Embedded contact homology and its applications to 3-dimensional Reeb flows

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Preface

Research on periodic orbits of Reeb flows on three-dimensional contact manifolds is a fertile field involving symplectic geometry, dynamical systems, contact geometry, gauge theory, and low-dimensional topology. Among these, Embedded Contact Homology (ECH) is a central tool. In this thesis, we mainly use Embedded contact homology to investigate properties of periodic orbits of Reeb flows on three-dimensional contact manifolds. Let us briefly explain the historical background of ECH.

In the 1990s, the Seiberg-Witten equations were introduced to four-dimensional topology [Wit]. The equations are defined on four-dimensional spin-c manifolds, and the Seiberg-Witten invariants are defined by counting solutions to the equations. Seiberg-Witten invariants have since become a remarkable tool in the study of low-dimensional topology.

On the other hand, on symplectic manifolds, invariants based on counting pseudoholomorphic curves have been considered, and various studies have been conducted. Taubes showed that in four-dimensional symplectic manifolds, certain invariants defined by counting pseudoholomorphic curves called Gromov invariants and Seiberg-Witten invariants are equivalent [T1].

Now, consider the three-dimensional analog of these invariants. In 2000s, Kronheimer and Mrowka defined a three-dimensional version of the Seiberg-Witten invariants, called Monopole Floer Homology (HM), which is a kind of Floer homology [KM]. Embedded contact Homology (ECH) is an analogue of Floer homology for Gromov invariants on three-dimensional contact manifolds, introduced by M. Hutchings (c.f. [H3]). Similar to the four-dimensional case, Taubes showed that ECH and HM are isomorphic [T3]. Currently, ECH plays a significant role in the study of three-dimensional Reeb flows.

Here, we will explain the structure of this thesis. This thesis consists of three chapters. The following is the abstract of each chapter.

Chapter1: In this chapter, the basic concepts related to three-dimensional contact manifolds and ECH are summarized. The proofs are basically omitted. The content summarized here will be used throughout this entire thesis.

Chapter2: In this chapter, as a refinement of the Weinstein conjecture, the existence of a positive hyperbolic periodic orbit is discussed. At first, §2.1 provides a summary of the history of the Weinstein conjecture as background. Next, we explain the main result of this chapter. The main result gives the relationship between the existence of elliptic periodic orbits and a positive hyperbolic periodic orbit. In addition, its applications to lens spaces are discussed. The content of this chapter is based on [Shi1, Shi2].

Chapter3: Prior to the introduction of ECH, many significant researches were conducted. One of which was carried out by Hofer, Wysocki, and Zehnder. They constructed a Birkhoff section of disk type using pseudoholomorphic curves on a dynamically convex contact 3-sphere. In this chapter we discuss the existence of Birkhoff sections of disk type and their applications. Specifically, we show the existence of Birkhoff sections in certaine lens spaces and discuss the relation between Birkhoff sections and the ECH spectrum. In addition, we give their applications. The content of this chapter is based on [AsaShi, Shi3, Shi4]. Here, [AsaShi] is a joint work with Masayuki Asaoka.

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Contents

Chapter 1

3-dimensional Reeb flows and Embedded contact homology

1.1 Contact manifolds and Reeb flows

Here, we list basic definitions involving contact manifolds.

Definition 1.1.1. *A contact manifold (Y,λ) is a pair of oriented 2n+1 dimensional manifold Y and a 1-form λ on Y such that $\lambda \wedge (d\lambda)^n > 0$. We write $\xi = \mathrm{Ker}\lambda \subset TY$ and ξ is called the contact structure.*

In this thesis, we assume that Y is closed.

Definition 1.1.2. *Let (Y,λ) be a contact manifold. Then there is an unique vector field X_λ satisfying $\lambda(X_\lambda) = 1$ and $i_{X_\lambda} d\lambda = 0$. X_λ is called the Reeb vector field (or Reeb flow).*

Definition 1.1.3. *Let (Y,λ) be a contact manifold. A periodic orbit is a map $\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y$ for some $T_\gamma > 0$ such that $\dot{\gamma} = X_\lambda \circ \gamma$. T_γ is called the period of γ and satisfies $T_\gamma = \int_\gamma \lambda$.*

In this thesis, we assume two periodic orbits to be equivalent if they are the same as currents.

Let ϕ^t denote the flow of X_λ.

Definition 1.1.4. *Let (Y,λ) be a contact manifold. Let $\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y$ be a periodic orbit of X_λ.*

1. *γ is simple if the map $\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y$ is embedding.*

2. *γ is non-degenerate if the derivative of the first return map along the contact structure $d\phi^{T_\gamma}|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ has no eigenvalue 1.*

3. *γ is called elliptic if the all eigenvalues of $d\phi^{T_\gamma}|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ are on the unit circle $\{|z| = 1 | z \in \mathbb{C}\}$.*

4. *γ is called positive hyperbolic if the all eigenvalues of $d\phi^{T_\gamma}|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ are on $\mathbb{R}_{>0}$.*

5. *γ is called negative hyperbolic if the all eigenvalues of $d\phi^{T_\gamma}|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ are on $\mathbb{R}_{<0}$.*

Note that the map $d\phi^{T_\gamma}|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ as above is symplectic with respect to the symplectic form $d\lambda$ on $\xi_{\gamma(0)}$.

Definition 1.1.5. *A contact manifold (Y, λ) is non-degenerate if all periodic orbits of X_γ are non-degenerate.*

If (Y, λ) is of 3-dimension, a periodic orbit γ is either positive hyperbolic, negative hyperbolic or elliptic. In addition, if γ is non-degenerate, these conditions do not overlap each other.

1.2 Conley-Zehnder index and winding number

From now on, we assume (Y, λ) to be 3-dimensional.

Here, we recall Conley-Zehnder index and its properties introduced in [HWZ2]. We focus on the Conley-Zehnder index with respect to 2-dimensional symplectic paths. The contents are based on [HWZ2, HWZ4].

Consider a smooth arc $[0,1] \ni t \mapsto S(t)$ of 2-dimensional symmetric matrices. Then we can define a self-adjoint operator L_S in $L^2(S^1, \mathbb{C})$ whose domain is $W^{1,2}(S^1, \mathbb{C})$ by

$$L_S = -i\partial_t - S(t). \tag{1.1}$$

Since $W^{1,2}(S^1, \mathbb{C}) \hookrightarrow L^2(S^1, \mathbb{C})$ is compact, its spectrum $\sigma(L_S)$ consists of eigenvalues of L_S and is on $\mathbb{R}$. In addition, $\sigma(L_S)$ is a countable set and has no accumulation point other than $\pm\infty$.

For $\eta \in \sigma(L_S)$, take an eigenfunction $e_\eta : S^1 \to \mathbb{C}$. Let $w(S,\eta) \in \mathbb{Z}$ denote the winding number of e_η. Note that $w(S,\eta)$ is independent of e_η. If $\eta_1 \leq \eta_2$, then $w(S,\eta_1) \leq w(S,\eta_2)$. Moreover for any $k \in \mathbb{Z}$, there are precisely two eigenvalues (multiplicities counted) with winding number k. In particular, this means that $w(S,\cdot) : \sigma(L_S) \to \mathbb{Z}$ is monotone and surjective.

Consider a a smooth path $\varphi : \mathbb{R} \to Sp(1)$ in 2-dimensional symplectic matrices with $\varphi(0) = id$ and $\varphi(t+1) = \varphi(t)\varphi(1)$ for any $t \in \mathbb{R}$. Define a smooth arc $[0,1] \ni t \mapsto S_\varphi(t)$ of symmetric matrices by $S_\varphi(t) = -i\dot{\varphi}(t)\varphi(t)$.

Definition 1.2.1. *For φ as above, let $\eta^< := \max\{\sigma(L_{S_\phi}) \cap (-\infty, 0)\}$ and $\eta^\geq := \min\{\sigma(L_{S_\varphi}) \cap [0, +\infty)\}$. The Conley-Zehnder index of φ is defined as*

$$\mu_{CZ}(\phi) := w(S_\varphi, \eta^<) + w(S_\varphi, \eta^\geq). \tag{1.2}$$

We note that it is obvious that μ_{CZ} is lower semi-continuous.

For $k \in \mathbb{Z}_{>0}$, define $\rho_k : \mathbb{R} \to \mathbb{R}$ as $t \mapsto kt$.

Proposition 1.2.2. *Consider a smooth path $\varphi : \mathbb{R} \to Sp(1)$ in symplectic matrices with $\varphi(0) = id$ and $\varphi(t+1) = \varphi(t)\varphi(1)$ for any $t \in \mathbb{R}$.*

(1). If $\mu_{CZ}(\varphi) = 2n$ for $n \in \mathbb{Z}$, then $\mu_{CZ}(\varphi \circ \rho_k) = 2kn$ for every $k \in \mathbb{Z}_{>0}$.

(2). If $\mu_{CZ}(\varphi) \geq 3$, then $\mu_{CZ}(\varphi \circ \rho_k) \geq 2k + 1$ for every $k \in \mathbb{Z}_{>0}$.

Proof of Proposition 1.2.2. Here, we give a proof of (1). For the proof of (2), see [HWZ4, Theorem 3.6] Let $e_\eta : S^1 \to \mathbb{C}$ be an eigenfunction of L_{S_φ} with eigenvalue $\eta \in \sigma(L_{S_\varphi})$. Then it is obvious that $L_{S_{\varphi\circ\rho_k}} e_\eta(kt) = \eta e_\eta(kt)$. Therefore $\sigma(L_{S_\varphi}) \subset \sigma(L_{S_{\varphi\circ\rho_k}})$ and $w(S_{\varphi\circ\rho_k}, \eta) = kw(S_\varphi, \eta)$ for every $\eta \in \sigma(L_{S_\varphi})$. In addition, we note that $w(S_\phi, \eta^<) = w(S_\varphi, \eta^\geq)$ or $w(S_\phi, \eta^<) + 1 = w(S_\varphi, \eta^\geq)$ because the map $w(S_\varphi, \cdot) : \sigma(L_{S_\varphi}) \to \mathbb{Z}$ is monotone and surjective.

Supppose that $\mu_{CZ}(\varphi) = 2n$ for $n \in \mathbb{Z}$. Since $\mu_{CZ}(\varphi) = w(S_\varphi, \eta^<) + w(S_\varphi, \eta^\geq)$ is even, we have $w(S_\varphi, \eta^<) = w(S_\varphi, \eta^\geq) = n$. Fix $k \in \mathbb{Z}_{>0}$. Then we have $\eta^< := \max\{\sigma(L_{S_{\varphi\circ\rho_k}}) \cap (-\infty, 0)\} = \max\{\sigma(L_{S_\varphi}) \cap (-\infty, 0)\}$ and $\eta^\geq := \min\{\sigma(L_{S_{\varphi\circ\rho_k}}) \cap [0, +\infty)\} = \min\{\sigma(L_{S_\varphi}) \cap [0, +\infty)\}$. This follows easily from $w(S_{\varphi\circ\rho_k}, \eta^<) = w(S_{\varphi\circ\rho_k}, \eta^\geq) = kn$ and the monotonicity of $w(S_{\varphi\circ\rho_k}, \cdot)$. Therefore we have $\mu_{CZ}(\varphi \circ \rho_k) = 2kn$.

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Consider a periodic orbit $\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y$ of (Y, λ) and a symplectic trivialization $\tau : \gamma^*\xi \to \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{C}$. Then we have a symplectic path $\mathbb{R} \ni t \mapsto \phi_{\gamma,\tau}(t) := \tau(\gamma(T_\gamma t)) \circ d\phi^{tT_\gamma}|_\xi \circ \tau^{-1}(\gamma(0))$ which satisfies $\phi_{\gamma,\tau}(t+1) = \phi_{\gamma,\tau}(t)\phi_{\gamma,\tau}(1)$ for any $t \in \mathbb{R}$.

Now, we define the Conley-Zender index of γ with respect to a trivialization τ as

$$\mu_\tau(\gamma) := \mu_{CZ}(\phi_{\gamma,\tau}). \tag{1.3}$$

Note that μ_τ is independent of the choice of a trivialization in the same homotopy class of τ.

Let $\mathcal{P}(\gamma)$ denote the set of homotopy classes of symplectic trivializations $\gamma^*\xi \to \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}^2$. Note that a symplectic trivialization $\tau \in \mathcal{P}(\gamma)$ induces naturally a symplectic trivialization on $(\gamma^p)^*\xi$ and we use the same notation $\tau \in \mathcal{P}(\gamma^p)$ for the induced trivialization on $(\gamma^p)^*\xi$ if there is no confusion.

For each $\tau \in \mathcal{P}(\gamma)$, take sections $Z_\tau, W_\tau : \mathbb{R}/T_\gamma\mathbb{Z} \to \gamma^*\xi$ so that the map $\gamma^*\xi \ni aZ_\tau + bW_\tau \mapsto a + ib \in \mathbb{C}$ gives a symplectic trivialization of the homotopy class τ. For $\tau, \tau' \in \mathcal{P}(\gamma)$, we define $\mathrm{wind}(\tau, \tau') \in \mathbb{Z}$ as follows. Let $a_{\tau,\tau'}, b_{\tau,\tau'} : \mathbb{R}/T_\gamma\mathbb{Z} \to \mathbb{R}$ be continuous functions such that $Z_\tau(t) = a_{\tau,\tau'}(t)Z_{\tau'}(t) + b_{\tau,\tau'}(t)W_{\tau'}(t)$ for $t \in \mathbb{R}/T_\gamma\mathbb{Z}$. Let $\theta : [0, T_\gamma] \to \mathbb{R}$ be a continuous function so that $a_{\tau,\tau'}(t) + ib_{\tau,\tau'}(t) \in \mathbb{R}_+e^{i\theta(t)}$ for $t \in [0, T_\gamma]$. Then we define

$$\mathrm{wind}(\tau, \tau') := \frac{\theta(T_\gamma) - \theta(0)}{2\pi} \in \mathbb{Z}. \tag{1.4}$$

It is obvious that $\mathrm{wind}(\tau, \tau')$ is independent of the choices $Z_{\tau^{(\prime)}}, W_{\tau^{(\prime)}}$. The following property is well-known and important.

Proposition 1.2.3. *Let γ be a periodic orbit in (Y, λ). For $\tau \in \mathcal{P}(\gamma)$. Then for any $\tau, \tau' \in \mathcal{P}(\gamma)$, $\mu_\tau(\gamma) + 2\mathrm{wind}(\tau, \tau') = \mu_{\tau'}(\gamma)$.*

If γ^n is non-degenerate for every $n \in \mathbb{Z}_{>0}$, the Conley-Zehnder index behaves good as follows.

Proposition 1.2.4. *Let γ be a orbit such that γ^n is non-degenerate for every $n \in \mathbb{Z}_{>0}$. Fix a trivialization τ of the contact plane over γ. Consider the Conley-Zehnder indices of the multiple covers with respect to τ. Write $\mu_\tau(\gamma^n) := \mu_{CZ}(\phi_{\gamma,\tau} \circ \rho_n)$.*

(1). If γ is hyperbolic, $\mu_\tau(\gamma^n) = n\mu_\tau(\gamma)$ for every $n \in \mathbb{Z}_{>0}$.

(2). If γ is elliptic, there is $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_\tau(\gamma^n) = 2\lfloor n\theta \rfloor + 1$ for every $n \in \mathbb{Z}_{>0}$.

We call θ the monodromy angle of γ with respect to τ.

For more properties of the Conley-Zehnder index, see [HWZ2, HWZ4].

1.3 Embedded contact homology

Here, we recall the basic construction and properties of ECH.

1.3.1 The definitions and properties of Embedded contact homology

Let (Y, λ) be a non-degenerate contact three manifold. For $\Gamma \in H_1(Y;\mathbb{Z})$, Embedded contact homology $\mathrm{ECH}(Y, \lambda, \Gamma)$ is defined. Here, we define the chain complex $(\mathrm{ECC}(Y, \lambda, \Gamma), \partial)$. For the purpose, at first we introduce the notions of ECH generator and ECH index.

In this thesis, we consider ECH over $\mathbb{Z}/2\mathbb{Z} = \mathbb{F}$.

Definition 1.3.1 ([H1, Definition 1.1]). *An orbit set $\alpha = \{(\alpha_i, m_i)\}$ is a finite pair of distinct simple periodic orbit α_i with positive integer m_i. If $m_i = 1$ whenever α_i is hyperboric orbit, then $\alpha = \{(\alpha_i, m_i)\}$ is called an ECH generator.*

Set $[\alpha] = \sum m_i[\alpha_i] \in H_1(Y)$. For two orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ with $[\alpha] = [\beta]$, we define $H_2(Y, \alpha, \beta)$ to be the set of relative homology classes of 2-chains Z in Y with $\partial Z = \sum_i m_i\alpha_i - \sum_j m_j\beta_j$. This is an affine space over $H_2(Y)$. From now on. we fix a trivialization of contact plane ξ over each simple orbit and write it by τ.

Definition 1.3.2 ([H1, Definition2.2]). *Let $Z \in H_2(Y; \alpha, \beta)$. A representative of Z is an immersed oriented compact surface S in $[0, 1] \times Y$ such that:*

1. *∂S consists of positively oriented (resp. negatively oriented) covers of $\{1\} \times \alpha_i$ (resp. $\{0\} \times \beta_j$) whose total multiplicity is m_i (resp. n_j).*

2. *$[\pi(S)] = Z$, where $\pi : [0, 1] \times Y \to Y$ denotes the projection.*

3. *S is embedded in $(0, 1) \times Y$, and S is transverse to $\{0, 1\} \times Y$.*

Definition 1.3.3 ([H1, §8.2]). *Let α_1, β_1, α_2 and β_2 be orbit sets with $[\alpha_1] = [\beta_1]$ and $[\alpha_2] = [\beta_2]$. For a trivialization τ, we can define*

$$Q_\tau : H_2(Y; \alpha_1, \beta_1) \times H_2(Y; \alpha_2, \beta_2) \to \mathbb{Z} \tag{1.5}$$

by $Q_\tau(Z_1, Z_2) = -l_\tau(S_1, S_2) + \#(S_1 \cap S_2)$ where S_1, S_2 are representatives of Z_1, Z_2 for $Z_1 \in H_2(Y; \alpha_1, \beta_1)$, $Z_2 \in H_2(Y; \alpha_2, \beta_2)$ respectively, $\#(S_1 \cap S_2)$ is their algebraic intersection number and l_τ is a kind of crossing number (see [H1, §8.3] for details).

Definition 1.3.4 ([H1, Definition 1.5]). *For $Z \in H_2(Y, \alpha, \beta)$, we define*

$$I(\alpha, \beta, Z) := c_1(\xi|_Z, \tau) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} \mu_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} \mu_\tau(\beta_j^k). \tag{1.6}$$

We call $I(\alpha, \beta, Z)$ an ECH index. Here, μ_τ is the Conely Zhender index with respect to τ and $c_1(\xi|_Z, \tau)$ is a reative Chern number and $Q_\tau(Z) = Q_\tau(Z, Z)$. Moreover this is independent of τ (see [H1] for more details).

Proposition 1.3.5 ([H1, Proposition 1.6]). *The ECH index I has the following properties.*

1. *For orbit sets α, β, γ with $[\alpha] = [\beta] = [\gamma] = \Gamma \in H_1(Y)$ and $Z \in H_2(Y, \alpha, \beta)$, $Z' \in H_2(Y, \beta, \gamma)$,*

$$I(\alpha, \beta, Z) + I(\beta, \gamma, Z') = I(\alpha, \gamma, Z + Z'). \tag{1.7}$$

2. *For $Z, Z' \in H_2(Y, \alpha, \beta)$,*

$$I(\alpha, \beta, Z) - I(\alpha, \beta, Z') =< c_1(\xi) + 2\mathrm{PD}(\Gamma), Z - Z' > . \tag{1.8}$$

3. *If α and β are ECH generators,*

$$I(\alpha, \beta, Z) = \epsilon(\alpha) - \epsilon(\beta) \mod 2. \tag{1.9}$$

Here, $\epsilon(\alpha)$, $\epsilon(\beta)$ are the numbers of positive hyperbolic orbits in α, β respectively.

For $\Gamma \in H_1(Y)$, we define $\mathrm{ECC}(Y, \lambda, \Gamma)$ as freely generated module over $\mathbb{Z}/2$ by ECH generators α such that $[\alpha] = \Gamma$. That is

$$\mathrm{ECC}(Y, \lambda, \Gamma) := \bigoplus_{\alpha:\text{ECH generator with } [\alpha]=\Gamma} \mathbb{Z}_2\langle\alpha\rangle. \tag{1.10}$$

To define the differential $\partial : \mathrm{ECC}(Y, \lambda, \Gamma) \to \mathrm{ECC}(Y, \lambda, \Gamma)$, we pick a generic almost complex structure J on $\mathbb{R} \times Y$ which satisfies

1. $\mathbb{R}$-invariant
2. $J(\frac{d}{ds}) = X_\lambda$
3. $J\xi = \xi$

We consider J-holomorphic curves $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$ where the domain (Σ, j) is a punctured compact Riemann surface. Here the domain Σ is not necessarily connected. Let γ be a (not necessarily simple) Reeb orbit. If a puncture of u is asymptotic to $\mathbb{R} \times \gamma$ as $s \to \infty$, we call it a positive end of u at γ and if a puncture of u is asymptotic to $\mathbb{R} \times \gamma$ as $s \to -\infty$, we call it a negative end of u at γ (see [H1] for more details).

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$ be orbit sets. Let $\mathcal{M}^J(\alpha, \beta)$ denote the set of J-holomorphic curves with positive ends at covers of α_i with total covering multiplicity m_i, negative ends at covers of β_j with total covering multiplicity n_j, and no other punctures. Moreover, in $\mathcal{M}^J(\alpha, \beta)$, we consider two J-holomorphic curves to be equivalent if they represent the same current in $\mathbb{R} \times Y$.

For $u \in \mathcal{M}^J(\alpha, \beta)$, we naturally have $[u] \in H_2(Y; \alpha, \beta)$ and set $I(u) = I(\alpha, \beta, [u])$. Moreover we define

$$\mathcal{M}^J_k(\alpha, \beta) := \{\, u \in \mathcal{M}^J(\alpha, \beta) \,|\, I(u) = k \,\} \tag{1.11}$$

Under this notations, we define $\partial_J : \mathrm{ECC}(Y, \lambda, \Gamma) \to \mathrm{ECC}(Y, \lambda, \Gamma)$ as follows.

For an ECH generator α with $[\alpha] = \Gamma$, we define

$$\partial_J \alpha = \sum_{\beta\text{: ECH generator with } [\beta]=\Gamma} \#(\mathcal{M}^J_1(\alpha, \beta)/\mathbb{R}) \cdot \beta. \tag{1.12}$$

Note that the above counting is well-defined and $\partial_J \circ \partial_J$. We can see the reason of the former in Proposition 1.3.10 and the later was proved in [HT1] and [HT2]. Moreover, the homology defined by ∂_J does not depend on J (see Theorem 1.3.11, or see [T3]).

Before we get to the next subsection, recall (Fredholm) index.

For $u \in \mathcal{M}^J(\alpha, \beta)$, the its (Fredholm) index is defined by

$$\mathrm{ind}(u) := -\chi(u) + 2c_1(\xi|_{[u]}, \tau) + \sum_k \mu_\tau(\gamma_k^+) - \sum_l \mu_\tau(\gamma_l^-). \tag{1.13}$$

Here $\{\gamma_k^+\}$ is the set consisting of (not necessarilly simple) all positive ends of u and $\{\gamma_l^-\}$ is that one of all negative ends. Note that for generic J, if u is connected and somewhere injective, then the moduli space of J-holomorphic curves near u is a manifold of dimension $\mathrm{ind}(u)$ (see [HT1, Definition 1.3]).

1.3.2 J-holomorphic curves with small ECH index and partition conditions of multiplicities

For $\theta \in \mathbb{R}\backslash\mathbb{Q}$, we define S_θ to be the set of positive integers q such that $\frac{\lceil q\theta \rceil}{q} < \frac{\lceil q'\theta \rceil}{q'}$ for all $q' \in \{1,\ 2, ...,\ q-1\}$ and write $S_\theta = \{q_0 = 1,\ q_1,\ q_2,\ q_3, ...\}$ in increasing order. Also $S_{-\theta} = \{p_0 = 1,\ p_1,\ p_2,\ p_3, ...\}$.

Proposition 1.3.6 ([HT3, Proof of Lemma 3.3], and [H1, Proof of Remark 4.4]). *For $\theta \in \mathbb{R}\backslash\mathbb{Q}$,*

1. *$q_{i+1} - q_i$ (resp. $p_{i+1} - p_i$) are nondecreasing with respect to i and some elements of $S_{-\theta}$ (resp. S_θ).*
2. *$S_\theta \cap S_{-\theta} = \{1\}$,*
3. *$q_{i+1} - q_i \to \infty$ (resp. $p_{i+1} - p_i \to \infty$) as $i \to \infty$.*

Definition 1.3.7 ([HT1, Definition 7.1], or [H1, §4]). *For non negative integer M, we inductively define the incoming partition $P_\theta^{\mathrm{in}}(M)$ as follows.*

For $M = 0$, $P_\theta^{\mathrm{in}}(0) = \emptyset$ and for $M > 0$,

$$P_\theta^{\mathrm{in}}(M) := P_\theta^{\mathrm{in}}(M-a) \cup (a) \tag{1.14}$$

where $a := \max(S_\theta \cap \{1,\ 2, ...,\ M\})$. Define outgoing partition

$$P_\theta^{\mathrm{out}}(M) := P_{-\theta}^{\mathrm{in}}(M). \tag{1.15}$$

Definition 1.3.8 ([HT1, Definition 7.11], or cf.[H1, §4]). *For a simple Reeb orbit γ and positive integer M, define two partitions of M, the incoming partition $P_\gamma^{\mathrm{in}}(M)$ and the outgoing partition $P_\gamma^{\mathrm{out}}(M)$ as follows.*

1. *If γ is positive hyperbolic, then*

$$P_\gamma^{\mathrm{in}}(M) = P_\gamma^{\mathrm{out}} := (1, ..., 1) \tag{1.16}$$

2. *If γ is negative hyperbolic, then*

$$P_\gamma^{\mathrm{in}}(M) = P_\gamma^{\mathrm{out}}(M) := \begin{cases} (2, ..., 2) & \text{if } M \text{ is even,} \\ (2, ..., 2,\ 1) & \text{if } M \text{ is odd,} \end{cases} \tag{1.17}$$

3. If γ is elliptic, then

$$P_\gamma^{\text{in}}(M) := P_\theta^{\text{in}}(M), \quad P_\gamma^{\text{out}}(M) := P_\theta^{\text{out}}(M). \tag{1.18}$$

where θ is the rotation number of γ up to $\mathbb{Z}$.

The standard ordering convention for $P_\gamma^{\text{in}}(M)$ or $P_\gamma^{\text{out}}(M)$ is to list the entries in "nonincreasing" order.

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$. For $u \in \mathcal{M}^J(\alpha, \beta)$, it can be uniquely written as $u = u_0 \cup u_1$ where u_0 are unions of all components which maps to $\mathbb{R}$-invariant cylinders in u and u_1 is the rest of u.

For $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$, let $P_{\alpha_i}^+$ denote the set consisting of the multiplicities of the positive ends of u_1 at covers of α_i. Define $P_{\beta_j}^-$ analogously for the negative end.

Definition 1.3.9 ([HT1, Definition 7.13])**.** *$u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$ is admissible if*

1. *u_1 is embedded and does not intersect u_0.*

2. *For each simple Reeb orbit α_i in α (resp. β_j in β), under the standerd ordering convention, $P_{\alpha_i}^+$(resp. $P_{\beta_j}^-$) is an initial segment of $P_{\alpha_i}^{\text{out}}(m_i)$(resp. $P_{\beta_j}^{\text{in}}(n_j)$).*

Proposition 1.3.10 ([HT1, Proposition 7.15])**.** *Suppose that J is generic and $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$. Then*

1. *$I(u) \geq 0$*

2. *If $I(u) = 0$, then $u_1 = \emptyset$*

3. *If $I(u) = 1$, then u is admissible and* $\text{ind}(u_1) = 1$.

4. *If $I(u) = 2$ and α and β are ECH generators, then u is admissible and* $\text{ind}(u_1) = 2$.

1.3.3 The gradings and the U-map

By (1.28) and (1.27), we can see that if $c_1(\xi) + 2\text{PD}(\Gamma)$ is divisible by d, ECH generators have relative $\mathbb{Z}/d$-grading. So we can decompose $\text{ECC}(Y, \lambda, \Gamma)$

as direct sum by $\mathbb{Z}/d$-grading.

$$\mathrm{ECC}(Y,\lambda,\Gamma)=\bigoplus_{*:\,\mathbb{Z}/d\text{ grading}}\mathrm{ECC}_*(Y,\lambda,\Gamma). \tag{1.19}$$

Note that if $c_1(\xi)+2\mathrm{PD}(\Gamma)$ is torsion, there exists the relative $\mathbb{Z}$-grading. The same as (1.19), we can see that

$$\mathrm{ECH}(Y,\lambda,\Gamma):=\bigoplus_{*:\,\mathbb{Z}\text{ grading}}\mathrm{ECH}_*(Y,\lambda,\Gamma). \tag{1.20}$$

Let Y be connected. Then there is degree-2 map U.

$$U:\mathrm{ECH}_*(Y,\lambda,\Gamma)\to\mathrm{ECH}_{*-2}(Y,\lambda,\Gamma). \tag{1.21}$$

To define this, choose a generic base point $z\in Y$ which is espetially not on the image of any Reeb orbit and let J be generic. Then define a map

$$U_{J,z}:\mathrm{ECC}_*(Y,\lambda,\Gamma)\to\mathrm{ECC}_{*-2}(Y,\lambda,\Gamma). \tag{1.22}$$

by

$$U_{J,z}\langle\alpha\rangle=\sum_{\beta:\text{ ECHgenerator with }[\beta]=\Gamma}\#\{\,u\in\mathcal{M}_2^J(\alpha,\beta)/\mathbb{R})\,|\,(0,z)\in u\,\}\cdot\langle\beta\rangle. \tag{1.23}$$

The above map $U_{J,z}$ is a chain map, and we define the U map as the induced map on homology. Under the assumption, this map is independent on z (for a generic J). See [HT3, §2.5] for more details. Moreover, in the same reason as ∂, $U_{J,z}$ does not depend on J (see Theorem 1.3.11, and see [T3]).

In this paper, we choose a suitable generic J as necessary (Specifically, we choose a generic J so that $U_{J,z}$ is well-defined for some countable sequences z appearing in the future discussions).

The next isomorphism is important.

Theorem 1.3.11 ([T3])**.** *For each $\Gamma\in H_1(Y)$, there is an isomorphism*

$$\mathrm{ECH}_*(Y,\lambda,\Gamma)\cong\widetilde{HM}_*(-Y,\mathfrak{s}(\xi)+\mathrm{PD}(\Gamma))\cong\widehat{HM}^{-*}(Y,\mathfrak{s}(\xi)+\mathrm{PD}(\Gamma)) \tag{1.24}$$

of relatively $\mathbb{Z}/d\mathbb{Z}$-graded abelian groups. Here d is the divisibility of $c_1(\xi) + 2\mathrm{PD}(\Gamma)$ in $H_1(Y)$ mod torsion and $\mathfrak{s}(\xi)$ is the spin-c structure associated to the oriented 2–plane field as in [KM].

Moerover, the above isomorphism interchanges the map U in (1.21) with the map

$$U_{\dagger} : \widetilde{HM}_*(-Y, \mathfrak{s}(\xi) + \mathrm{PD}(\Gamma)) \longrightarrow \widetilde{HM}_{*-2}(-Y, \mathfrak{s}(\xi) + \mathrm{PD}(\Gamma)) \quad (1.25)$$

defined in [KM].

Here $\widetilde{HM}_*(-Y, \mathfrak{s}(\xi) + \mathrm{PD}(\Gamma))$ is a version of Seiberg-Witten Floer homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients defined by Kronheimer-Mrowka [KM].

1.3.4 J_0 index and topological complexity of J-holomorphic curve

Here, we recall the J_0 index.

Definition 1.3.12 ([HT3, §3.3]). *Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be orbit sets with $[\alpha] = [\beta]$. For $Z \in H_2(Y, \alpha, \beta)$, we define*

$$J_0(\alpha, \beta, Z) := -c_1(\xi|_Z, \tau) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i - 1} \mu_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j - 1} \mu_\tau(\beta_j^k). \quad (1.26)$$

Proposition 1.3.13 ([HT3, §3.3] [CHR, §2.6]). *The index J_0 has the following properties.*

1. *For orbit sets α, β, γ with $[\alpha] = [\beta] = [\gamma] = \Gamma \in H_1(Y)$ and $Z \in H_2(Y, \alpha, \beta)$, $Z' \in H_2(Y, \beta, \gamma)$,*

$$J_0(\alpha, \beta, Z) + J_0(\beta, \gamma, Z') = J_0(\alpha, \gamma, Z + Z'). \quad (1.27)$$

2. *For $Z, Z' \in H_2(Y, \alpha, \beta)$,*

$$J_0(\alpha, \beta, Z) - J_0(\alpha, \beta, Z') = < -c_1(\xi) + 2\mathrm{PD}(\Gamma), Z - Z' > . \quad (1.28)$$

Definition 1.3.14 ([H3, just befor Proposition 5.8]). *Let $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$. Suppose that u_1 is somewhere injective. Let n_i^+ be the number of positive ends of u_1 which are asymptotic to α_i, plus 1 if u_0 includes the trivial cylinder $\mathbb{R} \times \alpha_i$ with some multiplicity. Likewise, let n_j^- be the number of negative ends of u_1 which are asymptotic to β_j, plus 1 if u_0 includes the trivial cylinder $\mathbb{R} \times \beta_j$ with some multiplicity.*

Write $J_0(u) = J_0(\alpha, \beta, [u])$.

Proposition 1.3.15 ([HT3, Lemma 3.5] [H3, Proposition 5.8]). *Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be ECH generators, and let $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$. Then*

$$-\chi(u_1) + \sum_i (n_i^+ - 1) + \sum_j (n_j^- - 1) \leq J_0(u) \tag{1.29}$$

If u is counted by the ECH differential or the U-map, then the above equality holds. Note that $J_0(u) \geq -1$ in any case.

1.3.5 ECH spectrum and the Weyl law

The action of an orbit set $\alpha = \{(\alpha_i, m_i)\}$ is defined by

$$A(\alpha) = \sum m_i A(\alpha_i) = \sum m_i \int_{\alpha_i} \lambda. \tag{1.30}$$

For any $L > 0$, $\mathrm{ECC}^L(Y, \lambda, \Gamma)$ denotes the subspace of $\mathrm{ECC}(Y, \lambda, \Gamma)$ which is generated by ECH generators whose actions are less than L. In the same way, $(\mathrm{ECC}^L(Y, \lambda, \Gamma), \partial)$ becomes a chain complex and the homology group $\mathrm{ECH}^L(Y, \lambda, \Gamma)$ is obtained. Here, we use the fact that if two ECH generators $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$ have $A(\alpha) \leq A(\beta)$, then the coefficient of β in $\partial\alpha$ is 0 because of the positivity of J holomorphic curves over $d\lambda$ and the fact that $A(\alpha) - A(\beta)$ is equivalent to the integral value of $d\lambda$ over J-holomorphic punctured curves which is asymptotic to α at $+\infty$, β at $-\infty$.

It follows from the construction that there exists a canonical homomorphism $i_L : \mathrm{ECH}^L(Y, \lambda, \Gamma) \to \mathrm{ECH}(Y, \lambda, \Gamma)$. In addition, for non-degenerate contact forms λ, λ' with $\mathrm{Ker}\lambda = \mathrm{Ker}\lambda' = \xi$, there is a canonical isomorphism $\mathrm{ECH}(Y, \lambda, \Gamma) \to \mathrm{ECH}(Y, \lambda', \Gamma)$ defined by the cobordism maps for product cobordisms (see [H2]). Therefore we may consider a pair of a group $\mathrm{ECH}(Y, \xi, \Gamma)$ and maps $j_\lambda : \mathrm{ECH}(Y, \lambda, \Gamma) \to \mathrm{ECH}(Y, \xi, \Gamma)$ for any non-degenerate contact form λ with $\mathrm{Ker}\lambda = \xi$ such that $\{j_\lambda\}_\lambda$ is compatible with the canonical map $\mathrm{ECH}(Y, \lambda, \Gamma) \to \mathrm{ECH}(Y, \lambda', \Gamma)$.

Definition 1.3.16 ([H2, Definition 4.1, cf. Definition 3.4]). *Let Y be a closed oriented three manifold with a non-degenerate contact form λ with $\mathrm{Ker}\lambda = \xi$ and $\Gamma \in H_1(Y, \mathbb{Z})$. If $0 \neq \sigma \in \mathrm{ECH}(Y, \xi, \Gamma)$, define*

$$c_\sigma^{\mathrm{ECH}}(Y, \lambda) = \inf\{L > 0 \,|\, \sigma \in \mathrm{Im}(j_\lambda \circ i_L : \mathrm{ECH}^L(Y, \lambda, \Gamma) \to \mathrm{ECH}(Y, \xi, \Gamma))\}$$

If λ is degenerate, define

$$c_\sigma^{\mathrm{ECH}}(Y,\lambda) = \sup\{c_\sigma(Y, f_-\lambda)\} = \inf\{c_\sigma(Y, f_+\lambda)\}$$

where the supremum is over functions $f_- : Y \to (0,1]$ such that $f\lambda$ is non-degenerate and the infimum is over smooth functions $f_+ : Y \to [1,\infty)$ such that $f_-\lambda$ is non-degenerate. Note that $c_\sigma^{\mathrm{ECH}}(Y,\lambda) < \infty$ and this definition makes sense. See [H2, Definition 4.1, Definition 3.4, §2.3] for more details.

