 & & 1 & & & & & \\
 & 0 & & & 1 & & & \\
\eta q^{-1} & & 0 & & & & & \\
 & & & 0 & & & 1 & \\
 & \eta q^{-1} & & & 0 & & & \\
 & & & & & 0 & & 1 \\
 & & & \eta q^{-1} & & & 0 & \\
 & & & & & \eta q^{-1} & & 0
\end{pmatrix}$$

and

$$B(s) = \begin{pmatrix}
Xf(s) & 1 & & & & & & \\
q^{-1} & f(s) & & & & & & \\
 & & Xf(s) & 1 & & & & \\
 & & q^{-1} & f(s) & & & & \\
 & & & & Xf(s) & 1 & & \\
 & & & & q^{-1} & f(s) & & \\
 & & & & & & Xf(s) & 1 \\
 & & & & & & q^{-1} & f(s)
\end{pmatrix}.$$

When we consider

$$\begin{array}{c}
I_2(\chi, s) \subset \mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{s-\frac{1}{2}}, \chi|\cdot|^{s+\frac{1}{2}}) \\
\Big\downarrow M_{w_2}^{(s+\frac{1}{2})} \\
\mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{s-\frac{1}{2}}, \chi|\cdot|^{-s-\frac{1}{2}}) \\
\Big\downarrow M_{w_1}^{(2s)} \\
\mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{-s-\frac{1}{2}}, \chi|\cdot|^{s-\frac{1}{2}}) \\
\Big\downarrow M_{w_2}^{(s-\frac{1}{2})} \\
\mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{-s-\frac{1}{2}}, \chi|\cdot|^{-s+\frac{1}{2}}) \supset I_2(\chi, -s)
\end{array}$$

the matrix representation of the intertwining operator is $A(s+\frac{1}{2})B(2s)A(s-\frac{1}{2})$, thus we calculate the matrix $E_2^{B_2} = A(s+1/2)B(2s)A(s-1/2)$.

$$E_2^{B_2} = \begin{pmatrix} X^2 f(2s)\eta q^{-1}\cdot 1_2 & 0 & 0 & 1\cdot 1_2 \\ 0 & X^2 f(2s)\eta q^{-1}\cdot 1_2 & \eta q^{-1}\cdot 1_2 & 0 \\ 0 & \eta q^{-2}\cdot 1_2 & f(2s)\eta q^{-1}\cdot 1_2 & 0 \\ \eta^2 q^{-3}\cdot 1_2 & 0 & 0 & f(2s)\eta q^{-1}\cdot 1_2 \end{pmatrix}.$$

Lemma 4.1. *The following three equations hold;*

$$f_0 = f_0^{B_2} + f_1^{B_2}.$$
$$f_1 = f_2^{B_2} + f_3^{B_2} + f_4^{B_2} + f_5^{B_2}.$$
$$f_2 = f_6^{B_2} + f_7^{B_2}.$$

Proof. We show the first equation. As the space of the induced representation according to B_2 is spanned by the function $f_i^{B_2}$, there are some $c_0, c_1 \in \mathbb{C}$ such that $f_0 = c_0 f_0^{B_2} + c_1 f_1^{B_2}$. Considering the value at $\omega_0^{B_2}$ and $\omega_1^{B_2}$, it is obvious that $c_0 = f_0(\omega_0^{B_2}) = 1$ and $c_1 = f_0(\omega_1^{B_2}) = 1$. □

Now we can calculate the matrix $E_2^{(s),\chi}$.

$$\begin{aligned} M_{w_2}^{(s)} f_0 &= M_{w_2}^{(s)}(f_0^{B_2} + f_1^{B_2}) \\ &= (X^2 f(2s)\eta q^{-1} f_0^{B_2} + \eta^2 q^{-3} f_6^{B_2}) + (X^2 f(2s)\eta q^{-1} f_1^{B_2} + \eta^2 q^{-3} f_7^{B_2}) \\ &= X^2 f(2s)\eta q^{-1} f_0 + \eta^2 q^{-3} f_2 \\ &= \chi(-1)q^{-1}(1-q^{-1})\frac{q^{-2s}}{1-q^{-2s}} f_0 + q^{-3} f_2, \end{aligned}$$

and

$$\begin{aligned} M_{w_2}^{(s)} f_1 &= M_{w_2}^{(s)}(f_2^{B_2} + f_3^{B_2} + f_4^{B_2} + f_5^{B_2}) \\ &= (X^2 f(2s)\eta q^{-1} f_2^{B_2} + \eta q^{-2} f_4^{B_2}) + (X^2 f(2s)\eta q^{-1} f_3^{B_2} + \eta q^{-2} f_5^{B_2}) \\ &\quad + (f(2s)\eta q^{-1} f_4^{B_2} + \eta q^{-1} f_2^{B_2}) + (f(2s)\eta q^{-1} f_5^{B_2} + \eta q^{-1} f_3^{B_2}). \end{aligned}$$

Here we note that

$$X^2 f(2s)\eta q^{-1} + \eta q^{-1} = f(2s)\eta q^{-1} + \eta q^{-2} = \chi(-1)q^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}}$$

and we get

$$M_{w_2}^{(s)} f_1 = \chi(-1)q^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}} f_1.$$

Finally,

$$\begin{aligned} M_{w_2}^{(s)} f_2 &= M_{w_2}^{(s)}(f_6^{B_2} + f_7^{B_2}) \\ &= (f(2s)\eta q^{-1} f_6^{B_2} + f_0^{B_2}) + (f(2s)\eta q^{-1} f_7^{B_2} + f_1^{B_2}) \\ &= f(2s)\eta q^{-1} f_2 + f_0 \\ &= \chi(-1)q^{-1}(1-q^{-1})\frac{1}{1-q^{-2s}} f_2 + f_0. \end{aligned}$$

In conclusion, we get the following theorem.

Theorem 4.1.

$$E_2^{(s),\chi} = \begin{pmatrix} \chi(-1)q^{-1}(1-q^{-1})\frac{q^{-2s}}{1-q^{-2s}} & 0 & 1 \\ 0 & \chi(-1)q^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}} & 0 \\ q^{-3} & 0 & \chi(-1)q^{-1}(1-q^{-1})\frac{1}{1-q^{-2s}} \end{pmatrix}.$$

4.3 The case where $n = 2$ and $\omega = \mathbf{1}$

When we calculate the matrix $E_2^{(s),\omega}$ when $\omega = \mathbf{1}$, we have

Theorem 4.2.

$$E_2^{(s),\mathbf{1}} = \begin{pmatrix} \frac{q^{-2s}(1-q^{-1})\left(1-q^{-s-\frac{3}{2}}\right)}{(1-q^{-2s})\left(1-q^{-s+\frac{1}{2}}\right)} & \frac{q^{-s+\frac{1}{2}}(1-q^{-2})}{1-q^{-s+\frac{1}{2}}} & 1 \\ \frac{q^{-s-\frac{3}{2}}(1-q^{-1})}{1-q^{-s+\frac{1}{2}}} & \frac{q^{-3s+1}+q^{-2s+\frac{5}{2}}(1-2q^{-1})-q^{-s+1}(2-q^{-1})+q^{\frac{3}{2}}}{q^{\frac{5}{2}}\left(1-q^{-s+\frac{1}{2}}\right)(1-q^{-2s})} & \frac{1-q^{-1}}{1-q^{-s+\frac{1}{2}}} \\ q^{-3} & \frac{1-q^{-2}}{q\left(1-q^{-s+\frac{1}{2}}\right)} & \frac{(1-q^{-1})\left(1-q^{-s-\frac{3}{2}}\right)}{(1-q^{-2s})\left(1-q^{-s+\frac{1}{2}}\right)} \end{pmatrix}.$$

Remark 4.2. The calculation of $(1,1)$ entry (the center) of the matrix $E_2^{(s),\mathbf{1}}$ is mistaken in our previous article; The above calculation is true.

4.4 The programming code of calculating the intertwining operator

By using the method of calculating the intertwining operator when $n = 2$, we can computationally calculate $E_n^{(s),\chi}$ when n is greater than 2. We note that the order of the Weyl group associated with the Borel subgroup W_n is $2^n \times n!$ and the number of multiplying rank 1 intertwining operator is $n(n+1)/2$. We have to multiply matrices (sparse matrices) of too big size many times.

We note that the Weyl group W_n is isomorphic to $\mathfrak{S}_n \rtimes \{\pm 1\}^n$, known as the signed permutation group. Later, we explain the case where n is 2. The generator of Weyl group, $w_1^{B_2}$ and $w_2^{B_2}$ is identified as

$$w_1^{B_2} \Longleftrightarrow \sigma_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad w_2^{B_2} \Longleftrightarrow \sigma_2 = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}.$$

In general n, the generator of signed permutation group is $(1\ 2), (2\ 3), \cdots, (n-1\ n)$, which correspond to short roots, and $(n\ -n)$, which corresponds to the long root.

Next, we translate the matrix $A(s)$ and $B(s)$ into the word of the signed permutation group. For $w, w' \in W_2$, it follows that

$$(A(s))_{w,w'} = \begin{cases} 1 & w' = w_2^{B_2} w,\ l(w_2^{B_2} w) > l(w) \\ \chi(-1) q^{-1} & w' = w_2^{B_2} w,\ l(w_2^{B_2} w) < l(w) \\ 0 & \text{otherwise.} \end{cases}$$

$$(B(s))_{w,w'} = \begin{cases} q^{-s} \dfrac{1-q^{-1}}{1-q^{-s}} & w' = w,\ l(w_1^{B_2} w) > l(w) \\ \dfrac{1-q^{-1}}{1-q^{-s}} & w' = w,\ l(w_1^{B_2} w) < l(w) \\ 1 & w' = w_1^{B_2} w,\ l(w_1^{B_2} w) > l(w) \\ q^{-1} & w' = w_1^{B_2} w,\ l(w_1^{B_2} w) < l(w) \\ 0 & \text{otherwise.} \end{cases}$$

Thus the calculation of $A(s+1/2)B(2s)A(s-1/2)$ is equivalent to taking the finite sum

$$(A(s+1/2)B(2s)A(s-1/2))_{w,w'} = \sum_{w_\alpha, w_\beta \in W_2} (A(s+1/2))_{w,w_\alpha} (B(2s))_{w_\alpha,w_\beta} (A(s-1/2))_{w_\beta,w'}.$$

We also note that, in the matrix $A(s)$, $(A(s))_{w,w'} = 0$ unless $w' = w_2^{B_2} w$. In $B(s)$, $(B(s))_{w,w'} = 0$ unless $w' = w$ or $w' = w_1^{B_2} w$. Therefore the summation with w_α, w_β is running over

$$w_\alpha = w_2^{B_2} w, \quad w_\beta = \begin{cases} w_1^{B_2} w_\alpha, \\ w_\alpha, \end{cases} \quad w' = w_2^{B_2} w_\alpha.$$

We identify the Weyl group and the signed permutation group. Each $w \in W_2$, corresponding signed permutation σ_w is charactered by $\sigma_w(1)$ and $\sigma_w(2)$ since $\sigma_w(-i) = -\sigma_w(i)$. Later we only consider the sequence $(\sigma_w(1), \sigma_w(2))$. Their sign is arbitrary, and without their signs, $(\sigma_w(1), \sigma_w(2))$ is an permutation of $\{1, 2\}$.

The following is elemental.

$$l(w_1^{B_2} w) > l(w) \Leftrightarrow \begin{cases} +1 \text{ is located on the left side of } +2, \\ -2 \text{ is located on the left side of } -1, \\ \text{the sign of 1 is } + \text{ and 2 is } -, \end{cases}$$
$$l(w_2^{B_2} w) > l(w) \Leftrightarrow \quad \text{the sign of 2 is } +.$$

In general n, the short root $w_1^{B_n}, \cdots, w_{n-1}^{B_n}$ satisfies the same length condition as $w_1^{B_2}$ and long root $w_n^{B_n}$ satisfies the same as $w_2^{B_2}$

4.4.1 The motivation of programming

We can construct the programming code to calculate $(A(s+1/2)B(2s)A(s-1/2))_{w,w'}$. Here, we use PARI/GP because it is easy to handle rational functions.

First, we define some notations. X, Q, Z is defined as

$$X = q^{-s}, \quad Q = q^{\frac{1}{2}}, \quad Z = \chi(-1).$$

Storing data of the signed permutation $\mathfrak{S}_n \rtimes \{\pm 1\}^n$ is a little difficult.

- Storing data as the permutation of order $2n$ (Identified $-1, \cdots, -n$ as $n+1, \cdots, 2n$) $\Rightarrow$ σ_w satisfies $\sigma_w(n+i) - \sigma_w(i) = \pm n$ $(1 \le i \le n)$ and writing this condition is difficult.
- Storing data as the sequence of length n $\Rightarrow$ Checking whether the sign-missing sequence is in $\mathfrak{S}_n$ or not is difficult, and writing this condition that the transposition (each short/long root) does not move signs is difficult. For example, if $\sigma_{w_1}(1) = -i$ and $w_2 = (i\ i+1)$, we easily show $\sigma_{w_2 w_1}(1) = -(i+1)$. If $\sigma_{w_1}(1) = i$, then $\sigma_{w_2 w_1}(1) = i+1$ and we have the data of w_2 contain $i \mapsto i+1$ and $-i \mapsto -(i+1)$. It is inconvenient.

Therefore we separate the information of signs and permutations, i.e., identify W_n as $\mathfrak{S}_n \times \{\pm 1\}^n$ as sets. (The group structure of permutations is preserved).

In PARI/GP, Vecsmall $[a_1, \cdots, a_n]$ represents the permutation $\begin{pmatrix} 1 & \cdots & n \\ a_1 & \cdots & a_n \end{pmatrix} \in \mathfrak{S}_n$. Vecsmall can be seen as vectors when there is no multiplying. If we use the inner product, we should use Vec. In this case, we also use Vecsmall to store the signs' datum. In the vectors of the datum of signs, 1 means the sign is $+1$, and 0 (not -1) means the sign is -1.

4.4.2 The example of code when $n = 2$

Listing 1: The calculation of intertwining operator

```
search(q,i)=for(j=1,2,if(q[j]-i,,return(j)));

change2(q,i)={
    if(i==1,
        Vecsmall([1-q[1],q[2]]),
        Vecsmall([q[1],1-q[2]])
    );
};

E2p(a1, a2, b1, b2, n) = {
    local(x);
    x = 0;
    for (i1 = 0, 1,
        A0 = Vecsmall([a1, a2, b1, b2]);
        p1 = Vecsmall([A0[3], A0[4]]);
        q1 = change2(p1, search(A0, 2));
        A1 = Vecsmall([A0[1], A0[2], q1[1], q1[2]]);
        p2 = Vecsmall([A1[1], A1[2]]);
        q2 = if(i1 == 0, p2, Vecsmall([2, 1]) * p2);
        A2 = Vecsmall([q2[1], q2[2], A1[3], A1[4]]);
        if(A2[3] + A2[4] == n,
            Q1 = if(A0[search(A0, 2) + 2] == 0, 1, Z * q^(-1));
            Q2 = if(i1 == 0,
                if(search(A1, 2) > search(A1, 1),
                    if(A1[search(A1, 1) + 2] == 0,
                        X * q^2 * (1 - q^(-1)) / (1 - X * q^2),
                        (1 - q^(-1)) / (1 - X * q^2)
                    ),
                    if(A1[search(A1, 2) + 2] == 0,
                        (1 - q^(-1)) / (1 - X * q^2),
                        X * q^2 * (1 - q^(-1)) / (1 - X * q^2)
                    );
                ),
                if(search(A1, 2) > search(A1, 1),
                    if(A1[search(A1, 1) + 2] == 0,
                        1,
                        q^(-1)
                    ),
                    if(A1[search(A1, 2) + 2] == 0,
                        q^(-1),
                        1
                    );
                );
            );
            Q3 = if(A2[search(A2, 2) + 2] == 0, 1, Z * q^(-1));
            x = x + Q1 * Q2 * Q3;
        );
    );
    return(x);
};

E2(m,n)={
    r1=if(m>=1,1,0);
    r2=if(m>=2,1,0);
```

```
55     E2p(1,2,r1,r2,n);
56 };
```

There are many conditional branches and nested if statements. There must be a more concise code, but we don't know. However, it is easy to extend this code to the case $n \geq 3$, and then we can calculate the intertwining operator for the case $n \geq 3$ this way.

