A summary of “Convergence of processes time-changed by Gaussian multiplicative chaos”

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This is a summary of the paper [17], entitled “Convergence of processes time-changed by Gaussian multiplicative chaos”.

In the paper, under certain technical assumptions, we proved the convergence of a process time-changed by Gaussian multiplicative chaos in the case the latter object is square integrable (the L^2-regime). As examples of the main result, we prove that, in the whole L^2-regime, the scaling limit of the Liouville simple random walk on $\mathbb{Z}^2$ is Liouville Brownian motion and, as $\alpha \to 1$, Liouville α-stable processes on $\mathbb{R}$ converge weakly to the Liouville Cauchy process.

There has been intensive research on Gaussian free fields, the Liouville measure and Liouville Brownian motion, due to their close relationship with important concepts in areas such as statistical mechanics, conformal field theory, random geometry and quantum gravity. Here, a massive Gaussian free field $X = \{X_f\}_{f\in\mathcal{S}(\mathbb{R}^d)}$ on $\mathbb{R}^d$ is a centred Gaussian system on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ whose covariance kernel is π times the λ-order Green's function of Brownian motion on $\mathbb{R}^d$.

For the massive Gaussian free field X on $\mathbb{R}^2$ and $\gamma \in (0, 2)$, a random measure formally represented by

$$\exp(\gamma^2 X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2])dx \tag{1}$$

is called the Liouville measure. This was first constructed rigorously by Kahane [14] and, since then, has been studied in many other articles, see [2, 19], for example. It is known that the Liouville measure is singular with respect to the Lebesgue measure for $\gamma \in (0, 2)$, see [14]. Moreover, it corresponds to the Riemann metric tensor of two-dimensional Liouville quantum gravity. See [5, 8] for details.

Liouville Brownian motion is the Brownian motion on $\mathbb{R}^2$ time-changed by the Liouville measure. This was constructed in [11] and similar results were proved in [3] simultaneously. We also refer the reader to [1, Appendix A] for the construction of Liouville Brownian motion. Liouville Brownian motion is the canonical diffusion process under Liouville quantum gravity, and it is conjectured that Liouville Brownian motion is the scaling limit of simple random walks on random planar maps, see [8, 11]. In [4, 12], it is proved that the scaling limit of random walks on mated-CRT planar maps is Liouville Brownian motion.

More general models have been studied. For example, fractional Gaussian fields are Gaussian fields whose covariance kernels can be represented in terms of fractional order Laplacians, and properties of these fields are summarized in [15]. Gaussian multiplicative chaos is a random measure formally represented as (1), with the Gaussian free field replaced by a more general Gaussian field. Berestycki [2] constructed Gaussian multiplicative chaos for log-correlated Gaussian fields using an elementary approach. Shamov [19] constructed Gaussian multiplicative chaos for Gaussian fields indexed by elements of Hilbert spaces, and also proved the convergence of Gaussian multiplicative chaos under some assumptions by introducing a randomized shift and a new definition of Gaussian multiplicative chaos. Hager and Neuman [13] proved the convergence of Gaussian multiplicative chaos for fractional Brownian fields by using Berestycki's method [2]. In summary, convergence of Gaussian fields and Gaussian multiplicative chaos have been proved in various specific situations. This leads to the following general question.

If a sequence of processes and Gaussian multiplicative chaoses converge respectively, then is it the case that the associated time-changed processes also converge?

In the paper [17], we answered this question under some assumptions on the Green's functions of the approximating processes on subsets of $\mathbb{R}$ and $\mathbb{R}^2$. In [7], the convergence of time-changed processes is studied in general cases where the limit process has positive capacity of a point. The novelty of our result is to establish general conditions when the convergence of time-changed processes by Gaussian multiplicative chaos hold in cases where the capacity of each point for the limit process is 0, so our result includes the case of Liouville Brownian motion.

We fix $d = 1$ or $d = 2$. Denote by m the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For $n \in \mathbb{N}$, we consider an m-symmetric Lévy process $Z^n = (\{Z^n_t\}_{t\geq 0}, \{\mathbb{P}^{Z^n}_x\}_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$. Denote by $Z^\infty = (\{Z^\infty_t\}_{t\geq 0}, \{\mathbb{P}^{Z^\infty}_x\}_{x\in\mathbb{R}^d})$ Brownian motion on $\mathbb{R}^2$ for $d = 2$, and the symmetric 1-stable process on $\mathbb{R}$, which is also called the Cauchy process, for $d = 1$. We consider the situation where one of the following conditions hold.

(A) For any $n \in \mathbb{N}$, Z^n has a continuous transition probability density function $p^n(t, x, y)$ with respect to m.