In the case that $c_1(\xi) + 2\mathrm{PD}(\Gamma)$ is torsion, the above spectrum recover the volume of $\mathrm{Vol}(Y,\lambda) = \int_Y \lambda \wedge d\lambda$.

Theorem 1.3.17 ([CHR, Theorem 1.3])**.** *Let Y be a closed connected three-manifold with a contact form λ, let $\Gamma \in H_1(Y)$ with $c_1(\xi) + 2\mathrm{PD}(\gamma)$ torsion, and let I be any refinement of the relative $\mathbb{Z}$-grading on $\mathrm{ECH}(Y,\lambda,\Gamma)$ to an absolute $\mathbb{Z}$-grading. Then for any sequence of nonzero homogeneous classes $\{\sigma_k\}_{1\geq k}$ in $\mathrm{ECH}(Y,\lambda,\Gamma)$ with $\lim_{k\to\infty} I(\sigma) = +\infty$, we have*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} = \mathrm{Vol}(Y,\lambda). \tag{1.31}$$

1.3.6 A sequence of ECH spectrum and lens spaces

Definition 1.3.18. *[H2, Definition 4.3] If (Y,λ) is a closed connected contact three-manifold with $c(\xi) \neq 0$, and if k is a nonnegative integer, define*

$$c_k^{\mathrm{ECH}}(Y,\lambda) := \min\{c_\sigma^{\mathrm{ECH}}(Y,\lambda) |\, \sigma \in \mathrm{ECH}(Y,\xi,0),\ U^k\sigma = c(\xi)\} \tag{1.32}$$

The sequence $\{c_k^{\mathrm{ECH}}(Y,\lambda)\}$ is called the ECH spectrum of (Y,λ).

Definition 1.3.19. *[H2, Subsection 2.2] Let Y be a closed oriented three manifold with a non-degenerate contact form λ with ξ. Then there is a canonical element called the ECH contact invariant,*

$$c(\xi) := \langle \emptyset \rangle \in \mathrm{ECH}(Y,\lambda,0). \tag{1.33}$$

where $\langle\emptyset\rangle$ is the equivalent class containing $\emptyset$. Note that $\partial_J\emptyset = 0$ because of the maximal principle. In addition, $c(\xi)$ depends only on the contact structure ξ.

Proposition 1.3.20. *Let (Y,λ) be a closed connected contact three manifold.*

(1).

$$0 = c_0^{\mathrm{ECH}}(Y,\lambda) < c_1^{\mathrm{ECH}}(Y,\lambda) \leq c_2^{\mathrm{ECH}}(Y,\lambda) ... \leq \infty.$$

(2). For any $a > 0$ and positive integer k,

$$c_k^{\mathrm{ECH}}(Y, a\lambda) = a c_k^{\mathrm{ECH}}(Y,\lambda).$$

(3). Let $f_1, f_2 : Y \to (0,\infty)$ be smooth functions with $f_1(x) \leq f_2(x)$ for every $x \in Y$. Then

$$c_k^{\mathrm{ECH}}(Y, f_1\lambda) \leq c_k^{\mathrm{ECH}}(Y, f_2\lambda).$$

(4). Suppose $c_k^{\mathrm{ECH}}(Y, f\lambda) < \infty$. The map

$$C^\infty(Y, \mathbb{R}_{>0}) \ni f \mapsto c_k^{\mathrm{ECH}}(Y, f\lambda) \in \mathbb{R}$$

is continuous in C^0-topology on $C^\infty(Y, \mathbb{R}_{>0})$.

Proof of Proposition 1.3.20. They follow from the properties of ECH. See [H2]. □

Now, we focus on lens spaces. Recall the standard contact structures on lens spaces. Let $p \geq q > 0$ be mutually prime. The standard contact structure ξ_{std} is defined as follows. Consider a contact 3-sphere $(\partial B(1), \lambda_0|_{\partial B(1)})$ where $\partial B(1) = \{(z_1,z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$, $\lambda_0 = \frac{i}{2}\sum_{i=1,2}(z_i d\bar{z}_i - \bar{z}_i dz_i)$. The action $(z_1,z_2) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2)$ preserves $(\partial B(1), \lambda_0|_{\partial B(1)})$ and the tight contact structure. Hence we have the quotient space which is a contact manifold and write $(L(p,q), \lambda_{p,q})$,

Since $H_2(L(p,q)) = 0$, we write ECH index of α, β as $I(\alpha,\beta)$ instead of $I(\alpha,\beta,Z)$ where $\{Z\} = H_2(L(p,q);\alpha,\beta)$.

Let $(L(p,q),\lambda)$ be a non-degenerate contact lens space and consider ECH of $0 \in H_1(L(p,q))$. Since $[\emptyset] = 0$, there is an absolute $\mathbb{Z}$-grading on $\mathrm{ECH}(L(p,q),\lambda,0) = \bigoplus_{k\in\mathbb{Z}} \mathrm{ECH}_k(L(p,q),\lambda,0)$ defined by ECH index relative to $\emptyset$ where $\mathrm{ECH}_k(L(p,q),\lambda,0)$ is as follows. Define

$$\mathrm{ECC}_k(L(p,q),\lambda,0) := \bigoplus_{\alpha:\text{ECH generator}, [\alpha]=0, I(\alpha,\emptyset)=k} \mathbb{F}\cdot\alpha.$$

Since ∂_J maps $\mathrm{ECC}_*(L(p,q),\lambda,0)$ to $\mathrm{ECC}_{*-1}(L(p,q),\lambda,0)$ and $U_{J,z}$ does $\mathrm{ECC}_*(L(p,q),\lambda,0)$ to $\mathrm{ECC}_{*-2}(L(p,q),\lambda,0)$, we have $\mathrm{ECH}_k(L(p,q),\lambda,0)$ and

$$U : \mathrm{ECH}_*(L(p,q),\lambda,0) \to \mathrm{ECH}_{*-2}(L(p,q),\lambda,0).$$

Based on these understandings, the next follows.

Proposition 1.3.21. *Let* $(L(p,q),\lambda)$ *be a non-degenerate and* $\mathrm{Ker}\lambda = \xi_{\mathrm{std}}$*.*

(1). If k is even and non-negative,

$$\mathrm{ECH}_k(L(p,q),\lambda,0) \cong \mathbb{F}.$$

If k is odd or negative, $\mathrm{ECH}_k(L(p,q),\lambda,0)$ *is zero. Moreover, for $n \geq 1$ the U-map*

$$U : \mathrm{ECH}_{2n}(L(p,q),\lambda,0) \to \mathrm{ECH}_{2(n-1)}(L(p,q),\lambda,0)$$

is isomorphism.

(2). $0 \neq c(\xi_{\mathrm{std}}) \in \mathrm{ECH}_0(L(p,q),\lambda,0)$*. Therefore, we can define the ECH spectrum (Definition 1.3.18).*

Proof of Proposition 1.3.21. (1) follows from the isomorphism between ECH and monopole floer homology. See [KM, T3]. (2) follows from the cobordism map between $(L(p,q),\lambda)$ and a quotient of an irrational ellipsoid. This is an analogue of (S^3,ξ_{std}) in [H2, Proof of Proposition 4.5]. $\square$

Chapter 2

Existence of positive hyperbolic orbits

2.1 Background

At first, we shall recall the historical background of the Weinstein conjecture. It should be noted that the researches mentioned here are merely a part of prior studies.

Originally, Weinstein showed in terms of a Hamiltonian dynamics that a strictly convex energy hypersurcface in $\mathbb{R}^{2n}$ carries a priodic orbit [Wei1]. After that, Rabinowitz [R] generalized it to a star-shaped energy hyper surface. As a generalization of Rabinowitz's result, Weinstein conjectured that any closed contact hypersurface with $H^1=0$ carries a periodic orbit [Wei2]. The original conjecture was proved by Viterbo [V] without the assumption $H^1=0$. Currently, the conjecture has been formulated as a statement on general closed contact manifolds and called the Weinstein conjecture.

Now, consider the Weinstein conjecture on the 3-dimensional contact manifolds. On the 3-dimensional Weinstein conjecture, one breakthrough result was achieved by Hofer [Ho]. Hofer proved by using J-holomorphic planes the Weinstein conjecture on a contact 3-manifold (Y,λ) under the assumptions that the contact structure is overtwisted or $\pi_2(Y)\neq 0$. After a series of stdies ([AbCiHo], [CoHon], etc...), the 3-dimensional Weinstein conecture was completely proved by Taubes [T2] by using Monopole Floer homology.

As a refinement of the Weinstein conjecture, it is natural to ask the number of periodic orbits and the existence of periodic orbits with certain properties. In relation to these questions, ECH has been used as a powerful tool. For instance, the following holds.

Theorem 2.1.1. *[HT3, CHP, CoDR] Let (Y, λ) be a non-degenerate closed contact 3-manifold. Then (Y, λ) satisfies on of the following;*

(1). There are infinitely many simple periodic orbits.

(2). Y is diffeomorphic to a lens space and there are exactly two simple elliptic orbits.

Remark 2.1.2. Let (Y, λ) be non-degenerate and $\mathrm{Ker}\lambda = \xi$. At first, Hutchings and Taubes [HT3] proved that if all periodic orbits of (Y, λ) are elliptic, then the number of simple periodic orbits are exactly two and Y is a lens space. In [CHP], Cristofaro-Gardiner, Hutchings and Pomerleano proved that if $c_1(\xi)$ is torsion, then there are infinitely many simple periodic orbits. After that, Colin, Dehornoy and Rechtman [CoDR] removed the assumption that $c_1(\xi)$ is torsion.

The next theorem was proved by Cristofaro-Gardiner, Hryniewicz, Hutchings and Liu.

Theorem 2.1.3. *[CHrHL1] Let (Y, λ) be a closed contact 3-manifold with exactly two simple periodic orbits. Then, (Y, λ) is non-degenerate and dynamically convex. It follows from Theorem 2.1.1 that Y is diffeomorphic to a lens space.*

Remark 2.1.4. See Definition 3.1.3 for the definition of dynamical convexity.

Remark 2.1.5. While writing this thesis, Cristofaro-Gardiner, Hryniewicz, Hutchings and Liu announced in [CHrHL2] the following; Let (Y, λ) be a (not necessarily non-degenerate) closed contact 3-manifold. If $c_1(\xi)$ is torsion, then the number of simple periodic orbits are infinity or exactly two.

2.2 Main results

Beside the number of the periodic orbits, we consider the existence of periodic orbits with certain properties.

In [CHP], Cristofaro-Gardiner, Hutchings and Pomerleano showed the following;

Theorem 2.2.1. *[CHP] Let (Y, λ) be non-degenerate closed contact 3-manifold with $b_1(Y) > 0$. Then, there is a simple positive hyperbolic orbit.*

The above theorem follows immediately from the isomorphism with monopole Floer homology. As a generalization, they asked the following;

Question 2.2.2. *Let (Y, λ) be non-degenerate closed contact 3-manifold with $b_1(Y) = 0$. If (Y, λ) is not a lens space with exactly two simple periodic orbits, then Does there exist a simple positive hyperbolic orbit?*

We recall that any periodic orbit can be classified into three types; negative hyperbolic, positive hyperbolic, or elliptic. Here we note that there are many non-degenerate closed contact 3-manifolds such that all periodic orbits are positive hyperbolic. For example, they can be achieved as Anosov contact manifolds such that the stable and unstable directions are orientable. It is known that such Anosov contact manifolds can be constructed in abundance [FH].

Therefore, it is the most reasonable to conjecture the existence of a positive hyperbolic periodic orbit as a refinement of the 3-dimensional Weinstein conjecture.

Our first result is as follows.

Theorem 2.2.3. *Let (Y, λ) be non-degenerate closed contact 3-manifold with $b_1(Y) = 0$. Suppose that (Y, λ) is not a lens space with exactly two simple periodic orbits. If there is an elliptic orbit, then there is a simple positive hyperbolic orbit.*

As will be mentioned in §3.2.2, the existence of an elliptic orbit on a given contact manifold has been studied. And it is a long standing conjecture whether a convex energy hypersurface in the standard symplectic Euclidean space carries an elliptic orbit, and it is partially solved. For example, it is proved in [DDE, AbMa1, HrS2] that if a convex energy hypersurface in the standard symplectic Euclidean space (more generally dynamically convex) is invariant under the invoution, it carries an elliptic orbit. As an application, we have

Corollary 2.2.4. *Assume that $(L(p, q)\lambda)$ is dynamically convex and non-degenerate. If p is even and $(L(p, q)\lambda)$ has at least 3 periodic orbits, then there is a simple positive hyperbolic orbit.*

We should remark that we can not say that a contact 3-manifold has a simple positive hyperbolic orbit even if a contact 3-manifold obtained as its finite cover does.

Another benefit of Theorem 2.2.3 is that to answer Question 2.2.2, it suffices to consider a contact 3-manifold such that all periodic orbits are hyperbolic. For instance, we will prove

Theorem 2.2.5. *Let $L(p,q)$ $(p \neq \pm 1)$ be a lens space with odd p. Then $L(p,q)$ can not admit a non-degenerate contact form λ whose all simple periodic orbits are negative hyperbolic.*

Theorem 2.2.5 will be proved in §2.9.

Immediately, we have the next corollary.

Corollary 2.2.6. *Let $(L(p,q),\lambda)$ $(p \neq \pm 1)$ be a lens space with a non-degenerate contact form λ. Suppose that p is odd and there are infinity many simple Reeb orbits. Then, $(L(p,q),\lambda)$ has a simple positive hyperbolic orbit.*

In addition to Theorem 2.2.5, the next Theorem 2.2.7 and Corollary 2.2.8 hold.

Theorem 2.2.7. *Let (S^3,λ) be a non-degenerate contact three sphere with a free $\mathbb{Z}/2\mathbb{Z}$ action. Suppose that (S^3,λ) has infinity many simple periodic orbits. Then (S^3,λ) has a simple positive hyperbolic orbit.*

Corollary 2.2.8. *Let (S^3,λ) be a non-degenerate contact three sphere with free action of a nontrivial finite group". Suppose that (S^3,λ) has infinity many simple periodic orbits. Then (S^3,λ) has a simple positive hyperbolic orbit.*

At last, we note that a result stated in §3.2.3 also involves the existence of infinitely many simple positive hyperbolic orbits.

2.2.1 Idea and the structure of this chapter

We will prove Theorem 2.2.3 from §2.3 to §2.8 by excluding two cases respectively. One is that (Y,λ) may have at least two simple elliptic orbits and the other is that (Y,λ) may have only one simple elliptic orbit.

In §2.3, we will prove the former case by contradiction. Suppose that there is no simple positive hyperbolic orbit. Then the boundary operator ∂

used to define ECH always vanishes because of the property of ECH index (1.9). Under the assumption, from $\partial = 0$ we will introduce some notations and use them to cause a contradiction with the Weyl law (Theorem 1.3.17).

In §2.4 and beyond, we will prove the main theorem by excluding the case that (Y, λ) may have only one simple elliptic orbit. This is the most difficult part containing many new essential ideas in this paper. Suppose that there is no positive hyperbolic orbit. Note that $\partial = 0$.

Throughout §2.4, the Weyl law with respect to ECH spectrum will also become essentially crucial. In [CHP], the Weyl law was used in fining sufficiently small energy $J_0 = 2$ holomorphic curves. In this paper, we will use the Weyl law in more technical ways that has been used ever before.

At first, in the first half of §2.4, we will show Proposition 2.4.2 which asserts that at the limit almost every orbit set satisfies some assumptions. All of the assumptions will be essential in future combinatorial arguments to control and restrict the existence and behaviors of certain J-holomorphic curves. For the purpose, we will introduce some notions about orbit sets which count the number of hyperbolic orbits and the multiplicity of the elliptic orbit and use the Weyl law in technical ways by combining them.

Next, in the later half of §2.4, we will show Proposition 2.4.4 under Lemma 2.4.5(which will be proved in §2.5) by combining Proposition 2.4.2. Proposition 2.4.4 asserts that there are sufficiently large consecutive $J_0 = 2$ holomorphic curves counted by the U-map between two elements satisfying some assumptions especially whose energies are sufficiently small. As the former part in this section, we will use the Weyl law. Here we note that the way to find consecutive small energy $J_0 = 2$ holomorphic curves is inspired by [CHP].

In §2.5 we will prove Lemma 2.4.5 pending in §2.4. Lemma 2.4.5 asserts that there is no $J_0 \leq 1$ holomorphic curve counted by the U-map between two elements satisfying some assumptions. We will prove this by excluding the possibilities one by one. In the proof and also in §2.6 and beyond, the next ideas will be important.

(a) To consider the boundaries of the moduli spaces of holomorphic curves counted by the U-map.

(b) Delayedness of J_0 index compared to ECH index under the operation adding the elliptic orbit to orbit sets one by one.

(c) To solve simultaneous approximate equations coming from the splitting of small energy J-holomorphic curves.

(d) To compare the solutions obtained in (c) and combinatrial properties of $S_{\pm\theta}$ (Proposition 1.3.6) where θ is the rotation number of the elliptic orbit.

We note that the U-map is defined by counting J-holomorphic curves through a fixed generic point $z \in Y$ and J_0 index restricts topological types of them. In particular, a generic $z \in Y$ is not in any orbits and non-vanishingness of U-map between two orbit sets implies an existence of certain J-holomorphic curve (more precisely, this existence is ensured by the isomorphism between Seiberg-Witten Floer homology and ECH). Such J-holomorphic curves have been useful in various situations. For example, in [CHP], they constructed a global surface of section for the Reeb vector field from certain J-holomorphic curves counted by the U-map and derived a contradiction. In this paper, we will use such J-holomorphic curves in the different way.

At first, we will consider sequences of holomorphic curves as $z \to$ some periodic orbits and its limit in terms of SFT compactness. Then, by considering their topological types and properties of ECH index, we will have certain splitting curves as the boundary elements. This is what (a) means. And under the assumption, some topological types will cause contradictions and be excluded.

Next, for excluding the rest topological types, we will introduce an operation which adds the elliptic orbit to some orbits one by one and see the behaviors of J_0 index and ECH index. In this situation, the J_0 index will change later than the ECH index. This phenomena will give other well-controlled J-holomorphic curves and we will apply (a) to these curves. Then we will obtain simultaneous approximate equations coming from the splitting of small energy J-holomorphic curves and solve them. This will give contradictions. This is what (c) and (d) mean.

In §2.6 and beyond, we will focus on the consecutive $J_0 = 2$ holomorphic curves and apply (a), (b), (c) and (d) to them in more technical ways.

In §2.6 and §2.7, we will state and prove Proposition 2.6.1. Proposition 2.6.1 asserts that such $J_0 = 2$ holomorphic curves obtained in Proposition 2.4.4 can be classified into six types. In particular, each type has some approximate relations about actions of some orbits. In order to determine approximate relations, we will list all possibilities of their splitting ways and solve dozens of simultaneous approximate equations.

In §2.8, we will derive a contradiction from Proposition 2.6.1 and Proposition 2.4.4. In this section, the combinatrial properties of $S_{\pm\theta}$ will be essential. Because of the smallness of the energies of holomorphic curves, we will be able to find that the approximate actions of some orbits will be at sufficiently close to $\frac{1}{12}S_{\pm\theta}$ times the action of the elliptic orbit. This means that the approximate relations about actions of some orbits coming from the consecutive holomorphic curves are reduced to combinatrial relations of $S_{\pm\theta}$. We will check each one carefully and show that the relations exactly restrict the consecutiveness. As a result, the proof of Theorem 2.2.3 will be completed.

In §§2.9, we will prove Theorem 2.2.5 and Theorem 2.2.7. The idea of the proof is to observe the properties of ECH under taking a finite cover.

2.3 The case that the number of simple elliptic orbits is at least two.

Suppose that $b_1(Y) = 0$. In this situation, for any orbit sets α and β with $[\alpha] = [\beta]$, $H_2(Y, \alpha, \beta)$ consists of only one component since $H_2(Y) = 0$. So we may omit the homology component from the notation of ECH index I and J_0, that is, they are just written by $I(\alpha, \beta)$ and $J_0(\alpha, \beta)$ respectively. Moreover, for any orbit sets α with $[\alpha] = 0 \in H_1(Y)$, we set $I(\alpha, [\emptyset]) := I(\alpha)$ and also $J_0(\alpha) := J_0(\alpha, [\emptyset])$. This $I(\alpha)$ defines an absolute $\mathbb{Z}$ grading in $\mathrm{ECH}(Y, \lambda, 0)$. From now on, we suppose that $\mathrm{ECH}(Y, \lambda, 0)$ is graded in this way.

The aim of this section is to prove the next proposition.

Theorem 2.3.1. *Let (Y, λ) be a connected non-degenerate closed contact three manifold with $b_1(Y) = 0$. Assume that the number of simple elliptic orbits is at least two and the number of all simple orbit is infinity. then there exists at least one positive hyperbolic orbit.*

We prove this by contradiction. At first, we show the following lemma.

Lemma 2.3.2. *Let (Y, λ) be a connected non-degenerate closed contact three manifold with $b_1(Y) = 0$. Then for $\Gamma \in H_1(Y)$, there is some finite generated vector space E_Γ such that*

$$\mathrm{ECH}(Y, \lambda, \Gamma) \cong \mathbb{F}[U_\dagger^{-1}, U_\dagger]/U_\dagger\mathbb{F}[U_\dagger] \bigoplus E_\Gamma. \tag{2.1}$$

Morover, the above isomorphism interchanges the map U in (1.21) with the action of the product by $U_\dagger$ on $\mathbb{F}[U_\dagger^{-1}, U_\dagger]/U_\dagger\mathbb{F}[U_\dagger]$.

Proof of Lemma 2.3.2. There are three type homologies $\widetilde{HM}^*(Y,\mathfrak{s})$, $\widehat{HM}^*(Y,\mathfrak{s})$ and $\overline{HM}^*(Y,\mathfrak{s})$ in Seiberg-Witten Floer homologies and there exist an exact sequence.

$$... \longrightarrow \widehat{HM}^*(Y,\mathfrak{s}) \longrightarrow \widetilde{HM}^*(Y,\mathfrak{s}) \longrightarrow \overline{HM}^*(Y,\mathfrak{s}) \longrightarrow \widehat{HM}^{*+1}(Y,\mathfrak{s}) \longrightarrow ... \tag{2.2}$$

As [KM, Proposition 35.3.1],

$$\bigoplus_* \overline{HM}^*(Y,\mathfrak{s}) \cong \mathbb{F}[U_\dagger^{-1}, U_\dagger] \tag{2.3}$$

By construction [KM, Definition 14.5.2, Subsection 22.1 and Subsection 22.3], $\widehat{HM}^*(Y,\mathfrak{s})$ vanishes if its grading is sufficiently low. Moreover the image of

$$\bigoplus_* \widehat{HM}^*(Y,\mathfrak{s}) \longrightarrow \bigoplus_* \widetilde{HM}^*(Y,\mathfrak{s}) \tag{2.4}$$

is finite rank [KM, Proposition 22.2.3]. This finishes the proof of the lemma. □

From now on, in this section we suppose that there is no positive hyperbolic orbit.

For $M > 0$ and $\Gamma \in H_1(Y)$, we set

$$\Lambda(M,\Gamma) := \{ \ \alpha \mid \alpha \text{ is ECH generator such that } [\alpha] = \Gamma \text{ and } A(\alpha) < M \ \} \tag{2.5}$$

Lemma 2.3.3. *For every $\Gamma \in H_1(Y)$,*

$$\lim_{M\to\infty} \frac{M^2}{|\Lambda(M,\Gamma)|} = 2\mathrm{Vol}(Y,\lambda). \tag{2.6}$$

Proof of Lemma 2.3.3. Fix $\Gamma \in H_1(Y)$. By $\partial = 0$ and (2.1), there are ECH generators $\{\alpha_i^\Gamma\}_{0\le i}$ and $\{\beta_j^\Gamma\}_{0\le j\le m_\Gamma}$ with $[\alpha_i^\Gamma] = [\beta_j^\Gamma] = \Gamma$ satisfy

$$\mathrm{ECH}(Y,\lambda,\Gamma) \cong \bigoplus_{i=0}^{\infty} \mathbb{F}\langle \alpha_i^\Gamma \rangle \bigoplus \bigoplus_{j=1}^{m_\Gamma} \mathbb{F}\langle \beta_j^\Gamma \rangle \tag{2.7}$$

and $U\langle\alpha_i^\Gamma\rangle = \langle\alpha_{i-1}^\Gamma\rangle$ for $i \geq 1$. Moreover since $\partial = 0$, each ECH generator α with $[\alpha] = \Gamma$ is equal to either β_j^Γ for some $1 \leq j \leq m_\Gamma$ or α_k^Γ for some $0 \leq k$.

Note that

$$A(\alpha_l^\Gamma) - A(\alpha_k^\Gamma) > 0 \quad \text{if} \quad l > k, \tag{2.8}$$

$$I(\alpha_l^\Gamma) - I(\alpha_k^\Gamma) = 2(l-k). \tag{2.9}$$

Since $A(\alpha_k^\Gamma) \to \infty$ as $k \to \infty$, there is $k_0 > 0$ such that for every $k > k_0$ $A(\alpha_k^\Gamma) > A(\beta_j^\Gamma)$ for $1 \leq j \leq m_\Gamma$. By (2.8) and (2.9), we have $I(\alpha_k^\Gamma) = 2k + I(\alpha_0^\Gamma)$ and for sufficiently large k, $|\Lambda(A(\alpha_k^\Gamma), \Gamma)| = k + m$. So for sufficiently large k, we have

$$|2|\Lambda(A(\alpha_k^\Gamma), \Gamma)| - I(\alpha_k^\Gamma)| \leq 2m + I(\alpha_0^\Gamma) \tag{2.10}$$

By the above we have,

$$\lim_{k\to\infty} \frac{2|\Lambda(A(\alpha_k^\Gamma), \Gamma)|}{I(\alpha_k^\Gamma)} = 1. \tag{2.11}$$

Since $\partial = 0$ and the definition of the spectrum (1.3.16), $c_{\langle\alpha\rangle}(Y, \lambda) = A(\alpha)$. So by (1.31),

$$\lim_{k\to\infty} \frac{c_{\langle\alpha_k\rangle}(Y,\lambda)^2}{I(\alpha_k^\Gamma)} = \lim_{k\to\infty} \frac{A(\alpha_k^\Gamma)^2}{2|\Lambda(A(\alpha_k^\Gamma), \Gamma)|} = \mathrm{Vol}(Y, \lambda). \tag{2.12}$$

Note that for large $M > 0$, there is a large $k > 0$ such that $A(\alpha_k^\Gamma) \geq M \geq A(\alpha_{k-1}^\Gamma)$.

Therefore we complete the proof of Lemma 2.3.3. □

Proof of Theorem 2.3.1. We pick up two simple elliptic orbits γ_1, γ_2. Let s_1 and s_2 denote the orders of $[\gamma_1]$ and $[\gamma_2]$ in $H_1(Y)$ respectively.

Since $|H_1(Y)| < \infty$, we can choose infinity sequence of simple orbit $\{\delta_i\}_{i>0}$ satisfying

1. their homology classes $[\delta_i]$ are in a same one.
2. $A(\delta_i) < A(\delta_j)$ if $i < j$.
3. Any δ_i is not equivalent to γ_1 and γ_2

Let r be the order of $[\delta_i]$ in $H_1(Y)$. Then we define a sequence of ECH generators $\{\epsilon_n\}_{n>0}$ by $\epsilon_n := \{(\delta_{r(n-1)+i}, 1)\}_{1\leq i\leq r}$. By construction, $[\epsilon_n] = 0$ and $A(\epsilon_n) < A(\epsilon_{n+1})$.

For $t_1,\ t_2 \in \mathbb{Z}_{\geq 0}$, we set

$$\epsilon_{(t_1,t_2,n)} := (\gamma_1, t_1 s_1) \cup (\gamma_2, t_2 s_2) \cup \epsilon_n \tag{2.13}$$

Note that $\epsilon_{(t_1,t_2,n)}$ is an ECH generator with $[\epsilon_{(t_1,t_2,n)}] = 0$ and morover if $(t_1, t_2, n) \neq (t_1', t_2', n')$ then $\epsilon_{(t_1,t_2,n)} \neq \epsilon_{(t_1',t_2',n')}$.

Let $T_n := A(\epsilon_n)$ and $R_i = s_i A(\delta_i)$ for $i = 1, 2$, then

$$A(\epsilon_{(t_1,t_2,n)}) = t_1 R_1 + t_2 R_2 + T_n \tag{2.14}$$

So for n,

$$\{\ (t_1, t_2) \mid A(\epsilon_{(t_1,t_2,n)}) < M\ \} = \{\ (t_1, t_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid t_1 R_1 + t_2 R_2 < M - T_n\ \} \tag{2.15}$$

In general, for any $S_1,\ S_2,\ T > 0$, the number of

$$\{\ (t_1, t_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid t_1 S_1 + t_2 S_2 < T\ \} \tag{2.16}$$

is $\frac{(T)^2}{2S_1S_2} + O(T)$ (for example, we can see the same argument in [H2]). So the right hand side of (2.15) is equal to $\frac{(M-T_n)^2}{2R_1R_2} + O(M - T_n)$.

For any $N \in \mathbb{Z}_{>0}$, we pick a sufficiently large $M > 0$ satisfying $M > T_N$. Then

$$\begin{aligned} |\Lambda(M, 0)| >& |\{\ (t_1, t_2, n) \mid A(\epsilon_{(t_1,t_2,n)}) < M\ ,\ 1 \leq n \leq N\ \}| \\ =& \sum_{k=1}^{N} |\{\ (t_1, t_2) \mid t_1 R_1 + t_2 R_2 < M - T_k\ \}| \\ =& \sum_{k=1}^{N} \frac{(M - T_k)^2}{2R_1R_2} + O(M - T_k) \\ =& \frac{(M)^2 N}{2R_1R_2} + O(M) \end{aligned} \tag{2.17}$$

So

$$\lim_{M\to\infty} \frac{M^2}{|\Lambda(M, 0)|} < \frac{2R_1R_2}{N}. \tag{2.18}$$

Since we choose N arbitrarily, by (2.6) we can see that $\mathrm{Vol}(Y, \lambda) = 0$. This is a contradiction. We complete the proof of Theorem 2.3.1. $\square$

2.4 The case that the number of simple elliptic orbits is exactly one.

We use the rest of this paper to prove the next theorem.

Theorem 2.4.1. *Let Y be a closed connected three manifold with $b_1(Y) = 0$. Then, Y does not admit a non-degenerate contact form λ such that exactly one simple orbit is elliptic orbit and all the others are negative hyperbolic.*

We prove this by contradiction.

From now on, we assume that (Y, λ) is non-degenerate contact three manifold such that exactly one simple orbit is elliptic and all the others are negative hyperbolic.

Let γ be the simple elliptic orbit. Moreover, let $A(\gamma) = R$ and θ be the rotation number with respect to some fixed trivialization τ over γ. This means $e^{\pm 2\pi\theta}$ are eigenvalues of $d\phi^R|_\xi$ and for every $k \in \mathbb{Z}$, $\mu_\tau(\gamma^k) = 2\lfloor k\theta \rfloor + 1$.

2.4.1 Density of orbit sets with some properties

For an ECH generator α, Let $E(\alpha)$, $H(\alpha)$ be the multiplicity at γ in α and the number of hyperbolic orbits in α, respectively.

Recall that for $M \in \mathbb{R}$ and $\Gamma \in H_1(M)$,

$$\Lambda(M,\Gamma) = \{\, \alpha \,|\, \alpha \text{ is an ECH generator such that } [\alpha] = \Gamma \text{ and } A(\alpha) < M \,\}. \tag{2.19}$$

In addition to this, we introduce some notations as follows.

$$\Lambda_{(n,m)}(M,\Gamma) := \{\ \alpha \in \Lambda(M,\Gamma) \,|\, (E(\alpha), H(\alpha)) = (n,m) \,\} \tag{2.20}$$

$$\Lambda_{(n,\infty)}(M,\Gamma) := \bigcup_{m=0}^{\infty} \Lambda_{(n,m)}(M,\Gamma) \tag{2.21}$$

$$\Lambda_{(\infty,m)}(M,\Gamma) := \bigcup_{n=0}^{\infty} \Lambda_{(n,m)}(M,\Gamma) \tag{2.22}$$

$$\Lambda(M) := \bigcup_{\Gamma \in H_1(M)} \Lambda(M,\Gamma) \tag{2.23}$$

Note that if $M \leq 0$, the above sets become empty.

Proposition 2.4.2. *For every $\Gamma \in H_1(Y)$,*

1. *For every positive integer n,*

$$\lim_{M\to\infty} \frac{|\Lambda_{(n,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} = 0 \tag{2.24}$$

2. *For every positive integer m,*

$$\lim_{M\to\infty} \frac{|\Lambda_{(\infty,m)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} = 0 \tag{2.25}$$

3.

$$\lim_{M\to\infty} \frac{|\bigcup_{p_i\in S_\theta} \Lambda_{(p_i,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} = 0 \tag{2.26}$$

Before we try to prove Proposition 2.4.2, we show the next almost trivial claim but this makes the proof of Proposition 2.4.2 easier.

Claim 2.4.3. *For every $\Gamma \in H_1(Y)$, we have*

$$\lim_{M\to\infty} \frac{|\Lambda(M)|}{|\Lambda(M,\Gamma)|} = |H_1(Y)| \tag{2.27}$$

Proof of Claim 2.4.3. By the definition, we have

$$|\Lambda(M)| = \sum_{\Gamma\in H_1(Y)} |\Lambda(M,\Gamma)| \tag{2.28}$$

And since Lemma 2.3.3,

$$\lim_{M\to\infty} \frac{|\Lambda(M)|}{M^2} = \lim_{M\to\infty} \sum_{\Gamma\in H_1(Y)} \frac{|\Lambda(M,\Gamma)|}{M^2} = \frac{|H_1(Y)|}{2\mathrm{Vol}(Y,\lambda)}. \tag{2.29}$$

Hence

$$\lim_{M\to\infty} \frac{|\Lambda(M)|}{|\Lambda(M,\Gamma)|} = \lim_{M\to\infty} \frac{|\Lambda(M)|}{M^2} \frac{M^2}{|\Lambda(M,\Gamma)|} = |H_1(Y)|. \tag{2.30}$$

□

Proof of Proposition 2.4.2. Note that $|\Lambda_{(n,\infty)}(M,\Gamma)| = |\Lambda_{(n-k,\infty)}(M - kR, \Gamma - k[\gamma])|$. This is because the correspondence by adding (γ, k) from $\Lambda_{(n-k,\infty)}(M - kR, \Gamma - k[\gamma])$ to $\Lambda_{(n,\infty)}(M,\Gamma)$ is bijective. Hence

$$|\Lambda(M,\Gamma)| = \sum_{n=0}^{\infty} |\Lambda_{(n,\infty)}(M,\Gamma)| = \sum_{n=0}^{\infty} |\Lambda_{(0,\infty)}(M - nR, \Gamma - n[\gamma])|. \quad (2.31)$$

Since $\lim_{M\to\infty} \frac{|\Lambda(M-R,\Gamma-[\gamma])|}{|\Lambda(M,\Gamma)|} = \lim_{M\to\infty} \frac{(M-R)^2}{M^2} = 1$, we have

$$\begin{aligned}\lim_{M\to\infty} \frac{|\Lambda_{(0,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} =& 1 - \lim_{M\to\infty} \frac{\sum_{n=0}^{\infty} |\Lambda_{(0,\infty)}(M-(n+1)R, \Gamma-(n+1)[\gamma])|}{\sum_{n=0}^{\infty} |\Lambda_{(0,\infty)}(M - nR, \Gamma - n[\gamma])|} \\ =& 1 - \lim_{M\to\infty} \frac{|\Lambda(M-R,\Gamma-[\gamma])|}{|\Lambda(M,\Gamma)|} \\ =& 0.\end{aligned} \quad (2.32)$$

And hence for $n > 0$,

$$\begin{aligned}\lim_{M\to\infty} \frac{|\Lambda_{(n,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} &= \lim_{M\to\infty} \frac{|\Lambda_{(0,\infty)}(M-nR,\Gamma-n[\gamma])|}{|\Lambda(M-nR,\Gamma-n[\gamma])|} \frac{|\Lambda(M-nR,\Gamma-n[\gamma])|}{|\Lambda(M,\Gamma)|} \\ &= \lim_{M\to\infty} \frac{|\Lambda_{(0,\infty)}(M-nR,\Gamma-n[\gamma])|}{|\Lambda(M-nR,\Gamma-n[\gamma])|} \frac{(M-nR)^2}{M^2} = 0.\end{aligned} \quad (2.33)$$

This completes the proof of the first statement.

To prove the second statement, we change the denominator in the statement to $|\Lambda(M)|$. By Claim 2.4.3, it is sufficient to prove this version. We define the map

$$f : \Lambda_{(\infty,m)}(\frac{M}{2}, \Gamma) \times \Lambda_{(\infty,m)}(\frac{M}{2}, \Gamma) \to \Lambda(M) \quad (2.34)$$

as follows. Let α, $\beta \in I_{(\infty,m)}(\frac{M}{2})$. If α and β have no common negative hyperbolic orbit, we define $f(\alpha,\beta) = \alpha \cup \beta$. Otherwise, that is if α and β have some common negative hyperbolic orbit, we define $f(\alpha,\beta)$ by changing all multiplicities of negative hyperbolic orbits in $\alpha \cup \beta$ to one. This definition is well-defined.