We will write the meaning of the above code.

- line 1: search function
  In this function, q is some vector whose length is two or more, and i is 1 or 2. More precisely, we consider the case where $\{q[1], q[2]\} = \{1, 2\}$. j runs 1 to 2 and if $q[j] = i$, return the value of j.
  Hence this function gives where the number i is in the vector q.

- line 3-8: change function
  In this function, we consider the case where $q \in \{0, 1\}^2$ and i is 1 or 2. The i-th element of vector q changes from 0 to 1 or 1 to 0.

- line 10-50: main function
  In this function, we calculate the value

$$\sum_{w':\ \mathrm{rank}\ n} (A(s+1/2)B(2s)A(s-1/2))_{w,w'},$$

  where (a_1, a_2) is sign-missing permutation of w and (b_1, b_2) is the sign of w. We describe this main function more precisely.

  - line 11, 12, 46, 49: the summation for each w_α
    x is declared as a variable and since $x = x + Q1 * Q2 * Q3$ this term is added. Return the total sum x.
  - line 15-17: Change the sign corresponding to the number 2, i.e., change $+2$ to -2 or -2 to $+2$.
  - line 18-20: if $i1 = 0$ ($\Leftrightarrow w_\beta = w_\alpha$) then nothing happens, if $i = 1$ ($\Leftrightarrow w_\beta = w_1^{B_2} w_\alpha$) then the number 1 and 2 changes.
  - line 21: We note that the rank (of the lower-left 2×2 matrix of modulo $\mathfrak{p}$) equals the number of the signs of $+$. Therefore, we determined that 0 means the sign $-$. In line 21, we take a sum if the rank w' is equal to given n.
  - line 22-46: the value of $A(s)$ or $B(s)$ is calculated. In the term of $Q2$, it is difficult for us to write the condition down shortly whether $l(w_1^{B_2} w) > l(w)$.

- line 52-56: the calculation of the $(E_2^{(s),\chi})_{mn}$
  In this function, the initial value a_i, b_i is fixed. We fix $a_1 = 1$ and $a_2 = 2$, and we choose a non-negative number $r_1 + r_2 = m$. In this function, we get

$$(r_1, r_2) = \begin{cases} (0,0) & \text{if } m = 0, \\ (1,0) & \text{if } m = 1, \\ (1,1) & \text{if } m = 2. \end{cases}$$

  Therefore we can calculate the (m, n)-entry of the matrix.

4.5 The matrix of the intertwining operator of $n \leq 8$

In this section, we calculate the intertwining operator and write the eigenvalues and eigenvectors of the matrix when $n \leq 8$. We calculate these matrices with PARI/GP. When $n \geq 9$, it is difficult to calculate with our computer because of working too much time.

Here we write the function $a_{i0}f_0 + a_{i1}\beta_0 f_1 + \cdots + a_{in}\beta_0^n f_n$ as $[a_{i0}, a_{i1}, \cdots, a_{in}]$ and the eigenvector of the intertwining operator as $e_0^{(n)}, e_1^{(n)}, \cdots, e_n^{(n)}$. The corresponding eigenvalue of $e_i^{(n)}$ is written as $\lambda_i^{(n)}$. For example, when $n = 1$,

$$\lambda_0^{(1)} = \beta, \qquad e_0^{(1)} = [1, 1],$$
$$\lambda_1^{(1)} = -\beta, \qquad e_1^{(1)} = [1, -1].$$

and when $n = 2$,

$$\lambda_0^{(2)} = \beta^2 \frac{1 - q^{-2s-1}}{1 - q^{-2s}}, \qquad e_0^{(2)} = [1, 1, 1],$$
$$\lambda_1^{(2)} = \beta^2 \frac{1 - q^{-2s-1}}{1 - q^{-2s}}, \qquad e_1^{(2)} = [1, -1, 1],$$
$$\lambda_2^{(2)} = \beta^2 q^{-2s} \frac{1 - q^{2s-1}}{1 - q^{-2s}}, \qquad e_2^{(2)} = [q, 0, -1].$$

(i) $n = 3$

$$\begin{aligned} e_0^{(3)} &= [1, 1, 1, 1], \\ e_1^{(3)} &= [1, -1, 1, -1], \\ e_2^{(3)} &= [q^5 - q^2, q^4 - q^3, -q^2 + q, -q^3 + 1], \\ e_3^{(3)} &= [q^5 - q^2, -q^4 + q^3, -q^2 + q, q^3 - 1]. \end{aligned}$$

We note that, when $[a_{i0}, a_{i1}, \cdots, a_{in}]$ is an eigenvector, then $[a_{i0}, -a_{i1}, \cdots, (-1)^n a_{in}]$ is also eigenvector except for $e_{2m}^{(2m)}$. Later, we will not write $e_i^{(n)}$ when i is odd.

(ii) $n = 4$

$$\begin{aligned} e_0^{(4)} &= [1, 1, 1, 1, 1], \\ e_2^{(4)} &= [q^5 + q^3, q^4, q^3 - q^2, -q, -q^2 - 1], \\ e_4^{(4)} &= [q^7 - q^4, 0, -q^4 + q^3, 0, q^3 - 1]. \end{aligned}$$

(iii) $n = 5$

$$\begin{aligned} e_0^{(5)} &= [1, 1, 1, 1, 1, 1], \\ e_2^{(5)} &= [q^9 - q^4, q^8 - q^5, q^7 - q^5 - q^4 + q^3, q^6 - q^5 - q^4 + q^2, -q^4 + q, -q^5 + 1], \\ e_4^{(5)} &= [q^{11} - q^6, q^9 - q^8, -q^6 + q^5, -q^6 + q^5, q^3 - q^2, q^5 - 1]. \end{aligned}$$

(iv) $n = 6$

$$
\begin{aligned}
e_0^{(6)} &= [1,1,1,1,1,1,1],\\
e_2^{(6)} &= [q^{11}-q^5, q^{10}-q^6, q^9-q^6-q^5+q^4, q^8-q^6-q^5+q^3, q^7-q^6-q^5+q^2, -q^5+q, -q^6+1],\\
e_4^{(6)} &= [q^8(q^5-1)(q^4+q^2+1), q^{15}-q^{10}, q^{13}-q^{12}-q^{10}+q^9-q^8+q^7, -q^{10}+q^7,\\
&\quad -q^{10}+q^9-q^8+q^7+q^5-q^4, q^7-q^2, (q^5-1)(q^4+q^2+1)],\\
e_6^{(6)} &= [q^{14}-q^9, 0, -q^9+q^8, 0, q^6-q^5, 0, -q^5+1].
\end{aligned}
$$

(v) $n = 7$

$$
\begin{aligned}
e_0^{(7)} &= [1,1,1,1,1,1,1,1],\\
e_2^{(7)} &= [q^{13}-q^6, q^{12}-q^7, q^{11}-q^7-q^6+q^5, q^{10}-q^7-q^6+q^4,\\
&\quad q^9-q^7-q^6+q^3, q^8-q^7-q^6+q^2, -q^6+q, -q^7+1],\\
e_4^{(7)} &= [q^{10}(q^6-1)(q^7-1), q^{12}(q^3-1)(q^6-1), f_{4,2}^{(7)}(q), f_{4,3}^{(7)}(q),\\
&\quad f_{4,4}^{(7)}(q), f_{4,5}^{(7)}(q), q^2(q^3-1)(q^6-1), (q^6-1)(q^7-1)],\\
e_6^{(7)} &= [q^{12}(q^5-1)(q^7-1), q^{15}(q-1)(q^5-1), -q^{11}(q-1)(q^5-1), -q^{12}(q-1)(q^3-1),\\
&\quad q^8(q-1)(q^3-1), q^7(q-1)(q^5-1), -q^3(q-1)(q^5-1), -(q^5-1)(q^7-1)],
\end{aligned}
$$

where

$$
\begin{aligned}
f_{4,2}^{(7)}(q) &= q^9(q-1)(q^2-1)(q^7+q^6+q^5-q^2-1),\\
f_{4,3}^{(7)}(q) &= q^9(q^2-1)(q^3-1)(q^3-q^2-1),\\
f_{4,4}^{(7)}(q) &= -q^6(q^2-1)(q^3-1)(q^3+q-1),\\
f_{4,5}^{(7)}(q) &= -q^4(q-1)(q^2-1)(q^7+q^5-q^2-q-1).
\end{aligned}
$$

(vi) $n = 8$

$$
\begin{aligned}
e_0^{(8)} &= [1,1,1,1,1,1,1,1,1],\\
e_2^{(8)} &= [q^{15}-q^7, q^{14}-q^8, q^{13}-q^8-q^7+q^6, q^{12}-q^8-q^7+q^5, q^{11}-q^8-q^7+q^4,\\
&\quad q^{10}-q^8-q^7+q^3, q^9-q^8-q^7+q^2, -q^7+q, -q^8+1],\\
e_4^{(8)} &= [q^{12}(q^4+1)(q^7-1), q^{14}(q^7-1), q^{11}(q^8-q^5-q+1), q^{11}(q^6-q^4-q^3+1), f_{4,4}^{(8)}(q),\\
&\quad -q^6(q^6-q^3-q^2+1), -q^4(q^8-q^7-q^3+1), q^2(q^7-1), (q^4-1)(q^7-1)],\\
e_6^{(8)} &= [q^{15}(q^2+1)(q^4+1)(q^7-1), q^{18}(q^7-1), q^{14}(q-1)(q^7-q^4-q^2-1),\\
&\quad -q^{15}(q^3-1), -q^{11}(q-1)(q^2+1)(q^3-1), q^{10}(q^3-1),\\
&\quad q^6(q-1)(q^7+q^5+q^3-1), -q^3(q^7-1), -(q^2+1)(q^4+1)(q^7-1)],\\
e_8^{(8)} &= [q^{16}(q^5-1)(q^7-1), 0, -q^{15}(q-1)(q^5-1), 0, q^{12}(q-1)(q^3-1),\\
&\quad 0, -q^7(q-1)(q^5-1), 0, (q^5-1)(q^7-1)],
\end{aligned}
$$

where $f_{4,4}^{(8)}(q) = q^8(q-1)(q^6-q^4-2q^3-q^2+1)$.

Each eigenvalue is calculated as follows. When $n = 2m$ is even, i.e. $n = 4, 6$, and 8, we define for each $0 \leq i \leq m$,

$$\lambda_i^{(2m)} = (-1)^m \beta_0^{2m} q^{2mi-i^2} \frac{\prod_{k=1}^{m-i}(1-q^{-2s+1-2k}) \cdot \prod_{k=1}^{m}(1-q^{-2s+2m+1-2k})}{(1-q^{-2s})(1-q^{-2s-2}) \cdots (1-q^{-2s-2m+2})}$$

thus $\lambda_i^{(2m)}$ is the eigenvalue with respect to eigenvector $e_{2i}^{(2m)} (0 \leq i \leq m)$ and $e_{2i+1}^{(2m)} (0 \leq i \leq m-1)$.

When $n = 2m+1$ is odd, i.e., $n = 3, 5$, and 7, we define for each $0 \leq i \leq m$,

$$\lambda_i^{(2m+1)} = (-1)^m \beta_0^{2m+1} \frac{\prod_{k=1}^{m-i}(1-q^{-2s-2k}) \cdot \prod_{k=1}^{m}(1-q^{-2s-2m-2+2k})}{(1-q^{-2s+1})(1-q^{-2s+3}) \cdots (1-q^{-2s+2m-1})}$$

thus $\lambda_i^{(2m)}$ is the eigenvalue with respect to eigenvector $e_{2i}^{(2m)} (0 \leq i \leq m)$ and $-\lambda_i^{(2m)}$ is with $e_{2i+1}^{(2m)}$.

5 The functional equation of the Siegel series

5.1 The functional equation of the Whittaker functional

Let $Q \in \mathrm{Sym}_n(F)$ satisfying $\det Q \neq 0$. For each Q, we put $D_Q = (-4)^{[n/2]} \det Q$. For $\theta \in F^\times / F^{\times 2}$, we define the character χ_θ by $\chi_\theta(x) = \langle \theta, x \rangle$. Let $\alpha(x)$ denote the Weil constant $\alpha_\psi(x)$.

For a character ω of $F^\times$ (we may not assume that $\omega^2 = 1$), we write $\varepsilon(s, \omega, \psi)$ and $L(s, \omega)$ for the ε and L factor of ω, respectively. We also use the notation

$$\varepsilon'(s, \omega) = \varepsilon'(s, \omega, \psi) = \varepsilon(s, \omega, \psi) \frac{L(1-s, \omega^{-1})}{L(s, \omega)}.$$

We denote by η_Q the Hasse invariant of the Clifford algebra (resp. the even Clifford algebra) of Q if n is even (resp. odd).

Lemma 5.1. *Assume that the matrix Q is equivalent to the diagonal matrix* $\mathrm{diag}(q_1, q_2, \cdots, q_n)$. *If $n = 2m+1$ is odd, then*

$$\eta_Q = \langle -1, -1 \rangle^{m(m+1)/2} \langle (-1)^m, \det Q \rangle \varepsilon_Q.$$

If $n = 2m$ is even, then

$$\eta_Q = \langle -1, -1 \rangle^{m(m-1)/2} \langle (-1)^{m+1}, \det Q \rangle \varepsilon_Q.$$

Here ε_Q is defined as $\varepsilon_Q = \prod_{1 \leq i < j \leq n} \langle q_i, q_j \rangle$.

For $\Phi \in \mathcal{S}(\mathrm{Sym}_n(F))$, the Fourier transform $\hat{\Phi}$ is defined as

$$\hat{\Phi}(x) = \int_{\mathrm{Sym}_n(F)} \Phi(y) \psi(\mathrm{tr}(xy)) dy.$$

Here, dy is the self-dual measure for this Fourier transform. From the prehomogeneity of the space $\mathrm{Sym}_n(F)$, there exists a meromorphic function $c_Q(\omega, s)$ which satisfies

$$\int_{\mathrm{Sym}_n(F)} \Phi(X)\omega(\det X)|\det X|^{s-\frac{n+1}{2}}dX = \sum_{Q\in\mathcal{O}} c_Q(\omega,s)\int_{V_Q}\hat{\Phi}(X)\omega^{-1}(\det X)|\det X|^{-s}dX$$

where $\mathcal{O}$ is the set of open orbits of $\mathrm{Sym}_n(F)$ under the action of algebraic group GL_n, and V_Q is the open orbit containing Q.