(B) For any $n \in \mathbb{N}$, there exists a discrete subset $D_n \subset \mathbb{R}^d$ such that $\mathbb{P}^{Z^n}_0(Z^n_t \in D_n \text{ for any } t) = 1$. Moreover, there exists a surjective map $i_n : \mathbb{R}^d \to D_n$ satisfying $i_n(x) = x$ for any $x \in D_n$ and $\lim_{n\to\infty} |i_n(x) - x| = 0$ for any $x \in \mathbb{R}^d$. We also represent $i_n(x)$ as x_n. In this case, we define $p^n(t, x, y) := \mathbb{P}_{x_n}(Z^n_t = y_n)$.

In the case of (A), we define D_n to be $\mathbb{R}^d$. Set $D_n(z) := \{x \in \mathbb{R}^d : x_n = z\}$ for any $z \in \mathbb{R}^d$ and define $C_n(z)$ by

$$C_n(z) := \begin{cases} \dfrac{1}{m(D_n(z))} & \text{if } m(D_n(z)) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

For $\lambda > 0$, we define the λ-order Green's kernel g^n_λ by

$$g^n_\lambda(x, y) := C_n(y) \int_0^\infty e^{-\lambda t} p^n(t, x_n, y_n) dt$$

for $x, y \in \mathbb{R}^d$. Fpr $n \in \mathbb{N} \cup \{\infty\}$, denote X^n by the centred Gaussian field with covariance kernel πg^n_λ on a probability space $(\Omega^{X^n}, \mathcal{M}^{X^n}, \mathbb{P}^{X^n})$, μ^n by Gaussian multiplicative chaos (GMC in abbreviation) for $\gamma \in (0, \sqrt{2d})$, and $\hat{Z}^n$ by the time-changed process of Z^n by μ^n, respectively. For the existences and properties of X^n, GMC μ^n and $\hat{Z}^n$, see, for example, [9, 16, 18], [14, 2, 19], and [6, 10, 11, 1], respectively.

We make the following assumption.

Assumption 1.
(0) For each $n \in \mathbb{N}$, it holds that $g_\lambda^n(0,0) < \infty$
(1) The process Z^n converges weakly to Z^∞ in the Skorokhod space $D[0,\infty)$ equipped with J_1-topology as $n \to \infty$.
(2) For m-almost every $x, y \in \mathbb{R}^d$ and $t > 0$, it holds that

$$\lim_{n\to\infty} C_n(y)p^n(t,x,y) = p^\infty(t,x,y)$$

and

$$\lim_{n\to\infty} g_\lambda^n(x,y) = g_\lambda^\infty(x,y).$$

(3) There exist positive constants C^* and C such that, for any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$, it holds that

$$g_\lambda^n(x,y) \le C^* g_\lambda^\infty(x,y) + C$$

(4) For any $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$, it holds that $C_n(x) = C_n(y)$.
(5) For any $t \ge 0$ and $z \in \mathbb{R}^d$, there exists $q(t,0,z) \ge 0$ such that, for any $n \in \mathbb{N}$ and $T > 0$, it holds that

$$C_n(z)p^n(t,0,z) \le q(t,0,z)$$
$$\text{and} \quad \int_0^T \int_{\mathbb{R}^d} q(t,0,z)dm(z)dt < \infty.$$

Under Assumption 1, for $\gamma \in (0, \sqrt{2d/C^*})$, GMC μ^n converges weakly to μ^∞ in the space of Radon measures with vague topology as $n \to \infty$.

Then the main theorem of the paper is stated as follows:

Theorem 2. *Suppose Assumption 1 holds. For any $x \in \mathbb{R}^d$ and $\gamma \in (0, \sqrt{d/C^*})$, as $n \to \infty$, the time-changed process $\hat{Z}^n$ under $\mathbb{P}_{x_n}^{Z^n} \otimes \mathbb{P}^{X^n}$ converges weakly to $\hat{Z}^\infty$ under $\mathbb{P}_x^{Z^\infty} \otimes \mathbb{P}^{X^\infty}$ in the Skorokhod space $D[0,\infty)$ equipped with J_1-topology.*

We give two examples of Theorem 2. In the first one, we introduce Liouville α-stable process and state the convergence of these processes. In the second one, we state the scaling limit of Liouville simple random walk on $\mathbb{Z}^2$ is Liouville Brownian motion.

Example 1.

For $\alpha \in (0,2]$, let m be the Lebesgue measure on $\mathbb{R}^d$, $Z^H = (\{Z_t^H\}_t, \{\mathbb{P}_x^{Z^H}\}_x)$ be an m-symmetric α-stable process on $\mathbb{R}$ on a probability space Ω^{Z^H}, where $H := \alpha/2 - 1/2$ is a Hurst parameter. Denote by $p^H(t,x,y)$ the continuous transition density function of Z^H and $g_\lambda^H(x,y) := \int_0^\infty p^H(t,x,y)e^{-\lambda t}dt$ a λ-order Green's function of Z^H for $\lambda > 0$. Let X^H be a Gaussian field on $\mathbb{R}$ on a probability space $(\Omega^{X^H}, \mathbb{P}^{X^H})$ whose covariance kernel is πg_λ^H. We define $\hat{Z}^H$ as the time-changed process of Z^H by μ^H and we call $\hat{Z}^H$ *Liouville α-stable process*. In particular, we call $\hat{Z}^0$ on $\mathbb{R}$ *Liouville Cauchy process*.