Let δ be an element in the image of f. Then the number of multiplicity at γ in δ is at most $\frac{M}{R}$. So by combinatorial arguments, we can find that for every m, there is C_m such that $\frac{C_m M}{R} > |f^{-1}(\delta)|$ for any $\delta \in \Lambda(M)$. So,

$$|\Lambda(M)| > \frac{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|^2}{\frac{C_m M}{R}} \tag{2.35}$$

thus

$$\frac{\frac{C_m M}{R}}{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|} > \frac{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|}{|\Lambda(M)|}. \tag{2.36}$$

Suppose that there are $\epsilon > 0$ and $M_k \to \infty$ such that $\frac{|\Lambda_{(\infty,m)}(\frac{M_k}{2},\Gamma)|}{|\Lambda(M_k)|} > \epsilon$ then,

$$\begin{aligned}
\lim_{k\to\infty} \frac{|\Lambda_{(\infty,m)}(\frac{M_k}{2},\Gamma)|}{|\Lambda(M_k)|} &\leq \lim_{k\to\infty} \frac{C_m M_k}{R|\Lambda_{(\infty,m)}(\frac{M_k}{2},\Gamma)|} \\
&= \lim_{k\to\infty} \frac{C_m M_k}{R|\Lambda(M_k)|} \frac{|\Lambda(M_k)|}{|\Lambda_{(\infty,m)}(\frac{M_k}{2})|} \\
&\leq \lim_{k\to\infty} \frac{C_m M_k^2}{\epsilon R|\Lambda(M_k)|} \cdot \frac{1}{M_k} \\
&= \frac{2C_m \mathrm{Vol}(Y,\lambda)}{\epsilon R|H_1(Y)|} \lim_{k\to\infty} \frac{1}{M_k} = 0.
\end{aligned} \tag{2.37}$$

This contradicts $\frac{|\Lambda_{(\infty,m)}(\frac{M_k}{2},\Gamma)|}{|\Lambda(M_k)|} > \epsilon$. Hence $\lim_{M\to\infty} \frac{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|}{|\Lambda(M)|} = 0$ and so

$$\begin{aligned}
\lim_{M\to\infty} \frac{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|}{|\Lambda(\frac{M}{2})|} &= \lim_{M\to\infty} \frac{|\Lambda_{(\infty,m)}(\frac{M}{2})|}{|\Lambda(M)|} \frac{|\Lambda(M)|}{|\Lambda(\frac{M}{2})|} \\
&= \lim_{M\to\infty} \frac{|\Lambda_{(\infty,m)}(\frac{M}{2},\Gamma)|}{|\Lambda(M)|} \frac{M^2}{(\frac{M}{2})^2} = 0
\end{aligned} \tag{2.38}$$

This completes the proof of the second statement.

Finally, we prove the third statement. Note that for $p_i \in S_\theta$, the sequence of $p_{i+1} - p_i$ is monotone increasing with respect to i and diverges to infinity as $i \to \infty$ (Proposition 1.3.6). Let $s \in \mathbb{Z}_{>0}$ be the order of $[\gamma]$ in $H_1(Y)$. Then for every $N \in \mathbb{Z}_{>0}$, there is $l \in \mathbb{Z}_{>0}$ such that $p_{i+1} - p_i > sN$ for every $i > l$. By the first statement,

$$\lim_{M\to\infty} \frac{|\bigcup_{p_i\in S_\theta,\, p_i<p_l} \Lambda_{(p_i,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|} = 0. \tag{2.39}$$

Since $|\Lambda_{(n,\infty)}(M,\Gamma)| \leq |\Lambda_{(n',\infty)}(M,\Gamma-(n-n')[\gamma])|$ for $n>n'$, if $p_i \geq p_l$,

$$\sum_{n=p_i+1}^{p_{i+1}} |\Lambda_{(n,\infty)}(M,\Gamma)| \geq N|\Lambda_{(p_{i+1},\infty)}(M,\Gamma)| \tag{2.40}$$

and then

$$|\Lambda(M,\Gamma)| \geq N|\bigcup_{p_i\in S_\theta,\ p_l\leq p_i} \Lambda_{(p_i,\infty)}(M,\Gamma)|. \tag{2.41}$$

By combining with (2.39), we have

$$\frac{1}{N} \geq \lim_{M\to\infty} \frac{|\bigcup_{p_i\in S_\theta} \Lambda_{(p_i,\infty)}(M,\Gamma)|}{|\Lambda(M,\Gamma)|}. \tag{2.42}$$

Since we can pick up N arbitrary large, we complete the proof of the third statement. □

2.4.2 Proof of Proposition 2.4.4 under Lemma 2.4.5

Recall that since Lemma 2.3.2 there is an isomorphism

$$\mathrm{ECH}(Y,\lambda,0) \cong \bigoplus_{k=0}^{\infty} \mathbb{F}\langle\alpha_k\rangle \bigoplus \bigoplus_{j=1}^{m} \mathbb{F}\langle\beta_j\rangle \tag{2.43}$$

with $U\langle\alpha_k\rangle = \langle\alpha_{k-1}\rangle$ for $k\geq 1$ (note that $U\langle\alpha_0\rangle$ is not necessarily 0). Moreover all but finite ECH generators are in $\{\alpha_k\}_{k\in\mathbb{Z}_{\geq 0}}$. Note that $A(\alpha_k) > A(\alpha_l)$ if and only if $k>l$. Here we omit some notations from (2.7) and from now on, we do under this notations unless there is confusion.

The aim of this subsection is the proof of the next proposition under Lemmma 2.4.5.

Proposition 2.4.4. *For every $\epsilon>0$ and positive integer l, there is $k\in\mathbb{Z}_{>0}$ which satisfies the following condition.*

The $l+1$ consecutive orbit sets $\alpha_k,\ \alpha_{k+1},....\ \alpha_{k+l}$ satisfies for every $0\leq i\leq l$,

1. $J(\alpha_{k+i+1},\alpha_{k+i}) = 2$
2. $A(\alpha_{k+i+1}) - A(\alpha_{k+i}) < \epsilon$
3. $E(\alpha_{k+i}) \notin S_\theta \cup S_{-\theta}$

4. $E(\alpha_{k+i}) > p_1,\ q_1$

5. $H(\alpha_{k+i}) > 4$

6. *In the notation of (2.7), for any* $\Gamma \in H_1(Y)$ *and* $1 \le j \le m_\Gamma$, $A(\alpha_{k+i}) > A(\gamma) + A(\beta_j^\Gamma)$.

The next lemma plays an important role in the proof of Proposition 2.4.4.

Lemma 2.4.5. *For any sufficiently small* $\epsilon > 0$, *there is no positive integer* k *which satisfies*

1. $J(\alpha_{k+1}, \alpha_k) \le 1$

2. $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$

3. $E(\alpha_{k+1}),\ E(\alpha_k) > p_1,\ q_1$

4. $E(\alpha_{k+1}),\ E(\alpha_k) \notin S_\theta \cup S_{-\theta}$

5. $H(\alpha_{k+1}),\ H(\alpha_k) > 4$

6. *In the notation of (2.7), for any* $\Gamma \in H_1(Y)$ *and* $1 \le j \le m_\Gamma$, $A(\alpha_k) > A(\gamma) + A(\beta_j^\Gamma)$.

Here we note that the number of k *which does not satisfy the sixth condition is finite.*

Before proving Lemma 2.4.5, we will give the proof of Proposition 2.4.4 under Lemma 2.4.5. We will prove Lemma 2.4.5 in the next section.

To prove Proposition 2.4.4, we introduce some notations as follows.

For positive integer k and $\epsilon > 0$, we set

$$\hat{I}_{(n,m)}(k) := \{\ \alpha_{k'} \mid k' \le k,\ (E(\alpha_{k'}), H(\alpha_{k'})) = (n,m)\ \} \tag{2.44}$$

$$\hat{I}_{(n,\infty)}(k) := \bigcup_{m=0}^{\infty} \hat{I}_{(n,m)}(k) \tag{2.45}$$

$$\hat{I}_{(\infty,m)}(k) := \bigcup_{n=0}^{\infty} \hat{I}_{(n,m)}(k) \tag{2.46}$$

$$\hat{I}_{\ge\epsilon}(k) := \{\ \alpha_{k'} \mid k' \le k,\ A(\alpha_{k'+1}) - A(\alpha_{k'}) \ge \epsilon\ \} \tag{2.47}$$

$$\hat{I}_{=2,<\epsilon}(k) := \{ \alpha_{k'} \mid k' \le k,\ J(\alpha_{k'+1},\alpha_{k'}) = 2,\ \alpha_{k'+1},\ \alpha_{k'} \text{ satisfy } 2,3,4,5,6 \text{ in Lemma 2.4.5}\} \tag{2.48}$$

$$\hat{I}_{>2,<\epsilon}(k) := \{ \alpha_{k'} \mid k' \le k,\ J(\alpha_{k'+1},\alpha_{k'}) > 2,\ \alpha_{k'+1},\ \alpha_{k'} \text{ satisfy } 2,3,4,5,6 \text{ in Lemma 2.4.5}\} \tag{2.49}$$

Proof of Proposition 2.4.4 under Lemma 2.4.5. For large n, there is a $k_n \in \mathbb{Z}$ such that $\alpha_{k_n} = \{(\gamma, sn)\}$ where $s \in \mathbb{Z}_{>0}$ is the order of $[\gamma]$ in $H_1(Y)$. Here we use $[(\gamma, sn)] = 0$ and (2.43).

Claim 2.4.6. *1. For every positive integer n, $\lim_{k\to\infty} \frac{|\hat{I}_{(n,\infty)}(k)|}{k} = 0$*

2. For every positive integer m, $\lim_{k\to\infty} \frac{|\hat{I}_{(\infty,m)}(k)|}{k} = 0$

3. $\lim_{k\to\infty} \frac{|\bigcup_{n\in S_\theta} \hat{I}_{(n,\infty)}(k)|}{k} = 0$

4. $\lim_{n\to\infty} \frac{|\hat{I}_{\ge\epsilon}(k_n)|}{k_n} = 0$

Proof of Claim 2.4.6. The first three statement is just restatements of Proposition 2.4.2. So we have only to prove the forth statement.

Since $I(\alpha_{k_n}) = 2k_n + I(\alpha_0)$, $\lim_{n\to\infty} \frac{(snR)^2}{2k_n} = \mathrm{Vol}(Y,\lambda)$ and so $n < C\sqrt{k_n}$ for some $C > 0$. So by the definition, for some $C > 0$,

$$\epsilon|\hat{I}_{\ge\epsilon}(k_n)| < A(\alpha_{k_n}) - A(\alpha_0) = snR - A(\alpha_0) < C\sqrt{k_n} \tag{2.50}$$

Thus

$$0 = \lim_{n\to\infty} \frac{C\sqrt{k_n}}{\epsilon k_n} \ge \lim_{n\to\infty} \frac{|\hat{I}_{\ge\epsilon}(k_n)|}{k_n}. \tag{2.51}$$

This finishes the proof of Claim 2.4.6. □

By definition,

$$I(\alpha_{k_n}) - J_0(\alpha_{k_n}) = 2nc_{s\gamma} + 2\lfloor n\theta \rfloor + 1 \tag{2.52}$$

where $c_{s\gamma} = c_1(\xi|_*, \tau)$ with $\{*\} = H_2(Y;(\gamma,s),\emptyset)$ and $J_0(\alpha_{k_n}) = J_0(\alpha_{k_n},\emptyset)$. So there is $C > 0$ such that $|I(\alpha_{k_n}) - J_0(\alpha_{k_n})| < Cn$ and then

$$J_0(\alpha_{k_n}) = \sum_{i=0}^{k_n-1} J_0(\alpha_{i+1},\alpha_i) < 2k_n + C\sqrt{k_n}. \tag{2.53}$$

for some $C > 0$. By considering $J_0(\alpha_{i+1}, \alpha_i) \geq -1$ and Lemma 2.4.5, for some $C > 0$ we have

$$\begin{aligned} 2|\hat{I}_{=2,<\epsilon}(k_n)| + 3|\hat{I}_{>2,<\epsilon}(k_n)| \leq & 2k_n + C\sqrt{k_n} + |\bigcup_{p_i \in S_\theta} \hat{I}_{(p_i,\infty)}(k_n)| \\ & + |\bigcup_{q_i \in S_{-\theta}} \hat{I}_{(q_i,\infty)}(k_n)| + \sum_{i=0}^{\max(p_1,q_1)} |\hat{I}_{(i,\infty)}(k_n)| \\ & + \sum_{i=0}^{4} |\hat{I}_{(\infty,i)}(k_n)| + |\hat{I}_{\geq\epsilon}(k_n)| \end{aligned} \tag{2.54}$$

Since the right hand side over k_n converges to 2 as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{2|\hat{I}_{=2,<\epsilon}(k_n)| + 3|\hat{I}_{>2,<\epsilon}(k_n)|}{k_n} \leq 2. \tag{2.55}$$

On the other hand, we have

$$\begin{aligned} k_n \geq |\hat{I}_{=2,<\epsilon}(k_n)| + |\hat{I}_{>2,<\epsilon}(k_n)| \geq & k_n - C\sqrt{k_n} - |\bigcup_{p_i \in S_\theta} \hat{I}_{(p_i,\infty)}(k_n)| \\ & - |\bigcup_{q_i \in S_{-\theta}} \hat{I}_{(q_i,\infty)}(k_n)| - \sum_{i=0}^{\max(p_1,q_1)} |\hat{I}_{(i,\infty)}(k_n)| \\ & - \sum_{i=0}^{4} |\hat{I}_{(\infty,i)}(k_n)| - |\hat{I}_{\geq\epsilon}(k_n)|. \end{aligned} \tag{2.56}$$

And so we have

$$\lim_{n\to\infty} \frac{|\hat{I}_{=2,<\epsilon}(k_n)| + |\hat{I}_{>2,<\epsilon}(k_n)|}{k_n} = 1. \tag{2.57}$$

From (2.55) and (2.57), we have

$$\lim_{n\to\infty} \frac{|\hat{I}_{=2,<\epsilon}(k_n)|}{k_n} = 1. \tag{2.58}$$

This implies that for every positive integer l, if n is sufficiently large, we can pick up $l+1$ consecutive orbit sets α_k, α_{k+1},...α_{k+l} which satisfy 2, 3, 4, 5 in Lemma 2.4.5 and $J(\alpha_{k+i+1}, \alpha_{k+i}) = 2$ for $0 \leq i \leq l-1$. We complete the proof of Proposition 2.4.4. □

2.5 Proof of Lemma 2.4.5

For our purpose, we choose $\epsilon > 0$ such that $\epsilon < \frac{1}{10^5}\min\{A(\alpha)\,|\alpha \text{ is a Reeb orbit}\}$ and make it smaller as needed.

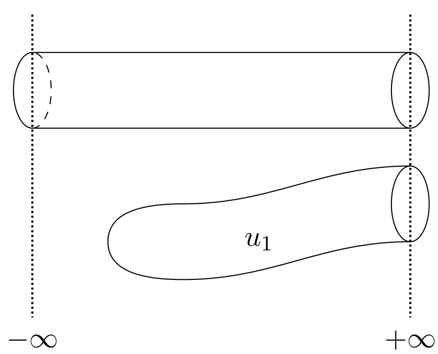
Since $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$, there is at least one J-holomorphic curve whose ECH index is equal to 2, and we write $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha_{k+1}, \alpha_k)$. Note that u_1 is through a fixed generic point z.

To prove Lemma 2.4.5, we prepare some notations as follows. Let g, k and l be the genus of u_1, the number of punctures of u_1 and $\sum_i(n_i^+ - 1) + \sum_j(n_j^- - 1)$ respectively. In this notation, $J_0(\alpha_{k+1}, \alpha_k) = -2 + 2g + k + l$. Note that k is definitely positive and u_1 has at least one positive end because of the maximum principle. In the proof of Lemma 2.4.5, we have only to consider the cases $J_0 = -1,\ 0,\ 1$. To make the proof easier to understand, we make a list of their topological types as follows.

Case $J_0 = -1$

In this case, $(g, k, l) = (0, 1, 0)$ may appear as J-holomorphic curves counted by U-map.

- $(g, k, l) = (0, 1, 0)$

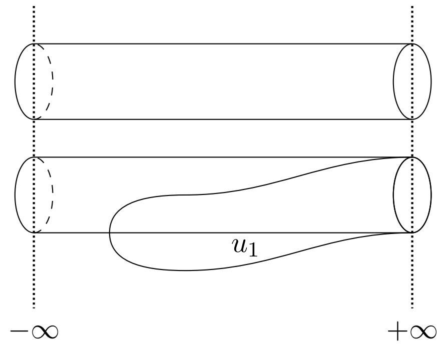


Case $J_0 = 0$

In this case, $(g, k, l) = (0, 1, 1)$, $(0, 2, 0)$ may appear as J- holomorphic curves counted by the U-map.

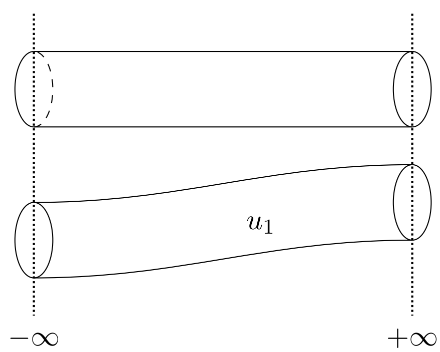
- $(g, k, l) = (0, 1, 1)$

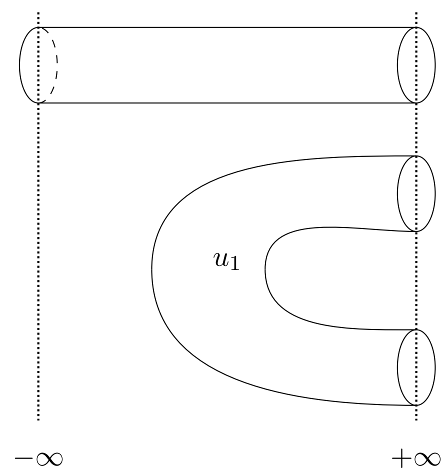
This case has only one type as follows.



- $(g, k, l) = (0, 2, 0)$

  This case has two types as follows.



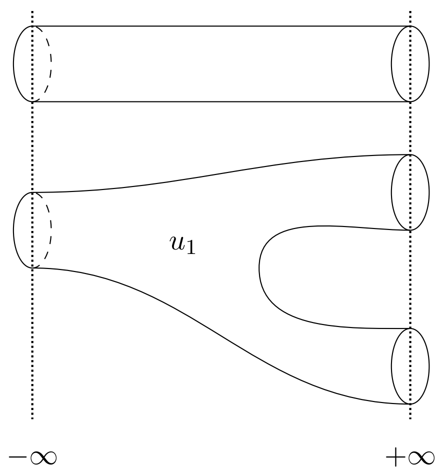


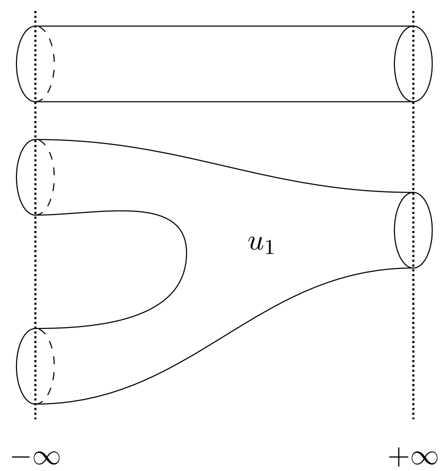
Case $J_0 = 1$

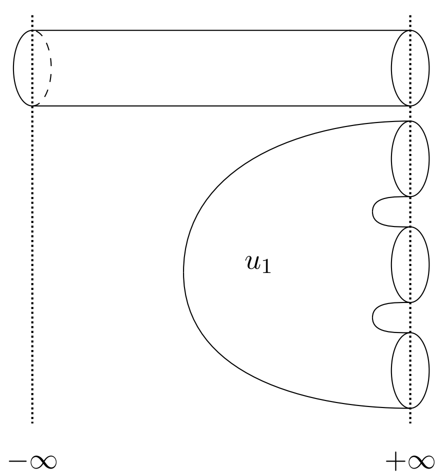
In this case, $(g, k, l) = (0, 3, 0)$, $(0, 2, 1)$, $(1, 1, 0)$ may appear as J-holomorphic curves counted by the U-map. Note that we can see from the definitions of g, k, l and geometric observation that the case $(g, k, l) = (0, 1, 2)$ satisfies the equation $J_0 = -2 + 2g + k + l = 1$ but this can not appear as a J-holomorphic curve.

- $(g, k, l) = (0, 3, 0)$

  This case has three types as follows.

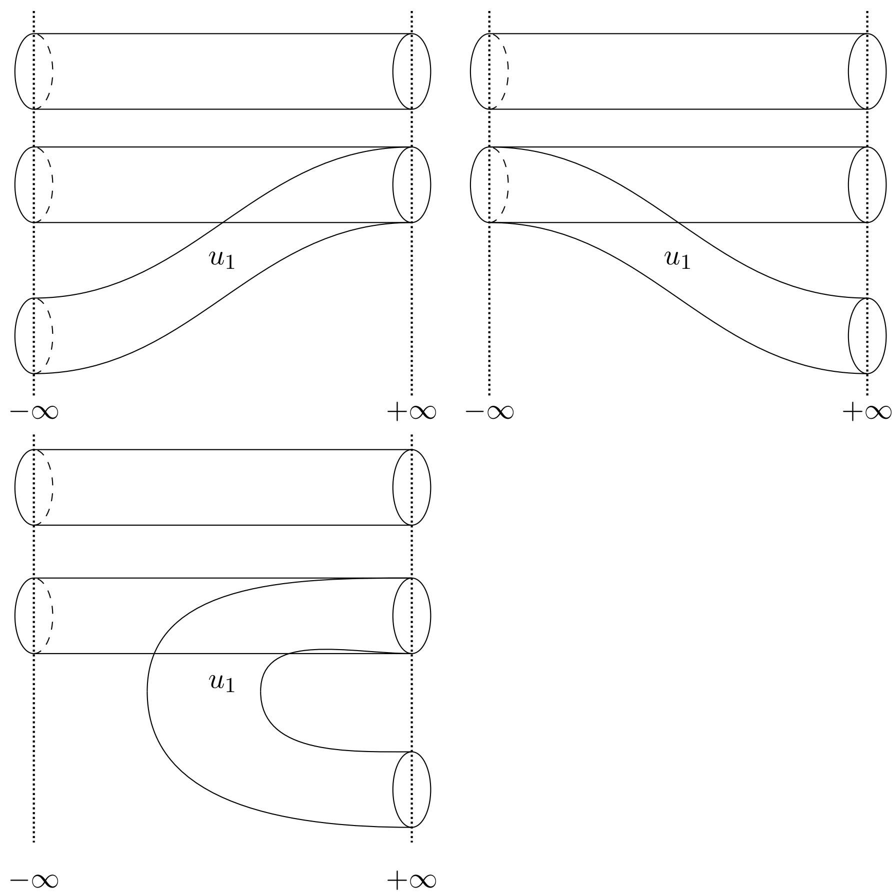






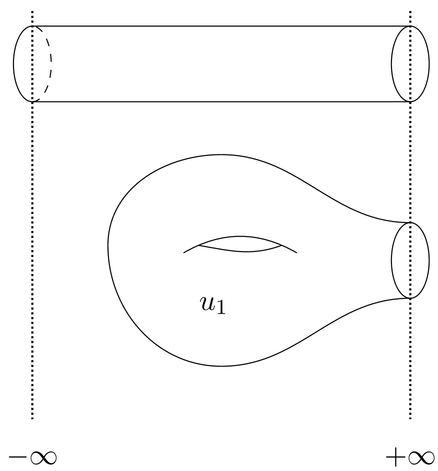
- $(g, k, l) = (0, 2, 1)$

This case has three types as follows.



- $(g, k, l) = (1, 1, 0)$

This case has only one type as follows.



Proof of Lemma 2.4.5. Since $J_0 \geq -1$, we prove Lemma 2.4.5 by dividing into three cases $J_0(\alpha_{k+1}, \alpha_k) = -1,\ 0,\ 1$.

Case1. $J_0(\alpha_{k+1}, \alpha_k) = -1$

In this case, only $(g, k, l) = (0, 1, 0)$ satisfies this equation. The integral value of this J-holomorphic curve over $d\lambda$ is equal to $A(\alpha_{k+1}) - A(\alpha_k)$ by Stoke's theorem and so moreover equal to some action of a Reeb orbit. This contradicts $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon < \frac{1}{10^5}\max\{A(\alpha)\ |\alpha \text{ is a Reeb orbit }\}$.

Case2. $J_0(\alpha_{k+1}, \alpha_k) = 0$

In this case, $(g, k, l) = (0, 2, 0)$, $(0, 1, 1)$. For the same reason as Case1, we have only to consider the case $(g, k, l) = (0, 2, 0)$ and u_1 has both positive and negative ends and their two orbits are different each other. Since $E(\alpha_{k+1}),\ E(\alpha_k) \notin S_\theta \cup S_{-\theta}$, $l = 0$ and the partition conditions of the ends of admissible curves (Definition 1.3.8, Definition 1.3.9 and Proposition 1.3.10), u_1 has no end asymptotic to γ. Moreover since $E(\alpha_{k+1}),\ E(\alpha_k) > p_1,\ q_1 > 1$, u_0 contains some covering of $\mathbb{R} \times \gamma$ and so we have $E(\alpha_{k+1}) = E(\alpha_k)$. Let δ_1 and δ_2 be the Reeb orbits where the positive and negative end of u_1 are asymptotic respectively. We set $E(\alpha_{k+1}) = E(\alpha_k) = M$. Then we can describe α_{k+1}, α_k as $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M) \cup (\delta_1, 1)$ and $\alpha_k = \hat{\alpha} \cup (\gamma, M) \cup (\delta_2, 1)$ respectively where $\hat{\alpha}$ is an ECH generator consisting of some negative hyperbolic orbits which do not contain δ_1, δ_2.

By the above argument, we can see that for any generic $z \in Y$, non trivial parts of J-holomorhic curves counted by $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$ through z are in $\mathcal{M}^J(\delta_1, \delta_2)$ and of genus zero.

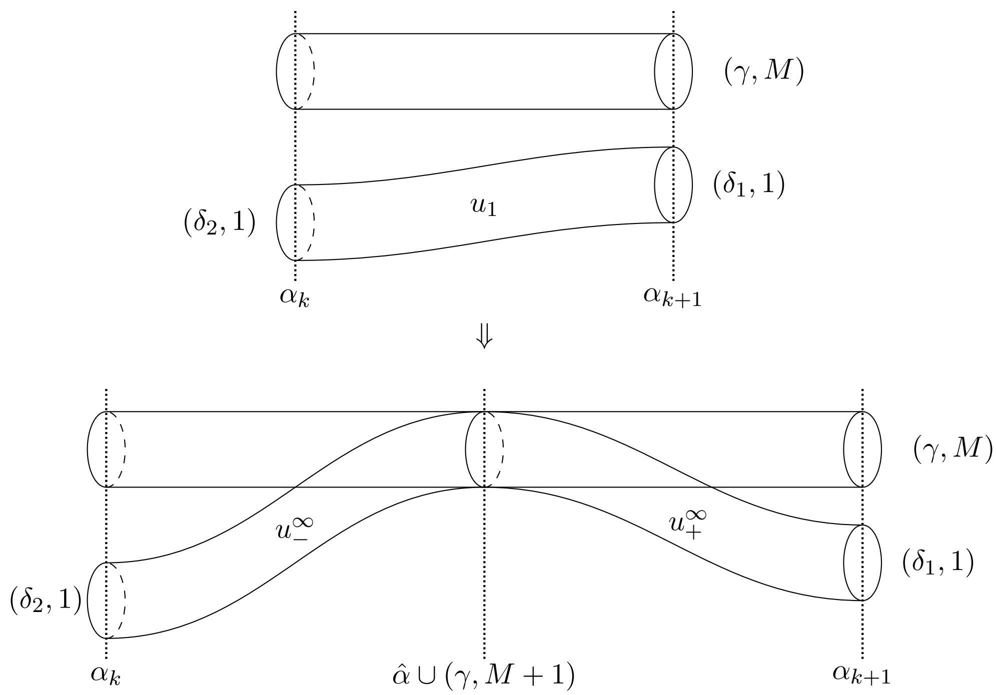
Now, we consider the behaviors of such J-holomorphic curves as $z \to \gamma$. By compactness argument (in this situation, for example see [H1, §9]), its some subsequence have a limiting J-holomorphic curve u_1^{∞} up to $\mathbb{R}$-action which may be splitting into two floors.

Suppose that $u_1^{\infty} \in \mathcal{M}^J(\delta_1, \delta_2)$, then by construction, it intersects with $\mathbb{R} \times \gamma$ but this contradicts the admissibility of curves of ECH index 2. So we may assume that the limiting curve has two floors. By construction, both have ends asymptotic to γ and same multiplicity.

we set $u_1^{\infty} = (u_-^{\infty}, u_+^{\infty})$ where $u_{\pm}^{\infty}$ are top and bottom curves up to $\mathbb{R}$-action respectively (see the below figure). The additivity and non negativeness of ECH index, we have $I(u_-^{\infty} \cup u_0) = I(u_+^{\infty} \cup u_0) = 1$ and thus the multiplicity of positive or negative ends of $u_{\pm}^{\infty}$ are one since $S_\theta \cap S_{-\theta} = \{1\}$. Then we have $u_-^{\infty} \cup u_0 \in \mathcal{M}^J(\hat{\alpha} \cup (\gamma, M+1), \alpha_k)$ and $u_+^{\infty} \cup u_0 \in \mathcal{M}^J(\alpha_{k+1}, \hat{\alpha} \cup (\gamma, M+1))$.

By definition assumption, $\hat{\alpha} \cup (\gamma, M+1)$ is an ECH generator and have no positive hyperbolic orbit. So by (1.9), $I(u_-^{\infty} \cup u_0) = I(u_+^{\infty} \cup u_0) = 0 \bmod 2$. This is a contradiction.

Here, we introduce another way to derive a contradiction from $u_-^{\infty} \cup u_0 \in \mathcal{M}^J(\hat{\alpha} \cup (\gamma, M+1), \alpha_k)$, $u_+^{\infty} \cup u_0 \in \mathcal{M}^J(\alpha_{k+1}, \hat{\alpha} \cup (\gamma, M+1))$ and $I(u_-^{\infty} \cup u_0) = I(u_+^{\infty} \cup u_0) = 1$. From the partition condition of admissible J-holomorphic curve at γ, we have $1 = \max(S_{-\theta} \cap \{1,\ 2, ...,\ M+1\}) = \max(S_\theta \cap \{1,\ 2, ...,\ M+1\})$. But by the assumption $M = E(\alpha_{k+1}) = E(\alpha_k) > p_1,\ q_1$, we have $\max(S_{-\theta} \cap \{1,\ 2, ...,\ M+1\}), \max(S_\theta \cap \{1,\ 2, ...,\ M+1\}) > p_1,\ q_1 > 1$. This is a contradiction.



Case3. $J_0(\alpha_{k+1}, \alpha_k) = 1$

In this case, $(g, k, l) = (0, 3, 0)$, $(0, 2, 1)$, $(1, 1, 0)$.

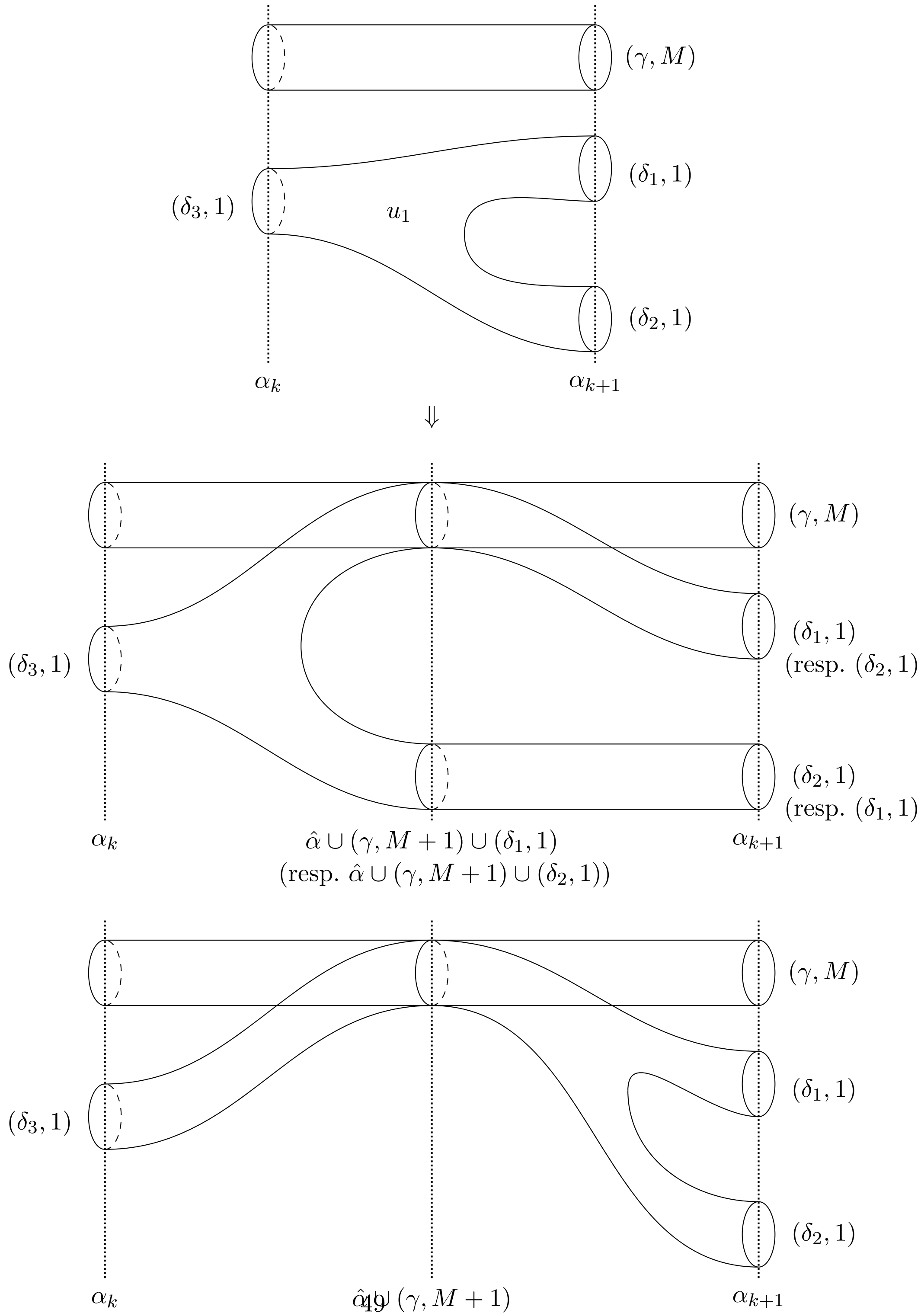
We can exclude the case $(g, k, l) = (1, 1, 0)$ in the same way as Case1. If $(g, k, l) = (0, 3, 0)$, we have $E(\alpha_{k+1}) = E(\alpha_k)$ just like the way explained in Case2. On the other hand, If $(g, k, l) = (0, 2, 1)$, the image of u_0 contains $\mathbb{R} \times \gamma$ and also one positive end or negative end of u_1 is asymptotic to γ and thus $E(\alpha_{k+1}) \neq E(\alpha_k)$. This implies that the pair α_{k+1}, α_k which $(g, k, l) = (0, 3, 0)$ or $(g, k, l) = (0, 2, 1)$ types J-holomorphic curves by the U-map can occur are mutually exclusive.

1. If $(g, k, l) = (0, 3, 0)$.

Let $u_0 \cup u_1 \in \mathcal{M}^J(\alpha_{k+1}, \alpha_k)$ be a J-holomorphic curve counted the by U-map. Since $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$, u_1 has either two positive ends and one negative end or one positive end and two negative ends. Without loss of generality, we may assume that u_1 has two positive ends asymptotic to δ_1, δ_2 and one negative end asymptotic to δ_3. Note that δ_1, δ_2 and δ_3 are mutually different because of the smallness of $A(\alpha_{k+1}) - A(\alpha_k)$.

In this notation. we have $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\delta_2, 1), (\delta_3, 1))$ for any non-trivial parts of J-holomorphic curve counted by $U\langle \alpha_{k+1} \rangle = \langle \alpha_k \rangle$ and also we write $\alpha_{k+1} = \hat{\alpha} \cup (\delta_2, 1) \cup (\delta_1, 1) \cup (\gamma, M)$, $\alpha_k = \hat{\alpha} \cup (\delta_3, 1) \cup (\gamma, M)$.

As $z \to \gamma$, we have three possibilities of splitting of u_1(see the below figure). In any case, this is a contradiction in the same reason as Case2.



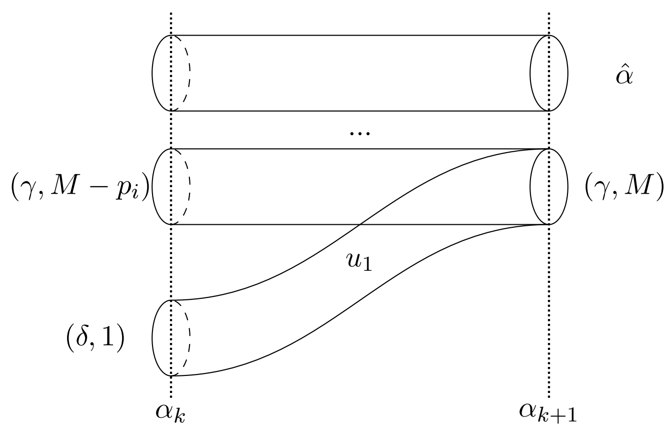
(γ, M)
$(\delta_1, 1)$
$(\delta_3, 1)$
u_1
$(\delta_2, 1)$
α_k
α_{k+1}
$\Downarrow$
(γ, M)
$(\delta_1, 1)$
(resp. $(\delta_2, 1)$)
$(\delta_3, 1)$
$(\delta_2, 1)$
(resp. $(\delta_1, 1)$)
α_k
$\hat{\alpha} \cup (\gamma, M+1) \cup (\delta_1, 1)$
(resp. $\hat{\alpha} \cup (\gamma, M+1) \cup (\delta_2, 1)$)
α_{k+1}
(γ, M)
$(\delta_1, 1)$
$(\delta_3, 1)$
$(\delta_2, 1)$
α_k
$\hat{\alpha} \cup (\gamma, M+1)$
α_{k+1}

2. If $(g,k,l)=(0,2,1)$.

From now on, we consider $(g,k,l)=(0,2,1)$. This case is more complicated but the way in this case also play an important role in §2.6 and beyond.