We define degenerate Whittaker functional Wh_B as

$$M_{w_n}^{(s)}f(g) = \int_{\mathrm{Sym}_n(F)} f(w_n n(x)g)dx,$$
$$\mathrm{Wh}_B(s)f = \int_{\mathrm{Sym}_n(F)} f(w_n n(x))\overline{\psi(\mathrm{tr}(Bx))}dx.$$

It is well known that these integrals are absolutely convergent for $\mathrm{Re}(s) \gg 0$, and they can be meromorphically continued to the whole $s \in \mathbb{C}$.

Theorem 5.1 ([15] Lemma 3.1)**.** *The following functional equation of the Whittaker functional holds.*

$$\mathrm{Wh}_B(-s)\circ M_{w_n}^{(s)} = \omega^{-1}(\det B)|\det B|^{-s}c_B(\omega,s)\mathrm{Wh}_B(s).$$

Theorem 5.2 ([15] Theorem 2.1)**.** *If $n = 2m+1$, then we have*

$$c_Q(\omega,s) = \varepsilon'(s-m,\omega)^{-1}\prod_{r=1}^{m}\varepsilon'(2s-2m-1+2r,\omega^2)^{-1}$$
$$\times |2|^{-2ms+\frac{m(2m+1)}{2}}\omega^{-m}(4)\eta_Q.$$

If $n = 2m$, then we have

$$c_Q(\omega,s) = \varepsilon'(s-m+\frac{1}{2},\omega)^{-1}\prod_{r=1}^{m}\varepsilon'(2s-2m+2r,\omega^2)^{-1}$$
$$\times |2|^{-2ms+\frac{m(2m-1)}{2}}\omega^{-m}(4)\frac{\alpha(D_Q)}{\alpha(1)}\varepsilon'(s+\frac{1}{2},\omega\chi_{D_Q}).$$

We need some lemma relating to ε'-factor.

Lemma 5.2. *When ω is a character over $F^\times$ and ψ is an additive character of order 0, we have*

$$\varepsilon'(s,\omega,\psi) = \begin{cases} q^{(\frac{1}{2}-s)c}\varepsilon'(\frac{1}{2},\omega,\psi) & \omega : \text{ramified},\\ \dfrac{1-\omega(\pi)q^{-s}}{1-\omega^{-1}(\pi)q^{s-1}} & \omega : \text{unramified}.\end{cases}$$

Here c denotes the conductor of the character ω.

Proof. This lemma is well-known for the theory of L-factor and ε-factor of the local zeta integral. See [35]. □

We note that, when χ is a ramified character on $F^\times$ which satisfies $\chi^2 = 1$ (and q is odd), the conductor c is equal to 1.

Lemma 5.3. *When $\omega = \chi_\theta$ and $s = \frac{1}{2}$, the value of ε'-factor is represented by Weil constants as follows:*

$$\varepsilon'\left(\frac{1}{2}, \chi_\theta, \psi\right) = \frac{\alpha_\psi(1)}{\alpha_\psi(\theta)}.$$

Proof. By taking the residue of Lemma 1.3 in [15], the statement holds. □

From the above lemma, we know that $\varepsilon'\left(\frac{1}{2}, \chi, \psi\right) = \alpha_\psi(\pi)\chi(\pi)$ since $\chi = \chi_\pi$.

5.2 The functional equations

Later each function $f_t = f_t^{(s)}$ is taken in the space $I_n(\chi, s)^{\Gamma,\chi}$. We write the function $f_\beta^{(s)} = f_{\beta,n}^{(s)}$ as $f_\beta^{(s)} = \sum_{t=0}^{n} \beta_0^t f_t^{(s)}$. Then

Theorem 5.3. *The functional equations of the ramified Siegel series are written as follows:*

(1) If n is odd,

$$\mathrm{Wh}_B(-s)f_\beta^{(-s)} = \chi(-D_B)\eta_B q^{s(\sum e_j+1)} \frac{(1-q^{2s-2})(1-q^{2s-4})\cdots(1-q^{2s-n+1})}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})} \mathrm{Wh}_B(s)f_\beta^{(s)}.$$

(2) If n is even and $\sum e_j$ is even,

$$\mathrm{Wh}_B(-s)f_\beta^{(-s)} = q^{s\sum e_j} \frac{(1-q^{2s-1})(1-q^{2s-3})\cdots(1-q^{2s-n+1})}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} \mathrm{Wh}_B(s)f_\beta^{(s)}.$$

(3) If n is even and $\sum e_j$ is odd,

$$\mathrm{Wh}_B(-s)f_\beta^{(-s)} = q^{s(\sum e_j+1)} \frac{1-\chi(-D_B)q^{-s-\frac{1}{2}}}{1-\chi(-D_B)q^{s-\frac{1}{2}}} \frac{(1-q^{2s-1})(1-q^{2s-3})\cdots(1-q^{2s-n+1})}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} \mathrm{Wh}_B(s)f_\beta^{(s)}.$$

We show the proof of the functional equations later.

Definition 5.1. We define the function $F(s) = F_B(s)$ as

$$F(s) = \begin{cases} \dfrac{\mathrm{Wh}_B(s)f_n^{(s)}}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})} & n \text{ is odd,} \\ \dfrac{\mathrm{Wh}_B(s)f_n^{(s)}}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} & n \text{ is even and } \sum_{k=1}^n e_k \text{ is even,} \\ \dfrac{(1-\chi(-D_B)q^{-s-\frac{1}{2}})\mathrm{Wh}_B(s)f_n^{(s)}}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})} & n \text{ is even and } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

Then the functional equation is written as follows:

$$F(-s) = \begin{cases} \eta_B\chi(-D_B)q^{s(\sum_{k=1}^n e_k+1)}F(s) & n \text{ is odd,} \\ q^{s\sum_{k=1}^n e_k}F(s) & n \text{ is even and } \sum_{k=1}^n e_k \text{ is even,} \\ q^{s(\sum_{k=1}^n e_k+1)}F(s) & n \text{ is even and } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

Later, we further define the function $\tilde{F}(s) = \tilde{F}_B(s)$ as

$$\tilde{F}(s) = \begin{cases} q^{\frac{s}{2}(\sum_{k=1}^n e_k+1)}F(s) & n \text{ is odd,} \\ q^{\frac{s}{2}\sum_{k=1}^n e_k}F(s) & n \text{ is even and } \sum_{k=1}^n e_k \text{ is even,} \\ q^{\frac{s}{2}(\sum_{k=1}^n e_k+1)}F(s) & n \text{ is even and } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

5.3 The proof of the functional equation

In this section, we are going to prove Theorem 5.3. First we have to prove the following lemma.

Lemma 5.4. *The function* $f_\beta = f_\beta^{(s)}$ *satisfies*

$$M_{w_n}^{(s)} f_\beta^{(s)} = \beta_0^n \prod_{i=1}^{[n/2]} \frac{1-q^{-2s-n-1+2i}}{1-q^{-2s+n-2i}} f_\beta^{(-s)}.$$

Proof. We consider the following intertwining operators of rank 1, in the same method as $n=2$.

$$\begin{array}{c}
I_n(\chi,s) \subset \mathrm{Ind}_{B_n}^G(\chi|\cdot|^{s-\frac{n-1}{2}}, \chi|\cdot|^{s-\frac{n-3}{2}}, \cdots, \chi|\cdot|^{s+\frac{n-1}{2}}) \\
\Big\downarrow \widetilde{M}_{w_n}^{(s+\frac{n-1}{2})} \\
\mathrm{Ind}_{B_n}^G(\chi|\cdot|^{s-\frac{n-1}{2}}, \chi|\cdot|^{s-\frac{n-3}{2}}, \cdots, \chi|\cdot|^{s+\frac{n-3}{2}}, \chi|\cdot|^{-s-\frac{n-1}{2}}) \\
\Big\downarrow \widetilde{M}_{w_{n-1}}^{(2s+n-2)} \\
\mathrm{Ind}_{B_n}^G(\chi|\cdot|^{s-\frac{n-1}{2}}, \chi|\cdot|^{s-\frac{n-3}{2}}, \cdots, \chi|\cdot|^{-s-\frac{n-1}{2}}, \chi|\cdot|^{s+\frac{n-3}{2}}) \\
\Big\downarrow \cdots \\
\Big\downarrow \widetilde{M}_{w_1}^{(2s)} \\
\mathrm{Ind}_{B_n}^G(\chi|\cdot|^{-s-\frac{n-1}{2}}, \chi|\cdot|^{s-\frac{n-1}{2}}, \cdots, \chi|\cdot|^{s+\frac{n-5}{2}}, \chi|\cdot|^{s+\frac{n-3}{2}}) \\
\Big\downarrow M(n-1, s-\tfrac{1}{2}) \\
\mathrm{Ind}_{B_n}^G(\chi|\cdot|^{-s-\frac{n-1}{2}}, \chi|\cdot|^{-s-\frac{n-3}{2}}, \cdots, \chi|\cdot|^{-s+\frac{n-3}{2}}, \chi|\cdot|^{-s+\frac{n-1}{2}}).
\end{array}$$

We note that the space $I_n(\chi,-s)$ is contained in the space at the bottom of the table above. Here notations are defined as

- B_n: the Borel subgroup of $\mathrm{Sp}_n(F)$.
- $w_i^{B_n}$: In this section it means the generators of the Weyl group, which corresponds to the simple root $\alpha_i = x_i - x_{i+1}$ $(1 \le i \le n-1)$ and $\alpha_n = 2x_n$ $(i=n)$; in other words,

$$(w_i^{B_n})_{jk} = \begin{cases} 1 & \text{if } j=k \text{ and } j \neq i, i+1, n+i, n+i+1 \\ 1 & \text{if } (j,k) = (i+1,i), (n+i+1, n+i) \\ -1 & \text{if } (j,k) = (i,i+1), (n+i, n+i+1) \\ 0 & \text{otherwise.} \end{cases} \quad (1 \le i \le n-1)$$

$$(w_n^{B_n})_{jk} = \begin{cases} 1 & \text{if } j=k \text{ and } j \neq n, 2n \\ 1 & \text{if } (j,k) = (2n,n) \\ -1 & \text{if } (j,k) = (n,2n) \\ 0 & \text{otherwise.} \end{cases}$$

We can show that

$$w_n^{B_n}\cdot w_{n-1}^{B_n}w_n^{B_n}\cdot w_{n-2}^{B_n}w_{n-1}^{B_n}w_n^{B_n}\cdots(w_1^{B_n}w_2^{B_n}\cdots w_n^{B_n})$$
$$=\left(\begin{array}{cccc|cccc} &&&&&&& (-1)^n \\ &&&&&& \cdot^{\cdot^{\cdot}} & \\ &&&&& 1 && \\ &&&& -1 &&& \\ \hline &&& (-1)^{n-1} &&&& \\ && \cdot^{\cdot^{\cdot}} &&&&& \\ & -1 &&&&&& \\ 1 &&&&&&& \end{array}\right)$$

and that the determinant of the lower left block is

$$\det\begin{pmatrix} &&& (-1)^{n-1} \\ && \cdot^{\cdot^{\cdot}} & \\ & -1 && \\ 1 &&& \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}\cdot(-1)^{\left[\frac{n}{2}\right]} = 1.$$

- $\mathrm{Ind}_{B_n}^G(\chi|\cdot|^{t_1},\cdots,\chi|\cdot|^{t_n})$: the space of C^∞ functions on G satisfying, for any $b=(b_{ij})\in B_n$,
$$f(bg)=\chi(b_{11})|b_{11}|^{t_1+n}\cdots\chi(b_{nn})|b_{nn}|^{t_n+1}f(g).$$
- $M^{(s)}_{(w_i^{B_n})}=\widetilde{M}^{(s)}_{w_i}$: the intertwining operator induced by that of rank 1, which corresponds to the element w_i of the Weyl group.
- $M(n,s)$: the intertwining operator $I_n(\chi,s)\to I_n(\chi,-s)$ with respect to the Borel subgroup.

Hence we have

$$M(n,s)=M(n-1,s-\frac{1}{2})\widetilde{M}^{(2s)}_{w_1}\widetilde{M}^{(2s+1)}_{w_2}\cdots\widetilde{M}^{(2s+n-2)}_{w_{n-1}}\widetilde{M}^{(s+\frac{n-1}{2})}_{w_n}.$$

Let $A_i(s)=(A_i(s)_{w,w'})_{w,w'\in W}$ be the square matrix of size $2^n\times n!$ representing the operator $\widetilde{M}^{(s)}_{w_i}$. As in Section 4.4, we have

$$(A_i(s))_{w,w'}=\begin{cases} q^{-s}\dfrac{1-q^{-1}}{1-q^{-s}} & w'=w,\ l(w_i^{B_n}w)>l(w) \\ \dfrac{1-q^{-1}}{1-q^{-s}} & w'=w,\ l(w_i^{B_n}w)<l(w) \\ 1 & w'=w_i^{B_n}w,\ l(w_i^{B_n}w)>l(w) \\ q^{-1} & w'=w_i^{B_n}w,\ l(w_i^{B_n}w)<l(w) \\ 0 & \text{otherwise,} \end{cases}\qquad(1\le i\le n-1)$$
$$(A_n(s))_{w,w'}=\begin{cases} 1 & w'=w_n^{B_n}w,\ l(w_n^{B_n}w)>l(w) \\ \chi(-1)q^{-1}(=\beta_0^2) & w'=w_n^{B_n}w,\ l(w_n^{B_n}w)<l(w) \\ 0 & \text{otherwise.} \end{cases}$$

For each $w \in W$, we define the *rank* of w (different from the usual meaning) as the number of the long root $w_n^{B_n}$ contained in the minimal representation of w. In other words, when we write $w \in W$ as the element of signed permutation, the rank is equal to the number of the set $\{i \mid 1 \leq i \leq n, w(i) < 0\}$.