Then the following holds.

Theorem 3. *For $d = 1$, $\gamma \in [0,1)$ and any $x \in \mathbb{R}$, time-changed process $\hat{Z}^H$ converges weakly to $\hat{Z}^0$ under $\mathbb{P}^{X^H} \otimes \mathbb{P}_x^{Z^H}$ with respect to J_1-topology as $H \searrow 0$.*

GMC μ^H is non-degenerate for any $\gamma \in [0, \sqrt{2d})$ if and only if $H \geq 0$ holds. So we only consider cases of $d = 1, 2$ and $\alpha \in [d, 2]$. Moreover, if $d = 2$, there is the only case of $\alpha = 2$, so we may assume $d = 1$ because we will consider convergence as the parameter changes.

For $H > 0$, Z^H has positive capacity for a point. Then X^H has a bounded covariance kernel, so X^H is not only a random distribution, but also a random function. Moreover μ^H is absolutely continuous with respect to the Lebesgue measure, $\mathbb{P}^{X^H}$-almost surely. On the other hand, Z^0 is recurrent simply, so X^0 does not have a bounded covariance kernel, X^0 is not a random function and μ^H is singular with respect to the Lebesgue measure, $\mathbb{P}^{X^0}$-almost surely. Properties change significantly when $H = 0$, so we consider the convergence to $H = 0$.

Example 2.

Let $d = 2$, m be the Lebesgue measure on $\mathbb{R}^2$ and Z^∞ be Brownian motion on $\mathbb{R}^2$. Moreover, let $D_n := \frac{1}{\sqrt{n}}\mathbb{Z}^2$ and $\{\xi_i\}_{i=1}^\infty$ be random variables on $\mathbb{Z}^2$ having independent identically distributions with

$$\mathbb{P}(\xi_1 = (1,1)) = \mathbb{P}(\xi_1 = (1,-1)) = \mathbb{P}(\xi_1 = (-1,1)) = \mathbb{P}(\xi_1 = (-1,-1)) = \frac{1}{4}.$$

For convenience, let $\xi_0 := (0,0)$. Denote by $\{N^n\}_n$ an independent Poisson processes with rates n independent of $\{\xi_i\}_i$. For any $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2$, we define

$$x_n := \left(\frac{\lfloor \sqrt{n} x^{(1)} \rfloor}{\sqrt{n}}, \frac{\lfloor \sqrt{n} x^{(2)} \rfloor}{\sqrt{n}} \right),$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to a. Then $C_n(x) = n$ for any $x \in \mathbb{R}^2$.

For $n \in \mathbb{N}$, we define the process Z^n on $(\Omega^{Z^n}, \{\mathbb{P}_{x_n}^{Z^n}\}_{x \in \mathbb{R}^2})$ by

$$Z_t^n := \frac{1}{\sqrt{n}} \sum_{i=0}^{N_t^n} \xi_i + x$$

for a starting point $x \in \mathbb{R}^2$. The restriction of Z^n to the space $D_n + Z_0^n$ is called a continuous-time random walk.

The transition density functions p^n and the λ-order Green's functions g_λ^n of Z^n are written as

$$p^n(t, x_n, y_n) = \mathbb{P}_{x_n}(Z_t^n = y_n),$$
$$g_\lambda^n(x, y) = n \int_0^\infty e^{-\lambda t} \mathbb{P}_{x_n}(Z_t^n = y_n) dt,$$

for $n \in \mathbb{N}, \lambda > 0$ and $x, y \in \mathbb{R}^2$.

Denote by X^n the centred Gaussian field on $\mathbb{R}^2$ on a probability space $(\Omega^{X^n}, \mathbb{P}^{X^n})$ having the covariance kernel $\pi g_\lambda^n(x, y)$, and μ^n by its GMC with $\gamma \in [0, 2)$ for $n \in \mathbb{N} \cup \{\infty\}$. Let $\hat{Z}^n$ be the time-changed process of Z^n by μ^n and we call $\hat{Z}^n$ Liouville simple random walk on $\mathbb{Z}^2$.

Then the following theorem holds.

Theorem 4. *For any $x \in \mathbb{R}^2$ and $\gamma \in [0, \sqrt{2})$, time-changed processes $\hat{Z}^n$ converge weakly to Liouville Brownian motion $\hat{Z}^\infty$ under $\mathbb{P}^{X^n} \otimes \mathbb{P}_x^{Z^n}$ with the local uniform topology.*

Acknowledgements

I would like to express my deepest gratitude to Professor Takashi Kumagai and Professor David Alexander Croydon for continuous discussions and helpful advice. Without their guidance and encouragement, this work would not have existed.

This work was supported by JSPS KAKENHI Grant Number JP21J20251 and the Research Institute for Mathematical Sciences.

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