Since $A(\alpha_{k+1})-A(\alpha_k)<\epsilon$, u_1 has both positive and negative end. Moreover by definition, u_0 contains some covering of $\mathbb{R}\times\gamma$ and either positive or negative end of u_1 asymptotic to γ. Because symmetry allows the same argument, here we consider only the case that the positive end of u_1 is asymptotic to γ (see the below figure). Let $E(\alpha_{k+1})=M$. Then, By the admissibility of u, the multiplicity of positive end of u_1 at γ is $p_i:=\max(S_{-\theta}\cap\{1,\ 2,...,\ M\})$ and so $E(\alpha_k)=M-p_i$. Let δ be the orbit where the negative end of u_1 is asymptotic . By the discussion so far, we can see that for any generic point $z\in Y$, any non trivial parts of J-holomorphic curves through z counted by the U-map are in $\mathcal{M}^J((\gamma,p_i),(\delta,1))$.

We set $\alpha_{k+1}=\hat{\alpha}\cup(\gamma,M)$ and then $\alpha_k=\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i)$ where $\hat{\alpha}$ only contains negative hyperbolic orbits.



Claim 2.5.1. *In the above notation,*

$$I(\hat{\alpha}\cup(\gamma,M-1),\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1))=2 \tag{2.59}$$

Proof of Claim 2.5.1. Suppose that this claim is false. Since $A(\alpha_{k+1})-A(\alpha_k)=A(\hat{\alpha}\cup(\gamma,M-1))-A(\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1))>0$ and (2.43), we have

$$I(\hat{\alpha}\cup(\gamma,M-1),\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1))>2. \tag{2.60}$$

By considering the sixth condition in Lemma 2.4.5, there is an ECH generator ζ with $[\zeta]=[\hat{\alpha}\cup(\gamma,M-1)]=[\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1)]$ such that

$U\langle\hat{\alpha}\cup(\gamma,M-1)\rangle=\langle\zeta\rangle$ and $A(\hat{\alpha}\cup(\gamma,M-1))>A(\zeta)>A(\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1))$. This implies that $A(\alpha_{k+1})>A(\zeta\cup(\gamma,1))>A(\alpha_k)$. This contradicts (2.7).

□

Remark 2.5.2. In essence, Claim 2.5.1 and Claim 2.5.3 come from only their topological conditions and the properties of ECH index (in particular the equation (2.67); For example see [H1, Proof of Proposition 7.1]). But to understand the proof only with the facts in this paper as much as possible, we prove them by using some special conditions.

Claim 2.5.3. *In the above notation, for any $p_i\le N<p_{i+1}$*

$$I(\hat{\alpha}\cup(\gamma,N),\hat{\alpha}\cup(\delta,1)\cup(\gamma,N-p_i))=2. \tag{2.61}$$

Moreover

$$I(\hat{\alpha}\cup(\gamma,p_{i+1}),\hat{\alpha}\cup(\delta,1)\cup(\gamma,p_{i+1}-p_i))=4. \tag{2.62}$$

And for any $p_i<N\le p_{i+1}$,

$$J_0(\hat{\alpha}\cup(\gamma,N),\hat{\alpha}\cup(\delta,1)\cup(\gamma,N-p_i))=1, \tag{2.63}$$

Proof of Claim 2.5.3. Let $\{Z\}=H_2(Y;\hat{\alpha}\cup(\gamma,p_i),\hat{\alpha}\cup(\delta_1,1))$ and $\{Z_\gamma\}=H_2(Y;\gamma,\gamma)$ respectively. Then by the definition, we have

$$\begin{aligned}2=I(\alpha_{k+1},\alpha_k)=&I(\hat{\alpha}\cup(\gamma,M),\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i))\\=&c_1(\xi|_Z,\tau)+Q_\tau(Z,Z)+2(M-p_i)Q_\tau(Z,Z_\gamma)\\&+\sum_{k=M-p_i+1}^{M}(2\lfloor k\theta\rfloor+1)-\mu_\tau(\delta).\end{aligned} \tag{2.64}$$

Here, we use the property $Q_\tau(Z_1+Z_2,Z_1+Z_2)=Q_\tau(Z_1,Z_1)+2Q_\tau(Z_1,Z_2)+Q_\tau(Z_2,Z_2)$ in Definition 1.3.1 and $Q_\tau(Z_\gamma)=0$ and $Q_\tau(Z,mZ_\gamma)=mQ_\tau(Z,Z_\gamma)$. These properties easily follows from the definition of Q_τ (see [H1, Lemma 8.5]).

And also

$$\begin{aligned}2=I(\hat{\alpha}\cup(\gamma,M-1),&\hat{\alpha}\cup(\delta,1)\cup(\gamma,M-p_i-1))\\&=c_1(\xi|_Z,\tau)+Q_\tau(Z,Z)+2(M-p_i-1)Q_\tau(Z,Z_\gamma)\\&\quad+\sum_{k=M-p_i}^{M-1}(2\lfloor k\theta\rfloor+1)-\mu_\tau(\delta).\end{aligned} \tag{2.65}$$

By taking the difference from the above equations, we have

$$2Q_\tau(Z, Z_\gamma) + 2(\lfloor M\theta \rfloor - \lfloor (M - p_i)\theta \rfloor) = 0 \tag{2.66}$$

Note that for any $p_i \leq N < p_{i+1}$, $\lfloor N\theta \rfloor - \lfloor (N - p_i)\theta \rfloor = \lfloor p_i\theta \rfloor$ and moreover $\lfloor p_{i+1}\theta \rfloor - \lfloor (p_{i+1} - p_i)\theta \rfloor = \lfloor p_i\theta \rfloor + 1$. These facts are special cases of [H1, Lemma 4.5]. Hence

$$2Q_\tau(Z, Z_\gamma) + 2\lfloor p_i\theta \rfloor = 0 \tag{2.67}$$

On the other hand, in the same way, for any $p_i < N \leq p_{i+1}$ we have

$$\begin{aligned} I(\hat{\alpha} \cup (\gamma, N), &\hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i)) \\ &- I(\hat{\alpha} \cup (\gamma, N - 1), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i - 1)) \\ =&2Q_\tau(Z, Z_\gamma) + 2(\lfloor N\theta \rfloor - \lfloor (N - p_i)\theta \rfloor) \\ =&2(\lfloor N\theta \rfloor - \lfloor (N - p_i)\theta \rfloor - \lfloor p_i\theta \rfloor). \end{aligned} \tag{2.68}$$

This implies that for any $p_i \leq N < p_{i+1}$, $I(\hat{\alpha} \cup (\gamma, N), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i))$ are equal to each other and hence 2, moreover we have $I(\hat{\alpha} \cup (\gamma, p_{i+1}), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i)) = 4$.

In the same way, we have

$$\begin{aligned} J_0(\hat{\alpha} \cup (\gamma, N), &\hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i)) \\ &- J_0(\hat{\alpha} \cup (\gamma, N - 1), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i - 1)) \\ =&2Q_\tau(Z, Z_\gamma) + 2(\lfloor (N-1)\theta \rfloor - \lfloor (N - p_i - 1)\theta \rfloor) \\ =&2(\lfloor (N-1)\theta \rfloor - \lfloor (N - p_i - 1)\theta \rfloor - \lfloor p_i\theta \rfloor). \end{aligned} \tag{2.69}$$

This implies that for any $p_i < N \leq p_{i+1}$, $J_0(\hat{\alpha} \cup (\gamma, N), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, N - p_i)) = 1$.

We complete the proof of Claim 2.5.3. □

Since $I(\hat{\alpha} \cup (\gamma, p_{i+1}), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i)) = 4$, there is an unique ECH generator ζ such that $I(\hat{\alpha} \cup (\gamma, p_{i+1}), \zeta) = 2 = I(\zeta, \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i))$. Note that $U\langle \hat{\alpha} \cup (\gamma, p_{i+1}) \rangle = \langle \zeta \rangle$, $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i) \rangle$.

Claim 2.5.4. *The above ζ satisfies $E(\zeta) = 0$.*

Proof of Claim 2.5.4. Suppose that $E(\zeta) > 0$. Since $A(\hat{\alpha} \cup (\gamma, p_{i+1})) > A(\zeta) > A(\hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i))$, we also have $A(\hat{\alpha} \cup (\gamma, p_{i+1} - 1)) > A(\zeta - (\gamma, 1)) > A(\hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i - 1))$. Since (2.43), this indicates $I(\hat{\alpha} \cup (\gamma, p_{i+1} - 1), \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i - 1)) > 2$. This contradicts Claim 2.5.3. Therefore we complete the proof of Claim 2.5.4. □

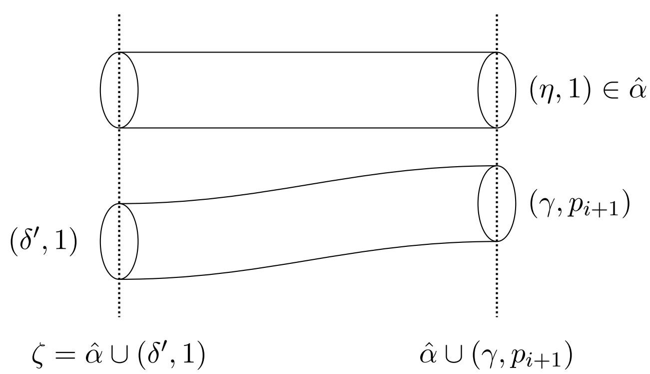
Since $J_0 \geq -1$ and additivity of J_0, we have $(J_0(\hat{\alpha}\cup(\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)))$=$(2, -1)$, $(1, 0)$, $(0, 1)$ or $(-1, 2)$.

If $(J_0(\hat{\alpha}\cup(\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)))$=$(2, -1)$ or $(-1, 2)$, we can derive a contradiction in the same way as Case1 since $A(\hat{\alpha}\cup(\gamma, p_{i+1}))-A(\zeta) < \epsilon$ and $A(\zeta) - A(\hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)) < \epsilon$. This is because $A(\hat{\alpha}\cup(\gamma, p_{i+1})) - A(\hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)) = A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$. Thus we have only to consider the case $(J_0(\hat{\alpha}\cup(\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)))$=$(1, 0)$ or $(0, 1)$. Here we note that the assumption $H(\alpha_k)$, $H(\alpha_{k+1}) > 4$ in Proposition 2.4.4 implies that $\hat{\alpha}$ contains more than four negative hyperbolic orbits. In these cases, we derive contradictions by using the splitting behaviors of J-holomorphic curves as follows.

(i). $(J_0(\hat{\alpha}\cup(\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i)))$=$(0, 1)$

From $A(\hat{\alpha}\cup(\gamma, p_{i+1})) - A(\zeta) < \epsilon$ and $E(\zeta) = 0$ and the partition condition of end, we have that there is a negative hyperbolic orbit δ' with $\delta' \notin \hat{\alpha}$ such that $\zeta = \hat{\alpha}\cup(\delta', 1)$. Moreover the nontrivial parts of any J-holomorphic curve counted by $U\langle\hat{\alpha}\cup(\gamma, p_{i+1})\rangle = \langle\zeta\rangle$ are of genus 0 and in $\mathcal{M}^J((\gamma, p_{i+1}), (\delta', 1))$.

2.5



Let us consider the behaviors of such curves as $z \to \eta$. In the same way as Case2, the limiting curve of such sequence splits and each of them has end at η. Furthermore both ECH indexes are one. Its multiplicities are two because of the admissibility of curves of ECH index 1. This implies that $|2A(\eta) - p_{i+1}R| < \epsilon$.

Consider back to the J-homolorphic curves of $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$. Its nontrivial parts are in $\mathcal{M}^J((\gamma, p_i), (\delta, 1))$. By considering the behaviors of this curves as $z \to \eta$, we have $|2A(\eta) - p_iR| < \epsilon$. By combining with $|2A(\eta) - p_{i+1}R| < \epsilon$, we have $(p_{i+1} - p_i)R < 2\epsilon$ and so $p_{i+1} = p_i$. This is a contradiction.

(ii). If $(J_0(\hat{\alpha} \cup (\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i)))$=(1,0)

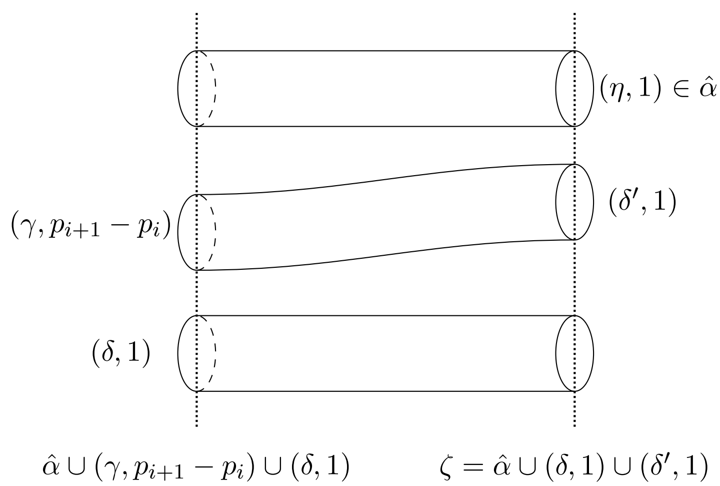
Consider the J-holomorphic curves counted by $U\langle\zeta\rangle = \langle\hat{\alpha}\cup(\delta, 1) \cup (\gamma, p_{i+1} - p_i))\rangle$. In the same way as the above argument, we can find that there is an hyperbolic orbit δ' such that $\zeta = \hat{\alpha}\cup(\delta, 1)\cup(\delta', 1)$ and that nontrivial parts of any J-holomorphic curve counted by $U\langle\zeta\rangle = \langle\hat{\alpha} \cup (\delta, 1) \cup (\gamma, p_{i+1} - p_i))\rangle$ are of genus 0 and in $\mathcal{M}^J((\delta', 1), (\gamma, p_{i+1} - p_i))$. And also we have $|2A(\eta) - p_iR| < \epsilon$ and $|2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$. Note that since $E(\alpha_{k+1}),\ E(\alpha_k) > p_1,\ q_1$, we have $p_i > 1$.

The next claim is obvious but often used later on. So we state here.

Claim 2.5.5. *Suppose that $q_i \in S_\theta$ (resp. $p_i \in S_{-\theta}$). If $q_i > 1$ (resp. $p_i > 1$), then $q_i \neq q_{i+1} - q_i$ (resp. $p_i \neq p_{i+1} - p_i$).*

Proof of Claim 2.5.5. Suppose that $q_i \in S_\theta$, then by Proposition 1.3.6, $q_{i+1} - q_i \in S_{-\theta}$ and since $S_\theta \cap S_{-\theta} = \{1\}$, if $q_i > 1$, we have $q_i \neq q_{i+1} - q_i$. We can do the same in the case p_i. □

From $|2A(\eta) - p_iR| < \epsilon$ and $|2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$, we have $|p_iR - (p_{i+1} - p_i)R| < 2\epsilon$. This implies that $p_i = p_{i+1} - p_i$ but this contradicts Claim 2.5.5.



Combining the above arguments, we complete the proof of Lemma 2.4.5. □

2.6 The properties of certain $J_0 = 2$ curves

To derive a contradiction from Proposition 2.4.4, at first we state Proposition 2.6.1 and use §2.6 and §2.7 to prove Proposition 2.6.1.

Notation

For any $a,\ b \in \mathbb{R}$, we write $a \approx b$ if $|a-b| < \frac{1}{100}\min\{A(\alpha)\,|\alpha \text{ is a Reeb orbit}\}$. In this notation, for $n,\ m \in \mathbb{Z}$ and $\tau > \frac{1}{100}$, if $n\tau R \approx m\tau R$, then $n = m$.

Proposition 2.6.1. *Let $\epsilon < \frac{1}{10^5}\min\{A(\alpha)\ |\alpha \text{ is a Reeb orbit }\}$ and k sufficiently large. Suppose that two ECH generators α_{k+1} and α_k with $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$ satisfy the following conditions.*

1. $J(\alpha_{k+1}, \alpha_k) = 2$
2. $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$
3. $E(\alpha_{k+1}),\ E(\alpha_k) \notin S_\theta \cup S_{-\theta}$
4. $E(\alpha_{k+1}),\ E(\alpha_k) > p_1,\ q_1$
5. $H(\alpha_{k+1}),\ H(\alpha_k) > 4.$

Let $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha_{k+1}, \alpha_k)$ be any J-holomorphic curve counted by the U-map.

Then one of the following conditions holds.

(a). *Let $E(\alpha_{k+1}) = M$ and $p_i := \max(S_{-\theta} \cap \{1,\ 2, ...,\ M\})$. Then there are two negative hyperbolic orbits δ_1, δ_2 and an ECH generator $\hat{\alpha}$ consisting of negative hyperbolic orbits such that $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M) \cup (\delta_1, 1)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_2, 1)$ and $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\gamma, p_i), (\delta_2, 1))$.*

Moreover, $A(\delta_1) \approx (p_{i+1} - p_i)R$, $A(\delta_2) \approx p_{i+1}R$ and for each $\eta \in \hat{\alpha}$, either $A(\eta) \approx \frac{1}{2}p_{i+1}R$ or $A(\eta) \approx \frac{1}{2}(p_{i+1} - p_i)R$.

(a'). *Let $E(\alpha_k) = M$ and $q_i := \max(S_\theta \cap \{1,\ 2, ...,\ M\})$. Then there are two negative hyperbolic orbits δ_1, δ_2 and an ECH generator $\hat{\alpha}$ consisting of negative hyperbolic orbits such that $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M) \cup (\delta_2, 1)$ and $u_1 \in \mathcal{M}^J((\delta_1, 1), (\delta_2, 1) \cup (\gamma, q_i))$.*

Moreover, $A(\delta_1) \approx q_{i+1}R$, $A(\delta_2) \approx (q_{i+1} - q_i)R$ and for each $\eta \in \hat{\alpha}$, either $A(\eta) \approx \frac{1}{2}q_{i+1}R$ or $A(\eta) \approx \frac{1}{2}(q_{i+1} - q_i)R$.

(b). *Let* $E(\alpha_{k+1}) = M$ *and* $p_i := \max(S_{-\theta} \cap \{1,\ 2, ...,\ M\})$*. Then* $\frac{3}{2}p_i = p_{i+1}$ *and there are two negative hyperbolic orbits* δ_1*,* δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M)$*,* $\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$ *and* $u_1 \in \mathcal{M}^J((\gamma, p_i), (\delta_1, 1) \cup (\delta_2, 1))$*.*

Moreover, $A(\delta_1) \approx \frac{1}{2}p_i R$*,* $A(\delta_2) \approx \frac{1}{2}p_i R$ *and for each* $\eta \in \hat{\alpha}$*, either* $A(\eta) \approx \frac{1}{2}p_i R$ *or* $A(\eta) \approx \frac{1}{4}p_i R$*.*

(b'). *Let* $E(\alpha_k) = M$ *and* $q_i := \max(S_\theta \cap \{1,\ 2, ...,\ M\})$*. Then* $\frac{3}{2}q_i = q_{i+1}$ *and there are two negative hyperbolic orbits* δ_1*,* δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$*,* $\alpha_k = \hat{\alpha} \cup (\gamma, M)$ *and* $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\delta_2, 1), (\gamma, q_i))$*.*

Moreover, $A(\delta_1) \approx \frac{1}{2}q_i R$*,* $A(\delta_2) \approx \frac{1}{2}q_i R$ *and for each* $\eta \in \hat{\alpha}$*, either* $A(\eta) \approx \frac{1}{2}q_i R$ *or* $A(\eta) \approx \frac{1}{4}q_i R$*.*

(c). *Let* $E(\alpha_{k+1}) = M$ *and* $p_i := \max(S_{-\theta} \cap \{1,\ 2, ...,\ M\})$*. Then* $\frac{4}{3}p_i = p_{i+1}$ *and there are two negative hyperbolic orbits* δ_1*,* δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M)$*,* $\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$ *and* $u_1 \in \mathcal{M}^J((\gamma, p_i), (\delta_1, 1) \cup (\delta_2, 1))$*.*

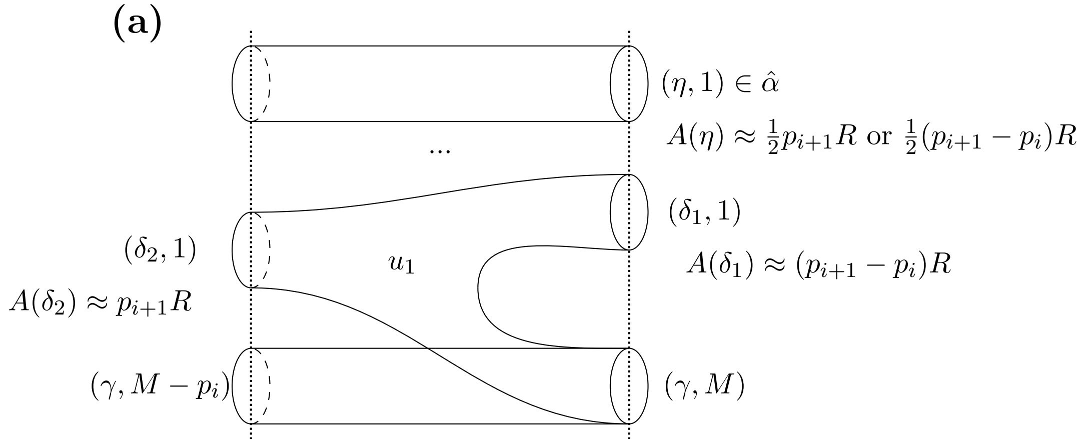
Moreover, $A(\delta_1) \approx \frac{2}{3}p_i R$*,* $A(\delta_2) \approx \frac{1}{3}p_i R$ *and for each* $\eta \in \hat{\alpha}$*, either* $A(\eta) \approx \frac{1}{2}p_i R$ *or* $A(\eta) \approx \frac{1}{6}p_i R$*.*

(c'). *Let* $E(\alpha_k) = M$ *and* $q_i := \max(S_\theta \cap \{1,\ 2, ...,\ M\})$*. Then* $\frac{4}{3}q_i = q_{i+1}$ *and there are and two negative hyperbolic orbits* δ_1*,* δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$*,* $\alpha_k = \hat{\alpha} \cup (\gamma, M)$ *and* $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\delta_2, 1), (\gamma, pq_i))$*. .*

Moreover, $A(\delta_1) \approx \frac{2}{3}q_i R$*,* $A(\delta_2) \approx \frac{1}{3}q_i R$ *and for each* $\eta \in \hat{\alpha}$*, either* $A(\eta) \approx \frac{1}{2}q_i R$ *or* $A(\eta) \approx \frac{1}{6}q_i R$*.*

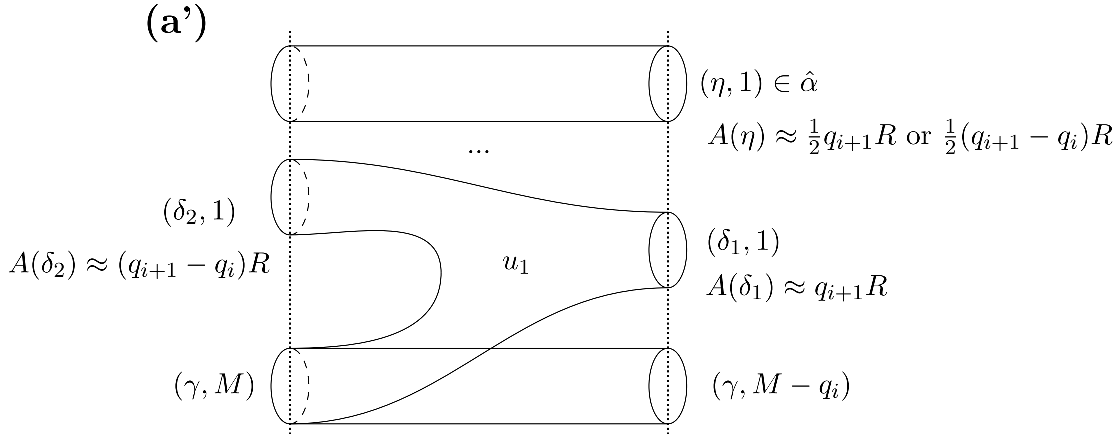
Note that ***(a)*** *and* ***(a')****,* ***(b)*** *and* ***(b')****,* ***(c)*** *and* ***(c')*** *are symmetrical respectively and,* ***(a)****,* ***(a')****,* ***(b)****,* ***(b')****,* ***(c)*** *and* ***(c')*** *are mutually exclusive.*

(a)



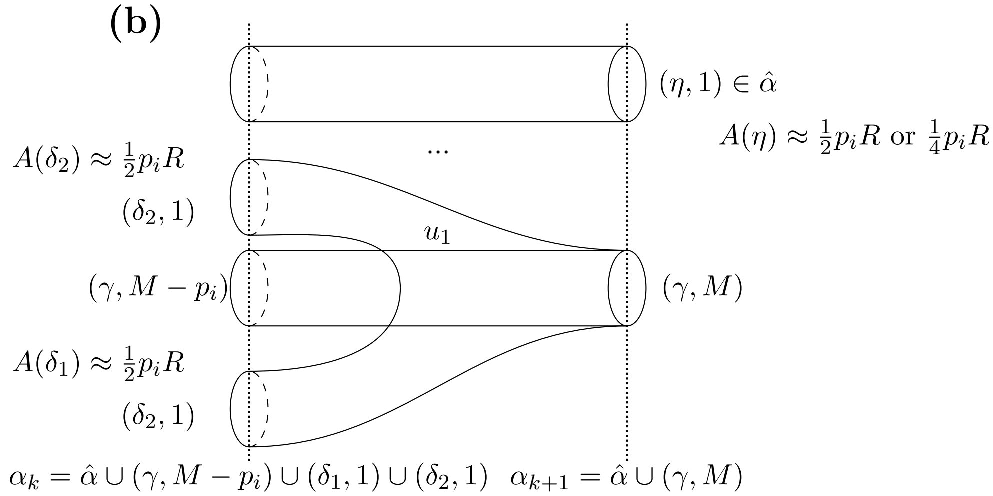
$\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_2, 1) \quad \alpha_{k+1} = \hat{\alpha} \cup (\gamma, M) \cup (\delta_1, 1)$

(a')



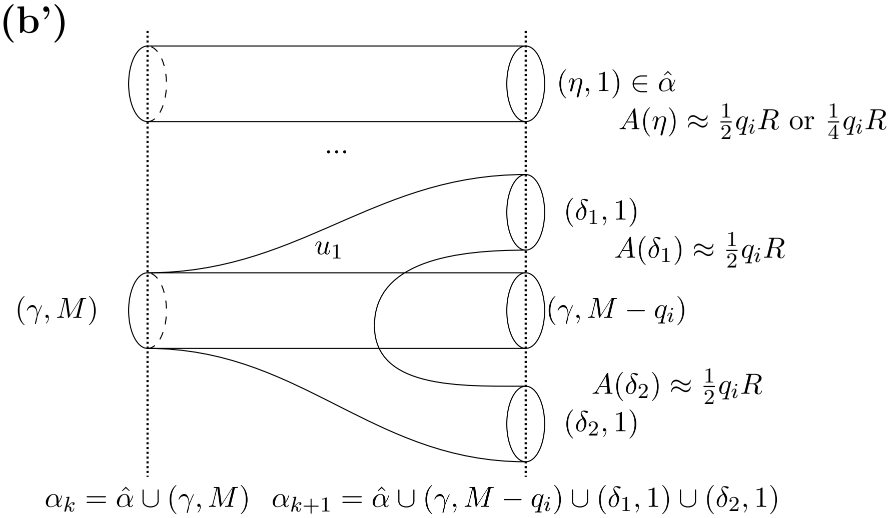
$\alpha_k = \hat{\alpha} \cup (\gamma, M) \cup (\delta_2, 1) \quad \alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1)$

(b)



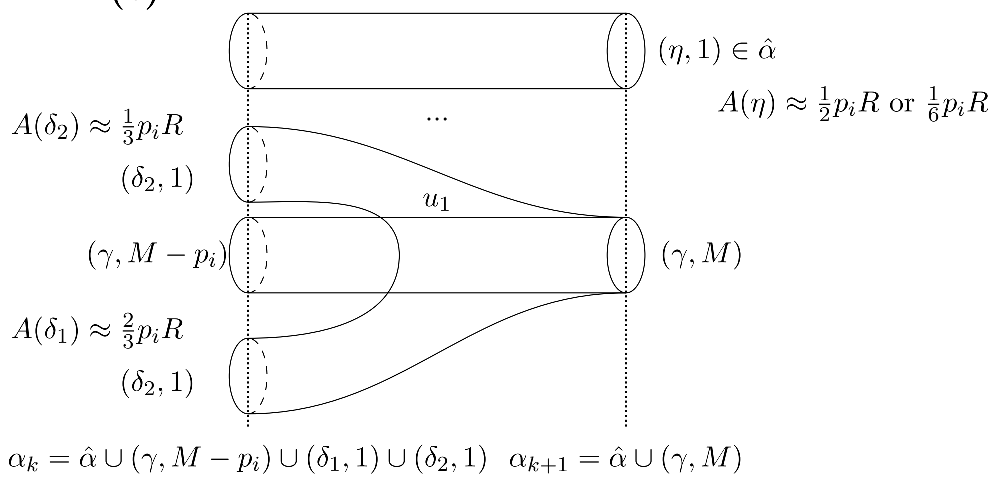
$(\eta, 1) \in \hat{\alpha}$
$A(\eta) \approx \frac{1}{2} p_i R$ or $\frac{1}{4} p_i R$
$A(\delta_2) \approx \frac{1}{2} p_i R$
...
$(\delta_2, 1)$
u_1
$(\gamma, M - p_i)$
(γ, M)
$A(\delta_1) \approx \frac{1}{2} p_i R$
$(\delta_2, 1)$
$\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$
$\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M)$

(b')

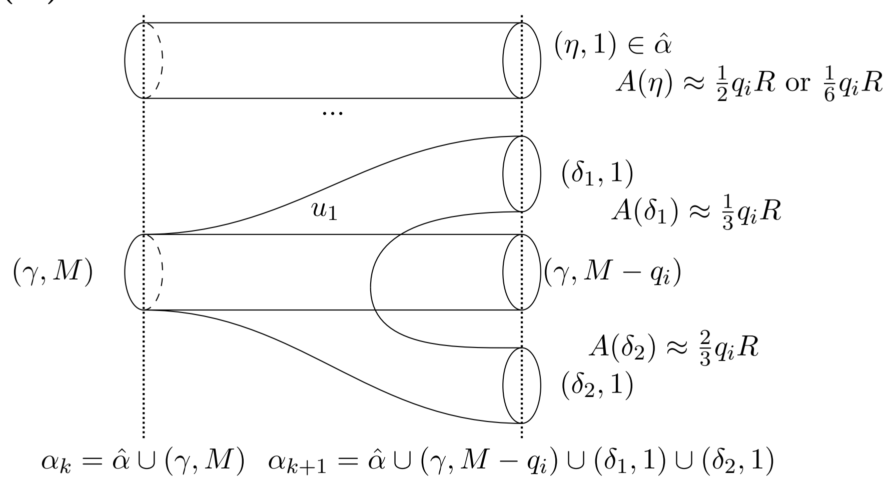


$(\eta, 1) \in \hat{\alpha}$
$A(\eta) \approx \frac{1}{2} q_i R$ or $\frac{1}{4} q_i R$
...
$(\delta_1, 1)$
u_1
$A(\delta_1) \approx \frac{1}{2} q_i R$
(γ, M)
$(\gamma, M - q_i)$
$A(\delta_2) \approx \frac{1}{2} q_i R$
$(\delta_2, 1)$
$\alpha_k = \hat{\alpha} \cup (\gamma, M)$
$\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$

(c)



(c')



2.6.1 Restriction of topological types of the J-holomorphic curves

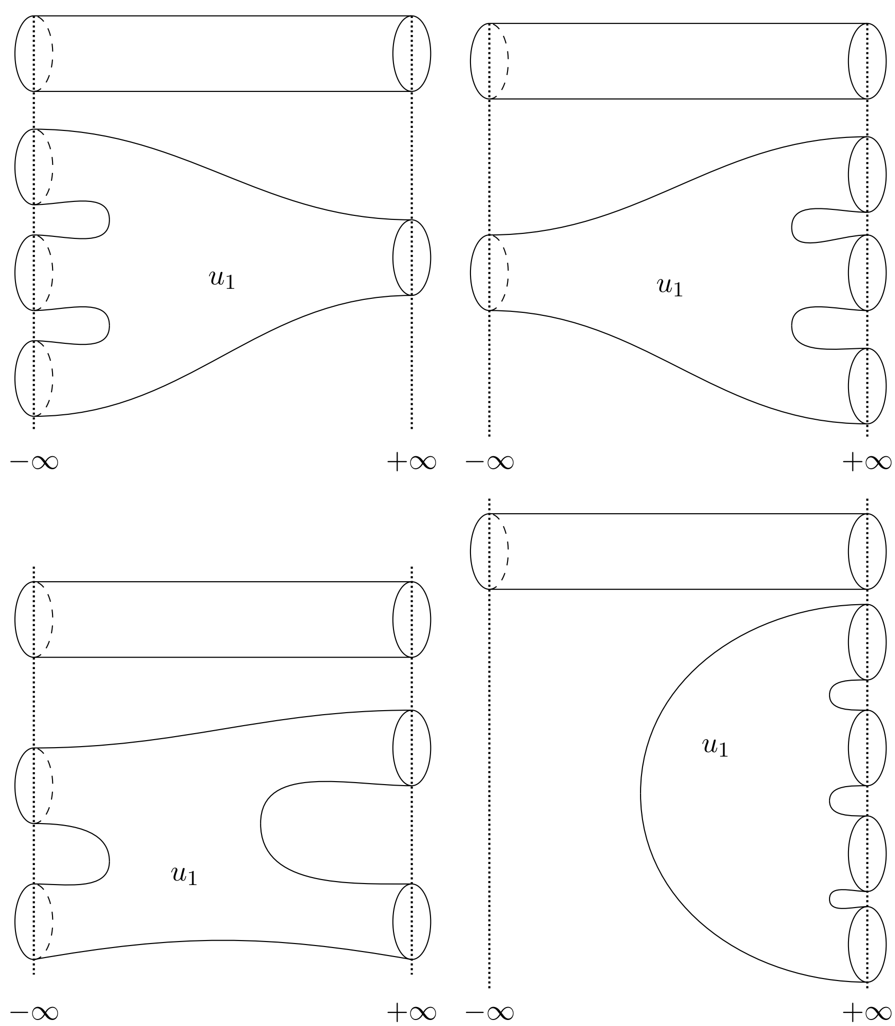
If $J_0(\alpha_{k+1},\alpha_k)=-2+2g+k+l=2$, each topological type of J-holomorphic curve counted by $U\langle\alpha_{k+1}\rangle=\langle\alpha_k\rangle$ is $(g,k,l)=(0,4,0)$, $(0,3,1)$, $(0,2,2)$,

$(1, 1, 1)$ or $(1, 2, 0)$. But in Proposition 2.6.1, only the type $(g, k, l) = (0, 3, 1)$ appears. So at first, we exclude the others.

As is the same with the case $J_0 \leq 1$, we make a list of topological types of $J_0 = 2$ as follows.

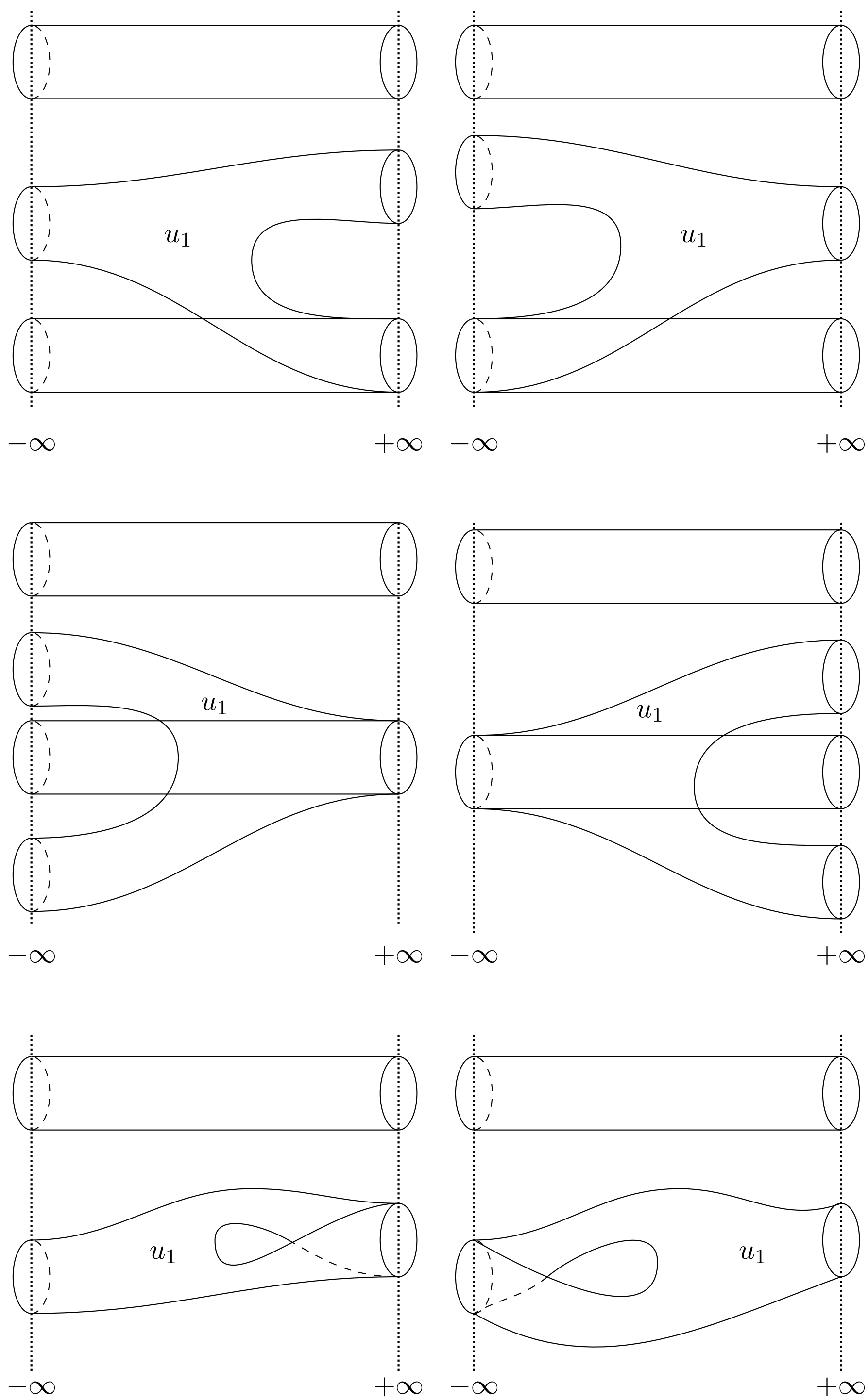
- $(g, k, l) = (0, 4, 0)$

  This case has four types as follows.