We define the vector $v = (v_w)_{w \in W}$ as $v_w = \beta_0^{\mathrm{rank} w}$. (Note that this vector v corresponds to the function f_β.) We consider this vector v as a column vector. Then it follows that

$$A_i(s)v = \frac{1-q^{-s-1}}{1-q^{-s}} v \ (1 \leq i \leq n-1), \quad A_n(v) = \beta_0 v. \tag{6}$$

In other words, the vector v is an eigenvector of all $A_i(s)$. We prove this equation (6). First, when $1 \leq i \leq n-1$, we fix an element $w \in W$. Then

$$\begin{aligned}(A_i(s)v)_w &= \sum_{w' \in W} (A_i(s))_{w,w'}(v)_{w'} \\ &= \begin{cases} q^{-s}\dfrac{1-q^{-1}}{1-q^{-s}}(v)_w + 1 \cdot (v)_{w_i w} & \text{if } l(w_i^{B_n} w) > l(w), \\ \dfrac{1-q^{-1}}{1-q^{-s}}(v)_w + q^{-1}(v)_{w_i w} & \text{if } l(w_i^{B_n} w) < l(w). \end{cases}\end{aligned}$$

Since $w_i^{B_n}$ is a short root of the Weyl group, the rank of $w_i^{B_n} w$ is the same as that of w. Hence

$$\begin{aligned}(A_i(s)v)_w &= \begin{cases} q^{-s}\dfrac{1-q^{-1}}{1-q^{-s}}(v)_w + 1 \cdot (v)_w & \text{if } l(w_i^{B_n} w) > l(w), \\ \dfrac{1-q^{-1}}{1-q^{-s}}(v)_w + q^{-1}(v)_w & \text{if } l(w_i^{B_n} w) < l(w) \end{cases} \\ &= \frac{1-q^{-s-1}}{1-q^{-s}}(v)_w.\end{aligned}$$

Secondly, we prove the $w_n^{B_n}$ case. The element $w \in W$ is fixed as above. Then

$$\begin{aligned}(A_n(s)v)_w &= \sum_{w' \in W} (A_n(s))_{w,w'}(v)_{w'} \\ &= \begin{cases} (v)_{w_n w} & \text{if } l(w_n^{B_n} w) > l(w), \\ \beta_0^2 (v)_{w_n w} & \text{if } l(w_n^{B_n} w) < l(w). \end{cases}\end{aligned}$$

When $l(w_n^{B_n} w) > l(w)$, we have $\mathrm{rank}(w_n^{B_n} w) = \mathrm{rank} w + 1$. Hence we have

$$(A_n(s)v)_w = (v)_{w_n^{B_n} w} = \beta_0^{\mathrm{rank}(w_n^{B_n} w)} = \beta_0^{\mathrm{rank} w + 1} = \beta_0 (v)_w.$$

When $l(w_n^{B_n} w) < l(w)$, we have $\mathrm{rank}(w_n^{B_n} w) = \mathrm{rank} w - 1$. Hence we have

$$(A_n(s)v)_w = \beta_0^2 (v)_{w_n^{B_n} w} = \beta_0^2 \beta_0^{\mathrm{rank}(w_n^{B_n} w)} = \beta_0^2 \beta_0^{\mathrm{rank} w - 1} = \beta_0 (v)_w.$$

Therefore we have $A_n(s)v = \beta_0 v$ and equation (6) holds.

Now we calculate the operator

$$M(n,s) = M(n-1, s-\frac{1}{2}) \widetilde{M}_{w_1}^{(2s)} \widetilde{M}_{w_2}^{(2s+1)} \cdots \widetilde{M}_{w_{n-1}}^{(2s+n-2)} \widetilde{M}_{w_n}^{(s+\frac{n-1}{2})}.$$

The operator $\widetilde{M}_{w_1}^{(2s)}\widetilde{M}_{w_2}^{(2s+1)}\cdots\widetilde{M}_{w_{n-1}}^{(2s+n-2)}\widetilde{M}_{w_n}^{(s+\frac{n-1}{2})}$ corresponds the matrix

$$A_1(2s)A_2(2s+1)\cdots A_{n-1}(2s+n-2)A_n(s+\frac{n-1}{2}).$$

From the equation (6), we have

$$\begin{aligned}&\left(A_1(2s)A_2(2s+1)\cdots A_{n-1}(2s+n-2)A_n(s+\frac{n-1}{2})\right)v\\ &=\frac{1-q^{-2s-1}}{1-q^{-2s}}\frac{1-q^{-2s-2}}{1-q^{-2s-1}}\cdots\frac{1-q^{-2s-n+1}}{1-q^{-2s-n+2}}\beta_0 v=\beta_0\frac{1-q^{-2s-n+1}}{1-q^{-2s}}v.\end{aligned}$$

We define the constant $c(n,s)$ as $M(n,s)v=c(n,s)v$. Thus $c(n,s)$ satisfies

$$c(n,s)=\beta_0\frac{1-q^{-2s-n+1}}{1-q^{-2s}}c(n-1,s-\frac{1}{2}),\quad c(1,s)=\beta_0,$$

and we have $c(n,s)=\beta_0^n\prod_{i=1}^{[n/2]}\frac{1-q^{-2s-n-1+2i}}{1-q^{-2s+n-2i}}$. Therefore Lemma 5.4 holds. □

We can prove the functional equation by using Theorem 5.1, 5.2, and Lemma 5.4. We divide by 3 cases.

(i) When n is odd, the value $c_B(\chi,s)$ is calculated as

$$\begin{aligned}c_B(\chi,s)&=\varepsilon'\left(s-\frac{n-1}{2},\ \chi\right)^{-1}\prod_{r=1}^{\frac{n-1}{2}}\varepsilon(2s-n+2r,1)^{-1}\eta_B\\ &=\left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1}\prod_{r=1}^{\frac{n-1}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}\eta_B.\end{aligned}$$

From Theorem 5.1, we have

$$\mathrm{Wh}_B(-s)M_{w_n}^{(s)}f_\beta^{(s)}=\chi^{-1}(\det B)|\det B|^{-s}c_B(\chi,s)\mathrm{Wh}_B(s)f_\beta^{(s)}.$$

From Theorem 5.2 and Lemma 5.4, it follows that

$$\begin{aligned}\mathrm{Wh}_B(-s)f_\beta^{(-s)}&=\left(\beta_0^n\prod_{i=1}^{\frac{n-1}{2}}\frac{1-q^{-2s-n-1+2i}}{1-q^{-2s+n-2i}}\right)^{-1}\chi^{-1}(\det B)|\det B|^{-s}\\ &\quad\times\left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1}\prod_{r=1}^{\frac{n-1}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}\eta_B\mathrm{Wh}_B(s)f_\beta^{(s)}\\ &=\chi(-1)^{\frac{n+1}{2}}\chi(\det B)\eta_B q^{s(\sum e_j+1)}\prod_{i=1}^{\frac{n-1}{2}}\frac{1-q^{2s-n-1+2i}}{1-q^{-2s-n-1+2i}}\mathrm{Wh}_B(s)f_\beta^{(s)}.\end{aligned}$$

Since $D_B=(-4)^{\frac{n-1}{2}}\det B$, it follows that $\chi(-1)^{\frac{n+1}{2}}\chi(\det B)=\chi(-D_B)$ and the functional equation holds.

(ii) When n is even and $\sum_{k=1}^{n} e_k$ is even, the value of $\alpha_\psi(D_B)$ and $\varepsilon'(s+\frac{1}{2},\chi\chi_{D_B})$ is

$$\begin{aligned}\alpha_\psi(D_B) &= 1,\\ \varepsilon'(s+1/2,\chi\chi_{D_B}) &= \varepsilon'(s+1/2,\chi_{D_B\pi})\\ &= q^{-s}\varepsilon'(1/2,\chi_{D_B\pi})\\ &= q^{-s}\alpha_\psi(D_B\pi)^{-1}\\ &= q^{-s}\chi(D_B)\alpha_\psi(\pi)^{-1}.\end{aligned}$$

Therefore the value of $c_B(\chi,s)$ is

$$\begin{aligned}c_B(\chi,s) &= \varepsilon'\left(s-\frac{n-1}{2},\ \chi\right)^{-1}\prod_{r=1}^{\frac{n}{2}}\varepsilon(2s-n+2r,1)^{-1}\alpha_\psi(D_B)\varepsilon'(s+1/2,\chi\chi_{D_B})\\ &= \left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1}\prod_{r=1}^{\frac{n}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}q^{-s}\chi(D_B)\alpha_\psi(\pi)^{-1}\end{aligned}$$

and combined with Theorem 5.1 and Lemma 5.4, we have

$$\begin{aligned}\mathrm{Wh}_B(-s)f_\beta^{(-s)} &= \left(\beta_0^n\prod_{i=1}^{\frac{n}{2}}\frac{1-q^{-2s-n-1+2i}}{1-q^{-2s+n-2i}}\right)^{-1}\chi^{-1}(\det B)|\det B|^{-s}\\ &\quad\times\left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1}\prod_{r=1}^{\frac{n}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}q^{-s}\chi(D_B)\alpha_\psi(\pi)^{-1}\mathrm{Wh}_B(s)f_\beta^{(s)}\\ &= \prod_{i=1}^{\frac{n}{2}}\frac{1-q^{2s-n-1+2i}}{1-q^{-2s-n-1+2i}}q^{s\sum e_j}\mathrm{Wh}_B(s)f_\beta^{(s)}.\end{aligned}$$

(iii) When n is even and $\sum_{k=1}^{n} e_k$ is odd, the value of $\alpha_\psi(D_B)$ and $\varepsilon'(s+\frac{1}{2},\chi\chi_{D_B})$ is

$$\begin{aligned}\alpha_\psi(D_B) &= \chi(D_B\pi)\alpha_\psi(\pi),\\ \varepsilon'(s+1/2,\chi\chi_{D_B}) &= \varepsilon'(s+1/2,\chi_{D_B\pi})\\ &= \frac{1-\chi_{D_B\pi}(\pi)q^{-s-\frac{1}{2}}}{1-\chi_{D_B\pi}(\pi)^{-1}q^{s-\frac{1}{2}}}.\end{aligned}$$

Therefore the value of $c_B(\chi,s)$ is

$$\begin{aligned}c_B(\chi,s) &= \varepsilon'\left(s-\frac{n-1}{2},\ \chi\right)^{-1}\prod_{r=1}^{\frac{n}{2}}\varepsilon(2s-n+2r,1)^{-1}\alpha_\psi(D_B)\varepsilon'(s+1/2,\chi\chi_{D_B})\\ &= \left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1}\prod_{r=1}^{\frac{n}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}\chi(D_B\pi)\alpha_\psi(\pi)\frac{1-\chi_{D_B\pi}(\pi)q^{-s-\frac{1}{2}}}{1-\chi_{D_B\pi}(\pi)^{-1}q^{s-\frac{1}{2}}}\end{aligned}$$

and combined with Theorem 5.1 and Lemma 5.4, we have

$$\begin{aligned}\mathrm{Wh}_B(-s)f_\beta^{(-s)} &= \left(\beta_0^n \prod_{i=1}^{\frac{n}{2}} \frac{1-q^{-2s-n-1+2i}}{1-q^{-2s+n-2i}}\right)^{-1} \chi^{-1}(\det B)|\det B|^{-s}\left(q^{\frac{n}{2}-s}\alpha_\psi(\pi)\chi(\pi)\right)^{-1} \\ &\times \prod_{r=1}^{\frac{n}{2}}\left(\frac{1-q^{-2s+n-2r}}{1-q^{2s-n+2r-1}}\right)^{-1}\chi(D_B\pi)\alpha_\psi(\pi)\frac{1-\chi_{D_B\pi}(\pi)q^{-s-\frac{1}{2}}}{1-\chi_{D_B\pi}(\pi)^{-1}q^{s-\frac{1}{2}}}\mathrm{Wh}_B(s)f_\beta^{(s)} \\ &= q^{s(\sum e_j+1)}\frac{1-\chi(-D_B)q^{-s-\frac{1}{2}}}{1-\chi(-D_B)q^{s-\frac{1}{2}}}\prod_{i=1}^{\frac{n}{2}}\frac{1-q^{2s-n-1+2i}}{1-q^{-2s-n-1+2i}}\mathrm{Wh}_B(s)f_\beta^{(s)}.\end{aligned}$$

6 The ramified Siegel series and the recursion formulas

6.1 e_n term of the ramified Siegel series

We put the diagonal matrix B' as

$$B' = \mathrm{diag}(\alpha_1\pi^{e_1}, \cdots, \alpha_{n-1}\pi^{e_{n-1}}, \alpha_n\pi^{e_n+2}).$$

In other words, we substitute e_n as e_n+2. We calculate the value of ${}^\Delta S_t(B,s) := S_t(B',s) - S_t(B,s)$.

First we assume that $e_n \gg e_{n-1}$. (More precisely, we assume $e_n - e_{n-1} \geq 2, 4,$ or 6 contextually.) When $e_n - e_{n-1}$ is smaller, see Remark 6.1.

We define the sets B^e and B^o as

$$\begin{aligned} B^e &:= \{k \mid 1 \leq k \leq n-1,\ e_n \equiv e_k \pmod 2\}, \\ B^o &:= \{k \mid 1 \leq k \leq n-1,\ e_n \not\equiv e_k \pmod 2\}. \end{aligned}$$

These sets satisfy the following lemma.

Lemma 6.1. *(1) $\#B^e + \#B^o = n-1$, i.e., the parity of n and $\#B^e + \#B^o$ are different.*

(2) If n is even, then $\sum_{k=1}^n e_k$ is even if and only if $\#B^o$ is even. If n is odd, then $\sum_{k=1}^{n-1} e_k$ is odd if and only if $\#B^o$ is odd.

We calculate ${}^\Delta S_t(B,s)$ divided by 4 cases, if n is even or odd, and if $\#B^o$ is even or odd.

Lemma 6.2. *In the formula of the ramified Siegel series $S_t(B,s)^\chi$, the sum of the term $\nu_0 \geq -e_{n-1}-1$ is independent of e_n.*

Proof. First we see that, since $\nu_0 \geq -e_{n-1}-1$, the set $\{\nu\}_k^t$ defined as

$$\left\{(\nu_0,\nu_1,\cdots,\nu_{k-1}) \in \mathbb{Z}\times\mathbb{Z}_{>0}^{k-1} \;\middle|\; -b_l(\sigma,B) \leq \nu_0+\nu_1+\cdots+\nu_l \leq -1\ (0 \leq l \leq k-1),\ n^{(k)} = t\right\}$$

is independent of e_n. Therefore each ν_i does not depend on e_{n-1}. Also, when $\nu_0 \geq -e_{n-1}-1$, it follows that $n \notin B_i(\lambda)$ and $\min\{e_n + e_{\sigma,i,n} + \lambda, 0\} = 0$. Since the term $\tilde{\rho}_{l,\lambda}(\sigma,B)$ and $\xi_{i,\lambda}(B)_\chi$ does not depend on e_n, the total sum of $S_t(B'',s)^\chi$ also does not depend on e_n. □

Remark 6.1. The above argument assumes that $e_n - e_{n-1} \geq 1$. It is because, when $e_n - e_{n-1} = 0$, the sum of the term $I_0 = \{n\}$ and $\sigma(n) = n$ vanishes, and the formula differs from that of $e_n - e_{n-1} \geq 1$. When $e_n - e_{n-1} = 0$, it satisfies the same theorem later (for example, Theorem 6.1), since the ramified Siegel series has an analytic continuation to the rational function of q^{-s} and each $q^{-e_i s}$. Here we need more precise discussion.

Theorem 6.1. *We put* ${}^{\Delta}S_t(B,s)^{\chi} = S_t(B',s)^{\chi} - S_t(B,s)^{\chi}$. *Then, it is calculated as follows.*

(1) $\#B^e$*: odd,* $\#B^o$ *: even*

$$ {}^{\Delta}S_t(B,s)^{\chi} = -\alpha_{\psi}(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} \times \left(q^{-2s+n}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}S_t(B^{(n-1)},s)^{\chi}. $$

(2) $\#B^e$*: even,* $\#B^o$ *: odd*

$$ \begin{aligned} {}^{\Delta}S_t(B,s)^{\chi} &= \alpha_{\psi}(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} \\ &\quad\times\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k}S_t(B^{(n-1)},s)^{\chi} \\ &\quad+\alpha_{\psi}(\pi)\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}\chi(\pi)^{e_n}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^{\chi}. \end{aligned} $$

(3) $\#B^e$, $\#B^o$ *: even*

$$ {}^{\Delta}S_t(B,s)^{\chi} = \alpha_{\psi}(\pi)\prod_{k\in B^e}\chi(\alpha_k)\chi(-\alpha_n)\chi(-1)^{\frac{\#B^e}{2}}\chi(\pi)^{e_n+1} \times\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k}S_t(B^{(n-1)},s)^{\chi}. $$

(4) $\#B^e$, $\#B^o$ *: odd*

$$ \begin{aligned} {}^{\Delta}S_t(B,s)^{\chi} &= -\alpha_{\psi}(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} \\ &\quad\times\left(q^{-2s+n}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}S_t(B^{(n-1)},s)^{\chi} \\ &\quad+\alpha_{\psi}(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^{\chi}. \end{aligned} $$

Here we note that, when $t = n$*, the term* $S_t(B^{(n-1)},s)^{\chi}$ *is considered as* 0.

6.1.1 n is even and $\#B^o$ is even

In this case, the number $\#B^e$ is odd and $\sum_{k=1}^{n}e_k$ is even. From the above lemma, we must calculate the sum of the term for $-e_n-1\le\nu_0\le -e_{n-1}-2$.

The Gauss sum $\xi_{i,\lambda}(B)_{\chi}$ is

$$ \xi_{n,\nu_0}(B)_{\chi} = \begin{cases} -\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}q^{-\frac{1}{2}} & \nu_0=-e_n-1, \\ \prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(1-q^{-1}) & \nu_0\ge -e_n,\ \nu_0\not\equiv e_n \mod 2, \\ 0 & \nu_0\ge -e_n,\ \nu_0\equiv e_n \mod 2, \end{cases} $$

since $B_n(\nu_0) = \begin{cases} B^o & \nu_0\equiv e_n \mod 2, \\ B^e & \nu_0\not\equiv e_n \mod 2. \end{cases}$

We note that, since $\nu_0 < -e_{n-1} - 1 \le \nu_0 + \nu_1$, it follows that $I_0 = \{n\}$ and $\sigma(n) = n$. Put $\sigma' := \sigma|_{[n-1]} \in \mathfrak{S}_{n-1}$.

First, we calculate the sum of the term with respect to $\nu_0 = -e_n - 1$. We call this sum $S_t(B,s)_0^\chi$. This sum is written as

$$\begin{aligned}
S_t(B,s)_0^\chi &= \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma^2=1,\sigma(n)=n}} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{\substack{I=I_0\cup\cdots\cup I_r \\ n^{(k)}=t, I_0=\{n\}}} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})} \frac{(1-q^{-1})^{\sum_{l=k}^r c_1^{(l)}(\sigma)} q^{n(k)}}{\prod_{l=k}^r (q^{n(l)}-1)} \\
&\quad \times \sum_{\substack{\{\nu\}_k^t \\ \nu_0=-e_n-1}} \prod_{l=0}^{k-1} \chi(\pi)^{\nu_l(n^{(l)}-n^{(k)})} q^{\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))+\tilde{\rho}_{l,\nu_0+\cdots+\nu_l}(\sigma;B)} \prod_{\substack{i\in I_l \\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_l}(B)_\chi \\
&= \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma'\in\mathfrak{S}_{n-1} \\ \sigma'^2=1}} (1-q^{-1})^{c_2(\sigma')} q^{-c_2(\sigma')} \sum_{\substack{I'=I'_0\cup\cdots\cup I'_{r-1} \\ n^{(k)}=t, I_0=\{n\}}} q^{-\tau(\{I'_i\})-t(\sigma',\{I'_i\})} \frac{(1-q^{-1})^{\sum_{l=k}^r c_1^{(l)}(\sigma)} q^{n(k)}}{\prod_{l=k}^r (q^{n(l)}-1)} \\
&\quad \times \sum_{\substack{\{\nu\}_k^t \\ \nu_0=-e_n-1}} \chi(\pi)^{\nu_0(n^{(0)}-n^{(k)})} q^{\nu_0((sn^{(0)}-n(0))-(sn^{(k)}-n(k)))+\tilde{\rho}_{0,\nu_0}(\sigma;B)} \xi_{n,\nu_0}(B)_\chi \\
&\quad \times \prod_{l=1}^{k-1} \chi(\pi)^{\nu_l(n^{(l)}-n^{(k)})} q^{\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k)))+\tilde{\rho}_{l,\nu_0+\cdots+\nu_l}(\sigma;B)} \prod_{\substack{i\in I_l \\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_l}(B)_\chi.
\end{aligned}$$

Here we use the fact that $c_2(\sigma') = c_2(\sigma)$. Put $I'_0 = I_1, I'_1 = I_2, \cdots, I'_{r-1} = I_r$. It is a σ'-invariant partition of $I' = \{1,2,\cdots,n-1\}$. The term $\tau(\{I'_i\})$ and $t(\sigma',\{I'_i\})$ is calculated as

$$\begin{aligned}
\tau(\{I_i\}) &= \sum_{l=1}^r \#\{(i,j)\in I_l\times(I_0\cup\cdots\cup I_{l-1}) \mid j<i\} \\
&= \sum_{l=2}^r \#\{(i,j)\in I_l\times(I_1\cup\cdots\cup I_{l-1}) \mid j<i\} = \tau(\{I'_i\}), \\
t(\sigma,\{I_i\}) &= \sum_{l=0}^r \#\{(i,j)\in I_l\times I_l \mid i<j<\sigma(i), \sigma(j)<\sigma(i)\} = t(\sigma',\{I'_i\}).
\end{aligned}$$

We put $l' = l-1$ and $k' = k-1$. Later we assume $t \neq n$ (if $t = n$ then $k = 0$). Here we note

$$n^{(k)} = t \Leftrightarrow \sum_{l=k}^r \#I_l = t \Leftrightarrow \sum_{l=k-1}^{r-1} \#I'_l = t \Leftrightarrow n'^{(k')} = t.$$

Also we put $\nu'_0, \nu'_1, \cdots, \nu'_{k-1}$ as

$$\nu'_0 = \nu_0 + \nu_1,\ \nu'_1 = \nu_2, \cdots, \nu'_{k'-1} = \nu_{k-1}.$$

ν'_0 and (ν_0,ν_1) is not 1 to 1, but if ν_0 is fixed, ν'_0 and ν_1 is 1 to 1. In other words, $\nu_1 = \nu'_0 - \nu_0$.

Therefore it follows that

$$
\begin{aligned}
S_t(B,s)_0^\chi &= \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma' \in \mathfrak{S}_{n-1} \\ \sigma'^2=1}} (1-q^{-1})^{c_2(\sigma')} q^{-c_2(\sigma')} \sum_{\substack{I'=I'_0\cup\cdots\cup I'_{r-1} \\ n'^{(k')}=t}} q^{-\tau(\{I'_i\})-t(\sigma',\{I'_i\})} \frac{(1-q^{-1})^{\sum_{l'=k'}^{r-1} c_1^{(l')}(\sigma')} q^{n'(k')}}{\prod_{l'=k'}^{r-1}(q^{n'(l')}-1)} \\
&\quad \times \sum_{\{\nu\}_{k'}^t} \chi(\pi)^{\nu_0(n^{(0)}-n'^{(k')})} q^{\nu_0((sn^{(0)}-n(0))-(sn'^{(k')}-n'(k')))+\tilde{\rho}_{0,\nu_0}(\sigma;B)} \xi_{n,\nu_0}(B)_\chi \\
&\quad \times \chi(\pi)^{\nu_1(n'^{(0)}-n'^{(k')})} q^{\nu_1((sn'^{(0)}-n'(0))-(sn'^{(k')}-n'(k')))+\tilde{\rho}_{0,\nu_0+\nu_1}(\sigma;B)} \xi_{n,\nu_0+\nu_1}(B)_\chi \\
&\quad \times \prod_{l'=1}^{k'-1} \chi(\pi)^{\nu'_{l'}(n'^{(l')}-n'^{(k')})} q^{\nu'_{l'}((sn'^{(l')}-n'(l'))-(sn'^{(k)}-n'(k)))+\tilde{\rho}_{l,\nu_0+\cdots+\nu_{l'}}(\sigma;B)} \prod_{\substack{i\in I'_l \\ \sigma'(i)=i}} \xi_{i,\nu_0+\cdots+\nu_{l'}}(B)_\chi \\
&= \alpha_\psi(\pi)\chi(\pi)^{\nu_0} q^{(s-n)\nu_0+\tilde{\rho}_{0,\nu_0}(\sigma;B)} \xi_{n,\nu_0}(B)_\chi \times S_t(B^{(n-1)},s)^\chi.
\end{aligned}
$$

Here we note that, the order of $\chi(\pi)$ is

$$
\begin{aligned}
&\nu_0(n^{(0)}-n'^{(k')}) + (\nu'_0-\nu_0)(n'^{(0)}-n'^{(k')}) + \sum_{l'=1}^{k'-1} \nu'_{l'}(n'^{(l')}-n'^{(k')}) \\
&= \nu_0(n^{(0)}-n'^{(0)}) + \sum_{l'=0}^{k'-1} \nu'_{l'}(n'^{(l')}-n'^{(k')}) = \nu_0 + \sum_{l'=0}^{k'-1} \nu'_{l'}(n'^{(l')}-n'^{(k')}),
\end{aligned}
$$

and the order of q is

$$
\begin{aligned}
&\nu_0((sn^{(0)}-n(0))-(sn'^{(k')}-n'(k'))) + (\nu'_0-\nu_0)((sn'^{(0)}-n'(0))-(sn'^{(k')}-n'(k'))) \\
&\quad + \sum_{l'=1}^{k'-1} \nu'_{l'}((sn'^{(l')}-n'(l'))-(sn'^{(k)}-n'(k))) \\
&= s\nu_0(n^{(0)}-n'^{(0)}) - \nu_0(n(0)-n'(0)) + \sum_{l'=0}^{k'-1} \nu'_{l'}((sn'^{(l')}-n'(l'))-(sn'^{(k)}-n'(k))) \\
&= (s-n)\nu_0 + \sum_{l'=0}^{k'-1} \nu'_{l'}((sn'^{(l')}-n'(l'))-(sn'^{(k)}-n'(k))).
\end{aligned}
$$

When n and $\#B^o$ is even and when $\nu_0=-e_n-1$, the value of $\tilde{\rho}_{0,\nu_0}(\sigma;B)$ is

$$
\tilde{\rho}_{0,\nu_0}(\sigma;B) = \frac{1}{2}\sum_{k=1}^n \min\{e_k+\nu_0,0\} = \frac{1}{2}\sum_{k=1}^n e_k - \frac{n(e_n+1)}{2},
$$

hence we have

$$
\begin{aligned}
S_t(B,s)_0^\chi &= \alpha_\psi(\pi)\chi(\pi)^{\nu_0} q^{(s-n)\nu_0+\frac{1}{2}\sum_{k=1}^n e_k-\frac{n(e_n+1)}{2}} \left(-\prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{1}{2}}\right) S_t(B^{(n-1)},s)^\chi \\
&= -\alpha_\psi(\pi)\chi(\pi)^{e_n+1} \prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k-\frac{1}{2}} S_t(B^{(n-1)},s)^\chi.
\end{aligned}
$$

Next, we calculate the sum of the term for $\nu_0 \geq -e_n,\ \nu_0 \not\equiv e_n \mod 2$. We call this sum $S_t(B,s)_1^\chi$. The range of ν_0 is $-e_n \leq \nu_0 \leq -e_{n-1}-2$. We note that if $\nu_0 \equiv e_n \mod 2$, the sum vanishes since $\xi_{n,\nu_0}(B)_\chi$ vanishes.

First, we assume $e_n - e_{n-1}$ is even. In this case, the sum $S_t(B,s)_1^\chi$ is written as $S_t(B,s)_{1,e}^\chi$. If $e_n - e_{n-1} = 2$, then ν_0 must be $-e_n$ and the sum vanishes. We suppose $e_n - e_{n-1} \geq 4$. In this case, we put

$$\nu_0 = -e_{n-1} + 1 - 2i \quad \left(2 \leq i \leq \frac{e_n - e_{n-1}}{2}\right).$$

$I', \sigma', k', l', \nu_i'$ is the same as above. We have

$$\begin{aligned}
S_t(B,s)_{1,e}^\chi &= \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^2=1, \sigma(n)=n}} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{\substack{I = I_0 \cup \cdots \cup I_r \\ n^{(k)}=t, I_0=\{n\}}} q^{-\tau(\{I_i\}) - t(\sigma, \{I_i\})} \frac{(1-q^{-1})^{\sum_{l=k}^r c_1^{(l)}(\sigma)} q^{n(k)}}{\prod_{l=k}^r (q^{n(l)}-1)} \\
&\times \sum_{\substack{\{\nu\}_k^t \\ \nu_0 \geq -e_n}} \prod_{l=0}^{k-1} \chi(\pi)^{\nu_l(n^{(l)}-n^{(k)})} q^{\nu_l((sn^{(l)}-n(l))-(sn^{(k)}-n(k))) + \tilde{\rho}_{l,\nu_0+\cdots+\nu_l}(\sigma;B)} \prod_{\substack{i \in I_l \\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_l}(B)_\chi \\
&= \alpha_\psi(\pi)^{n-t} \sum_{\substack{\sigma' \in \mathfrak{S}_{n-1} \\ \sigma'^2=1}} (1-q^{-1})^{c_2(\sigma')} q^{-c_2(\sigma')} \sum_{\substack{I' = I_0' \cup \cdots \cup I_{r-1}' \\ n'^{(k')}=t}} q^{-\tau(\{I_i'\}) - t(\sigma', \{I_i'\})} \frac{(1-q^{-1})^{\sum_{l'=k'}^{r-1} c_1^{(l')}(\sigma')} q^{n'(k')}}{\prod_{l'=k'}^{r-1} (q^{n'(l')}-1)} \\
&\times \sum_{i=2}^{\frac{1}{2}(e_n-e_{n-1})} \sum_{\{\nu\}_{k'}^t} \chi(\pi)^{\nu_0(n^{(0)}-n'^{(k')})} q^{\nu_0((sn^{(0)}-n(0))-(sn'^{(k')}-n'(k'))) + \tilde{\rho}_{0,\nu_0}(\sigma;B)} \xi_{n,\nu_0}(B)_\chi \\
&\times \chi(\pi)^{\nu_1(n'^{(0)}-n'^{(k')})} q^{\nu_1((sn'^{(0)}-n'(0))-(sn'^{(k')}-n'(k'))) + \tilde{\rho}_{0,\nu_0+\nu_1}(\sigma;B)} \xi_{n,\nu_0+\nu_1}(B)_\chi \\
&\times \prod_{l'=1}^{k'-1} \chi(\pi)^{\nu_{l'}'(n'^{(l')}-n'^{(k')})} q^{\nu_{l'}'((sn'^{(l')}-n'(l'))-(sn'^{(k)}-n'(k))) + \tilde{\rho}_{l,\nu_0+\cdots+\nu_{l'}}(\sigma;B)} \prod_{\substack{i \in I_l' \\ \sigma'(i)=i}} \xi_{i,\nu_0+\cdots+\nu_{l'}}(B)_\chi.
\end{aligned}$$