- $(g, k, l) = (0, 3, 1)$

  This case has eight types as follows.


u_1
u_1
$-\infty$
$+\infty$
$-\infty$
$+\infty$
u_1
u_1
$-\infty$
$+\infty$
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$+\infty$
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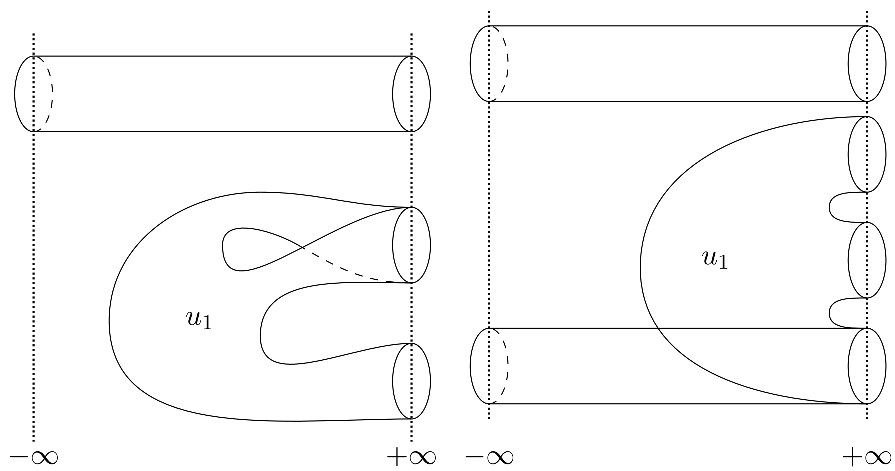
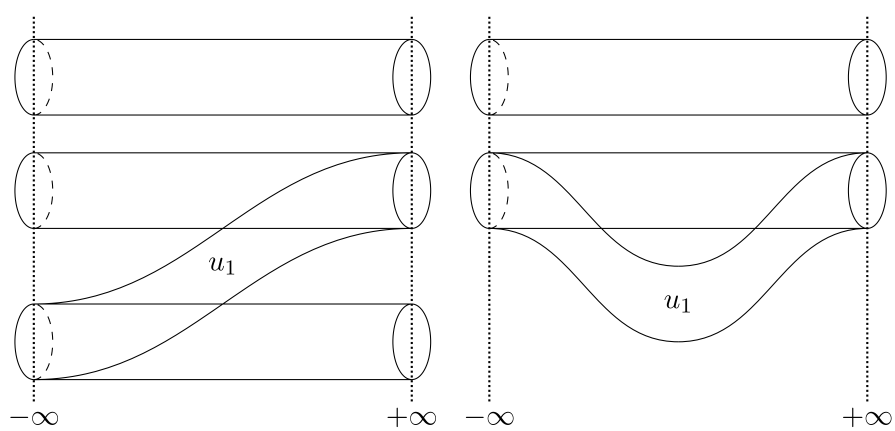
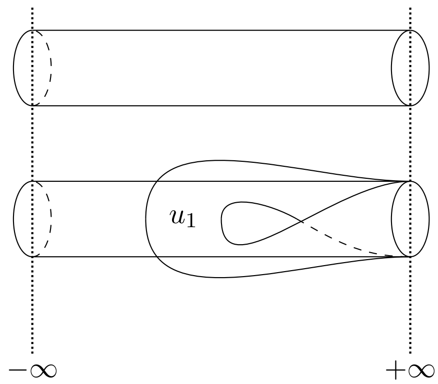
$-\infty$
$+\infty$

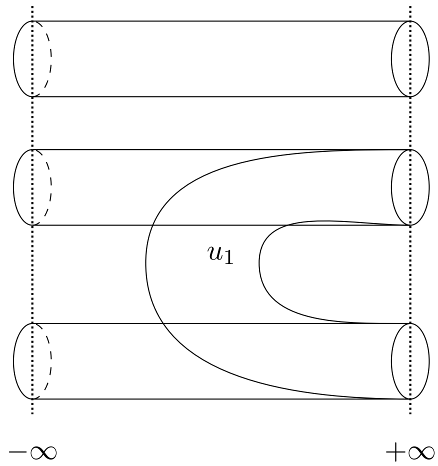


- $(g, k, l) = (0, 2, 2)$

This case has four types as follows. Note that under the assumption that there is only one simple elliptic orbit, the first and the fourth cases can not occur.

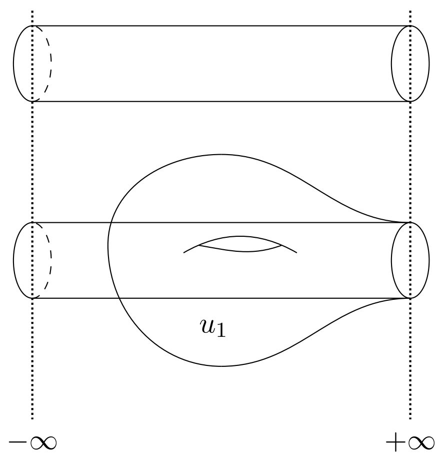






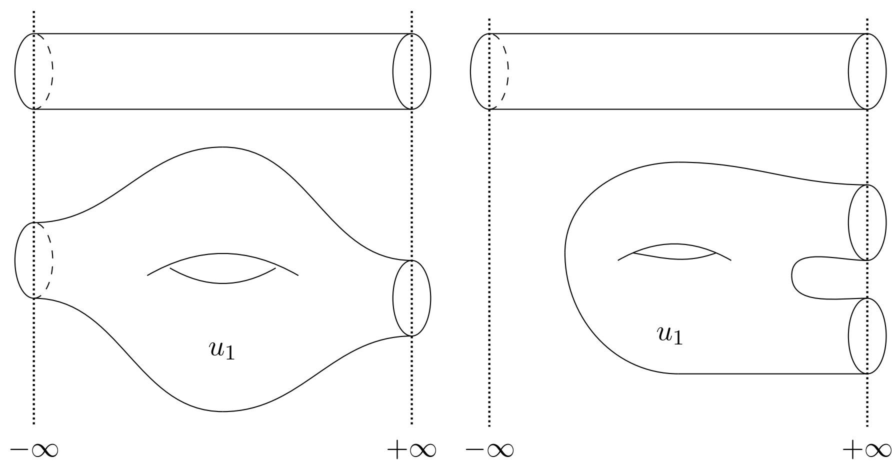
- $(g, k, l) = (1, 1, 1)$

  This case has only one type as follows.



- $(g, k, l) = (1, 2, 0)$

  This case has two types as follows.



Lemma 2.6.2. *Suppose that α_{k+1} and α_k satisfy the assumptions 1, 2, 3, 4 and 5 in Proposition 2.6.1. Let $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha_{k+1}, \alpha_k)$ be any J-holomorphic curve counted by $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$. Then u is $(g,k,l) = (0,3,1)$.*

Proof of Lemma 2.6.2. In the cases $(g,k,l) = (0,2,2)$, $(1,1,1)$, we can see from the topological types that $A(\alpha_{k+1}) - A(\alpha_k)$ have to be larger than some action of orbit. But this contradicts $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$.

From now on, we consider $(g,k,l) = (0,4,0)$, $(1,2,0)$. As a matter of fact, we can easily exclude these cases in almost the same way as Lemma 2.4.5. But to make sure, we explain how to do in detail.

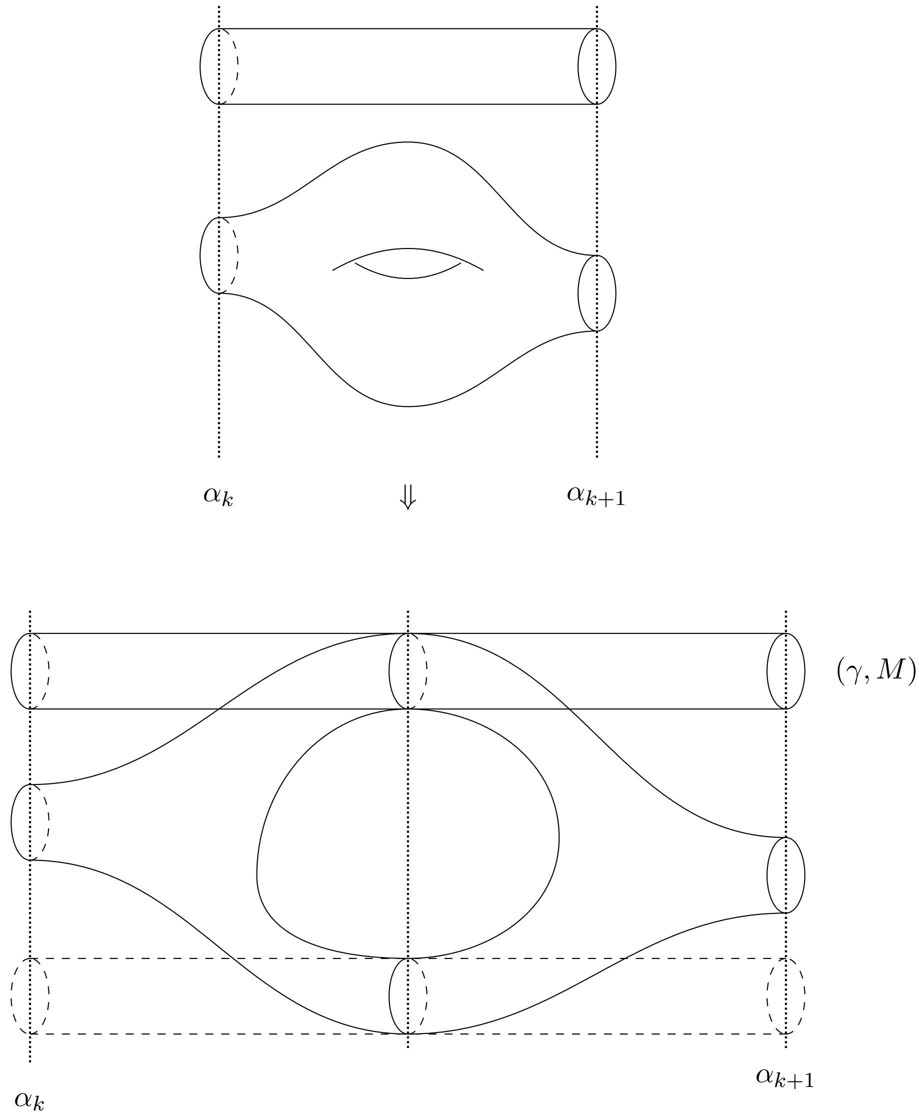
Since $l = 0$ and $E(\alpha_{k+1})$, $E(\alpha_k) \notin S_{-\theta} \cup S_\theta$, u_1 has no end asymptotic to γ and since $E(\alpha_{k+1})$, $E(\alpha_k) > p_1$, $q_1 > 0$, u_0 has some covering of $\mathbb{R} \times \gamma$. Let $z_i \to \gamma$. Then, we obtain a sequence of J-holomorphic curves u_1^i which are through z_i and either $(g,k) = (0,4)$ or $(1,2)$. Note that their topological types and orbit where their ends are asymptotic may change in the sequence.

At first, suppose that the sequence contains infinity many J-holomorphic curves whose topological types are $(g,k) = (0,4)$. By the compactness argument, there is an J- holomoprhic curve u_1^∞ which may be splitting into some floors. By its topological type and properties of ECH and Fredholm indexes, we can find that the number of floors are at most two. If u_1^∞ does not split, $\mathbb{R} \times \gamma \cap u_1^\infty \neq \emptyset$ and so $u_0 \cap u_1^\infty \neq \emptyset$. This contradict $I(u_0 \cup u_1^\infty) = 2$ and its admissibility. So we may assume that u_1^∞ has two floors and write $u_1^\infty = (u_-^\infty, u_+^\infty)$ up to $\mathbb{R}$-action. By the additivity of ECH index and Fredholm index, we have $I(u_0 \cup u_-^\infty) = I(u_0 \cup u_+^\infty) = 1$ and each non trivial part of $u_\pm^\infty$ are connected. Moreover, by the assumption of their topological type, one of the non trivial parts of $u^{\infty}_{\pm}$ is of genus 0 with one end of it asymptotic to γ. This indicates that the middle orbit set, that is, the orbit set consisting of orbits where positive ends of $u_0 \cup u^{\infty}_{-}$ are asymptotic is an ECH generator. This contradicts (1.9) and $I(u_0 \cup u^{\infty}_{-}) = I(u_0 \cup u^{\infty}_{+}) = 1$.

Next, suppose that the sequence contains infinity many J-holomorphic curves whose topological types are $(g, k) = (1, 2)$. In the same way, we have a splitting curve. Let $u^{\infty}_{\pm}$ be the curves in top and bottom floors of the splitting curve respectively. Then $I(u_0 \cup u^{\infty}_{-}) = I(u_0 \cup u^{\infty}_{+}) = 1$. By geometric observation, u^{∞}_{+} has one or two negative ends and moreover at least one of them is elliptic. Also u^{∞}_{-} has one or two positive ends and at least one of them is elliptic. If all negative ends of u^{∞}_{-} are elliptic, this contradicts (1.9). Also if all positive ends of u^{∞}_{+} are elliptic, this contradicts (1.9). This means that the number of negative ends of u^{∞}_{+} and that one of positive ends of u^{∞}_{-} are both two and only one of them is asymptotic to the elliptic orbit respectively. Moreover, their multiplicities are the same (see the below figure).

Let $E(\alpha_{k+1}) = E(\alpha_k) = M$. Then the total multiplicity of the middle orbit set is $M + 1$ and by the partition condition, we have $1 = \max(S_{-\theta} \cap \{1,\ 2, ...,\ M+1\}) = \max(S_{\theta} \cap \{1,\ 2, ...,\ M+1\})$. But this is a contradiction because of the assumption $M > q_1,\ p_1$.

Combining the above argument, we complete the proof of Lemma 2.6.2.



□

More precisely, we have the next lemma.

Lemma 2.6.3. *Suppose that α_{k+1} and α_k satisfy the assumptions 1, 2, 3, 4 and 5 in Proposition 2.6.1. Let $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha_{k+1}, \alpha_k)$ be any J-holomorphic curve counted by $U\langle \alpha_{k+1} \rangle = \langle \alpha_k \rangle$. Then α_{k+1}, α_k and u hold one of the following conditions.*

(A). *Let* $E(\alpha_{k+1}) = M$ *and* $p_i := \max(S_{-\theta} \cap \{1,\ 2, ...,\ M\})$. *Then there are two negative hyperbolic orbits* δ_1, δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M) \cup (\delta_1, 1)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_2, 1)$ *and* $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\gamma, p_i), (\delta_2, 1))$.

(A'). *Let* $E(\alpha_k) = M$ *and* $q_i := \max(S_{\theta} \cap \{1,\ 2, ...,\ M\})$. *Then there are two negative hyperbolic orbits* δ_1, δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M) \cup (\delta_2, 1)$ *and* $u_1 \in \mathcal{M}^J((\delta_1, 1), (\delta_2, 1) \cup (\gamma, q_i))$.

(B). *Let* $E(\alpha_{k+1}) = M$ *and* $p_i := \max(S_{-\theta} \cap \{1,\ 2, ...,\ M\})$. *There are two negative hyperbolic orbits* δ_1, δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$ *and* $u_1 \in \mathcal{M}^J((\gamma, p_i), (\delta_1, 1) \cup (\delta_2, 1))$.

(B'). *Let* $E(\alpha_k) = M$ *and* $q_i := \max(S_{\theta} \cap \{1,\ 2, ...,\ M\})$. *There are two negative hyperbolic orbits* δ_1, δ_2 *and an ECH generator* $\hat{\alpha}$ *consisting of negative hyperbolic orbits such that* $\alpha_{k+1} = \hat{\alpha} \cup (\gamma, M - q_i) \cup (\delta_1, 1) \cup (\delta_2, 1)$, $\alpha_k = \hat{\alpha} \cup (\gamma, M)$ *and* $u_1 \in \mathcal{M}^J((\delta_1, 1) \cup (\delta_2, 1), (\gamma, q_i))$.

Proof of Lemma 2.6.3. Since $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$, u_1 has at least one negative end. Moreover, at least one end of u_1 have to be asymptotic to some negative hyperbolic orbit because the fact causes a contradiction that if all ends are asymptotic to γ, the value $A(\alpha_{k+1}) - A(\alpha_k)$ have to be lager than or equal to $A(\gamma)$. From the assumptions $E(\alpha_{k+1})$, $E(\alpha_k) > p_1$, $q_1 > 0$ and the partition conditions of the ends, we have Lemma 2.6.3. □

2.6.2 Restriction of J_0 combinations

In the previous subsection, we decided the topological type of J-holomorphic curves counted by $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$. To prove Proposition 2.6.1, we have to decide the approximate relations in the actions of the orbits in α_{k+1} and α_k.

From now on, because symmetry allows the same argument, , we only consider the cases (A) and (B).

The next claim is almost the same as Claim 2.5.3.

Claim 2.6.4. *In the case of (A) in Lemma 2.6.3,*

(A). *For any* $p_i \leq N < p_{i+1}$,

$$I(\hat{\alpha} \cup (\gamma, N) \cup (\delta_1, 1), \hat{\alpha} \cup (\gamma, N - p_i) \cup (\delta_2, 1)) = 2. \tag{2.70}$$

Moreover

$$I(\hat{\alpha}\cup(\gamma,p_{i+1})\cup(\delta_1,1),\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))=4. \tag{2.71}$$

And for any $p_i < N \leq p_{i+1}$,

$$J_0(\hat{\alpha}\cup(\gamma,N)\cup(\delta_1,1),\hat{\alpha}\cup(\gamma,N-p_i)\cup(\delta_2,1))=1. \tag{2.72}$$

In the case of (B) in Lemma 2.6.3,

(B). *For any* $p_i \leq N < p_{i+1}$,

$$I(\hat{\alpha}\cup(\gamma,N),\hat{\alpha}\cup(\gamma,N-p_i)\cup(\delta_1,1)\cup(\delta_2,1))=2. \tag{2.73}$$

Moreover

$$I(\hat{\alpha}\cup(\gamma,p_{i+1}),\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1))=4. \tag{2.74}$$

And for any $p_i < N \leq p_{i+1}$,

$$J_0(\hat{\alpha}\cup(\gamma,N),\hat{\alpha}\cup(\gamma,N-p_i)\cup(\delta_1,1)\cup(\delta_2,1))=1. \tag{2.75}$$

Proof of Claim 2.6.4. We can prove this in the same way as Claim 2.5.1 and Claim 2.5.3. □

Claim 2.6.5. *In the case of (A) in Lemma 2.6.3,*

(A). *There is an ECH generator* ζ *with* $U\langle\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ *and* $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1)\rangle$. *Moreover,* $E(\zeta)=0$.

In the case of (B) in Lemma 2.6.3,

(B). *There is an ECH generator* ζ *with* $U\langle\hat{\alpha}\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ *and* $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1)\rangle$. *Moreover,* $E(\zeta)=0$.

Proof of Claim 2.6.5. In the same way as Claim 2.5.4 and just before that. □

Since $J_0\geq -1$, there are five possibilities, $(J_0(\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1}),\zeta),J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1)))$(resp. $(J_0(\hat{\alpha}\cup(\gamma,p_{i+1}),\zeta),J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1)))) = (3,-1)$, $(2,0)$, $(1,1)$, $(0,2)$ or $(-1,3)$. But except for $(1,1)$, the behaviors of J-holomorphic curves counted by the U-map cause contradictions.

Now, we use the rest of §2.6 to prove the next lemma.

Lemma 2.6.6. *Under the notation in Claim 2.6.5, J-holomorphic curves counted by the U-map cause contradictions except for* $(J_0(\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1}),\zeta), J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))) = (J_0(\hat{\alpha}\cup(\gamma,p_{i+1}),\zeta), J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1))) = (1,1)$.

Proof of Lemma 2.6.6. At first, we can easily exclude the cases $(J_0(\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1}),\zeta), J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))) = (3,-1)$, $(-1,3)$ because of their smallness of the difference of their actions.

Next, we will consider the cases (A), (B) respectively.

Case (A).

At first, we consider the splitting behaviors of J-holomorphic curve counted by $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$ as $z\to\eta$ for some fixed $\eta\in\hat{\alpha}$. Then there are three possibilities of splitting of holomorphic curves and also we have three possibilities of approximate relations as follows.

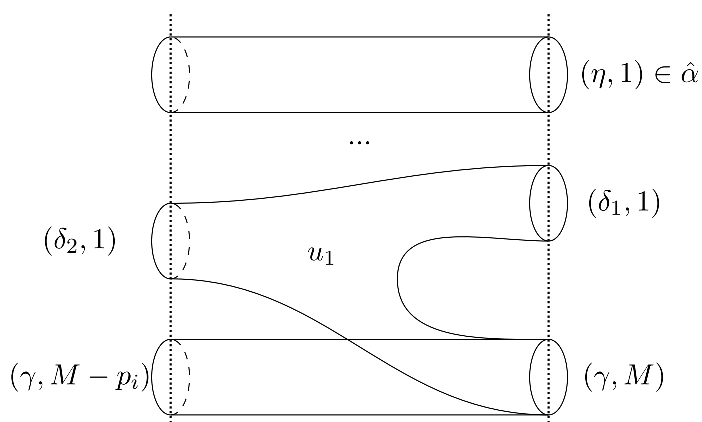
$(\mathfrak{a}_1)$. $|A(\delta_1)-2A(\eta)|<\epsilon$

$(\mathfrak{a}_2)$. $|A(\delta_2)-2A(\eta)|<\epsilon$

$(\mathfrak{a}_3)$. $|p_iR-2A(\eta)|<\epsilon$.

Moreover, we always have

(♣) $A(\alpha_{k+1})-A(\alpha_k)=|A(\delta_1)+p_iR-A(\delta_2)|<\epsilon$.

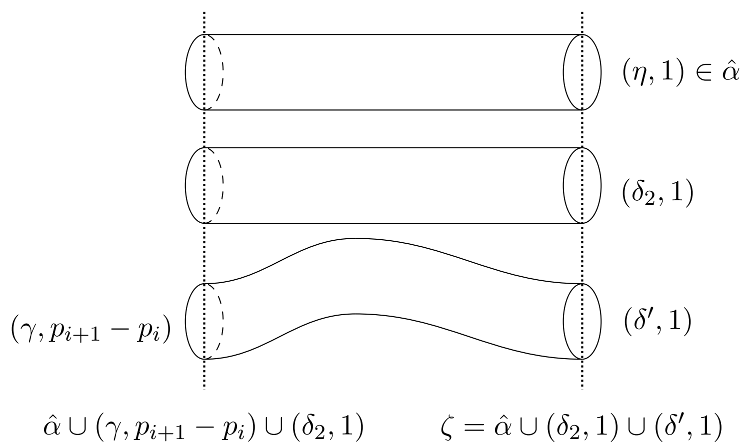


$\alpha_k=\hat{\alpha}\cup(\gamma,M-p_i)\cup(\delta_2,1)$ $\alpha_{k+1}=\hat{\alpha}\cup(\gamma,M)\cup(\delta_1,1)$

(i). If $(J_0(\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1}),\zeta), J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))) = (2,0)$

Note that genus of each J-holomorphic curve counted by $U\langle\zeta\rangle = \langle\hat{\alpha}\cup(\gamma, p_{i+1}-p_i)\cup(\delta_2,1))\rangle$ are 0. Moreover, each curve has both negative end covering at γ with multiplicity $p_{i+1}-p_i$ and positive end covering at negative hyperbolic orbit $(\delta',1)$ which is not equivalent to δ_2 because $A(\zeta)-A(\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))<\epsilon$ and so in $\mathcal{M}^J((\delta',1),(\gamma,p_{i+1}-p_i))$.

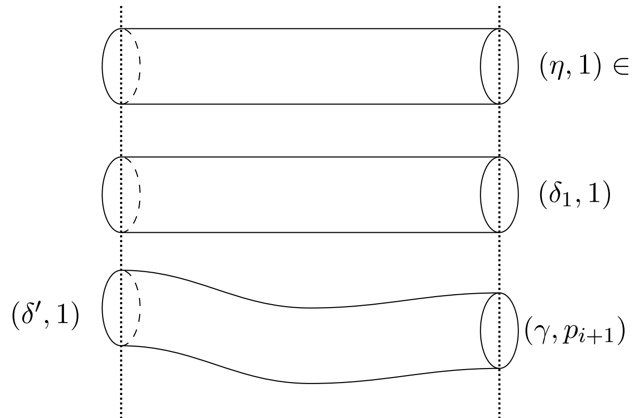
Then from the splitting behavior as $z\to\delta_2$, we have $|2A(\delta_2)-(p_{i+1}-p_i)R|<\epsilon$ and also as $z\to\eta\in\hat{\alpha}$, we have $|2A(\eta)-(p_{i+1}-p_i)R|<\epsilon$. These two relations indicate that $|A(\eta)-A(\delta_2)|<\epsilon$. Since $|A(\delta_1)+p_iR-A(\delta_2)|<\epsilon$, we have $A(\delta_2)>p_iR$, $A(\delta_1)$ and hence $A(\eta)>p_iR$, $A(\delta_1)$. These relations contradict $(\mathfrak{a}_1)$, $(\mathfrak{a}_2)$ and $(\mathfrak{a}_3)$ in any case. Therefore, this case can not occur.



(ii). If $(J_0(\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1}),\zeta), J_0(\zeta,\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1)))=(0,2)$

In the same way as above, J-holomorphic curves counted by $U\langle\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ are of genus 0 and have both negative end covering at γ with multiplicity p_{i+1} and positive end covering one negative hyperbolic orbit $(\delta',1)$ which is not equivalent to δ_1 and so in $\mathcal{M}^J((\gamma,p_{i+1}),(\delta',1))$.

Then from the splitting behavior as $z\to\delta_1$, we have $|2A(\delta_1)-p_{i+1}R|<\epsilon$ and also as $z\to\eta$, we have $|2A(\eta)-p_{i+1}R|<\epsilon$. Since $(\clubsuit).|A(\delta_1)+p_iR-A(\delta_2)|<\epsilon$, we have $|(\frac{1}{2}p_{i+1}+p_i)R-A(\delta_2)|<\frac{3}{2}\epsilon$. These relations contradict $(\mathfrak{a}_1)$, $(\mathfrak{a}_2)$ and $(\mathfrak{a}_3)$ in any case. Here, we use Claim 2.5.5 implicitly.



$\zeta = \hat{\alpha} \cup (\delta_1, 1) \cup (\delta', 1)$ $\qquad$ $\hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1)$

By the arguments so far, we can see that only the case $(J_0(\hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))) = (1, 1)$ may occur.

Case (B). At first, we consider the splitting behaviors of J-holomorphic curves counted by $U\langle \alpha_{k+1} \rangle = \langle \alpha_k \rangle$ as $z \to \eta$ for some $\eta \in \hat{\alpha}$. Then there are three possibilities of splitting of holomorphic curve and also we have three possibilities of approximate relations as follows.

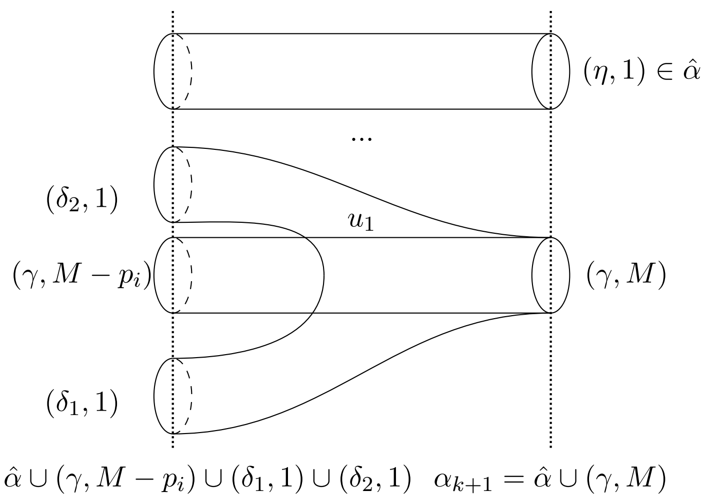
$(\mathfrak{b}_1)$. $|A(\delta_1) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_2)$. $|A(\delta_2) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_3)$. $|p_i R - 2A(\eta)| < \epsilon$.

Moreover, we always have

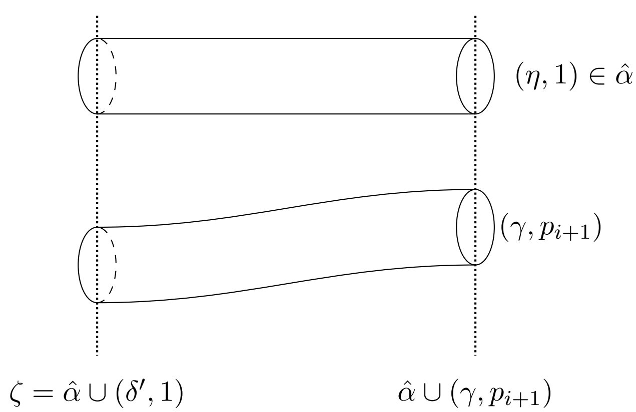
$(\spadesuit)$. $A(\alpha_{k+1}) - A(\alpha_k) = |p_i R - (A(\delta_1) + A(\delta_2))| < \epsilon$.



(i). If $(J_0(\hat{\alpha} \cup (\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1))) = (0, 2)$

In the same as before, J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\gamma, p_{i+1}) \rangle = \langle \zeta \rangle$ are of genus 0 and in $\mathcal{M}^J((\gamma, p_{i+1}), (\delta', 1))$ for some negative hyperbolic orbit δ' with $\zeta = \hat{\alpha} \cup (\delta', 1)$.

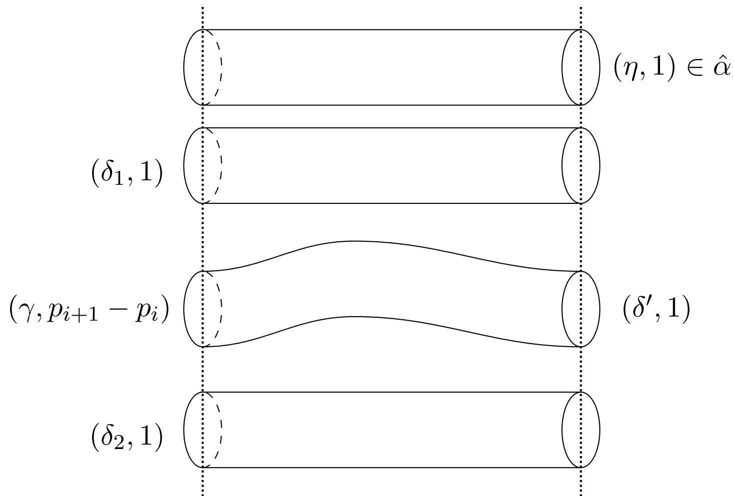
Then from the splitting behavior as $z \to \eta \in \hat{\alpha}$, we have $|2A(\eta) - p_{i+1}R| < \epsilon$. Since (♠). $|p_i R - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $p_i R > A(\delta_1)$, $A(\delta_2)$. So we can easily see that the relations contradict $(\mathfrak{b}_1)$, $(\mathfrak{b}_2)$ and $(\mathfrak{b}_3)$ in any case. Therefore this case can not occur.



(ii). If $(J_0(\hat{\alpha} \cup (\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1))) = (2, 0)$

In the same as before, J-holomorphic curves counted by $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)) \rangle$ are of genus 0 and in $\mathcal{M}^J((\delta', 1), (\gamma, p_{i+1} - p_i))$ for some negative hyperbolic orbit δ' with $\zeta = \hat{\alpha} \cup (\delta', 1) \cup (\delta_1, 1) \cup (\delta_2, 1)$.

Then from the splitting behavior as $z \to \delta_1,\ \delta_2$, we have $|2A(\delta_1) - (p_{i+1} - p_i)R| < \epsilon$ and $|2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$. From these relations, we also have $|(A(\delta_1) + A(\delta_2)) - (p_{i+1} - p_i)R| < \epsilon$. By combining with (♠).$|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $|p_iR - (p_{i+1} - p_i)R| < \epsilon$ and so $p_i = p_{i+1} - p_i$. This contradicts Claim 2.5.5.



$\hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)\ \ \zeta = \hat{\alpha} \cup (\delta_1, 1) \cup (\delta_2, 1) \cup (\delta', 1)$

By the arguments so far, we can see that only the case $(J_0(\hat{\alpha} \cup (\gamma, p_{i+1}), \zeta), J_0(\zeta, \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1))) = (1, 1)$ may occur. □

2.7 Calculations of the approximate values of the actions of the orbits

In this section, we compute the approximate values of the actions of the orbits and complete the proof of Proposition 2.6.1 under the result obtained so far.

In the same way as before, the splitting behaviors of J-holomorphic curves counted by the U-map play an important role.

In this section, we consider (A) and (B) in Lemma 2.6.3 respectively.

2.7.1 Type (A)

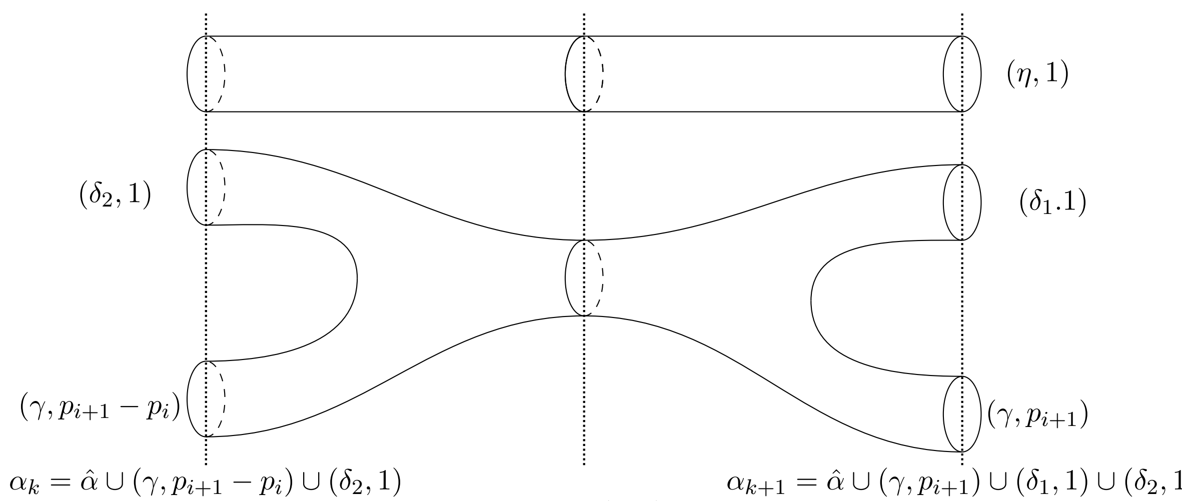
Since $E(\zeta) = 0$, $p_{i+1} \in S_{-\theta}$ and $p_{i+1} - p_i \in S_\theta$, the topological types of J-holomoprhic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ are both $(g, k, l) = (0, 3, 0)$.

By considering which ends of J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ correspond to ones of J-holomorphic curves counted by $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ respectively, we can see that there are three pairs of ζ, J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ as follows.