We note that, when $\nu_0 = -e_{n-1} + 1 - 2i$, the value of $\tilde{\rho}_{0,\nu_0}(\sigma;B)$ is

$$\tilde{\rho}_{0,\nu_0}(\sigma;B) = \frac{1}{2} \sum_{k=1}^n \min\{e_k + \nu_0, 0\} = \frac{1}{2} \sum_{k=1}^{n-1} e_k + \frac{\nu_0(n-1)}{2},$$

therefore the sum is

$$\begin{aligned}
S_t(B,s)_{1,e}^\chi &= \alpha_\psi(\pi) \sum_{i=2}^{\frac{1}{2}(e_n-e_{n-1})} \chi(\pi)^{\nu_0} q^{(s-n)\nu_0 + \tilde{\rho}_{0,\nu_0}(\sigma;B)} \xi_{n,\nu_0}(B)_\chi \times S_t(B^{(n-1)},s)^\chi \\
&= \alpha_\psi(\pi)\chi(\pi)^{e_n+1} \sum_{i=2}^{\frac{1}{2}(e_n-e_{n-1})} q^{(s-n)\nu_0 + \frac{1}{2}\sum_{k=1}^{n-1} e_k + \frac{\nu_0(n-1)}{2}} \\
&\times \left(\prod_{k \in B^e} \chi(\alpha_k) \chi(-1)^{\frac{\#B^e+1}{2}} (1-q^{-1}) \right) S_t(B^{(n-1)},s)^\chi \\
&= \alpha_\psi(\pi)\chi(\pi)^{e_n+1} \prod_{k \in B^e} \chi(\alpha_k) \chi(-1)^{\frac{\#B^e+1}{2}}
\end{aligned}$$

$$\times \sum_{i=2}^{\frac{1}{2}(e_n-e_{n-1})} q^{\frac{-e_{n-1}+1-2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi.$$

Secondly, we assume e_n-e_{n-1} is odd. In this case, the sum $S_t(B,s)_1^\chi$ is written as $S_t(B,s)_{1,o}^\chi$. We put

$$\nu_0 = -e_{n-1}-2i, \quad \left(1 \le i \le \frac{e_n-e_{n-1}-1}{2}\right).$$

The sum $S_t(B,s)_{1,o}^\chi$ is given as

$$\begin{aligned} S_t(B,s)_{1,o}^\chi &= \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} \\ &\times \sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1}-1)} q^{\frac{-e_{n-1}-2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

We calculate the value of ${}^\Delta S_t(B,s)^\chi$. From the above calculations, we have

$$S_t(B,s)^\chi = (\text{terms not depend on } e_n) + \begin{cases} S_t(B,s)_0^\chi + S_t(B,s)_{1,e}^\chi & e_n-e_{n-1}\text{: even,} \\ S_t(B,s)_0^\chi + S_t(B,s)_{1,o}^\chi & e_n-e_{n-1}\text{: odd.} \end{cases}$$

Therefore, to calculate ${}^\Delta S_t(B,s)^\chi$, we need ${}^\Delta S_t(B,s)_0^\chi$ and ${}^\Delta S_t(B,s)_1^\chi$. The first one is

$$\begin{aligned} {}^\Delta S_t(B,s)_0^\chi &= S_t(B',s)_0^\chi - S_t(B,s)_0^\chi \\ &= -\alpha_\psi(\pi)\chi(\pi)^{e_n+3}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+3}{2}(2s-n)+\frac{1}{2}\left(\sum_{k=1}^n e_k+2\right)-\frac{1}{2}} S_t(B^{(n-1)},s)^\chi \\ &\quad + \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k-\frac{1}{2}} S_t(B^{(n-1)},s)^\chi \\ &= -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n+1}-1) q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k-\frac{1}{2}} S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

The second one is

$$\begin{aligned} {}^\Delta S_t(B,s)_{1,e}^\chi &= S_t(B',s)_{1,e}^\chi - S_t(B,s)_{1,e}^\chi \\ &= \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+1}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1} e_k}(1-q^{-1}) S_t(B^{(n-1)},s)^\chi \end{aligned}$$

and odd case is the same: ${}^\Delta S_t(B,s)_{1,o}^\chi = {}^\Delta S_t(B,s)_{1,e}^\chi$.

In conclusion,

$$\begin{aligned} {}^\Delta S_t(B,s)^\chi &= {}^\Delta S_t(B,s)_0^\chi + {}^\Delta S_t(B,s)_1^\chi \\ &= -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n+1}-1) q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k-\frac{1}{2}} S_t(B^{(n-1)},s)^\chi \\ &\quad + \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+1}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1} e_k}(1-q^{-1}) S_t(B^{(n-1)},s)^\chi \\ &= -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n}-1) q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k+\frac{1}{2}} S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

6.1.2 n is even and $\#B^o$ is odd

In this case, the number $\#B^e$ is even and $\sum_{k=1}^n e_k$ is odd. The Gauss sum $\xi_{n,\nu_0}(B)_\chi$ is

$$\xi_{n,\nu_0}(B)_\chi = \begin{cases} \prod_{k\in B^e} \chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} & \nu_0 = -e_n - 1, \\ \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}(1-q^{-1}) & \nu_0 \geq -e_n,\ \nu_0 \equiv e_n \mod 2, \\ 0 & \nu_0 \geq -e_n,\ \nu_0 \not\equiv e_n \mod 2. \end{cases}$$

The notations of $S_t(B,s)_0^\chi$, $S_t(B,s)_{1,e}^\chi$ and $S_t(B,s)_{1,o}^\chi$ are the same as above. Since the method of the the calculation is the same, we only show the value of each sum. The first one is

$$S_t(B,s)_0^\chi = \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1}q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k}S_t(B^{(n-1)},s)^\chi,$$

and the second one is calculated as follows. When $e_n - e_{n-1}$ is even, we put

$$\nu_0 = -e_{n-1} - 2i \quad \left(1 \leq i \leq \frac{e_n - e_{n-1}}{2}\right),$$

and we have

$$\begin{aligned} S_t(B,s)_{1,e}^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} \\ & \times \sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1})} q^{-\frac{e_{n-1}+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

When $e_n - e_{n-1}$ is odd, we put

$$\nu_0 = -e_{n-1} - 1 - 2i \quad \left(1 \leq i \leq \frac{e_n - e_{n-1} - 1}{2}\right),$$

hence we have

$$\begin{aligned} S_t(B,s)_{1,o}^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} \\ & \times \sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1}-1)} q^{-\frac{e_{n-1}+1+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

Thus the value of ${}^\Delta S_t(B,s)_0^\chi$ and ${}^\Delta S_t(B,s)_1^\chi$ is

$$\begin{aligned} {}^\Delta S_t(B,s)_0^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} \\ & \times (q^{-2s+n+1}-1)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k}S_t(B^{(n-1)},s)^\chi, \\ {}^\Delta S_t(B,s)_1^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

6.1.3 n is odd and $\#B^o$ is even

In this case, the value $\#B^e$ is even and $\sum_{k=1}^{n-1} e_k$ is even. The Gauss sum $\xi_{n,\nu_0}(B)_\chi$ is

$$\xi_{n,\nu_0}(B)_\chi = \begin{cases} \prod_{k\in B^e} \chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} & \nu_0 = -e_n - 1, \\ 0 & \nu_0 \geq -e_n. \end{cases}$$

The sum $S_t(B,s)_0^\chi$, the sum of the term with respect to $\nu_0 = -e_n - 1$, is

$$S_t(B,s)_0^\chi = \alpha_\psi(\pi)\chi(\pi)^{e_n+1} \prod_{k\in B^e} \chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k} S_t(B^{(n-1)},s)^\chi,$$

and we have

$$\begin{aligned} &{}^\Delta S_t(B,s)^\chi = {}^\Delta S_t(B,s)_0^\chi \\ &= \alpha_\psi(\pi)\chi(\pi)^{e_n+1} \prod_{k\in B^e} \chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} (q^{-2s+n+1} - 1) q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k} S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

6.1.4 n is odd and $\#B^o$ is odd

In this case, the value $\#B^e$ is odd and $\sum_{k=1}^{n-1} e_k$ is odd. The Gauss sum $\xi_{n,\nu_0}(B)_\chi$ is

$$\xi_{n,\nu_0}(B)_\chi = \begin{cases} -\prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^e}{2}+1} q^{-\frac{1}{2}} & \nu_0 = -e_n - 1, \\ \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} (1-q^{-1}) & \nu_0 \geq -e_n,\ \nu_0 \equiv e_n \mod 2, \\ \prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} (1-q^{-1}) & \nu_0 \geq -e_n,\ \nu_0 \not\equiv e_n \mod 2. \end{cases}$$

The notation of the sum of 3 cases is written as $S_t(B,s)_0^\chi$, $S_t(B,s)_1^\chi$ and $S_t(B,s)_2^\chi$. The first one is the same for the first case;

$$S_t(B,s)_0^\chi = -\alpha_\psi(\pi)\chi(\pi)^{e_n+1} \prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k - \frac{1}{2}} S_t(B^{(n-1)},s)^\chi.$$

The second one is calculated as follows. When $e_n - e_{n-1}$ is even, we put $\nu_0 = -e_{n-1} - 2i$ $(1 \leq i \leq \frac{e_n - e_{n-1}}{2})$, thus

$$\begin{aligned} S_t(B,s)_{1,e}^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n} \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} \\ & \times \sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1})} q^{-\frac{e_{n-1}+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1} e_k} (1-q^{-1}) S_t(B^{(n-1)},s)^\chi, \end{aligned}$$

and when $e_n - e_{n-1}$ is odd, we put $\nu_0 = -e_{n-1} - 1 - 2i$ $(1 \leq i \leq \frac{e_n - e_{n-1} - 1}{2})$, thus

$$\begin{aligned} S_t(B,s)_{1,o}^\chi = {} & \alpha_\psi(\pi)\chi(\pi)^{e_n} \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} \\ & \times \sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1}-1)} q^{-\frac{e_{n-1}+1+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1} e_k} (1-q^{-1}) S_t(B^{(n-1)},s)^\chi. \end{aligned}$$

The third one is calculated similarly.

$$S_t(B,s)^\chi_{2,e} = \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}$$
$$\times\sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1})} q^{-\frac{e_{n-1}+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi,$$
$$S_t(B,s)^\chi_{2,o} = \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}$$
$$\times\sum_{i=1}^{\frac{1}{2}(e_n-e_{n-1}-1)} q^{-\frac{e_{n-1}+1+2i}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S_t(B^{(n-1)},s)^\chi.$$

Then, we have the following equations.

$$^\Delta S_t(B,s)^\chi_0 = -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n+1}-1)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k-\frac{1}{2}}S_t(B^{(n-1)},s)^\chi$$

Therefore, it follows that

$$^\Delta S_t(B,s)^\chi = -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n}-1)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}S_t(B^{(n-1)},s)^\chi$$
$$+\alpha_\psi(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}S_t(B^{(n-1)},s)^\chi,$$

and the theorem is proved.

Remark 6.2. When $t=n$, the term of $^\Delta S_t(B,s)^\chi$ is 0 since $S_t(B,s)^\chi = 1$. However, we assume $S_t(B^{(n-1)},s)^\chi = 0$ and the theorem holds.

Remark 6.3. The ramified Siegel series $S(B,s)^\chi = \sum_{t=0}^{n}\beta^t S_t(B,s)^\chi$ satisfies the same equation of the above theorem. For example, when n is even and $\#B^o$ is even, we recall

$$^\Delta S_t(B,s)^\chi = -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n}-1)q^{-\frac{(e_n+1)}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}S_t(B^{(n-1)},s)^\chi.$$

From the above remark, this satisfies when $t=n$. Therefore we have

$$^\Delta S(B,s)^\chi = -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s+n}-1)q^{-\frac{(e_n+1)}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}S(B^{(n-1)},s)^\chi.$$

6.2 The calculation of the Clifford invariant

Theorem 6.2. *(1) When n is odd, the Clifford invariant η_B of the matrix B is*

$$\eta_B = \begin{cases}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}} & \sum_{k=1}^{n-1}e_k : \text{even},\\ \prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e+1}{2}} & \sum_{k=1}^{n-1}e_k : \text{odd}.\end{cases}$$

(2) When n is even, the Clifford invariant $\eta_{B^{(n-1)}}$ of the matrix $B^{(n-1)}$ is

$$\eta_{B^{(n-1)}} = \begin{cases} \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}} & \sum_{k=1}^n e_k : \textit{even}, \\ \prod_{k\in B^e} \chi(\alpha_k)\chi(-1)^{\frac{\#B^e}{2}} & \sum_{k=1}^n e_k : \textit{odd}. \end{cases}$$

Proof. First, we calculate the case when n is odd (1). η_B is defined as

$$\eta_B = \langle -1,-1\rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, \det B\rangle \varepsilon_B, \quad (n = 2m+1),$$
$$\varepsilon_B = \prod_{1\le i<j\le n} \langle \alpha_i \pi^{e_i}, \alpha_j \pi^{e_j}\rangle.$$

The Hilbert symbol is calculated as, when $\varepsilon_i \in \mathfrak{o}^\times$ and $n_i \ge 0$,

$$\langle \varepsilon_1\pi^{n_1}, \varepsilon_2\pi^{n_2}\rangle = \begin{cases} 1 & n_1 \text{ and } n_2 \text{ are even} \\ \chi(\varepsilon_1) & n_1 \text{ is even and } n_2 \text{ is odd} \\ \chi(\varepsilon_2) & n_1 \text{ is odd and } n_2 \text{ is even} \\ \chi(-\varepsilon_1\varepsilon_2) & n_1 \text{ and } n_2 \text{ are odd} \end{cases}$$
$$= \chi(-1)^{n_1 n_2}\chi(\varepsilon_1)^{n_2}\chi(\varepsilon_2)^{n_1},$$

because

$$\langle \alpha, \beta\rangle = \frac{\alpha_\psi(\alpha)\alpha_\psi(\beta)}{\alpha_\psi(\alpha\beta)\alpha_\psi(1)} \quad (\alpha, \beta \in F^\times),$$
$$\alpha_\psi(\varepsilon\pi^n) = \begin{cases} 1 & n \text{ is even} \\ \chi(\varepsilon)\alpha_\psi(\pi) & n \text{ is odd.} \end{cases} \quad (\varepsilon \in \mathfrak{o}^\times,\ n \ge 0).$$

Therefore we have, when n is odd and $\sum_{k=1}^{n-1} e_k$ is even,

$$\begin{aligned} \eta_B &= \chi((-1)^m)^{\sum_{k=1}^n e_k} \cdot \prod_{1\le i<j\le n} \chi(-1)^{e_i e_j}\chi(\alpha_i)^{e_j}\chi(\alpha_j)^{e_i} \\ &= \chi(-1)^{m\sum_{k=1}^n e_k + \sum_{1\le i<j\le n} e_i e_j} \prod_{i=1}^n \chi(\alpha_i)^{\sum_{k=1}^n e_k - e_i} \\ &= \chi(-1)^{\frac{n-1}{2}e_n + \sum_{1\le i<j\le n} e_i e_j} \prod_{i=1}^n \chi(\alpha_i)^{e_i + e_n}. \end{aligned}$$

The term $\prod_{k\in B^o}\chi(\alpha_k)$ is written as

$$\prod_{k\in B^o}\chi(\alpha_k) = \prod_{k=1}^{n-1}\chi(\alpha_k)^{e_k+e_n} = \prod_{k=1}^{n}\chi(\alpha_k)^{e_k+e_n},$$

therefore, we have to prove that

$$\begin{aligned} &\frac{\#B^o}{2} \equiv \frac{n-1}{2}e_n + \sum_{1\le i<j\le n} e_i e_j \pmod 2 \\ \Leftrightarrow \quad &\#B^o \equiv (n-1)e_n + 2\sum_{1\le i<j\le n} e_i e_j \pmod 4. \end{aligned}$$

It is easy to show that

$$2\sum_{1\le i<j\le n} e_ie_j = \sum_{1\le i,\, j\le n} e_ie_j = \left(\sum_{k=1}^{n} e_k\right)^2 - \sum_{k=1}^{n} e_k^2$$
$$\equiv e_n^2 - \sum_{k=1}^{n} e_k^2 = -\sum_{k=1}^{n-1} e_k^2 \equiv \sum_{k=1}^{n-1} e_k^2 \pmod 4,$$

and $e_k^2 \equiv 0$ or 1 whether e_k is even or odd. Since we assume that $\sum_{k=1}^{n-1} e_k$ is even, we know that $\sum_{k=1}^{n-1} e_k^2$ is also even, and that the last equation $-\sum_{k=1}^{n-1} e_k^2 \equiv \sum_{k=1}^{n-1} e_k^2 \pmod 4$ holds.

We define the number t as the number of e_i $(1 \le i \le n-1)$, which is odd. From the assumption that $\sum_{k=1}^{n-1} e_k$ is even, the number t is also even and we have

$$t \equiv 2\sum_{1\le i<j\le n} e_ie_j \pmod 4.$$

When e_n is even, $\#B^o = t$ and $(n-1)e_n$ is divided by 4. When e_n is odd, we have $\#B^o = (n-1) - t$ therefore

$$\#B^o - \left((n-1)e_n + 2\sum_{1\le i<j\le n} e_ie_j\right)$$
$$\equiv (n-1)(1-e_n) - 2t \equiv 0 \pmod 4.$$

In conclusion, we have

$$\#B^o \equiv (n-1)e_n + 2\sum_{1\le i<j\le n} e_ie_j \pmod 4$$

and it follows that $\eta_B = \prod_{k\in B^o} \chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}}$.

When n is odd, and $\sum_{k=1}^{n-1} e_k$ is odd, and when n is even, we can prove the statement with the same method. □

6.3 Main Theorem: Recursion formulas

We define the rational functions $C(e,\tilde{e},\xi;Y,X)$ and $D(e,\tilde{e},\xi;Y,X)$ as follows.

Definition 6.1. Let $e,\tilde{e}$ be integers, and let ξ be a real number.

The rational functions $C(e,\tilde{e},\xi;Y,X)$ and $D(e,\tilde{e},\xi;Y,X)$ in $Y^{\frac{1}{2}}$ and $X^{\frac{1}{2}}$ are defined as

$$C(e,\tilde{e},\xi;Y,X) = \frac{Y^{\tilde{e}/2}X^{-(e-\tilde{e})/2-1}(1-\xi Y^{-1}X)}{X^{-1}-X},$$
$$D(e,\tilde{e},\xi;Y,X) = \frac{Y^{\tilde{e}/2}X^{-(e-\tilde{e})/2}}{1-\xi X}.$$

For a positive integer i, we define the rational function $C_i(e,\tilde{e},\xi;Y,X)$ as

$$C_i(e,\tilde{e},\xi;Y,X) = \begin{cases} C(e,\tilde{e},\xi;Y,X) & i\text{: even},\\ D(e,\tilde{e},\xi;Y,X) & i\text{: odd}.\end{cases}$$

Definition 6.2. Let $\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ be a sequence of integers. For an integer i which satisfies $1 \le i \le n$, we define $\mathfrak{e}_i = \mathfrak{e}_i(\underline{a})$ as

$$\mathfrak{e}_i = \begin{cases} a_1 + a_2 + \cdots + a_i & \text{if } i \text{ is even and } \sum_{k=1}^n a_k \text{ is even,} \\ a_1 + a_2 + \cdots + a_i + 1 & \text{otherwise.} \end{cases}$$

We put $D_B = (-4)^{[\frac{n}{2}]} \det B$ and define ξ_B as

$$\xi_B = \begin{cases} 0 & \text{if } \sum_{k=1}^n e_k \text{ is even,} \\ \chi(-D_B) & \text{if } \sum_{k=1}^n e_k \text{ is odd.} \end{cases}$$

The main theorem of this article is the recursion formula of the ramified Siegel series. It is written as follows.

Theorem 6.3. *We put $X = q^{-s}$ and $Y = q^{\frac{1}{2}}$ and write $\tilde{F}_B(q^{-s}) = \tilde{F}_B(s)$. The function $\tilde{F}_B(X)$ satisfies the following recursion formula.*

$$\begin{aligned} \tilde{F}_B(X) = {} & \beta_0 C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X) \tilde{F}_{B^{(n-1)}}(YX) \\ & + \beta_0 \zeta_i C_i(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X^{-1}) \tilde{F}_{B^{(n-1)}}(YX^{-1}) \end{aligned}$$

where ζ_i and ξ are defined as

$$\zeta_i = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \eta_B \chi(-D_B) & \text{if } n \text{ is odd,} \end{cases} \qquad \xi = \begin{cases} \chi(-D_B) & \text{if } n \text{ is even,} \\ \chi(-D_{B^{(n-1)}}) & \text{if } n \text{ is odd.} \end{cases}$$

When we explicitly write down the function $C_i(e, \tilde{e}, \xi; Y, X)$, the above theorem can be expressed as follows.

Theorem 6.4. *The ramified Siegel series satisfies the following equations.*

(1) When n is even and $\sum_{k=1}^n e_k$ is even,

$$\begin{aligned} \tilde{F}_B(s) = {} & \frac{1}{1 - q^{-2s}} \beta_0 q^{\frac{s}{2} \sum_{k=1}^n e_k - \frac{s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} \tilde{F}_{B^{(n-1)}}(s - 1/2) \\ & + \frac{1}{1 - q^{2s}} \beta_0 q^{-\frac{s}{2} \sum_{k=1}^n e_k - \frac{-s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} \tilde{F}_{B^{(n-1)}}(-s - 1/2). \end{aligned}$$

(2) When n is even and $\sum_{k=1}^n e_k$ is odd,

$$\begin{aligned} \tilde{F}_B(s) = {} & \beta_0 \frac{1 - \chi(-D_B) q^{-s - \frac{1}{2}}}{1 - q^{-2s}} q^{\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} F_{B^{(n-1)}}(s - 1/2) \\ & + \beta_0 \frac{1 - \chi(-D_B) q^{s - \frac{1}{2}}}{1 - q^{2s}} q^{-\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{-s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} F_{B^{(n-1)}}(-s - 1/2). \end{aligned}$$

(3) When n is odd and $\sum_{k=1}^{n-1} e_k$ is even,

$$\begin{aligned} \tilde{F}_B(s) = {} & \beta_0 q^{\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{s - \frac{1}{2}}{2} \sum_{k=1}^{n-1} e_k} \tilde{F}_{B^{(n-1)}}(s - 1/2) \\ & + \beta_0 \eta_B \chi(-D_B) q^{-\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{-s - \frac{1}{2}}{2} \sum_{k=1}^{n-1} e_k} \tilde{F}_{B^{(n-1)}}(-s - 1/2). \end{aligned}$$

(4) When n is odd and $\sum_{k=1}^{n-1} e_k$ is odd,

$$\begin{aligned} \tilde{F}_B(s) = {} & \frac{1}{1 - \chi(-D_{B^{(n-1)}}) q^{-s}} \beta_0 q^{\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} \tilde{F}_{B^{(n-1)}}(s - 1/2) \\ & + \frac{1}{1 - \chi(-D_{B^{(n-1)}}) q^{s}} \beta_0 \eta_B \chi(-D_B) q^{-\frac{s}{2} (\sum_{k=1}^n e_k + 1) - \frac{-s - \frac{1}{2}}{2} (\sum_{k=1}^{n-1} e_k + 1)} \tilde{F}_{B^{(n-1)}}(-s - 1/2). \end{aligned}$$

6.4 Proof of the recursion formulas

We show the induction formulas of the ramified Siegel series by using the above theorems. We calculate by dividing into 4 cases, considering the parity of n and $\#B^o$.

6.4.1 n: even, $\sum_{k=1}^n e_k$: even ($\Leftrightarrow \#B^e$: odd, $\#B^o$: even)

In this case, it follows that

$$\begin{aligned}{}^{\Delta}S(B,s)^{\chi} &= -\alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}} \\ &\times\left(q^{-2s+n}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k+\frac{1}{2}}S(B^{(n-1)},s)^{\chi}, \qquad (7)\end{aligned}$$

and the functional equation is

$$F(-s)=q^{s\sum_{k=1}^n e_k}F(s).$$

We know that the Clifford invariant $\eta_{B^{(n-1)}}$ is calculated as

$$\eta_{B^{(n-1)}}=\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}}$$

from Theorem 6.2. From the fact that

$$\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}\cdot\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}}=\prod_{k=1}^{n-1}\chi(\alpha_k)\chi(-1)^{\frac{n}{2}}=\chi(-D_{B^{(n-1)}}\pi^{e_n}),$$

(7) is written as

$${}^{\Delta}S(B,s)^{\chi}=-\alpha_\psi(\pi)\chi(\pi)\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})\left(q^{-2s+n}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k+\frac{1}{2}}S(B^{(n-1)},s)^{\chi},$$

and combined with $\mathrm{Wh}_B(s)f_n^{(s)}=S(B,s+\frac{n+1}{2})^{\chi}$, we get

$${}^{\Delta}\mathrm{Wh}_B(s)=\alpha_\psi(\pi)\chi(\pi)\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})(1-q^{-2s-1})q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}}\mathrm{Wh}_{B^{(n-1)}}(s+1/2).$$

We write $\mathrm{Wh}_B(s)f_n^{(s)}$ as $\mathrm{Wh}_B(s)$ when there is no confusion.

Recall that $\beta_0=\alpha_\psi(\pi)\chi(\pi)q^{-\frac{1}{2}}$. The function F is calculated as

$$\begin{aligned}F_B(s)&=\frac{\mathrm{Wh}_B(s)}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})},\\ F_{B^{(n-1)}}(s+1/2)&=\frac{\mathrm{Wh}_{B^{(n-1)}}(s+1/2)}{(1-q^{-2s-3})(1-q^{-2s-5})\cdots(1-q^{-2s-n+1})},\end{aligned}$$

and the above equation is equal to

$${}^{\Delta}F_B(s)=\beta_0\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+1}F_{B^{(n-1)}}(s+1/2). \qquad (8)$$

We write four equations, (8), (8) substituted s into $-s$, the functional equations for B and B',

$$\begin{aligned}{}^{\Delta}F_B(s)&=\beta_0\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+1}F_{B^{(n-1)}}(s+1/2),\\ {}^{\Delta}F_B(-s)&=\beta_0\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(-2s+1)+1}F_{B^{(n-1)}}(-s+1/2),\\ F_{B'}(-s)&=q^{s(\sum_{k=1}^n e_k+2)}F_{B'}(s),\\ F_B(-s)&=q^{s\sum_{k=1}^n e_k}F_B(s).\end{aligned}$$

From these equations, we make the equation of $F_B(s)$ and $F_{B^{(n-1)}}(\pm s+1/2)$, by eliminating $F_B(-s)$ and $F_{B'}(\pm s)$.

$$\begin{aligned}F_B(s) &= -\frac{1}{1-q^{-2s}}\beta_0\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+1}F_{B^{(n-1)}}(s+1/2)\\ &\quad -\frac{1}{1-q^{2s}}\beta_0\eta_{B^{(n-1)}}\chi(-D_{B^{(n-1)}})q^{(-s+\frac{1}{2})\sum_{k=1}^n e_k-\frac{e_n+1}{2}(-2s+1)+1}F_{B^{(n-1)}}(-s+1/2).\end{aligned}$$

Combined with the functional equations for $B^{(n-1)}$, we get

$$\begin{aligned}F_B(s) &= \frac{1}{1-q^{-2s}}\beta_0 F_{B^{(n-1)}}(s-1/2)\\ &\quad +\frac{1}{1-q^{2s}}\beta_0 q^{-s\sum_{k=1}^n e_k}F_{B^{(n-1)}}(-s-1/2)\end{aligned}$$

and

$$\begin{aligned}\tilde{F}_B(s) &= \frac{1}{1-q^{-2s}}\beta_0 q^{\frac{s}{2}\sum_{k=1}^n e_k-\frac{s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1}e_k+1)}\tilde{F}_{B^{(n-1)}}(s-1/2)\\ &\quad +\frac{1}{1-q^{2s}}\beta_0 q^{-\frac{s}{2}\sum_{k=1}^n e_k-\frac{-s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1}e_k+1)}\tilde{F}_{B^{(n-1)}}(-s-1/2).\end{aligned}$$

6.4.2 n: even, $\sum_{k=1}^n e_k$: odd ($\Leftrightarrow \#B^e$: even, $\#B^o$: odd)

In this case, it follows that

$$\begin{aligned}{}^{\Delta}S(B,s)^\chi &= \alpha_\psi(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1}\\ &\quad\times\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k}S(B^{(n-1)},s)^\chi\\ &\quad+\alpha_\psi(\pi)\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}\chi(\pi)^{e_n}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S(B^{(n-1)},s)^\chi,\end{aligned}$$

and the functional equation is

$$F_B(-s) = q^{s(\sum_{k=1}^n e_k+1)}F_B(s).$$

The Clifford invariant is $\eta_{B^{(n-1)}} = \prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e}{2}}$ and we note that

$$\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1}\cdot\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}} = \prod_{k=1}^n\chi(\alpha_k)\chi(-1)^{\frac{n}{2}+1} = \chi(-D_B\pi).$$

Hence we have

$$\begin{aligned}{}^{\Delta}S(B,s)^\chi &= \alpha_\psi(\pi)\chi(-\alpha_n)\chi(\pi)^{e_n+1}\eta_{B^{(n-1)}}\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^n e_k}S(B^{(n-1)},s)^\chi\\ &\quad+\alpha_\psi(\pi)\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}\chi(-D_B\pi)q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S(B^{(n-1)},s)^\chi.\end{aligned}$$

From the fact that $\mathrm{Wh}_B(s) = S(B, s+\frac{n+1}{2})^\chi$, we have

$$\begin{aligned}{}^{\Delta}\mathrm{Wh}_B(s) &= \beta_0\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}(q^{-2s}-1)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^n e_k+\frac{1}{2}}\mathrm{Wh}_{B^{(n-1)}}(s+1/2)\\ &\quad+\beta_0\chi(-\alpha_n)\chi(\pi)^{e_n+1}\eta_{B^{(n-1)}}\chi(-D_B\pi)(q-1)q^{\frac{1}{2}\sum_{k=1}^{n-1}e_k-(e_n+2)s-\frac{1}{2}}\mathrm{Wh}_{B^{(n-1)}}(s+1/2)\\ &= \beta_0\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}\left\{(q^{-2s}-1)+\chi(-D_B)(q-1)q^{-s-\frac{1}{2}}\right\}\\ &\quad\times q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^n e_k+\frac{1}{2}}\mathrm{Wh}_{B^{(n-1)}}(s+1/2).\end{aligned}$$