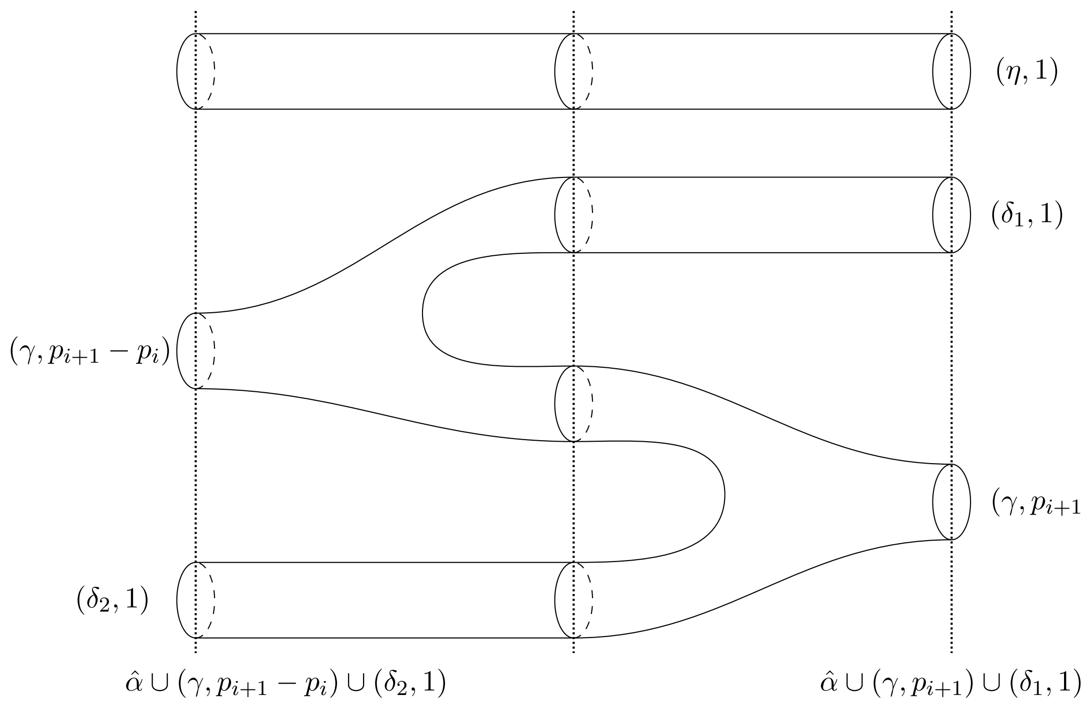
Type (A_1) There is a negative hyperbolic orbit δ' with $\zeta = \hat{\alpha} \cup (\delta', 1)$. Moreover any J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ are in $\mathcal{M}^J((\delta_1, 1) \cup (\gamma, p_{i+1}), (\delta', 1))$ and $\mathcal{M}^J((\delta', 1), (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))$ respectively.

Type (A_2) There is a negative hyperbolic orbit δ' with $\zeta = \hat{\alpha} \cup (\delta', 1) \cup (\delta_1, 1) \cup (\delta_2, 1)$. Moreover any J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ are in $\mathcal{M}^J((\gamma, p_{i+1}), (\delta', 1) \cup (\delta_2, 1))$ and $\mathcal{M}^J((\delta', 1) \cup (\delta_1), (\gamma, p_{i+1} - p_i))$ respectively.

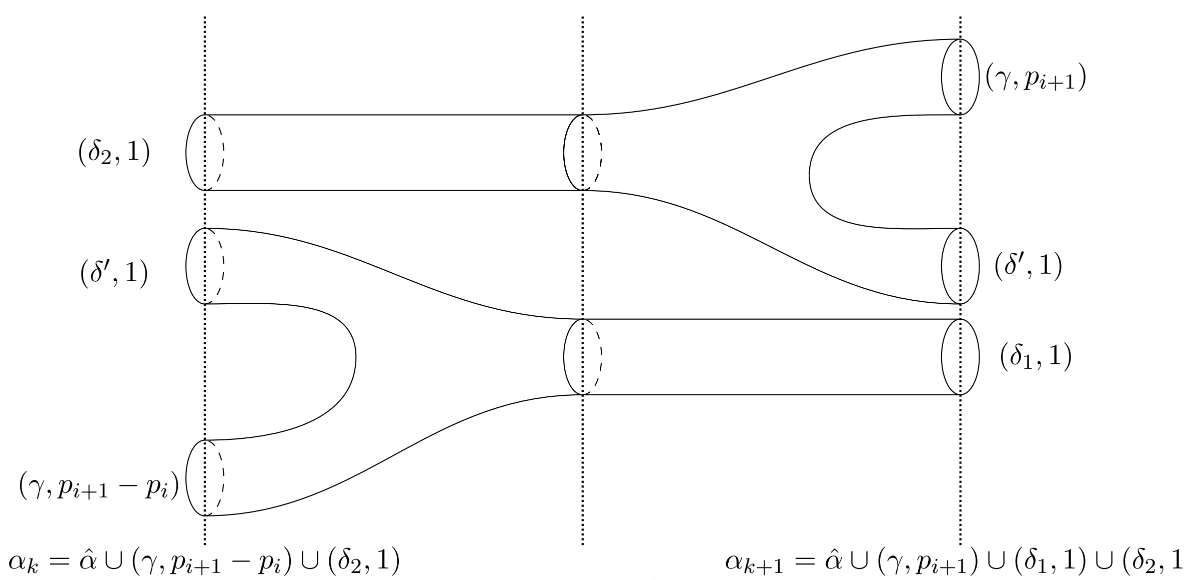
Type (A_3) There is a negative hyperbolic orbit δ' with $\delta' \in \hat{\alpha}$ such that any J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ are in $\mathcal{M}^J((\gamma, p_{i+1}) \cup (\delta', 1), (\delta_2, 1))$ and in $\mathcal{M}^J((\delta_1), (\gamma, p_{i+1} - p_i) \cup (\delta', 1))$ respectively.



Type (A_1)



Type (A_2)



Type (A_3)

To prove Proposition 2.6.1, at first, we calculate the approximate actions of δ_1, δ_2 and $\eta \in \hat{\alpha}$ from (A_1) by using the splitting behavior of J-holomorphic curves. Next, we will show that any splitting behaviors of (A_2) and (A_3) cause contradictions.

Type (A_1)

Lemma 2.7.1. *Suppose that the pair of ζ, J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ is Type (A_1). Then, we have $A(\delta_1) \approx (p_{i+1} - p_i)R$, $A(\delta_2) \approx p_{i+1}R$ and for each $\eta \in \hat{\alpha}$, either $A(\eta) \approx \frac{1}{2}p_{i+1}R$ or $A(\eta) \approx \frac{1}{2}(p_{i+1} - p_i)R$.*

Proof of Lemma 2.7.1. We will consider the behavior of J-holomorphic curves counted by U-map as $z \to \eta$. Then each J-holomorphic curve counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$ have three possibilities of splitting and we have three estimates respectively as follows.

From $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$, we have

$(\mathfrak{c}_1)$. $|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$

$(\mathfrak{c}_2)$. $|A(\delta_2) - 2A(\eta)| < \epsilon$

$(\mathfrak{c}_3)$. $|(p_{i+1} - p_i)R + A(\delta_2) - 2A(\eta)| < \epsilon$

From $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle \zeta \rangle$, we have

$(\mathfrak{d}_1)$. $|A(\delta_1) - 2A(\eta)| < \epsilon$

$(\mathfrak{d}_2)$. $|p_{i+1}R - 2A(\eta)| < \epsilon$

$(\mathfrak{d}_3)$. $|p_{i+1}R + A(\delta_1) - 2A(\eta)| < \epsilon$

Here recall that we have from $U\langle \alpha_{k+1} \rangle = \langle \alpha_k \rangle$,

$(\mathfrak{a}_1)$. $|p_iR - 2A(\eta)| < \epsilon$

$(\mathfrak{a}_2)$. $|A(\delta_1) - 2A(\eta)| < \epsilon$

$(\mathfrak{a}_3)$. $|A(\delta_2) - 2A(\eta)| < \epsilon$

and moreover we always have

$(\clubsuit).|A(\delta_1) + p_i R - A(\delta_2)| < \epsilon.$

At first, we prove the next lemma.

Lemma 2.7.2. *Any above pair* $((\mathfrak{c}_{i_1}),(\mathfrak{d}_{i_2}),(\mathfrak{a}_{i_3}))$ *causes a contradiction except for the following two cases.*

$(i_1, i_2, i_3) = (1, 1, 2), (2, 2, 3).$

In the next claim, we exclude pairs $((\mathfrak{c}_{i_1}),(\mathfrak{d}_{i_2}))$ such that we can derive contradictions by only them.

Claim 2.7.3. *Any pair* $((\mathfrak{c}_{i_1}),(\mathfrak{d}_{i_2}))$ *causes a contradiction except for the following cases*

$(i_1, i_2) = (1, 1), (2, 2), (3, 3).$

Proof of Claim 2.7.3. We have to derive a contradiction from $(i_1, i_2) = (1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 3)$, $(3, 1)$, $(3, 2)$ respectively.

Case $(i_1, i_2) = (1, 2)$

By cancelling the terms $p_{i+1}R$ and $2A(\eta)$ from $(\mathfrak{c}_1).|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$ and $(\mathfrak{d}_2).|p_{i+1}R - 2A(\eta)| < \epsilon$, we have $|p_i R| < 2\epsilon$. This is a contradiction since ϵ is sufficiently small.

Case $(i_1, i_2) = (1, 3)$

By cancelling the terms $p_{i+1}R$ and $2A(\eta)$ from $(\mathfrak{c}_1).|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$ and $(\mathfrak{d}_3).|p_{i+1}R + A(\delta_1) - 2A(\eta)| < \epsilon$, we have $|A(\delta_1) + p_i R| < 2\epsilon$. This is a contradiction.

Case $(i_1, i_2) = (2, 1)$

From $(\mathfrak{c}_2).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{d}_1).|A(\delta_1) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|p_i R + A(\delta_1) - A(\delta_2)| < \epsilon$, we have $|p_i R| < 3\epsilon$. This is a contradiction.

Case $(i_1, i_2) = (2, 3)$

In the same way, from $(\mathfrak{c}_2).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{d}_3).|p_{i+1}R + A(\delta_1) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|p_i R + A(\delta_1) - A(\delta_2)| < \epsilon$, we have $|(p_{i+1} - p_i)R| < 3\epsilon$. This is a contradiction.

Case $(i_1, i_2) = (3, 1)$

By cancelling the terms $A(\delta_1)$, $A(\delta_2)$ and $2A(\eta)$ from $(\mathfrak{c}_3).|(p_{i+1}-p_i)R+A(\delta_2)-2A(\eta)|<\epsilon$, $(\mathfrak{d}_1).|A(\delta_1)-2A(\eta)|<\epsilon$ and $(\clubsuit).|p_iR+A(\delta_1)-A(\delta_2)|<\epsilon$, we have $|p_{i+1}R|<3\epsilon$. This is a contradiction.

Case $(i_1,i_2)=(3,2)$

By cancelling the terms $p_{i+1}R$ and $2A(\eta)$ from $(\mathfrak{c}_3).|p_{i+1}R+A(\delta_1)-2A(\eta)|<\epsilon$ and $(\mathfrak{d}_2).|p_{i+1}R-2A(\eta)|<\epsilon$, we have$|A(\delta_1)|<2\epsilon$. This is a contradiction.

By the above arguments, we complete the proof of Claim 2.7.3. □

Proof of Lemma 2.7.2. By Claim 2.7.3, we can see that the rest cases which we have to exclude are $(i_1,i_2,i_3)=(1,1,1)$, $(1,1,3)$, $(2,2,1)$, $(2,2,2)$, $(3,3,1)$, $(3,3,2)$, $(3,3,3)$.

Case $(i_1,i_2,i_3)=(1,1,1)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{c}_1).|(p_{i+1}-p_i)R-2A(\eta)|<\epsilon$ and $(\mathfrak{a}_1).|p_iR-2A(\eta)|<\epsilon$, we have $|(p_{i+1}-2p_i)R|<\epsilon$ and thus $p_i=p_{i+1}-p_i$. This contradicts Claim 2.5.5.

Case $(i_1,i_2,i_3)=(1,1,3)$, (resp. $(2,2,2)$)

By cancelling the term $2A(\eta)$ from $(\mathfrak{d}_1)$.(resp. $(\mathfrak{a}_2)$.)$|A(\delta_1)-2A(\eta)|<\epsilon$ and $(\mathfrak{a}_3)$.(resp. $(\mathfrak{c}_2)$.)$|A(\delta_2)-2A(\eta)|<\epsilon$, we have $|A(\delta_1)-A(\delta_2)|<2\epsilon$. This contradicts $(\clubsuit).|p_iR+A(\delta_1)-A(\delta_2)|<\epsilon$.

Case $(i_1,i_2,i_3)=(2,2,1)$

By cancelling the terms $2A(\eta)$, $A(\delta_2)$ and p_iR from $(\mathfrak{c}_2).|A(\delta_2)-2A(\eta)|<\epsilon$, $(\mathfrak{a}_1).|p_iR-2A(\eta)|<\epsilon$ and $(\clubsuit).|p_iR+A(\delta_1)-A(\delta_2)|<\epsilon$, we have $|A(\delta_1)|<3\epsilon$. This is a contradiction.

Case $(i_1,i_2,i_3)=(3,3,1)$

By cancelling the terms $2A(\eta)$ from $(\mathfrak{d}_3).|p_{i+1}R+A(\delta_1)-2A(\eta)|<\epsilon$ and $(\mathfrak{a}_1).|p_iR-2A(\eta)|<\epsilon$, we have $|(p_{i+1}-p_i)R+A(\delta_1)|<2\epsilon$. This is a contradiction.

Case $(i_1,i_2,i_3)=(3,3,2)$

$(\mathfrak{d}_3).|p_{i+1}R+A(\delta_1)-2A(\eta)|<\epsilon$ and $(\mathfrak{a}_2).|A(\delta_1)-2A(\eta)|<\epsilon$ indicate $|p_{i+1}R|<2\epsilon$ and so $p_{i+1}=0$. This is a contradiction.

Case $(i_1,i_2,i_3)=(3,3,3)$

In the same way, $(\mathfrak{a}_3).|A(\delta_2)-2A(\eta)|<\epsilon$ and $(\mathfrak{c}_3).|(p_{i+1}-p_i)R+A(\delta_2)-2A(\eta)|<\epsilon$ indicate $p_{i+1}=p_i$. This is a contradiction.

Combining the above argument, we complete the proof of Lemma 2.7.2.

□

By the discussions so far, the rest cases are $(i_1,i_2,i_3)=(1,1,2)$ and $(2,2,3)$.

Case $(i_1,i_2,i_3)=(1,1,2)$.

From $(\mathfrak{c}_1).|(p_{i+1}-p_i)R-2A(\eta)|<\epsilon$, $(\mathfrak{d}_1).|A(\delta_1)-2A(\eta)|<\epsilon$ and $(\clubsuit).|p_iR+A(\delta_1)-A(\delta_2)|<\epsilon$, we have

$$A(\delta_1)\approx(p_{i+1}-p_i)R,\ A(\delta_2)\approx p_{i+1}R,\ A(\eta)\approx\frac{1}{2}(p_{i+1}-p_i)R. \tag{2.76}$$

Case $(i_1,i_2,i_3)=(2,2,3)$.

From $(\mathfrak{c}_2).|A(\delta_2)-2A(\eta)|<\epsilon$, $(\mathfrak{d}_2).|p_{i+1}R-2A(\eta)|<\epsilon$ and $(\clubsuit).|p_iR+A(\delta_1)-A(\delta_2)|<\epsilon$, we have

$$A(\delta_1)\approx(p_{i+1}-p_i)R,\ A(\delta_2)\approx p_{i+1}R,\ A(\eta)\approx\frac{1}{2}p_{i+1}R. \tag{2.77}$$

Combining the arguments, we complete the proof of Lemma 2.7.1. □

Type (A_2)

Lemma 2.7.4. *Suppose that the pair of ζ, J-holomorphic curves counted by $U\langle\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ and $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))\rangle$ is Type (A_2). Then, the J-holomorphic curves counted by the U-map cause a contradiction.*

Proof of Lemma 2.7.4. Consider the behaviors of J-holomorphic curves counted by U-map as $z\to\eta$. Then we obtain three possibilities from $U\langle\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$.

$(\mathfrak{e}_1)$. $|p_{i+1}R-2A(\eta)|<\epsilon$

($\mathfrak{e}_2$). $|A(\delta_2) - 2A(\eta)| < \epsilon$

($\mathfrak{e}_3$). $|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$

By the same way, we obtain three possibilities from $U\langle\zeta\rangle = \langle\hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$.

($\mathfrak{f}_1$). $|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$

($\mathfrak{f}_2$). $|A(\delta_1) - 2A(\eta)| < \epsilon$

($\mathfrak{f}_3$). $|A(\delta_1) + 2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$

Also as $z \to \delta_2$. then by the splitting behaviors of the curves counted by $U\langle\zeta\rangle = \langle\hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_2, 1))\rangle$, we have

($\mathfrak{g}_1$). $|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$

($\mathfrak{g}_2$). $|A(\delta_1) - 2A(\delta_2)| < \epsilon$

($\mathfrak{g}_3$). $|A(\delta_1) + 2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$

Also as $z \to \delta_1$. then From $U\langle\hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1})\rangle = \langle\zeta\rangle$, we have

($\mathfrak{h}_1$). $|p_{i+1}R - 2A(\delta_1)| < \epsilon$

($\mathfrak{h}_2$). $|A(\delta_2) - 2A(\delta_1)| < \epsilon$

($\mathfrak{h}_3$). $|A(\delta_2) + 2A(\delta_1) - p_{i+1}R| < \epsilon$

Recall that we always have ($\clubsuit$).$|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, since $A(\alpha_{k+1}) - A(\alpha_k) < \epsilon$.

We exclude the pairs (($\mathfrak{e}_{i_1}$),($\mathfrak{f}_{i_2}$),($\mathfrak{g}_{i_3}$),($\mathfrak{h}_{i_4}$)) in the same way as Type (A_1).

At first, we prove the next lemma.

Lemma 2.7.5. *Any above pairs ((*$\mathfrak{e}_{i_1}$*),(*$\mathfrak{f}_{i_2}$*),(*$\mathfrak{g}_{i_3}$*),(*$\mathfrak{h}_{i_4}$*)) causes a contradiction except for the following cases.*

(i_1, i_2, i_3, i_4)=$(3, 3, 1, 2)$, $(3, 3, 1, 3)$, $(3, 3, 3, 2)$.

To prove Lemma 2.7.5, first of all, we will exclude pairs (($\mathfrak{e}_{i_1}$),($\mathfrak{f}_{i_2}$)) such that we can derive contradictions by only them.

Claim 2.7.6. *The following each pair ((*$\mathfrak{e}_{i_1}$*),(*$\mathfrak{f}_{i_2}$*)) causes a contradiction.*

(i_1, i_2)=$(1, 1)$, $(3, 1)$, $(2, 2)$, $(1, 3)$.

Proof of Claim 2.7.6.

Case (i_1, i_2)=$(1, 1)$

This is trivial.

Case (i_1, i_2)=$(3, 1)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{e}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$ and $(\mathfrak{f}_1).|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$, we have $|p_iR - A(\delta_2)| < 3\epsilon$. But this contradicts $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$.

Case (i_1, i_2)=$(2, 2)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{e}_2).|A(\delta_2)-2A(\eta)| < \epsilon$ and $(\mathfrak{f}_2).|A(\delta_1)- 2A(\eta)| < \epsilon$, we have $|A(\delta_1) - A(\delta_2)| < 2\epsilon$. This obviously contradict $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$.

Case (i_1, i_2)=$(1, 3)$

$(\mathfrak{e}_1).|p_{i+1}R-2A(\eta)| < \epsilon$ and $(\mathfrak{f}_3).|A(\delta_1)+2A(\eta)-(p_{i+1}-p_i)R| < \epsilon$ imply that $|A(\delta_1) + p_iR| < 2\epsilon$. This is a contradiction. □

Claim 2.7.7. *Any pairs* $((\mathfrak{e}_{i_1}),(\mathfrak{f}_{i_2}),(\mathfrak{g}_{i_3}))$ *causes a contradiction except for the following cases.*

(i_1, i_2, i_3)=$(3, 3, 1)$, $(3, 3, 3)$.

Proof of Claim 2.7.7. At first, note that $(\mathfrak{g}_2).|A(\delta_1) - 2A(\delta_2)| < \epsilon$ obviously contradicts $(\clubsuit).|A(\delta_1)+p_iR-A(\delta_2)| < \epsilon$. So we have only to consider the cases $i_3 = 1$, 2.

Hence by Claim 2.7.6, we can find that it is sufficient to exclude the cases $(i_1, i_2, i_3) = (2, 1, 1)$, $(2, 1, 3)$, $(1, 2, 1)$, $(1, 2, 3)$, $(3, 2, 1)$, $(3, 2, 3)$, $(2, 3, 1)$, $(2, 3, 3)$.

Case (i_1, i_2, i_3)=$(2, 1, 1)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{e}_2).|A(\delta_2)-2A(\eta)| < \epsilon$ and $(\mathfrak{f}_1).|(p_{i+1}- p_i)R - 2A(\eta)| < \epsilon$, we have $|A(\delta_2) - (p_{i+1} - p_i)R| < 2\epsilon$. This contradicts $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(2, 1, 3)$

In the same way as above, by cancelling the term $2A(\eta)$ from $(\mathfrak{e}_2).|A(\delta_2)- 2A(\eta)| < \epsilon$ and $(\mathfrak{f}_1).|(p_{i+1}-p_i)R-2A(\eta)| < \epsilon$, we have $|A(\delta_2)-(p_{i+1}-p_i)R| < 2\epsilon$. This contradicts $(\mathfrak{g}_3).|A(\delta_1) + 2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$.

Case (i_1, i_2, i_3)=$(1, 2, 1)$

By cancelling the terms $2A(\eta)$ and $A(\delta_1)$ from $(\mathfrak{e}_1).|p_{i+1}R - 2A(\eta)| < \epsilon$, $(\mathfrak{f}_2).|A(\delta_1) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have $|A(\delta_2) - (p_{i+1} + p_i)R| < 3\epsilon$. This contradicts $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(1, 2, 3)$

In the same way as above, by cancelling the terms $2A(\eta)$ and $A(\delta_1)$ from $(\mathfrak{e}_1).|p_{i+1}R - 2A(\eta)| < \epsilon$, $(\mathfrak{f}_2).|A(\delta_1) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have $|A(\delta_2) - (p_{i+1} + p_i)R| < 3\epsilon$. This contradicts $(\mathfrak{g}_3).|A(\delta_1) + 2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$.

Case (i_1, i_2, i_3)=$(3, 2, 1)$

By cancelling the terms $2A(\eta)$ and $A(\delta_1)$ from $(\mathfrak{e}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{f}).|A(\delta_i) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have $|2A(\delta_2) - (p_{i+1} + p_i)R| < 3\epsilon$. This contradicts $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(3, 2, 3)$

In the same way as above, by cancelling the terms $2A(\eta)$ and $A(\delta_1)$ from $(\mathfrak{e}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{f}).|A(\delta_i) - 2A(\eta)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have $|2A(\delta_2) - (p_{i+1} + p_i)R| < 3\epsilon$. This contradicts $(\mathfrak{g}_3).|A(\delta_1) + 2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$.

Case (i_1, i_2, i_3)=$(2, 3, 1)$

$(\mathfrak{e}_2).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{f}_3).|A(\delta_1) + 2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$ and $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$ imply that $|A(\delta_1) - A(\delta_2)| < 3\epsilon$. This contradicts $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(2, 3, 3)$

$(\mathfrak{e}_2).|A(\delta_2) - 2A(\eta)| < \epsilon$ and $(\mathfrak{f}_3).|A(\delta_1) + 2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$ imply $|A(\delta_1) + A(\delta_2) - (p_{i+1} - p_i)R| < 2\epsilon$. This obviously contradicts $(\mathfrak{g}_3).|A(\delta_1) + 2A(\delta_2) - (p_{i+1} - p_i)R| < \epsilon$.

□

Proof of Lemma 2.7.5. By the above arguments, we can find that the remaining cases are $(i_1, i_2, i_3, i_4) = (3, 3, 1, 1)$, $(3, 3, 1, 2)$, $(3, 3, 1, 3)$, $(3, 3, 3, 1)$, $(3, 3, 3, 2)$, $(3, 3, 3, 3)$.

To prove the lemma, we will exclude the cases $(i_1, i_2, i_3, i_4) = (3, 3, 1, 1)$, $(3, 3, 1, 3)$ $(3, 3, 3, 1)$, $(3, 3, 3, 3)$ as follows.

Case $(i_1, i_2, i_3, i_4) = (3, 3, 1, 1)$

By cancelling the term $p_{i+1}R$ from $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$ and $(\mathfrak{h}_1).|p_{i+1}R - 2A(\delta_1)| < \epsilon$, we have $|A(\delta_1) - \frac{1}{2}p_iR - A(\delta_2)| < \epsilon$. This contradicts $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$

Case $(i_1, i_2, i_3, i_4) = (3, 3, 3, 1)$

By cancelling the term $A(\delta_1)$ from $(\mathfrak{g}_3).|A(\delta_1)+2A(\delta_2)-(p_{i+1}-p_i)R| < \epsilon$ and $(\mathfrak{h}_1)$. $|p_{i+1}R - 2A(\delta_1)| < \epsilon$, we have $|4A(\delta_2) - (p_{i+1} - 2p_i)R| < 3\epsilon$. Also from $(\mathfrak{h}_1)$. $|p_{i+1}R - 2A(\delta_1)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have $|2A(\delta_2) - (p_{i+1} + 2p_i)R| < 3\epsilon$.

$|4A(\delta_2) - (p_{i+1} - 2p_i)R| < 3\epsilon$ and $|2A(\delta_2) - (p_{i+1} + 2p_i)R| < 3\epsilon$ imply $|(p_{i+1} + 6p_i)R| < 9\epsilon$. This is a contradiction.

Case $(i_1, i_2, i_3, i_4) = (3, 3, 3, 3)$

By cancelling the term $p_{i+1}R$ from $(\mathfrak{g}_3).|A(\delta_1)+2A(\delta_2)-(p_{i+1}-p_i)R| < \epsilon$ and $(\mathfrak{h}_3).|A(\delta_2) + 2A(\delta_1) - p_{i+1}R| < \epsilon$, we have $|A(\delta_2) - A(\delta_1) + p_iR| < 2\epsilon$. This contradicts $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$.

Combining the above consequences, we finish the proof of Lemma 2.7.5.

□

By Lemma 2.7.5, we still have the following pairs.

(i_1, i_2, i_3, i_4)=$(3, 3, 1, 2)$, $(3, 3, 1, 3)$, $(3, 3, 3, 2)$.

From these pairs. we can decide the approximate relations as follows.

Case (i_1, i_2, i_3, i_4)=$(3, 3, 1, 2)$

From $(\mathfrak{e}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{f}_3).|A(\delta_1) + 2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$, $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$, $(\mathfrak{h}_2).|A(\delta_2) - 2A(\delta_1)| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have

$$A(\delta_1) \approx p_iR,\ A(\delta_2) \approx 2p_iR,\ A(\eta) \approx \frac{3}{2}p_iR \tag{2.78}$$

Case (i_1, i_2, i_3, i_4)=$(3, 3, 1, 3)$

From $(\mathfrak{e}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{f}_3).|A(\delta_1) + 2A(\eta) - (p_{i+1} - p_i)R| < \epsilon$, $(\mathfrak{g}_1).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$, $(\mathfrak{h}_3).|A(\delta_2) + 2A(\delta_1) - p_{i+1}R| < \epsilon$ and $(\clubsuit).|A(\delta_1) + p_iR - A(\delta_2)| < \epsilon$, we have

$$A(\delta_1) \approx 2p_iR,\ A(\delta_2) \approx 3p_iR,\ A(\eta) \approx 2p_iR. \tag{2.79}$$

Case (i_1, i_2, i_3, i_4)=$(3, 3, 3, 2)$

From $(\mathfrak{e}_3).|p_{i+1}R-(2A(\eta)+A(\delta_2))|<\epsilon$, $(\mathfrak{f}_3).|A(\delta_1)+2A(\eta)-(p_{i+1}-p_i)R|<\epsilon$, $(\mathfrak{g}_3).|A(\delta_1)+2A(\delta_2)-(p_{i+1}-p_i)R|<\epsilon$, $(\mathfrak{h}_2).|A(\delta_2)-2A(\delta_1)|<\epsilon$ and $(\clubsuit).|A(\delta_1)+p_iR-A(\delta_2)|<\epsilon$, we have

$$A(\delta_1)\approx p_iR,\ A(\delta_2)\approx 2p_iR,\ A(\eta)\approx 2p_iR \tag{2.80}$$

Recall that from the splitting behaviors of J-holomorphic curve counted by $U\langle\alpha_{k+1}\rangle=\langle\alpha_k\rangle$ as $z\to\eta$, we have three possibilities of splitting of holomorphic curve and also three possibilities of approximate relations as follows.

$(\mathfrak{a}_1)$. $|p_iR-2A(\eta)|<\epsilon$

$(\mathfrak{a}_2)$. $|A(\delta_1)-2A(\eta)|<\epsilon$

$(\mathfrak{a}_3)$. $|A(\delta_2)-2A(\eta)|<\epsilon$

But it is easy to check that (2.78), (2.79), (2.80) can not hold any these relation. We complete the proof of Lemma 2.7.4. □

Type (A_3)

Lemma 2.7.8. *Suppose that the pair of ζ, J-holomorphic curves counted by $U\langle\hat{\alpha}\cup(\delta_1,1)\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ and $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))\rangle$ is Type (A_3). Then, the J-holomorphic curves counted by the U-map cause a contradiction.*

Proof of Lemma 2.7.8. Consider the behaviors of J-holomorphic curves counted by U-map as $z\to\delta_2$.

Then we obtain three possibilities of the approximate relations in the actions from the splitting behaviors of the curves counted by $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_2,1))\rangle$.

$(\mathfrak{i}_1)$. $|A(\delta_1)-(p_{i+1}-p_i)R-2A(\delta_2)|<\epsilon$

$(\mathfrak{i}_2)$. $|(p_{i+1}-p_i)R-2A(\delta_2)|<\epsilon$

$(\mathfrak{i}_3)$. $|A(\delta_1)-2A(\delta_2)|<\epsilon$.

But Every $(\mathfrak{i}_1)$, $(\mathfrak{i}_2)$ and $(\mathfrak{i}_3)$ causes a contradiction as follows

1. Case ($\mathfrak{i}_1$)

   Recall (♣).$|A(\delta_1) + p_i R - A(\delta_2)| < \epsilon$. This obviously contradicts $|A(\delta_1) - (p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$. In fact, by cancelling the term $A(\delta_1)$ from them, we have $|p_{i+1}R + A(\delta_2)| < 2$. This is a contradiction.

2. Case ($\mathfrak{i}_2$)

   Since any J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\delta_1, 1) \cup (\gamma, p_{i+1}) \rangle = \langle \zeta \rangle$ $\delta' \in \hat{\alpha}$ are in $\mathcal{M}^J((\gamma, p_{i+1}) \cup (\delta', 1), (\delta_2, 1))$, we have $|p_{i+1}R + A(\delta') - A(\delta_2)| < \epsilon$.

   By cancelling the term $A(\delta_2)$ from the above inequality and ($\mathfrak{i}_2$).$|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$, we have $|\frac{1}{2}(p_{i+1} + p_i)R + A(\delta')| < \frac{3}{2}\epsilon$. This is a contradiction.

3. Case ($\mathfrak{i}_3$)

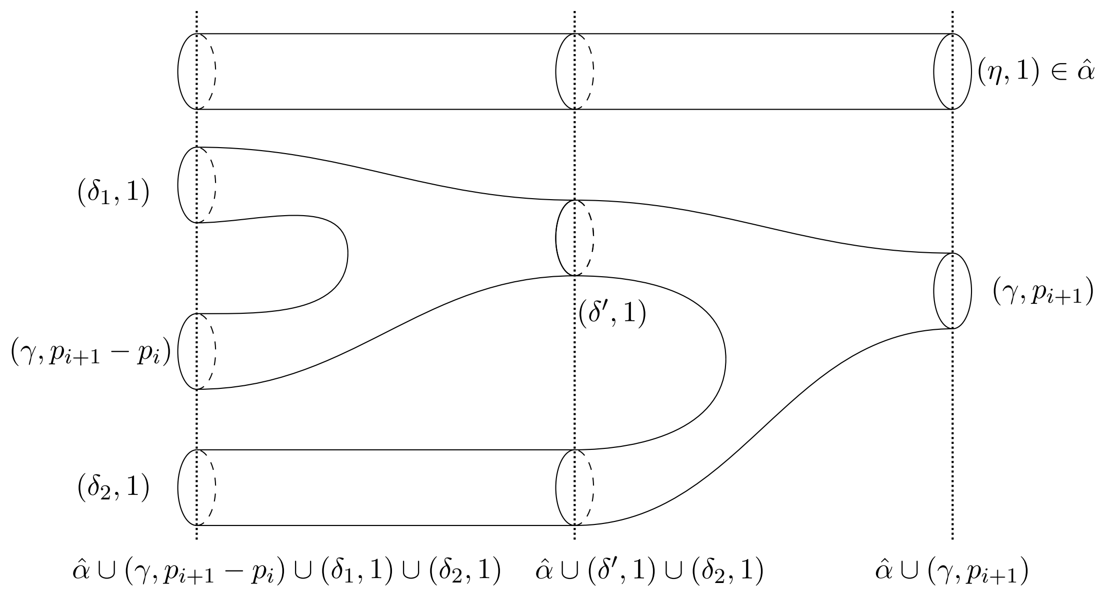
   Recall (♣).$|A(\delta_1) + p_i R - A(\delta_2)| < \epsilon$. This obviously contradicts ($\mathfrak{i}_3$).$|A(\delta_1) - 2A(\delta_2)| < \epsilon$.

Combining the above arguments, we can see that Type (A_3) can not occur. □

2.7.2 Type (B)

In the same way as Type (A), the topological types of J-holomoprhic curves counted by $U\langle \hat{\alpha} \cup (\gamma, p_{i+1}) \rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)) \rangle$ are both $(g, k, l) = (0, 3, 0)$.

And also we can see that there is a negative hyperbolic orbit δ' such that $\zeta = \hat{\alpha} \cup (\delta', 1) \cup (\delta_2, 1)$ and, any J-holomorphic curves counted by $U\langle \hat{\alpha} \cup (\gamma, p_{i+1}) \rangle = \langle \zeta \rangle$ and $U\langle \zeta \rangle = \langle \hat{\alpha} \cup (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1) \cup (\delta_2, 1)) \rangle$ are in $\mathcal{M}^J((\gamma, p_{i+1}), (\delta', 1) \cup (\delta_2, 1))$ and $\mathcal{M}^J((\delta', 1), (\gamma, p_{i+1} - p_i) \cup (\delta_1, 1))$ respectively.



Type (B)

Lemma 2.7.9. *Suppose that the pair of ζ, J-holomorphic curves counted by $U\langle\hat{\alpha}\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$ and $U\langle\zeta\rangle=\langle\hat{\alpha}\cup(\gamma,p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1))\rangle$ are above. Then, one of the following holds.*

(Δ_1). $\frac{3}{2}p_i=p_{i+1}$. Moreover $A(\delta_1)\approx\frac{1}{2}p_iR$, $A(\delta_2)\approx\frac{1}{2}p_iR$ and for each $\eta\in\hat{\alpha}$, either $A(\eta)\approx\frac{1}{2}p_iR$ or $A(\eta)\approx\frac{1}{4}p_iR$.

(Δ_2). $\frac{4}{3}p_i=p_{i+1}$. Moreover, $A(\delta_1)\approx\frac{2}{3}p_iR$, $A(\delta_2)\approx\frac{1}{3}p_iR$ and for each $\eta\in\hat{\alpha}$, either $A(\eta)\approx\frac{1}{2}p_iR$ or $A(\eta)\approx\frac{1}{6}p_iR$.

Proof of Proposition 2.6.1. Since (Δ_1) and (Δ_2) are correspond to (b) and (c) respectively, combine with the result of Type (A), we complete the proof of Proposition 2.6.1. □

Proof of Lemma 2.7.9. Let $z\to\eta$. Then we obtain three possibilities from the splitting behaviors of the curves counted by $U\langle\hat{\alpha}\cup(\gamma,p_{i+1})\rangle=\langle\zeta\rangle$.

(j$_1$). $|A(\delta_2)-2A(\eta)|<\epsilon$

(j$_2$). $|p_{i+1}R-2A(\eta)|<\epsilon$

(j$_3$). $|p_{i+1}R-(2A(\eta)+A(\delta_2))|<\epsilon$

By the same way, we have three possibilities of the approximate actions in the orbits from the splitting behaviors of the curves counted by $U\langle\zeta\rangle = \langle\hat{\alpha}\cup(\gamma, p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1))\rangle$.

($\mathfrak{k}_1$). $|A(\delta_1)-2A(\eta)|<\epsilon$

($\mathfrak{k}_2$). $|(p_{i+1}-p_i)R-2A(\eta)|<\epsilon$

($\mathfrak{k}_3$). $|(p_{i+1}-p_i)R+A(\delta_1)-2A(\eta)|<\epsilon$

Also let $z\to\delta_2$. then by the splitting behaviors of the curves counted by $U\langle\zeta\rangle = \langle\hat{\alpha}\cup(\gamma, p_{i+1}-p_i)\cup(\delta_1,1)\cup(\delta_2,1))\rangle$, we have

($\mathfrak{l}_1$). $|A(\delta_1)-2A(\delta_2)|<\epsilon$

($\mathfrak{l}_2$). $|(p_{i+1}-p_i)R-2A(\delta_2)|<\epsilon$

($\mathfrak{l}_3$). $|(p_{i+1}-p_i)R+A(\delta_1)-2A(\delta_2)|<\epsilon$

Recall that we always have ($\spadesuit$).$|A(\delta_1)+A(\delta_2)-p_iR|<\epsilon$, since $A(\alpha_{k+1})-A(\alpha_k)<\epsilon$.

As is the same with so far, Some pair (($\mathfrak{j}_{i_1}$),($\mathfrak{k}_{i_2}$),($\mathfrak{l}_{i_3}$)) cause contradictions as follows.

Lemma 2.7.10. *Any above pairs (($\mathfrak{j}_{i_1}$),($\mathfrak{k}_{i_2}$),($\mathfrak{l}_{i_3}$)) causes a contradiction except for the following cases.*

$(i_1,i_2,i_3)=(1,1,3),(1,2,1),(1,2,3),(3,3,1),(3,3,3)$.

Proof of Lemma 2.7.10. At first, we will exclude pairs (($\mathfrak{j}_{i_1}$),($\mathfrak{k}_{i_2}$)) such that we can derive a contradiction by only them.

Claim 2.7.11. *Each of the following pairs (($\mathfrak{j}_{i_1}$),($\mathfrak{k}_{i_2}$)) causes a contradiction.*

(i_1,i_2)=(2, 1), (2, 2), (2, 3), (3, 1), (3, 2).