We note that

$$\left\{(q^{-2s}-1)+\chi(-D_B)(q-1)q^{-s-\frac{1}{2}}\right\} = -\left(1+\chi(-D_B)q^{-s-\frac{1}{2}}\right)\left(1-\chi(-D_B)q^{-s+\frac{1}{2}}\right).$$

The function F is calculated as

$$F_B(s) = \frac{(1-\chi(-D_B)q^{-s-\frac{1}{2}})\mathrm{Wh}_B(s)}{(1-q^{-2s-1})(1-q^{-2s-3})\cdots(1-q^{-2s-n+1})},$$
$$F_{B^{(n-1)}}(s+1/2) = \frac{\mathrm{Wh}_{B^{(n-1)}}(s+1/2)}{(1-q^{-2s-3})(1-q^{-2s-5})\cdots(1-q^{-2s-n+1})},$$

and we have

$$^{\Delta}F_B(s) = -\beta_0\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}\left(1-\chi(-D_B)q^{-s+\frac{1}{2}}\right)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(s+1/2).$$

We write four equations as well as in the above case.

$$\begin{aligned}
^{\Delta}F_B(s) &= -\beta_0\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}\left(1-\chi(-D_B)q^{-s+\frac{1}{2}}\right)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(s+1/2),\\
^{\Delta}F_B(-s) &= -\beta_0\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}\left(1-\chi(-D_B)q^{s+\frac{1}{2}}\right)q^{-\frac{e_n+1}{2}(-2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(-s+1/2),\\
F_{B'}(-s) &= q^{s(\sum_{k=1}^{n}e_k+3)}F_{B'}(s),\\
F_B(-s) &= q^{s(\sum_{k=1}^{n}e_k+1)}F_B(s).
\end{aligned}$$

From these equations, we make the equation of $F_B(s)$ and $F_{B^{(n-1)}}(\pm s+1/2)$ as above.

$$\begin{aligned}
F_B(s) = \beta_0&\frac{1-\chi(-D_B)q^{-s+\frac{1}{2}}}{1-q^{-2s}}\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(s+1/2)\\
&+\beta_0\frac{1-\chi(-D_B)q^{s+\frac{1}{2}}}{1-q^{2s}}\chi(-\alpha_n)\chi(\pi)^{e_n}\eta_{B^{(n-1)}}q^{-\frac{e_n+1}{2}(-2s+1)+(-s+\frac{1}{2})\sum_{k=1}^{n}e_k-s+\frac{1}{2}}F_{B^{(n-1)}}(-s+1/2).
\end{aligned}$$

Combined with the functional equation for $B^{(n-1)}$, it follows that

$$\begin{aligned}
F_B(s) = \beta_0&\frac{1-\chi(-D_B)q^{-s-\frac{1}{2}}}{1-q^{-2s}}F_{B^{(n-1)}}(s-1/2)\\
&+\beta_0\frac{1-\chi(-D_B)q^{s-\frac{1}{2}}}{1-q^{2s}}q^{-s(\sum_{k=1}^{n}e_k+1)}F_{B^{(n-1)}}(-s-1/2).
\end{aligned}$$

We note that

$$\chi(D_{B^{(n-1)}}D_B) = \chi\left((-4)^{\frac{n-2}{2}}\det B^{(n-1)}(-4)^{\frac{n}{2}}\det B\right) = \chi(-\alpha_n\pi^{e_n}).$$

Therefore we have

$$\begin{aligned}
\tilde{F}_B(s) = \beta_0&\frac{1-\chi(-D_B)q^{-s-\frac{1}{2}}}{1-q^{-2s}}q^{\frac{s}{2}(\sum_{k=1}^{n}e_k+1)-\frac{s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1}e_k+1)}F_{B^{(n-1)}}(s-1/2)\\
&+\beta_0\frac{1-\chi(-D_B)q^{s-\frac{1}{2}}}{1-q^{2s}}q^{-\frac{s}{2}(\sum_{k=1}^{n}e_k+1)-\frac{-s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1}e_k+1)}F_{B^{(n-1)}}(-s-1/2).
\end{aligned}$$

6.4.3 n: odd, $\sum_{k=1}^{n-1} e_k$: even ($\Leftrightarrow \#B^e$: even, $\#B^o$: even)

In this case, it follows that

$$\begin{aligned}{}^{\Delta}S_t(B,s)^{\chi} &= \alpha_{\psi}(\pi)\prod_{k\in B^e}\chi(\alpha_k)\chi(-\alpha_n)\chi(-1)^{\frac{\#B^e}{2}}\chi(\pi)^{e_n+1}\\ &\times\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k}S_t(B^{(n-1)},s)^{\chi},\end{aligned}$$

and the functional equation is

$$F_B(-s) = \eta_B\chi(-D_B)q^{s(\sum e_j+1)}F_B(s).$$

η_B is calculated as $\eta_B = \prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}}$, and we note that

$$\prod_{k\in B^e}\chi(\alpha_k)\chi(-\alpha_n)\chi(-1)^{\frac{\#B^e}{2}}\cdot\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o}{2}} = \prod_{k=1}^{n}\chi(\alpha_k)\chi(-1)^{\frac{n+1}{2}} = \chi(-D_B\pi^{e_n}).$$

The function F is calculated as

$$\begin{aligned}F_B(s) &= \frac{\mathrm{Wh}_B(s)}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})},\\ F_{B^{(n-1)}}(s+1/2) &= \frac{\mathrm{Wh}_{B^{(n-1)}}(s+1/2)}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})},\end{aligned}$$

and it follows that

$${}^{\Delta}F_B(s) = -\beta_0\eta_B\chi(-D_B)(1-q^{-2s})q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}.$$

We write four equations as well as in the above case.

$$\begin{aligned}{}^{\Delta}F_B(s) &= -\beta_0\eta_B\chi(-D_B)(1-q^{-2s})q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(s+1/2),\\ {}^{\Delta}F_B(-s) &= -\beta_0\eta_B\chi(-D_B)(1-q^{2s})q^{-\frac{e_n+1}{2}(-2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(-s+1/2),\\ F_{B'}(-s) &= \eta_B\chi(-D_B)q^{s(\sum_{k=1}^{n}e_k+3)}F_{B'}(s),\\ F_B(-s) &= \eta_B\chi(-D_B)q^{s(\sum_{k=1}^{n}e_k+1)}F_B(s).\end{aligned}$$

From these equations, we make the equation of $F_B(s)$ and $F_{B^{(n-1)}}(\pm s+1/2)$. It is calculated as

$$\begin{aligned}F_B(s) &= \beta_0\eta_B\chi(-D_B)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}F_{B^{(n-1)}}(s+1/2)\\ &+\beta_0 q^{-\frac{e_n+1}{2}(-2s+1)+(-s+\frac{1}{2})\sum_{k=1}^{n}e_k-s+\frac{1}{2}}F_{B^{(n-1)}}(-s+1/2).\end{aligned}$$

Combined with the functional equation for $B^{(n-1)}$, it follows that

$$\begin{aligned}F_B(s) &= \beta_0F_{B^{(n-1)}}(s-1/2)\\ &+\beta_0\eta_B\chi(-D_B)q^{-s(\sum_{k=1}^{n}e_k+1)}F_{B^{(n-1)}}(-s-1/2),\end{aligned}$$

and

$$\begin{aligned}\tilde{F}_B(s) &= \beta_0q^{\frac{s}{2}(\sum_{k=1}^{n}e_k+1)-\frac{s-\frac{1}{2}}{2}\sum_{k=1}^{n-1}e_k}\tilde{F}_{B^{(n-1)}}(s-1/2)\\ &+\beta_0\eta_B\chi(-D_B)q^{-\frac{s}{2}(\sum_{k=1}^{n}e_k+1)-\frac{-s-\frac{1}{2}}{2}\sum_{k=1}^{n-1}e_k}\tilde{F}_{B^{(n-1)}}(-s-1/2).\end{aligned}$$

6.4.4 n: odd, $\sum_{k=1}^{n-1} e_k$: odd ($\Leftrightarrow \#B^e$: odd, $\#B^o$: odd)

In this case, it follows that

$$\begin{aligned}
{}^{\Delta}S(B,s)^{\chi} &= \alpha_{\psi}(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e}{2}+1} \\
&\quad\times\left(q^{-2s+n+1}-1\right)q^{-\frac{e_n+1}{2}(2s-n)+\frac{1}{2}\sum_{k=1}^{n}e_k}S(B^{(n-1)},s)^{\chi} \\
&\quad+\alpha_{\psi}(\pi)\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}\chi(\pi)^{e_n}q^{-\frac{e_n+2}{2}(2s-n-1)+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})S(B^{(n-1)},s)^{\chi},
\end{aligned}$$

and the functional equation is

$$F_B(-s)=\eta_B\chi(-D_B)q^{s(\sum e_j+1)}F_B(s).$$

η_B is calculated as $\eta_B=\prod_{k\in B^e}\chi(\alpha_k)\chi(\alpha_n)\chi(-1)^{\frac{\#B^e+1}{2}}$ and we note that

$$\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}\cdot\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}=\prod_{k=1}^{n-1}\chi(\alpha_k)\chi(-1)^{\frac{n+1}{2}}=\chi(-D_{B^{(n-1)}}\pi).$$

Therefore we get

$$\begin{aligned}
{}^{\Delta}\mathrm{Wh}_B(s) &= -\alpha_{\psi}(\pi)\chi(\pi)^{e_n+1}\prod_{k\in B^e}\chi(\alpha_k)\chi(-1)^{\frac{\#B^e+1}{2}}(q^{-2s-1}-1)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+\frac{1}{2}}\mathrm{Wh}_{B^{(n-1)}}(s+1/2) \\
&\quad+\alpha_{\psi}(\pi)\chi(\pi)^{e_n}\prod_{k\in B^o}\chi(\alpha_k)\chi(-1)^{\frac{\#B^o+1}{2}}q^{-(e_n+2)s+\frac{1}{2}\sum_{k=1}^{n-1}e_k}(1-q^{-1})\mathrm{Wh}_{B^{(n-1)}}(s+1/2) \\
&= -\beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B(q^{-2s-1}-1)q^{-\frac{e_n+1}{2}(2s+1)+\frac{1}{2}\sum_{k=1}^{n}e_k+1}\mathrm{Wh}_{B^{(n-1)}}(s+1/2) \\
&\quad+\beta_0\chi(\pi)^{e_n+1}\chi(\alpha_n)\eta_B\chi(-D_{B^{(n-1)}}\pi)(q-1)q^{\frac{1}{2}\sum_{k=1}^{n}e_k-\frac{e_n+1}{2}(2s+1)+1}q^{-s-1}\mathrm{Wh}_{B^{(n-1)}}(s+1/2) \\
&= \beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B\left\{-(q^{-2s-1}-1)+\chi(-D_{B^{(n-1)}})(q-1)q^{-s-1}\right\} \\
&\quad\times q^{\frac{1}{2}\sum_{k=1}^{n}e_k-\frac{e_n+1}{2}(2s+1)+1}\mathrm{Wh}_{B^{(n-1)}}(s+1/2)
\end{aligned}$$

and that

$$\left\{-(q^{-2s-1}-1)+\chi(-D_{B^{(n-1)}})(q-1)q^{-s-1}\right\}=\left(1+\chi(-D_{B^{(n-1)}})q^{-s}\right)\left(1-\chi(-D_{B^{(n-1)}})q^{-s-1}\right).$$

The function F is calculated as

$$\begin{aligned}
F_B(s) &= \frac{\mathrm{Wh}_B(s)}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})}, \\
F_{B^{(n-1)}}(s+1/2) &= \frac{(1-\chi(-D_{B^{(n-1)}})q^{-s-1})\mathrm{Wh}_{B^{(n-1)}}(s+1/2)}{(1-q^{-2s-2})(1-q^{-2s-4})\cdots(1-q^{-2s-n+1})},
\end{aligned}$$

and we have

$${}^{\Delta}F_B(s)=\beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B\left(1+\chi(-D_{B^{(n-1)}})q^{-s}\right)q^{\frac{1}{2}\sum_{k=1}^{n}e_k-\frac{e_n+1}{2}(2s+1)+1}F_{B^{(n-1)}}(s+1/2).$$

We write four equations as well as in the above case.

$$\begin{aligned}
{}^{\Delta}F_B(s) &= \beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B\left(1+\chi(-D_{B^{(n-1)}})q^{-s}\right)q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+1}F_{B^{(n-1)}}(s+1/2),\\
{}^{\Delta}F_B(-s) &= \beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B\left(1+\chi(-D_{B^{(n-1)}})q^{s}\right)q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(-2s+1)+1}F_{B^{(n-1)}}(-s+1/2),\\
F_{B'}(-s) &= \eta_B\chi(-D_B)q^{s(\sum_{k=1}^n e_k+3)}F_{B'}(s),\\
F_B(-s) &= \eta_B\chi(-D_B)q^{s(\sum_{k=1}^n e_k+1)}F_B(s).
\end{aligned}$$

From these equations, we make the equation of $F_B(s)$ and $F_{B^{(n-1)}}(\pm s+1/2)$ as above.

$$\begin{aligned}
F'_B(s) = &-\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{-s}}\beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\eta_B q^{\frac{1}{2}\sum_{k=1}^n e_k-\frac{e_n+1}{2}(2s+1)+1}F'_{B^{(n-1)}}(s+1/2)\\
&-\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{s}}\beta_0\chi(\pi)^{e_n}\chi(\alpha_n)\chi(-D_B)q^{(-s+\frac{1}{2})\sum_{k=1}^n e_k-\frac{e_n+1}{2}(-2s+1)-s+1}F'_{B^{(n-1)}}(-s+1/2).
\end{aligned}$$

Combined with the functional equation for $B^{(n-1)}$, it follows that

$$\begin{aligned}
F_B(s) = &\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{-s}}\beta_0 F_{B^{(n-1)}}(s-1/2)\\
&+\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{s}}\beta_0\eta_B\chi(-D_B)q^{-s(\sum_{k=1}^n e_k+1)}F_{B^{(n-1)}}(-s-1/2).
\end{aligned}$$

Here, we note that

$$\chi(D_B D_{B^{(n-1)}}) = \chi\left((-4)^{\frac{n-1}{2}}\det B(-4)^{\frac{n-1}{2}}\det B^{(n-1)}\right) = \chi(\alpha_n\pi^{e_n}).$$

We conclude

$$\begin{aligned}
\tilde{F}_B(s) = &\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{-s}}\beta_0 q^{\frac{s}{2}(\sum_{k=1}^n e_k+1)-\frac{s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1} e_k+1)}\tilde{F}_{B^{(n-1)}}(s-1/2)\\
&+\frac{1}{1-\chi(-D_{B^{(n-1)}})q^{s}}\beta_0\eta_B\chi(-D_B)q^{-\frac{s}{2}(\sum_{k=1}^n e_k+1)-\frac{-s-\frac{1}{2}}{2}(\sum_{k=1}^{n-1} e_k+1)}\tilde{F}_{B^{(n-1)}}(-s-1/2).
\end{aligned}$$

Remark 6.4. When the character *chi* is trivial, the same method can also be used to show a recursion formula for the Siegel series from the explicit formula of the Siegel series. This recursion formula is compatible with Katsurada's result [19].

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