Proof of Claim 2.7.11.

Case (i_1,i_2)=(2, 1)

By cancelling the term $2A(\eta)$ from ($\mathfrak{j}_2$).$|p_{i+1}R-2A(\eta)|<\epsilon$ and ($\mathfrak{k}_1$).$|A(\delta_1)-2A(\eta)|<\epsilon$, we have $|p_{i+1}R-A(\delta_1)|<2\epsilon$. This contradicts ($\spadesuit$).$|p_iR-(A(\delta_1)+A(\delta_2))|<\epsilon$.

Case (i_1, i_2)=$(2, 2)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{j}_2).|p_{i+1}R - 2A(\eta)| < \epsilon$ and $(\mathfrak{k}_2).|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$, we have $|p_iR| < 2\epsilon$. This is a contradiction.

Case (i_1, i_2)=$(2, 3)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{j}_2).|p_{i+1}R - 2A(\eta)| < \epsilon$ and $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$, we have $|p_iR - A(\delta_1)| < 2\epsilon$. This contradicts $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$.

Case (i_1, i_2)=$(3, 1)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{j}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$ and $(\mathfrak{k}_1).|A(\delta_1) - 2A(\eta)| < \epsilon$, we have $|p_{i+1}R - (A(\delta_1) + A(\delta_2))| < 2\epsilon$. But this contradicts $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$.

Case (i_1, i_2)=$(3, 2)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{j}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$ and $(\mathfrak{k}_2).|(p_{i+1} - p_i)R - 2A(\eta)| < \epsilon$, we have $|p_iR - A(\delta_2)| < 2\epsilon$. But this contradicts $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$. □

By the arguments so far, we have only to show that each of the following pairs $((\mathfrak{j}_{i_1}),(\mathfrak{k}_{i_2}),(\mathfrak{l}_{i_3}))$ causes a contradiction.

(i_1, i_2, i_3)=$(1, 1, 1)$, $(1, 1, 2)$, $(1, 3, 1)$, $(1, 3, 2)$, $(1, 3, 3)$, $(1, 2, 2)$, $(3, 3, 2)$.

Case (i_1, i_2, i_3)=$(1, 1, 1)$

Obviously, $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$ and $(\mathfrak{k}_1).|A(\delta_1) - 2A(\eta)| < \epsilon$ contradict $(\mathfrak{l}_1).|A(\delta_1) - 2A(\delta_2)| < \epsilon$

Case (i_1, i_2, i_3)=$(1, 1, 2)$

By cancelling the terms $A(\delta_1)$, $A(\delta_2)$ and $2A(\eta)$ from $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{k}_1).|A(\delta_1) - 2A(\eta)| < \epsilon$ $(\mathfrak{l}_2).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$, and $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $|p_iR - (p_{i+1} - p_i)R| < 4\epsilon$. Hence $p_i = p_{i+1} - p_i$. This is a contradiction.

Case (i_1, i_2, i_3)=$(1, 3, 1)$

By cancelling the terms $A(\delta_1)$ and $2A(\eta)$ from $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$ and $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $|p_{i+1}R - 2A(\delta_2)| < 3\epsilon$ and hence $|(p_i - \frac{1}{2}p_{i+1})R - A(\delta_1)| < \frac{5}{2}\epsilon$.

By combining with $(\mathfrak{l}_1).|A(\delta_1) - 2A(\delta_2)| < \epsilon$, we have $|(\frac{3}{2}p_{i+1} - p_i)R| < \frac{13}{2}\epsilon$ This is a contradiction.

Case (i_1, i_2, i_3)=$(1,3,2)$

In the same way as above, from $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$, and $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $|p_{i+1}R - 2A(\delta_2)| < 3\epsilon$. This obviously contradicts $(\mathfrak{l}_2).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(1,3,3)$

In the same way as above, from $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$, and $(\spadesuit).|p_iR - (A(\delta_1) + A(\delta_2))| < \epsilon$, we have $|p_{i+1}R - 2A(\delta_2)| < 3\epsilon$ and $|(p_i - \frac{1}{2}p_{i+1})R - A(\delta_1)| < \frac{5}{2}\epsilon$.

By combining with $(\mathfrak{l}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\delta_2)| < \epsilon$, we have $|\frac{1}{2}p_{i+1}R| < \frac{13}{2}\epsilon$.

Case (i_1, i_2, i_3)=$(1,2,2)$

By cancelling the term $2A(\eta)$ from $(\mathfrak{j}_1).|A(\delta_2)-2A(\eta)| < \epsilon$ and $(\mathfrak{j}_2).|p_{i+1}R-2A(\eta)| < \epsilon$, we have $|p_{i+1}R - A(\delta_2)| < 2\epsilon$. This obviously contradicts $(\mathfrak{l}_2).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$.

Case (i_1, i_2, i_3)=$(3,3,2)$

From $(\mathfrak{j}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{l}_2).|(p_{i+1} - p_i)R - 2A(\delta_2)| < \epsilon$ and $(\spadesuit).|p_iR - A(\delta_1) - A(\delta_2)| < \epsilon$, we have

$$(\frac{3}{2}p_i - \frac{1}{2}p_{i+1})R \approx A(\delta_1),\ \frac{1}{2}(p_{i+1} - p_i)R \approx A(\delta_2),\ \frac{1}{4}(p_{i+1} + p_i)R \approx A(\eta). \tag{2.81}$$

Recall that when we consider the splitting behaviors of J-holomorphic curve counted by $U\langle\alpha_{k+1}\rangle = \langle\alpha_k\rangle$ as $z \to \eta$ for some $\eta \in \hat{\alpha}$. Then there are three possibilities of splitting of holomorphic curve and also we have three possibilities of the approximate relation in the actions as follows.

$(\mathfrak{b}_1)$. $|A(\delta_1) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_2)$. $|A(\delta_2) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_3)$. $|p_iR - 2A(\eta)| < \epsilon$.

But obviously, (2.81) contradicts the above relations in any case.

Combining the arguments, we complete the proof of Lemma 2.7.10. □

To complete the proof of Lemma 2.7.9, it is sufficient to compute the approximate relations from the rest cases $(i_1, i_2, i_3) = (1,1,3)$, $(1,2,1)$, $(1,2,3)$, (3, 3, 1), (3, 3, 3).

Case $(i_1, i_2, i_3) = (1, 1, 3)$

From $(\mathfrak{j}_1).|A(\delta_2) - 2A(\eta)| < \epsilon$, $(\mathfrak{k}_1).|A(\delta_1) - 2A(\eta)| < \epsilon$, $(\mathfrak{l}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\delta_2)| < \epsilon$ and $(\spadesuit).|A(\delta_1) + A(\delta_2) - p_iR| < \epsilon$, we have

$$A(\delta_1) \approx \frac{1}{2}p_iR,\ A(\delta_2) \approx \frac{1}{2}p_iR,\ p_{i+1} = \frac{3}{2}p_i,\ A(\eta) \approx \frac{1}{4}p_iR. \tag{2.82}$$

Case $(i_1, i_2, i_3) = (1, 2, 1)$

From $(\mathfrak{j}_1).|A(\delta_2)-2A(\eta)| < \epsilon$, $(\mathfrak{k}_2).|(p_{i+1}-p_i)R-2A(\eta)| < \epsilon$, $(\mathfrak{l}_1).|A(\delta_1)-2A(\delta_2)| < \epsilon$ and $(\spadesuit).|A(\delta_1) + A(\delta_2) - p_iR| < \epsilon$. we have

$$A(\delta_1) \approx \frac{2}{3}p_iR,\ A(\delta_2) \approx \frac{1}{3}p_iR,\ p_{i+1} = \frac{4}{3}p_i,\ A(\eta) \approx \frac{1}{6}p_iR. \tag{2.83}$$

Case $(i_1, i_2, i_3) = (1, 2, 3)$

From $(\mathfrak{j}_1).|A(\delta_2)-2A(\eta)| < \epsilon$, $(\mathfrak{k}_2).|(p_{i+1}-p_i)R-2A(\eta)| < \epsilon$, $(\mathfrak{l}_3).|(p_{i+1}-p_i)R + A(\delta_1) - 2A(\delta_2)| < \epsilon$ and $(\spadesuit).|A(\delta_1) + A(\delta_2) - p_iR| < \epsilon$, we have

$$A(\delta_1) \approx \frac{1}{2}p_iR,\ A(\delta_2) \approx \frac{1}{2}p_iR,\ p_{i+1} = \frac{3}{2}p_i,\ A(\eta) \approx \frac{1}{4}p_iR. \tag{2.84}$$

Case $(i_1, i_2, i_3) = (3, 3, 1)$

Recall that by considering the splitting behavior of J-holomorphic curves counted by U-map from $\langle\alpha_{k+1}\rangle$ to $\langle\alpha_k\rangle$ as $z \to \eta$. Then we have three possibilties,

$(\mathfrak{b}_1)$. $|A(\delta_1) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_2)$. $|A(\delta_2) - 2A(\eta)| < \epsilon$

$(\mathfrak{b}_3)$. $|p_iR - 2A(\eta)| < \epsilon$

The first inequality contradicts $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$ and also the second one does $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$ and $|A(\delta_1) - 2A(\delta_2)| < \epsilon$. So only third one may be possible.

By combining with $(\mathfrak{j}_3).|p_{i+1}R-(2A(\eta)+A(\delta_2))| < \epsilon$, $(\mathfrak{k}_3).|(p_{i+1}-p_i)R+A(\delta_1)-2A(\eta)| < \epsilon$, $(\mathfrak{l}_1).|A(\delta_1)-2A(\delta_2)| < \epsilon$ and $(\spadesuit).|A(\delta_1)+A(\delta_2)-p_iR| < \epsilon$, we have

$$A(\delta_1) \approx \frac{2}{3}p_iR,\ A(\delta_2) \approx \frac{1}{3}p_iR,\ p_{i+1} = \frac{4}{3}p_i,\ A(\eta) \approx \frac{1}{2}p_iR. \tag{2.85}$$

Case $(i_1, i_2, i_3) = (3, 3, 3)$

From $(\mathfrak{j}_3).|p_{i+1}R - (2A(\eta) + A(\delta_2))| < \epsilon$, $(\mathfrak{k}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\eta)| < \epsilon$, $(\mathfrak{l}_3).|(p_{i+1} - p_i)R + A(\delta_1) - 2A(\delta_2)| < \epsilon$ and $(\spadesuit).|A(\delta_1) + A(\delta_2) - p_iR| < \epsilon$ we have

$$A(\delta_1) \approx \frac{1}{2}p_iR,\ A(\delta_2) \approx \frac{1}{2}p_iR,\ p_{i+1} = \frac{3}{2}p_i,\ A(\eta) \approx \frac{1}{2}p_iR. \tag{2.86}$$

Combining the arguments, we complete the proof of Lemma 2.7.9. □

2.8 Proof of Theorem 2.4.1

Suppose that α_k, α_{k+1} satisfy the assumptions 1, 2, 3, 4 and 5 in Proposition 2.6.1, then the pair (α_k, α_{k+1}) is one of the types of **(a)**, **(a')**, **(b)**, **(b')**, **(c)**, **(c')**. Moreover, we can see that any actions of negative hyperbolic orbits in α_{k+1}, α_k are in $(\frac{1}{12}R\mathbb{Z})_{\epsilon'} = \{\, x \in \mathbb{R} \,|\, \mathrm{dist}(x, \frac{1}{12}R\mathbb{Z}) < \epsilon' \,\}$ where $\epsilon' = \frac{1}{100}\max\{A(\alpha)\,|\alpha \text{ is a Reeb orbit}\,\}$. Therefore, we can define a map

$$f : \{\, \eta \in \alpha_k \,|\, \eta \text{ is negative hyperbolic} \,\} \to \frac{1}{12}R\mathbb{Z} \tag{2.87}$$

Here, $f(\eta) \in \frac{1}{12}R\mathbb{Z}$ is an image by the natural projection of $A(\eta)$ from $(\frac{1}{12}R\mathbb{Z})_{\epsilon'}$ to $\frac{1}{12}R\mathbb{Z}$. Note that $\frac{1}{12}R\mathbb{Z}$ is discrete.

Let α_{k+1}, α_k and α_{k-1} be such 3 consecutive ECH generators. then each type of the pairs (α_k, α_{k+1}) and (α_{k-1}, α_k) decides the action relations in α_k respectively. But sometimes, they may be incompatible. In particular, we can show that such fact induces contradictions with Proposition 2.4.4.

At first, we show the next lemma.

Lemma 2.8.1. *Suppose that α_{k-1}, α_k and α_{k+1} satisfy the assumptions 1, 2, 3, 4 and 5 in Proposition 2.6.1. Then the following pairs of types of (α_k, α_{k+1}) and (α_{k-1}, α_k) classified in Proposition 2.6.1 cause contradictions.*

(type of (α_{k-1}, α_k), type of (α_k, α_{k+1}))
*=(**a'**, **a**), (**b'**, **a'**), (**b'**, **a**), (**b'**, **b**), (**b'**, **b'**), (**b'**, **c**), (**b'**, **c'**), (**c'**, **a**), (**c'**, **a'**), (**c'**, **b'**), (**c'**, **c**), (**c'**, **c'**), (**a'**, **b**), (**b**, **b**), (**c'**, **b**), (**c**, **b**), (**a**, **c**), (**b**, **c**), (**c**, **c**), (**a**, **a**), (**a'**, **a'**), (**a**, **b**), (**a'**, **c**), (**a**, **a'**).*

Proof of Lemma 2.8.1. We derive a contradiction respectively.

Case (**b'**, **a**)(or by symmetry, (**a'**, **b**))

Since the type of (α_{k-1}, α_k) is (**b'**), the number of the inverse image of the largest value of $f : \{\, \eta \in \alpha_k \,|\, \eta \text{ is negative hyperbolic} \,\} \to \frac{1}{12}R\mathbb{Z}$ is at least two. But the assumption that the type of (α_k, α_{k+1}) is (**a**) indicates that the number of the inverse image of the largest value of $f : \{\, \eta \in \alpha_k \,|\, \eta \text{ is negative hyperbolic} \,\} \to \frac{1}{12}R\mathbb{Z}$ is one. This is a contradiction.

Case (**b'**, **a'**)(or by symmetry, (**a**, **b**))

Let $E(\alpha_{k-1}) = M$, then $E(\alpha_k) = M - q_i$ where $q_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$. Let $q_{i'} = \max(S_\theta \cap \{1,\ 2, ...,\ M - q_i\})$.

Since the type of (α_k, α_{k+1}) is (**a'**), the image of f on α_k is in $\{\, \frac{1}{2}q_{i'+1}R,\ \frac{1}{2}(q_{i'+1} - q_{i'})R,\ (q_{i'+1} - q_{i'})R \,\}$.

Since the type of (α_{k-1}, α_k) is (**a'**), we have $q_{i+1} = \frac{3}{2}q_i$. Hence

$$q_{i'} \leq M - q_i \leq q_{i+1} - q_i = \frac{1}{2}q_i < q_i. \tag{2.88}$$

Therefore $q_{i'+1} \leq q_i$.

If $q_{i'+1} < q_i$, the actions of each orbit in $\hat{\alpha}$ are less than $\frac{1}{2}q_iR$. The number of the positive ends of (**b'**) whose image of f are largest is at least two but that one of negative ends of (**a'**) whose image of f are largest is only δ_2. This is a contradiction.

If $q_{i'+1} = q_i$, by considering the correspondence of the approximate action values and Claim 2.5.5, we can see $\frac{1}{4}q_iR = (q_{i'+1} - q_{i'})R$ and so $\frac{1}{4}q_i = q_{i'+1} - q_{i'} = q_i - q_{i'}$ hence $\frac{3}{4}q_i = q_{i'}$. This contradict $q_{i'} \leq \frac{1}{2}q_i$.

Case (**b'**, **b**)

Let $E(\alpha_{k-1}) = M\ E(\alpha_{k+1}) = M'$, then $E(\alpha_k) = M - q_i = M' - p_{i'}$ where $q_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$ and $p_{i'} = \max(S_{-\theta} \cap \{1,\ 2, ...,\ M'\})$.

Since the type of (α_{k-1}, α_k) is (**b'**), the largest value of f on α_k is $\frac{1}{2}q_iR$ and Since the type of (α_k, α_{k+1}) is (**b**), the largest value of f on α_k is $\frac{1}{2}p_{i'}R$. Therefore we have $q_i = p_{i'}$. This contradicts $S_\theta \cap S_{-\theta} = \{1\}$.

Case (**b'**, **b'**) (or by symmetry, (**b**, **b**))

Let $E(\alpha_{k-1}) = M$, then $E(\alpha_k) = M - q_i < \frac{1}{2}q_i$. This implies that the largest approximate value decided by the positive ends of former (**b'**) is larger than that one decided by the negative ends of the later (**b'**) and so this is a contradiction.

Case (**b'**, **c**) (or by symmetry, (**c'**, **b**))

Since the type of (α_{k-1}, α_k) is (**b'**), the number of the image of f on α_k is at most 2. On the other hand, since the type of (α_k, α_{k+1}) is (**c**), the number of the image of f on α_k is at least three. This is a contradiction.

Case (**b'**, **c'**) (or by symmetry, (**c**, **b**))

Let $E(\alpha_{k-1}) = M$. Then $E(\alpha_k) = M - q_i < \frac{1}{2}q_i$. This implies that the largest approximate value decided by the action of the positive ends of (**b'**) is larger than that one decided by the action of the negative ends of (**c'**) and so this is a contradiction.

Case (**c'**, **a**) (or by symmetry, (**a'**, **c**))

Let $E(\alpha_{k-1}) = M$ $E(\alpha_{k+1}) = M'$, then $E(\alpha_k) = M - q_i = M' - p_{i'}$ where $q_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$ and $p_{i'} = \max(S_{-\theta} \cap \{1,\ 2, ...,\ M'\})$.

Since the type of (α_{k-1}, α_k) is (**c'**), the largest value of f on α_k is $\frac{2}{3}q_i R$ and Since the type of (α_k, α_{k+1}) is (**a**), the largest value of f on α_k is $p_{i'+1}R$. Therefore we have $\frac{2}{3}q_i = p_{i'+1}$. By considering the correspondence of the actions of the part of trivial cylinders, we have two possibilities, $\frac{1}{2}q_i = \frac{1}{2}(p_{i'+1} - p_{i'})$ or $\frac{1}{6}q_i = \frac{1}{2}(p_{i'+1} - p_{i'})$.

If $\frac{1}{2}q_i = \frac{1}{2}(p_{i'+1} - p_{i'})$, this contradicts $\frac{2}{3}q_i = p_{i'+1}$. If $\frac{1}{6}q_i = \frac{1}{2}(p_{i'+1} - p_{i'})$, we have $\frac{1}{2}p_{i'+1} = (p_{i'+1} - p_{i'})$ since $\frac{2}{3}q_i = p_{i'+1}$. This contradicts Claim 2.5.5.

Combining the above arguments, we can see that this case causes a contradiction.

Case (**c'**, **a'**) (or by symmetry, (**a**, **c**))

Let $E(\alpha_{k-1}) = M$. Then $E(\alpha_k) = M - q_i$ $E(\alpha_{k+1}) = M - q_i - q_{i'}$ where $q_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$ and $q_{i'} = \max(S_\theta \cap \{1,\ 2, ...,\ M - q_i\})$.

Moreover, since $\frac{4}{3}q_{i+1} = q_i$,

$$q_{i'} < E(\alpha_k) = M - q_i < \frac{1}{3}q_i < q_i \tag{2.89}$$

and so

$$q_{i'+1} \leq q_i. \tag{2.90}$$

Since the type of (α_{k-1}, α_k) is (**c'**), the largest value of f on α_k is $\frac{2}{3}q_i R$. On the other hand, Since the type of (α_{k-1}, α_k) is (**a'**), the largest value of f on α_k is $(q_{i'+1} - q_{i'})R$ or $\frac{1}{2}q_{i'+1}$. This indicates that $\frac{2}{3}q_i = q_{i'+1} - q_{i'}$ or $\frac{2}{3}q_i = \frac{1}{2}q_{i'+1}$ but the later case contradicts (2.90). So it is sufficient to consider the former case.

Suppose that $\frac{2}{3}q_i = q_{i'+1} - q_{i'}$. By considering the correspondence of the actions of the part of trivial cylinder, we have two possibilities, $\frac{1}{2}q_i = \frac{1}{2}q_{i'+1}$ or $\frac{1}{6}q_i = \frac{1}{2}q_{i'+1}$.

If $\frac{1}{2}q_i = \frac{1}{2}q_{i'+1}$, we can see $\frac{1}{3}q_i = q_{i'}$ by $\frac{2}{3}q_i = q_{i'+1} - q_{i'}$. But this contradict (2.89) $q_{i'} < \frac{1}{3}q_i$.

If $\frac{1}{6}q_i = \frac{1}{2}q_{i'+1}$, we can see $\frac{1}{3}q_i + q_{i'} = 0$ by $\frac{2}{3}q_i = q_{i'+1} - q_{i'}$. This is a contradiction.

Combining the above argument, we can see that this case cause a contradiction.

Case (**c'**, **b'**) (or by symmetry, (**b**, **c**))

Since the type of (α_{k-1}, α_k) is (**c'**), the number of the image of f on α_k is at least three. On the other hand, Since the type of (α_k, α_{k+1}) is (**b'**), the number of the image of f on α_k is at most two. This is a contradiction.

Case (**c'**, **c**)

Let $E(\alpha_{k-1}) = M$ and $E(\alpha_{k+1}) = M'$. Let $q_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$ and $p_{i'} = \max(S_{-\theta} \cap \{1,\ 2, ...,\ M'\})$.

Since the type of (α_{k-1}, α_k) is (**c'**), the largest value of f on α_k is $\frac{2}{3}q_i R$. On the other hand, Since the type of (α_k, α_{k+1}) is (**c**), the largest value of f on α_k is $\frac{2}{3}p_{i'} R$. Therefore $q_i = p_{i'}$. This contradicts $S_\theta \cap S_{-\theta} = \{1\}$.

Case (**c'**, **c'**) (or by symmetry, (**c**, **c**))

Since the type of (α_{k-1}, α_k) is (**c'**), the number of the image of f on α_k on α_k is at least three. On the other hand, Since the type of (α_k, α_{k+1}) is (**c'**), the number of the image of f on α_k is at most two. This is a contradiction.

Case (**a**, **a**) (or by symmetry, (**a'**, **a'**))

Let $E(\alpha_{k+1}) = M$. Then $E(\alpha_k) = M - p_i$ where $p_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$. Set $p_{i'} = \max(S_\theta \cap \{1,\ 2, ...,\ M - p_i\})$. By construction, $p_i \leq p_{i'}$ and so $p_{i+1} \leq p_{i'+1}$.

Since the type of (α_k, α_{k+1}) is (**a**), the largest value of f on α_k is $p_{i+1}R$. On the other hand, since the type of (α_{k-1}, α_k) is (**a**), the image of f on α_k is in $\{\frac{1}{2}p_{i'+1}R,\ \frac{1}{2}(p_{i'+1} - p_{i'})R,\ (p_{i'+1} - p_{i'})R\}$. This is obviously a contradiction.

Case (**a**, **a'**)

Let $E(\alpha_k) = M$, $p_i = \max(S_\theta \cap \{1,\ 2, ...,\ M\})$, $q_{i'} = \max(S_{-\theta} \cap$

$\{1,\ 2, ...,\ M\}$).

By considering the correspondence of the actions of the part of trivial cylinder and Claim 2.5.5, we have two possibilities, $\frac{1}{2}p_{i+1}=\frac{1}{2}(q_{i'+1}-q_{i'})$ or $\frac{1}{2}q_{i'+1}=\frac{1}{2}(p_{i+1}-p_i)$. Here we use $S_\theta \cap S_{-\theta}=\{1\}$ implicitly.

Since the type of (α_{k-1},α_k) is (**a**), the image of f on α_k is in $\{\ \frac{1}{2}p_{i+1}R,\ \frac{1}{2}(p_{i+1}-p_i)R,\ (p_{i+1}-p_i)R\,\}$.

Suppose that $\frac{1}{2}p_{i+1}=\frac{1}{2}(q_{i'+1}-q_{i'})$. Since the type of (α_k,α_{k+1}) is (**a'**), the image of f on α_k contains $(q_{i'+1}-q_{i'})R$. But since $\frac{1}{2}p_{i+1}=\frac{1}{2}(q_{i'+1}-q_{i'})$, then $(q_{i'+1}-q_{i'})R$ is larger than any $\{\ \frac{1}{2}p_{i+1}R,\ \frac{1}{2}(p_{i+1}-p_i)R,\ (p_{i+1}-p_i)R\,\}$. This is a contradiction.

In the case of $\frac{1}{2}q_{i'+1}=\frac{1}{2}(p_{i+1}-p_i)$, we can also derive a contradiction in the same way.

Case (a', a)

Since the type of (α_{k-1},α_k) is (**a'**), the largest value in the image of f on α_k is $q_{i+1}R$ and also since the type of (α_k,α_{k+1}) is (**a**), we have that the largest value in the image of f on α_k is $p_{i+1}R$. This indicates $p_{i+1}=q_{i+1}$ but this contradicts $S_\theta \cap S_{-\theta}=\{1\}$.

Combining the discussions so far, we complete the proof of Lemma. □

Proof of Theorem 2.4.1. By Lemma 2.8.1, we have the rest possibilities of pairs, (**b**, **a**), (**a**, **b'**), (**b**, **b'**), (**c**, **b'**), (**a'**, **b'**), (**c**, **a**), (**a**, **c'**), (**b**, **c'**), (**c**, **c'**), (**a'**, **c'**), (**b**, **a'**), (**c**, **a'**).

It is easy to check that we can not connect the above pairs more than two. This contradicts Proposition 2.4.4 and we complete the proof of Theorem 2.4.1.

□

2.9 Existence of a positive hyperbolic orbit on lens spaces

Proposition 2.9.1. *Let (Y,λ) be a non-degenerate connected contact three manifold with $b_1(Y)=0$. Let $\rho:\tilde{Y}\to Y$ be a p-fold cover with $b_1(\tilde{Y})=0$. Let $(\tilde{Y},\tilde{\lambda})$ be a non-degenerate contact three manifold induced by the covering map. Suppose that α and β be ECH generators in (Y,λ) consisting of only* *hyperbolic orbits. Then*

$$I(\rho^*\alpha, \rho^*\beta) = pI(\alpha, \beta) \tag{2.91}$$

where $\rho^*\alpha$ and $\rho^*\beta$ are inverse images of α, β and thus ECH generators in $(\tilde{Y}, \tilde{\lambda})$.

Proof of Proposition 2.9.1. Let τ be a fixed trivialization of ξ defined over every simple orbit γ in (Y, λ) and $\tilde{\tau}$ be its induced trivialization in $(\tilde{Y}, \tilde{\lambda})$. See just before Definition 1.3.3. For every hyperbolic orbit γ in (Y, λ), $\mu_\tau(\gamma^p) = p\mu_\tau(\gamma)$ and so $p\mu_\tau(\gamma) = \mu_{\tilde{\tau}}(\rho^*\gamma)$ (in the right hand side, if several orbits appear in $\rho^*\gamma$, we add their Conley-Zehnder indexes all together). Moreover, since the terms $c_1(\xi|_Z, \tau)$ and $Q_\tau(Z)$ in ECH index can be defined by counting some kind of intersection numbers, their induced numbers $c_1(\xi|_{\tilde{Z}}, \tilde{\tau})$ and $Q_{\tilde{\tau}}(\tilde{Z})$ in $(\tilde{Y}, \tilde{\lambda})$ where $\{\tilde{Z}\} = H_1(\tilde{Y}; \rho^*\alpha, \rho^*\beta)$ become p times. Under the assumptions, these properties imply that each term of ECH in (Y, λ) becomes p times in $(\tilde{Y}, \tilde{\lambda})$. We complete the proof of Proposition 2.9.1. □

Recall that there are isomorphisms as follows.

$$\mathrm{ECH}(S^3, \lambda, 0) = \mathbb{F}[U^{-1}, U]/U\mathbb{F}[U] \tag{2.92}$$

and for any $\Gamma \in H_1(L(p, q))$,

$$\mathrm{ECH}(L(p, q), \lambda, \Gamma) = \mathbb{F}[U^{-1}, U]/U\mathbb{F}[U]. \tag{2.93}$$

Lemma 2.9.2. *Suppose that all simple orbits in (S^3, λ) are negative hyperbolic. Then, there is a sequence of ECH generators $\{\alpha_i\}_{i=0,1,2,\dots}$ satisfying the following conditions.*

1. *For any ECH generator α, α is in $\{\alpha_i\}_{i=0,1,2,\dots}$.*
2. *$A(\alpha_i) < A(\alpha_j)$ if and only if $i < j$.*
3. *$I(\alpha_i, \alpha_j) = 2(i - j)$ for any i, j .*

Proof of Lemma 2.9.2. The assumption that there is no simple positive hyperbolic orbit means $\partial = 0$ because of (1.9). So the ECH is isomorphic to a free module generated by all ECH generators over $\mathbb{F}$. Moreover, from (2.92), we can see that for every two ECH generators α and β with $A(\alpha) > A(\beta)$, $U^k\langle\alpha\rangle = \langle\beta\rangle$ for some $k > 0$. So for every non negative even number $2i$, there is exactly one ECH generator α_i whose ECH index relative to $\emptyset$ is equal to $2i$. By considering these arguments, we obtain Lemma 2.9.2. □

In the same way as before, we also obtain the next Lemma.

Lemma 2.9.3. *Suppose that all simple orbits in $(L(p,q),\lambda)$ are negative hyperbolic. Then, for any $\Gamma \in H_1(L(p,q))$, there is a sequence of ECH generators $\{\alpha_i^{\Gamma}\}_{i=0,1,2,...}$ satisfying the following conditions.*

1. *For any $i=0,1,2,...$, $[\alpha_i^{\Gamma}]=\Gamma$ in $H_1(L(p,q))$.*

2. *For anyECH generator α with $[\alpha]=\Gamma$, α is in $\{\alpha_i^{\Gamma}\}_{i=0,1,2,...}$.*

3. *$A(\alpha_i^{\Gamma})<A(\alpha_j^{\Gamma})$ if and only if $i<j$.*

4. *$I(\alpha_i^{\Gamma},\alpha_j^{\Gamma})=2(i-j)$ for any $i,\ j$.*

Lemma 2.9.4. *Suppose that all simple orbits in $(L(p,q),\lambda)$ are negative hyperbolic. Then there is no contractible simple orbit.*

Proof of Lemma 2.9.4. Let $\rho:(S^3,\tilde{\lambda})\to(L(p,q),\lambda)$ be the covering map where $\tilde{\lambda}$ is the induced contact form of λ by ρ. Suppose that there is a contractible simple orbit γ in $(L(p,q),\lambda)$. Then the inverse image of γ by ρ consists of p simple negative hyperbolic orbits. By symmetry, they have the same ECH index relative to $\emptyset$. This contradicts the results in Lemma 2.9.2. □

Recall the covering map $\rho:(S^3,\tilde{\lambda})\to(L(p,q),\lambda)$. By Lemma 2.9.4, we can see that there is an one-to-one correspondence between periodic orbits in $(S^3,\tilde{\lambda})$ and ones in $(L(p,q),\lambda)$ under the assumptions. For simplify the notations, we distinguish orbits in $(S^3,\tilde{\lambda})$ from ones in $(L(p,q),\lambda)$ by adding tilde. That is, for each orbit γ in $(L(p,q),\lambda)$, $\tilde{\gamma}$ denotes the corresponding orbit in $(S^3,\tilde{\lambda})$. Furthermore, we also do the same way in orbit sets. That is, for each orbit set $\alpha=\{(\alpha_i,m_i)\}$ over $(L(p,q),\lambda)$, we set $\tilde{\alpha}=\{(\tilde{\alpha}_i,m_i)\}$.

Lemma 2.9.5. *Under the assumptions and notations in Lemma 2.9.3, there is a labelling $\{\Gamma_0,\Gamma_1,....,\Gamma_{p-1}\}=H_1(L(p,q))$ satisfying the following conditions.*

1. *$\Gamma_0=0$ in $H_1(L(p.q))$.*

2. *If $A(\tilde{\alpha}_i^{\Gamma_j})<A(\tilde{\alpha}_{i'}^{\Gamma_{j'}})$, then $i<i'$ or $j<j'$.*

3. *For any $i=0,1,2....$ and $\Gamma_j\in\{\Gamma_0,\Gamma_1,....,\Gamma_{p-1}\}$, $\frac{1}{2}I(\tilde{\alpha}_i{}^{\Gamma_j})=j$ in $\mathbb{Z}/p\mathbb{Z}$.*

Proof of Lemma 2.9.5. By Proposition 2.9.1, for each $\Gamma \in H_1(L(p,q))$, $\frac{1}{2}I(\tilde{\alpha}_i^{\Gamma})$ in $\mathbb{Z}/p\mathbb{Z}$ is independent of i. Moreover, by Lemma 2.9.4, every ECH generator of $(S^3, \tilde{\lambda})$ comes from some of $(L(p,q),\lambda)$ and so in the notation of Lemma 2.9.2 and Lemma 2.9.3, the set $\{\tilde{\alpha}_i^{\Gamma}\}_{i=0,1,...,\ \Gamma \in H_1(L(p,q))}$ is exactly equivalent to $\{\alpha_i\}$. These arguments imply Lemma 2.9.5 (see the below diagram).

$$\begin{array}{ccccccccc}
\langle \emptyset = \tilde{\alpha}_0^{\Gamma_0} \rangle & \xleftarrow{U} & \langle \tilde{\alpha}_0^{\Gamma_1} \rangle & \xleftarrow{U} & \langle \tilde{\alpha}_0^{\Gamma_2} \rangle & \xleftarrow{U} & \cdots\cdots & \xleftarrow{U} & \langle \tilde{\alpha}_0^{\Gamma_{p-1}} \rangle \\
 & & & & \nearrow U & & & & \\
\langle \tilde{\alpha}_1^{\Gamma_0} \rangle & \longleftarrow & \langle \tilde{\alpha}_1^{\Gamma_1} \rangle & \xleftarrow{U} & \langle \tilde{\alpha}_1^{\Gamma_2} \rangle & \xleftarrow{U} & \cdots\cdots & \xleftarrow{U} & \langle \tilde{\alpha}_1^{\Gamma_{p-1}} \rangle \\
 & & & & \nearrow U & & & & \\
\langle \tilde{\alpha}_2^{\Gamma_0} \rangle & \longleftarrow & \langle \tilde{\alpha}_2^{\Gamma_1} \rangle & \xleftarrow{U} & \langle \tilde{\alpha}_2^{\Gamma_2} \rangle & \xleftarrow{U} & \cdots\cdots & &
\end{array} \tag{2.94}$$

□

Lemma 2.9.6. *Suppose that all simple orbits in $(L(p,q),\lambda)$ are negative hyperbolic and p is prime. For $\Gamma \in H_1(L(p.q))$, we set $f(\Gamma_j) \equiv \frac{1}{2}I(\tilde{\alpha}_i^{\Gamma_j}) = j \in \mathbb{Z}/p\mathbb{Z}$ for some $i \geq 0$. Then this map has to be isomorphism as cyclic groups. Here we note that by Lemma 2.9.5, this map has to be well-defined and a bijective map from $H_1(L(p.q))$ to $\mathbb{Z}/p\mathbb{Z}$.*

Proof of Lemma 2.9.6. Since under the assumption there are infinity many simple orbits and $|H_1(L(p,q))| < \infty$, we can pick up p different simple periodic orbits $\{\gamma_1, \gamma_2, ..., \gamma_p\}$ in $(L(p,q),\lambda)$ with $[\gamma_1] = [\gamma_2] = ... = [\gamma_p] = \Gamma$ for some $\Gamma \in H_1(L(p,q))$. Since there is no contractible simple orbit (Lemma 2.9.4), $\Gamma \neq 0$ and so $f(\Gamma) \neq 0$. For $i = 1, 2, ..., p$, let $\tilde{\gamma}_i$ be the corresponding orbit in $(S^3, \tilde{\lambda})$ of γ_i and $C_{\tilde{\gamma}_i}$ be a representative of $Z_{\tilde{\gamma}_i}$ where $\{Z_{\tilde{\gamma}_i}\} = H_2(S^3; \tilde{\gamma}_i, \emptyset)$ (see Definition 1.3.2).

Claim 2.9.7. *Suppose that $1 \leq i, j \leq p$ and $i \neq j$. Then the intersection number $\#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_j})$ in $\mathbb{Z}/p\mathbb{Z}$ does not depend on the choice of i, j where $\#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_j})$ is the algebraic intersection number in $[0,1] \times Y$.*

Proof of Claim 2.9.7. By the definition, we have

$$\frac{1}{2}I(\tilde{\gamma}_i \cup \tilde{\gamma}_j, \tilde{\gamma}_i) = \frac{1}{2}I(\tilde{\gamma}_j) + \#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_j}) \tag{2.95}$$

So in $\mathbb{Z}/p\mathbb{Z}$,

$$\begin{aligned}\#([0,1]\times\tilde{\gamma}_i\cap C_{\tilde{\gamma}_j}) &= \frac{1}{2}I(\tilde{\gamma}_i\cup\tilde{\gamma}_j,\tilde{\gamma}_i)-\frac{1}{2}I(\tilde{\gamma}_j)\\ &=\frac{1}{2}I(\tilde{\gamma}_i\cup\tilde{\gamma}_j)-\frac{1}{2}I(\tilde{\gamma}_i)-\frac{1}{2}I(\tilde{\gamma}_j)=f(2\Gamma)-2f(\Gamma)\end{aligned} \tag{2.96}$$

This implies that the value $\#([0,1]\times\tilde{\gamma}_i\cap C_{\tilde{\gamma}_j})$ in $\mathbb{Z}/p\mathbb{Z}$ depends only on $f(2\Gamma)$ and $f(\Gamma)$. This complete the proof of Claim 2.9.7. □

Return to the proof of Lemma 2.9.6. we set $l:=\#([0,1]\times\tilde{\gamma}_i\cap C_{\tilde{\gamma}_j})\in\mathbb{Z}/p\mathbb{Z}$ for $i\neq j$.

In the same way as Claim 2.9.7, for $1\leq n\leq p$, we have

$$\frac{1}{2}I(\bigcup_{1\leq i\leq n}\tilde{\gamma}_i,\bigcup_{1\leq i\leq n-1}\tilde{\gamma}_i)=\frac{1}{2}I(\tilde{\gamma}_n)+\sum_{1\leq i\leq n-1}\#([0,1]\times\tilde{\gamma}_i\cap C_{\tilde{\gamma}_n}) \tag{2.97}$$

and so

$$f(n\Gamma)-f((n-1)\Gamma)=f(\Gamma)+(n-1)l\quad\text{in }\mathbb{Z}/p\mathbb{Z}. \tag{2.98}$$

Suppose that $l\neq 0$. Since p is prime, there is $1\leq k\leq p$ such that $f(\Gamma)+(k-1)l=0$. This implies that $f(k\Gamma)-f((k-1)\Gamma)=0$. But this contradicts the bijectivity of f. So $l=0$ and therefore $f(n\Gamma)=nf(\Gamma)$. Since $f(\Gamma)\neq 0$, we have that f is isomorphism. □

Lemma 2.9.8. *Suppose that all simple orbits in $(L(p,q),\lambda)$ are negative hyperbolic. Let $\gamma_{\min}$ and γ_{sec} be orbits with smallest and second smallest actions in $(L(p,q),\lambda)$ respectively. Then,*

$$I(\tilde{\gamma}_{\min})=I(\tilde{\gamma}_{\text{sec}},\tilde{\gamma}_{\min})=2 \tag{2.99}$$

and moreover,

$$6<I(\tilde{\gamma}_{\min}\cup\tilde{\gamma}_{\text{sec}})\leq 2p. \tag{2.100}$$

Proof of Lemma 2.9.8. Consider the diagram (2.94) and Lemma 2.9.2. As ECH generators, $\tilde{\gamma}_{\min}$ and $\tilde{\gamma}_{\text{sec}}$ correspond to $\tilde{\alpha}_0^{\Gamma_1}$ and $\tilde{\alpha}_0^{\Gamma_2}$ respectively. This implies (2.99).

Next, we show the inequality (2.100). See $\tilde{\alpha}_1^{\Gamma_0}$ in the diagram (2.94). By the diagram, $I(\tilde{\alpha}_1^{\Gamma_0})=2p$. Moreover, this comes from $\alpha_1^{\Gamma_0}$ with $[\alpha_1^{\Gamma_0}]=0\in H_1(L(p,q))$. By Lemma 2.9.4, $\alpha_1^{\Gamma_0}$ has to consist of at least two negative hyperbolic orbits. This implies that $A(\tilde{\gamma}_{\min}\cup\tilde{\gamma}_{\text{sec}})\leq A(\tilde{\alpha}_1^{\Gamma_0})$ and so by Lemma 2.9.2, $I(\tilde{\gamma}_{\min}\cup\tilde{\gamma}_{\text{sec}})\leq I(\tilde{\alpha}_1^{\Gamma_0})=2p$.

Considering the above argument and Lemma 2.9.2, it is enough to show that $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}) \neq 6$. We prove this by contradiction. Suppose that $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}) = 6$. Since $\tilde{\gamma}_{\mathrm{sec}}$ corresponds to $\tilde{\alpha}_0^{\Gamma_2}$, we have $I(\tilde{\gamma}_{\mathrm{sec}}) = 4$ and so $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}, \tilde{\gamma}_{sec}) = 2$. To consider the U-map, fix a generic almost complex structure J on $\mathbb{R} \times S^3$. Consider the U-map $U\langle \tilde{\alpha}_0^{\Gamma_1} = \tilde{\gamma}_{\min} \rangle = \langle \emptyset \rangle$. This implies that for each generic point $z \in S^3$, there is an embedded J-holomorphic curve $C_z \in \mathcal{M}^J(\tilde{\gamma}_{\min}, \emptyset)$ through $(0, z) \in \mathbb{R} \times S^3$. By using this C_z, we have

$$I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}, \tilde{\gamma}_{\mathrm{sec}}) = I(\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cup C_z). \tag{2.101}$$

Note that the right hand side of (2.101) is the ECH index of holomorphic curves $\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cup C_z$. Since $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}) = 2$ and Proposition 1.3.10, we have $\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cap C_z = \emptyset$.

Consider a sequence of holomorphic curves C_z as $z \to \tilde{\gamma}_{\mathrm{sec}}$. By the compactness argument, this sequence has a convergent subsequence and this has a limiting holomorphic curve C_∞ which may be splitting into more than one floor. But in this case, C_∞ can not split because the action of the positive end of C_∞ is smallest value $A(\tilde{\gamma}_{\min})$. This implies that $\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cap C_\infty \neq \emptyset$. Since $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}, \tilde{\gamma}_{\mathrm{sec}}) = I(\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cup C_\infty) = 2$, $\mathbb{R} \times \tilde{\gamma}_{\mathrm{sec}} \cap C_\infty \neq \emptyset$ contradicts the fourth statement in Proposition 1.3.10. Therefore, we have $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}) \neq 6$ and thus complete the proof of Lemma 2.9.8. □

Proof of Theorem 2.2.5. We may assume that p is prime because the condition that all simple periodic orbits are negative hyperbolic does not change under taking odd-fold covering. By Lemma 2.9.6 and (2.99), we have $[\gamma_{\mathrm{sec}}] = 2[\gamma_{\min}]$ and so $[\gamma_{\min} \cup \gamma_{\mathrm{sec}}] = 3[\gamma_{\min}]$ in $H_1(L(p,q))$. Since f is isomorphic, we have $\frac{1}{2}I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\mathrm{sec}}) = 3$ in $\mathbb{Z}/p\mathbb{Z}$. But this can not occur in the range of (2.100). This is a contradiction and we complete the proof of Theorem 2.2.5. □

Proof of Theorem 2.2.7. We prove this by contradiction. By Theorem 2.2.3, we may assume that there is no elliptic orbit and so that all simple orbits are negative hyperbolic. In the same as Lemma 2.9.2, there is exactly one ECH generator α_i whose ECH index relative to $\emptyset$ is equal to $2i$. If there is a non $\mathbb{Z}/2\mathbb{Z}$-invariant orbit γ, by symmetry, there is two orbit with the same ECH index relative to $\emptyset$. this is a contradiction. So we may assume that all simple orbits are $\mathbb{Z}/2\mathbb{Z}$-invariant.

Let $(\mathbb{RP}^3, \lambda')$ be the non-degenerate contact three manifold obtained as the quotient space of (S^3, λ) and γ be a $\mathbb{Z}/2\mathbb{Z}$-invariant periodic orbit. Then, this orbit corresponds to a double covering of a non-contractible orbit γ' in

$(\mathbb{RP}^3, \lambda')$. This implies that the eigenvalues of the return map of γ are square of the ones of γ'. This means that the eigenvalues of the return map of γ are both positive and so γ is positive hyperbolic. This is a contradiction and so we complete the proof of Theorem 2.2.7. □

Proof of Corollary 2.2.8. Note that if $(L(p,q), \lambda)$ has a simple positive hyperbolic orbit, then its covering space $(S^3, \tilde{\lambda})$ is also the same. Considering a non-trivial cyclic subgroup acting on the contact three sphere together with Corollary 2.2.6 and Theorem 2.2.7, we complete the proof of Corollary 2.2.8. □

Chapter 3

Birkhoff sections of disk-type, convex Reeb flows and Embedded contact homology

3.1 Background

At first, we recall some notions. Let $\mathbb{D}$ denote the unit closed disk.

Definition 3.1.1. *Let (Y, λ) be a contact three-manifold. A Birkhoff section of disk type for X_λ is a compact immersed surface $\mathbb{D} \to Y$ such that*

(1). $u(\mathbb{D} \backslash \partial \mathbb{D}) \subset Y \backslash u(\partial \mathbb{D})$ is embedded,

(2). X_λ is transversal to $u(\mathbb{D} \backslash \partial \mathbb{D})$,

(3). $u(\partial \mathbb{D})$ consists of a periodic orbit of X_λ,

(4). For every $x \in Y \backslash u(\partial \mathbb{D})$, there are $-\infty < t_x^- < 0 < t_x^+ < +\infty$ such that $\phi^{t_x^{\pm}}(x) \in u(\mathbb{D} \backslash \partial \mathbb{D})$ where ϕ^t is the flow of X_λ.

Consider Birkhoff sections of disk type. A remarkable benefit of the existence of a Berkhoff section is that the restriction of $d\lambda$ on $u(\mathring{\mathbb{D}})$ define a volume form and the first return map $\phi : (u(\mathring{\mathbb{D}}), d\lambda) \to (u(\mathring{\mathbb{D}}), d\lambda)$ is an orientation and volume preserving map. Hofer, Wysocki and Zehnder [HWZ4] constructed a Birkhoff section of disk type from a J-holomorphic curve in a dynamically convex contact 3-sphere and proved by applying the Franks' result and Brouwer's translation theorem that there are either infinitely many simple periodic orbits or exactly two simple periodic orbits in a dynamically convex contact 3-sphere. Here, the notion of dynamically convex was introduced in [HWZ4] as a generalization of strictly convex contact hypersurface in the 4-dimensional standard symplectic Euclidean space $(\mathbb{R}^4, \omega)$. In particular, strictly convex contact hypersurface in $(\mathbb{R}^4, \omega)$ is dynamically convex.

Remark 3.1.2. According to [HT3, CHrHL1] if a contact 3-manifold (Y, λ) has exactly two simple periodic orbit, then (Y, λ) is dynamically convex and both of them are non-degenerate elliptic orbit. In addition, Y is a lens space. If (Y, λ) is non-degenerate and not a lens space with exactly two simple orbits, there are infinitely many periodic orbits (see[CHP, CoDR]).

The next is the precise definition of dynamical convexity.

Definition 3.1.3 ([HWZ4])**.** *Assume that a contact three manifold (Y, λ) satisfies $c_1(\xi)|_{\pi_2(Y)} = 0$. λ is called dynamically convex if $\mu_{\text{disk}}(\gamma) \geq 3$ for any contractible periodic orbit γ, where $\mu_{\text{disk}}(\gamma)$ is defined as follows. Take a smooth map $u : \{|z| \leq 1 | z \in \mathbb{C}\} := \mathbb{D} \to Y$ with $u(e^{2\pi it}) = \gamma(T_\gamma t)$ and a global trivialization $\tau_{\text{disk}} : u^*\xi \to \mathbb{D} \times \mathbb{C}$. Then $\mu_{\text{disk}}(\gamma) := \mu_{\tau_{\text{disk}}}(\gamma)$. Note that $\mu_{\text{disk}}(\gamma)$ is independent of the choice of u since $c_1(\xi)|_{\pi_2(Y)} = 0$.*

We define a contact lens space to be stirictly convex as follows.

Definition 3.1.4. *(S^3, λ) is strictly convex if there is an embedding $i : S^3 \to \mathbb{C}^2$ such that $i(S^3) \subset \mathbb{C}^2$ is a strictly convex hypersurface surrounding 0 and $i^*\lambda_0 = \lambda$. More generally, consider $(L(p,q), \lambda)$ and the covering map $\rho : S^3 \to L(p,q)$. $(L(p,q), \lambda)$ is strictly convex if there is an embedding $i : S^3 \to \mathbb{C}^2$ with $i^*\lambda_0 = \rho^*\lambda$ such that $i(S^3) \subset \mathbb{C}^2$ is a strictly convex hypersurface surrounding 0 and in addition i is an equivalent map under $(z_1, z_2) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi iq}{p}} z_2)$.*

Theorem 3.1.5. *[HWZ4] If (S^3, λ) (resp. $(L(p,q), \lambda)$) is strictly convex, then (S^3, λ) (resp. $(L(p,q), \lambda)$) is dynamically convex.*

Remark 3.1.6. It follows from the definition that the condition of dynamical convexity is preserved under taking a finite cover.

In this chapter we focus on lens spaces $L(p,q)$ and the standard contact structure ξ_{std} on them. Recall the standard contact structures. Let $p \geq$

$q > 0$ be mutually prime. The standard contact structure ξ_{std} is defined as follows. Consider a contact 3-sphere $(\partial B(1), \lambda_0|_{\partial B(1)})$ where $\partial B(1) = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$, $\lambda_0 = \frac{i}{2} \sum_{i=1,2}(z_i d\bar{z}_i - \bar{z}_i dz_i)$. The action $(z_1, z_2) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2)$ preserves $(\partial B(1), \lambda_0|_{\partial B(1)})$ and the tight contact structure. Hence we have the quotient space which is a contact manifold and write $(L(p,q), \lambda_{p,q})$, $\xi_{\text{std}} = \text{Ker}\lambda_{p,q}$.

To explain results, we recall some notions.

Definition 3.1.7. *A knot $K \subset Y^3$ is called p-unknotted if there exists an immersion $u : \mathbb{D} \to Y$ such that $u(\mathring{\mathbb{D}}) \subset Y \backslash u(\partial \mathbb{D})$ is embedded and $u|_{\partial \mathbb{D}} : \partial \mathbb{D} \to K$ is a p-covering map.*

Remark 3.1.8. Let $K \subset Y$ be a p-unknotted knot and $u : \mathbb{D} \to Y$ be a immersed disk as above. Then the union of a neighborhood of $u(\mathbb{D})$ and K is diffeomorohic to $L(p,k) \backslash B^3$ for some k where B^3 is the 3-ball. Therefore, Y is $L(p,k) \# M$ for some k and a closed 3-manifold M. In particular, if a lens space admits a p-unknotted knot, then it is diffeomorphic to $L(p,k)$ for some k [BE, cf. Section 5].

Definition 3.1.9. *[BE, cf. Subsection 1.1] Let (Y, λ) be a contact 3-manifold with $\text{Ker}\lambda = \xi$. Assume that a knot $K \subset Y$ is p-unknotted, transversal to ξ and oriented by the co-orientation of ξ. Let $u : \mathbb{D} \to Y$ be an immersion such that $u|_{\text{int}(\mathbb{D})}$ is embedded and $u|_{\partial \mathbb{D}} : \partial \mathbb{D} \to K$ is a orientation preserving p-covering map. Take a non-vanishing section $Z : \mathbb{D} \to u^*\xi$ and consider the immersion $\gamma_\epsilon : t \in \mathbb{R}/\mathbb{Z} \to \exp_{u(e^{2\pi i t})}(\epsilon Z(u(e^{2\pi i t}))) \in Y \backslash K$ for small $\epsilon > 0$.*

Define the rational self-linking number $sl^{\mathbb{Q}}_{\xi}(K, u) \in \mathbb{Q}$ as

$$sl^{\mathbb{Q}}_{\xi}(K, u) = \frac{1}{p^2} \#(\text{Im}\gamma_\epsilon \cap u(\mathbb{D}))$$

where # counts the intersection number algebraically. If $c_1(\xi)|_{\pi_2(Y)} = 0$, $sl^{\mathbb{Q}}_{\xi}(K, u)$ is independent of u. Thus we write $sl^{\mathbb{Q}}_{\xi}(K)$.

Remark 3.1.10. In generall, (rational) self-linking number is defined for rationally null-homologous knot by using a (rational) Seifert surface. See [BE].

According to [HWZ3, HWZ4, HWZ5, Hr1], if a dynamically convex contact 3-manifold (Y, λ) with $\text{Ker}\lambda = \xi$ has a 1-unknotted simple orbit γ with self-linking number -1, then (Y, ξ) is contactomorphic to (S^3, ξ_{std}) and in addition there is a global surface of section of disk type binding γ. Moreover any dynamically convex contact form λ on S^3 has a 1-unknotted simple orbit γ with self-linking number -1 and $\mu_{\text{disk}}(\gamma) = 3$.

In [HrLS], Hryniewicz, Licata, and Salomão generalized the result to lens spaces as follows. Let $p \in \mathbb{Z}_{>0}$. A closed connected contact 3-manifold (Y, ξ) is contactomorphic to $(L(p,k), \xi_{\text{std}})$ for some k if and only if there is a dynamically convex contact form λ with $\text{Ker}\lambda = \xi$ such that X_λ has a p-unknotted simple orbit γ with self-linking number $-\frac{1}{p}$. In connection with these results, Hryniewicz and Salomão showed the following result.

Theorem 3.1.11. *[HrS2, Theorem 1.7, Corollary 1.8] If λ is any dynamically convex contact form on $L(p,q)$, then for every p-unknotted simple orbit γ with $sl_\xi^{\mathbb{Q}}(\gamma) = -\frac{1}{p}$, γ^p must bound a disk which is a global surface of section for the Reeb flow. Moreover, this disk is a page of a rational open book decomposition of $L(p,q)$ with binding γ such that all pages are disk-like global surfaces of section.*

It is natural to consider the next question.

Question 3.1.12. *Does any dynamically convex contact form λ on $L(p,q)$ with $\text{Ker}\lambda = \xi_{\text{std}}$ have a p-unknotted simple orbit with self-linking number $-\frac{1}{p}$?*

For $(L(p,q), \lambda)$, define

$$\mathcal{S}_p(L(p,q),\lambda) := \{\gamma \text{ simple orbit of } (L(p,q),\lambda) |\ p\text{−unknotted, } sl_\xi^{\mathbb{Q}}(\gamma) = -\frac{1}{p}\}.$$

We sometimes write $\mathcal{S}_p$ instead of $\mathcal{S}_p(L(p,q),\lambda)$ if there is no confusion.

Originally as mentioned, Hofer, Wysocki and Zehnder proved in [HWZ4] that any dynamically convex contact form λ on S^3 admits $\gamma \in \mathcal{S}_1$ with $\mu_{\text{disk}}(\gamma) = 3$. In particular, they used the compactification of pseudoholomorphic curves coming from an ellipsoid in a symplectic manifold which connects the symplectization of the ellipsoid with the one of (S^3, λ). After that, Hryniewicz and Salomão proved in [HrS2] that any dynamically convex contact form λ on $L(2,1)$ admits $\gamma \in \mathcal{S}_2$ with $\mu_{\text{disk}}(\gamma^2) = 3$ especially such orbits are elliptic. Recently, Schneider generalized in [Sch] the results to $(L(p,1), \lambda)$ with $\text{Ker}\lambda = \xi_{\text{std}}$.

In any case, the original ideas in [HWZ4] of considering the compactification of pseudoholomorphic curves coming from an ellipsoid are essentially used. Note that in lens spaces, we should consider the multiple covers of pseudoholomorphic curves and in [HrS2, Sch] they observe that such curves behave good in $(L(p,1),\xi_{\mathrm{std}})$.

In summary;

Theorem 3.1.13. *[HWZ4, HrS2, Sch] Let $(L(p,1),\lambda)$ be dynamically convex with* $\mathrm{Ker}\lambda=\xi_{\mathrm{std}}$. *Then, there is a simple orbit γ such that $\gamma\in\mathcal{S}_p$ and $\mu_{\mathrm{disk}}(\gamma^p)=3$.*

Organization of this chapter

In §3.2, we introduce main results. §3.2 consists of three parts. In §3.2.1, we will present the results concerning the existence of a Birkhoff section of disk type on convex $L(p,p-1)$ and its relation with the first ECH spectrum. The contents of §3.2.1 based on [Shi3]. Next, §3.2.2 will discuss the ellipticy of a rational unknotted Reeb orbit on a lens space and its relation with a Birkhoff section of disk type. In addition, we will also see the first ECH spectrum on convex $L(3,1)$. The contents of §3.2.2 are based on [Shi4]. Last, in §3.2.3, we will discuss area-preserving diffeomorphisms on the open unit disk with certain properties and its applications to the existence of infinitely simple positive hyperbolic orbits on a lens space with a Birkhoff section of disk type. §3.2.3 is based on [AsaShi] which is a joint work with Masayuki Asaoka.

3.2 Main results

3.2.1 Existence of Birkhoff section of disk type on convex $L(p,p-1)$ and the first ECH spectrum

The main result of §3.2.1 is to find a simple orbit $\gamma\in\mathcal{S}_p$ in $(L(p,p-1),\xi_{\mathrm{std}})$ by using Embedded contact homology and relate their periods to the first ECH spectrum.

According to [HrHuRa], the next Theorem holds.

Theorem 3.2.1. *[HrHuRa] Assume that (S^3,λ) is dynamically convex and $c_1^{\mathrm{ECH}}(S^3,\lambda)$ is the first ECH spectrum of (S^3,λ) (explained in the next subsection). Then*

$$c_1^{\mathrm{ECH}}(S^3,\lambda)=\inf_{\gamma\in\mathcal{S}_1}\int_\gamma\lambda$$

Consider $(L(p,p-1),\lambda)$ with $\mathrm{Ker}\lambda=\xi_{\mathrm{std}}$. Since ξ_{std} is trivial, we can take a global symplectic trivialization $\tau_{\mathrm{glob}}:\xi_{\mathrm{std}}\to L(p,p-1)\times\mathbb{C}$. For an orbit γ, define $\mu_{\mathrm{glob}}(\gamma)$ as the Conley-Zehnder index with respect to the global trivialization. We note that if γ is contractible, $\mu_{\mathrm{glob}}(\gamma)=\mu_{\mathrm{disk}}(\gamma)$.

Our main results are as follows.

Theorem 3.2.2.

(1). If $(L(2,1),\lambda)$ is strictly convex or non-degenerate dynamically convex, then

$$\inf_{\gamma\in\mathcal{S}_2,\mu_{\mathrm{glob}}(\gamma)=1}\int_\gamma\lambda=\frac{1}{2}c_1^{\mathrm{ECH}}(L(2,1),\lambda).$$

Moreover there exists $\gamma\in\mathcal{S}_2$ satisfying $\mu_{\mathrm{glob}}(\gamma)=1$ and

$$\int_\gamma\lambda=\frac{1}{2}c_1^{\mathrm{ECH}}(L(2,1),\lambda).$$

(2). Suppose that $(L(p,p-1),\lambda)$ is strictly convex. If $p=3$ or 4 or 6, there exists $\gamma\in\mathcal{S}_p$ satisfying $\mu_{\mathrm{glob}}(\gamma)=1$ and

$$\int_\gamma\lambda\leq\frac{1}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda).$$

(3). Suppose that $(L(p,p-1),\lambda)$ is non-degenerate dynamically convex.. Then for any p, there exists $\gamma\in\mathcal{S}_p$ satisfying $\mu_{\mathrm{glob}}(\gamma)=1$ and

$$\int_\gamma\lambda\leq\frac{1}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda).$$

Remark 3.2.3.

(1). In general, in order to prove something under degeneracy, we usually consider it as a limiting case of non-degenerate objects. The reason why we only consider $p=3,4,6$ in the Theorem 3.2.2 (3) is that otherwise the limit of simple orbits might not be simple.

(2). In contrast to Theorem 3.1.13, if $p\geq 3$ in $L(p,p-1)$, the orbit γ is $\mu_{\mathrm{glob}}(\gamma)=1$ but $\mu(\gamma^p)$ might not be equal to 3. The author thinks that this relates to the difficulty of applying the original method in [HWZ4] to $L(p,q)$ under $q\neq 1$. Note that Schneider mentions in [Sch, Remark 1.6] that even if we can find an orbit $\gamma\in\mathcal{S}_p$ in $L(p,q)$ under $q\neq 1$, $\mu_{\mathrm{glob}}(\gamma^p)$ might not be equal to 3.

As an immediate application of Theorem 3.2.2 and Theorem 3.1.11, we have

Corollary 3.2.4. *If $(L(p,p-1),\lambda)$ is non-degenerate dynamically convex, then X_λ admits a p-unknotted closed Reeb orbit γ with $\mu_{\mathrm{glob}}(\gamma)=1$ which is the binding of a rational open book decomposition. Each page of the open book is a rational global surface of section of disk type whose contact area is not larger than $\frac{p}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)$. Moreover, if $(L(p,p-1),\lambda)$ is strictly convex and $p=2,3,4,6$, we can exclude the condition of non-degeneracy.*

By combining with Theorem 3.2.1, we have

Corollary 3.2.5. *Assume that $(L(p,p-1),\lambda)$ is strictly convex or non-degenerate dynamically convex. Let $\rho : S^3 \to L(p,p-1)$ be the covering map. Then,*

$$c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda) \leq \frac{p}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda).$$

Proof of Corollary 3.2.5. If non-degenerate, it follows from Theorem 3.2.2 that there is $\gamma \in \mathcal{S}_p(L(p,p-1),\lambda)$ with $\mu(\gamma)=1$ and $\int_\gamma \lambda \leq \frac{1}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)$. Let $\tilde{\gamma}$ be the periodic orbit in $(S^3,\rho^*\lambda_n)$ such that $\rho|_{\tilde{\gamma}} : \tilde{\gamma} \to \gamma$ is a p-fold cover. It is obvious that $\tilde{\gamma} \in \mathcal{S}_1(S^3,\rho^*\lambda)$. It follows from Theorem 3.2.1 that

$$c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda) = \inf_{\gamma\in\mathcal{S}_1}\int_\gamma \lambda \leq \int_{\tilde{\gamma}}\rho^*\lambda = p\int_\gamma \lambda \leq \frac{p}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda). \quad (3.1)$$

Therefore, we have $c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda) \leq \frac{p}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)$.

If strictly convex, we consider a sequence of strictly convex and non-degenerate contact forms λ_n such that $\mathrm{Ker}\lambda_n = \xi_{\mathrm{std}}$ and $\lambda_n \to \lambda$ in C^∞-topology. Then it follows from the property of ECH spectrum that $c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda) \leq \frac{p}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)$ as the limit. We complete the proof. □

Remark 3.2.6. The Weyl law with respact to ECH spectrum proved in [CHR] says $\lim_{k\to\infty}\frac{c_k^{\mathrm{ECH}}(S^3,\rho^*\lambda)^2}{k} = 2\mathrm{Vol}(S^3,\rho^*\lambda)$ and $\lim_{k\to\infty}\frac{c_k^{\mathrm{ECH}}(L(p,p-1),\lambda)^2}{k} = 2\mathrm{Vol}(L(p,p-1),\lambda)$. This implies that

$$\lim_{k\to\infty}\frac{c_k^{\mathrm{ECH}}(S^3,\rho^*\lambda)}{c_k^{\mathrm{ECH}}(L(p,p-1),\lambda)} = \sqrt{\frac{\mathrm{Vol}(S^3,\rho^*\lambda)}{\mathrm{Vol}(L(p,p-1),\lambda)}} = \sqrt{p}. \quad (3.2)$$

But if λ is strictly convex, it follows from Corollary 3.2.5 that $\frac{c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda)}{c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)} \leq \frac{p}{2}$. This implies that if $p=2,3$, the ratio is not close to $\sqrt{p}$ when $k=1$.

Idea and outline of the proof

The basic idea is to find the orbits from the algebraic structure of ECH called U-map. From the behaviors of J-holomorphic curves and algebraic structure, we can find an ECH generator α satisfying $\langle U_{J,z}\alpha, \emptyset\rangle \neq 0$ and $A(\alpha) \leq c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)$. In addition, it follows that if $\langle U_{J_z}\alpha, \emptyset\rangle \neq 0$, α is described as either

(1). $\alpha = (\gamma, p)$ with $\gamma \in \mathcal{S}_p$, $\mu(\gamma^p) = 3$ and $\mu(\gamma) = 1$, or

(2). $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ with $\gamma_1, \gamma_2 \in \mathcal{S}_p$, $\mu(\gamma_1) = \mu(\gamma_2) = 1$.

In §3.3.1, we focus on non-degenerate cases and study the behaviors of Conley-Zehnder index and J-holomorphic curves. In §3.3.2, we complete the proof Theorem 3.2.2 under non-degeneracy. In §3.3.3, we extend the result of §3.3.2 to degenerate cases under strictly convex and $p = 2, 3, 4, 6$. In §3.3.4, we prove Theorem 3.3.10 used in §3.3.1.

3.2.2 Elliptic bindings on lens spaces and the first ECH spectrum on $L(3,1)$

Besides the existence of Birkhoff sections, there is a long-standing conjecture whether a convex energy hypersurface in the standard symplectic Euclidean space carries an elliptic orbit, and there are many previous studies under some additional assumptions (c.f. [AbMa1], [AbMa2], [DDE], [HuWa], [LoZ]). As one of them, it is natural to consider the quotient of a convex energy hypersurface by a cyclic group action which becomes a lens space ([AbMa1], [AbMa2]) . In this case, the result depends on the lens space.

Our first result in this subsection as follows is to introduce a sufficient condition for a given rational unknotted Reeb orbit γ in $L(p,q)$ to be elliptic by using the rational self-linking number $sl_{\xi}^{\mathbb{Q}}(\gamma)$ and the Conley-Zehnder index $\mu_{\mathrm{disk}}(\gamma^p)$.

Theorem 3.2.7. *Let $p > q > 0$ be mutually prime. Let λ be a contact form on $L(p,q)$ with $\lambda \wedge d\lambda > 0$. Let $\mathrm{Ker}\lambda = \xi$ and γ be a p-unknotted Reeb orbit in $(L(p,q),\lambda)$. Suppose that $-2r - 2p \cdot sl_{\xi}^{\mathbb{Q}}(\gamma) - \mu_{\mathrm{disk}}(\gamma^p)$ is not divisible by p for any $r \in \mathbb{Z}$ satisfying either $r = -q \mod p$ or $rq = -1 \mod p$. Then γ is elliptic.*

As will be mentioned, the sufficient condition is especially useful under the existence of Birkhoff sections.

We note that it is obvious that a periodic orbit with $\mu_{\text{disk}}(\gamma^2) = 3$ on dynamically convex $(L(2,1),\lambda)$ is elliptic as follows. As mentioned, $\text{Ker}\lambda = \xi$ is universally tight if $(L(2,1),\lambda)$ is dynamically convex. Therefore ξ is topologically trivial and we can take a global symplectic trivialization $\tau_{\text{glob}} : \xi \to L(2,1) \times \mathbb{R}^2$. If a periodic orbit γ is hyperbolic, we have $\mu_{\tau_{\text{glob}}}(\gamma^2) = 2\mu_{\tau_{\text{glob}}}(\gamma)$ (c.f. Proposition 1.2.4). On the other hand, since γ^2 is contractible, it follows from the definition that $\mu_{\tau_{\text{glob}}}(\gamma^2) = \mu_{\text{disk}}(\gamma^2)$. This means that $2\mu_{\tau_{\text{glob}}}(\gamma) = \mu_{\tau_{\text{glob}}}(\gamma^2) = \mu_{\text{disk}}(\gamma^2) = 3$. This contradicts $\mu_{\tau_{\text{glob}}}(\gamma) \in \mathbb{Z}$.

In general, the above argument can not be applied to $(L(p,1),\lambda)$ with $\lambda \wedge d\lambda > 0$ unlike $L(2,1)$ because the universally tight contact structure on $L(p,1)$ is not topologically trivial for $p > 2$. But it follows from Theorem 3.1.11 that the same result holds for $L(p,1)$ as follows.

Corollary 3.2.8. *Let $(L(p,1),\lambda)$ with $\lambda \wedge d\lambda > 0$ be dynamically convex. Let γ be a periodic orbit such that γ^p binds a Birkhoff section of disk type and $\mu_{\text{disk}}(\gamma^p) = 3$. Then γ is elliptic. Note that according to [Sch], such a periodic orbit always exists.*

Remark 3.2.9. It is proved in [AbMa1] that a dynamically convex $(L(p,1),\lambda)$ admits an elliptic orbit.

Proof of Corollary 3.2.8. We apply Theorem 3.1.11 to $L(p,1)$. In this case, $q = 1$ and hence it suffices to consider $r \in \mathbb{Z}$ satisfying $r = -1 \bmod p$. Since $r = -1 \bmod p$, $\mu_{\text{disk}}(\gamma^p) = 3$ and $sl_{\xi}^{\mathbb{Q}}(\gamma) = -\frac{1}{p}$ (Theorem 3.1.11), we have $-2r - 2p \cdot sl_{\xi}^{\mathbb{Q}}(\gamma) - \mu_{\text{disk}}(\gamma^p) = 1 \bmod p$. This means that γ is elliptic. $\square$

As a generalization, it is a natural to ask the following question.

Question 3.2.10. *Let $(L(p,q),\lambda)$ (including S^3 as $p = 1$, $q = 0$) be dynamically convex. Does there always exist a periodic orbit γ such that γ^p binds a Birkhoff section of disk type?*

Note that the author does not know whether the periodic orbits are elliptic obtained in Theorem 3.2.2 for dynamically convex $(L(p,p-1),\lambda)$ with $\lambda \wedge d\lambda > 0$.

Our second result in this subsection is to compute the first ECH spectrum on convex $L(3,1)$ as in Theorem 3.2.2. In the computation, our first result Theorem 3.2.7 plays an important role.

Theorem 3.2.11. *Let $(L(3,1)\lambda)$ be a strictly convex (or non-degenerate dynamically convex) contact 3-manifold. Then*

$$\frac{1}{3}c_1^{\mathrm{ECH}}(L(3,1)\lambda) = \inf_{\gamma\in\mathcal{S}_3,\ \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda.$$

As an immediate corollary, we have

Corollary 3.2.12. *Assume that $(L(3,1),\lambda)$ is strictly convex or non-degenerate dynamically convex. Let $\rho : S^3 \to L(3,1)$ be the covering map. Then,*

$$c_1^{\mathrm{ECH}}(S^3,\rho^*\lambda) \leq c_1^{\mathrm{ECH}}(L(3,1),\lambda).$$

The proof is completely the same with Corollary 3.2.5.

Idea and outline of the proof

At first, we prove Therorem 3.2.7. The idea is as follows. Let γ be a unkonotted orbit in a lens space. Take a tubular neighborhood, then we have a Heegaard splitting of genus 1. Consider the twist of the gluing map and compare the trivializations of the contact structure over the solid torus and a binding disk. By combining them with the properties of Conley-Zehnder index, we have Therorem 3.2.7. The observation of the proof is also essential in the following proof of Theorem 3.2.11.

Next, we consider Theorem 3.2.11.

The estimate $\inf_{\gamma\in\mathcal{S}_3,\ \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda \leq \frac{1}{3}c_1^{\mathrm{ECH}}(L(3,1)\lambda)$ is a hard part of the proof. At first, we conduct technical computations regarding indices present in ECH to clear which holomorphic curves appear as U-map to the empty set. As a result, it follows that any moduli space of holomorphic curves counted by U-map gives a structure of rational open book decomposition (Lemma 3.4.7 and Lemma 3.4.10). Next, I consider the rational open book decompositions. By considering which actually supports $(L(3,1),\xi_{\mathrm{std}})$, we can narrow down the list of holomorphic curves. Then it turn out that any binding of the rational open book decomposition coming from the U-map γ is in $\mathcal{S}_3$ and $\mu_{\mathrm{disk}}(\gamma^3)=3$ and in addition we can choose it so that $\int_\gamma \lambda \leq \frac{1}{3}c_1^{\mathrm{ECH}}(L(3,1),\lambda)$ (Proposition 3.4.6). In this manner, we have the theorem.

The estimate $\frac{1}{3}c_1^{\mathrm{ECH}}(L(3,1)\lambda) \leq \inf_{\gamma\in\mathcal{S}_3,\ \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda$ is almost the same with Theorem 3.2.2.

3.2.3 Area-preserving diffeomorphisms on the disk and positive hyperbolic orbits

Area-preserving diffeomorphisms on the open annulus or the open disk have been studied and play important roles in 3-dimentional dynamics. They frequently arise as return maps on Birkhoff sections. For instance, J. Franks [Fr2, Fr3] showed that an area-preserving homeomorphism of an open annulus which has at least one periodic point has infinitely many interior periodic points and as an application, proved that every smooth Riemannian metric on S^2 with positive scalar curvature has infinitely many distinct closed geodesics. In the context of 3-dimensional Reeb flows, Hofer, Wysocki and Zehnder [HWZ2] constructed a Birkhoff section of disk type from a J-holomorphic curve in a dynamically convex contact 3-sphere and proved by applying the Franks' result and Brouwer's translation theorem that there are either infinitely many simple periodic orbits or exactly two simple periodic orbits in a dynamically convex contact 3-sphere. The primary motivation of the paper is to refine the result of Hofer, Wysocki and Zehnder and to study periodic orbits in more detail. In particular, Our first theorem (Theorem 3.2.13) leads to the existence of infinitely many periodic orbits with the specific types called positive hyperbolic if there are at least 3 periodic orbits.

Let Σ be a surface. For a diffeomorphism f of Σ, we call $p \in \Sigma$ a periodic point with period $n(>0)$ if $p = f^n(p)$ and in addition $p \neq f^m(p)$ for any $0 < m < n$. f is called non-degenerate if for any n and any fixed point p of f^n, the map $df^n : T_p\Sigma \to T_p\Sigma$ has no eigenvalue 1.

Consider a volume form ω on Σ. Let f be a diffeomorphism of Σ with $f^*\omega = \omega$. A periodic point p with period n is called positive (resp. negative) hyperbolic if the eigenvalues of $df^n; T_p\Sigma \to T_p\Sigma$ are positive (resp. negative) real numbers and elliptic if the eigenvalues of $df^n; T_p\Sigma \to T_p\Sigma$ are of length 1. We note that since f is area-preserving, any periodic point is either positive/negative hyperbolic or elliptic and if f is non-degenerate, the conditions do not overlap each other.

Let $\mathbb{D}$ be the closed unit disk and $\mathring{\mathbb{D}}$ the interior. According to [Fr2, Fr3, HWZ2], it follows that an area and orientation preserving map on $\mathring{\mathbb{D}}$ with finite area has either exactly one periodic point or infinitely many periodic orbits.

Our first result is as follows.

Theorem 3.2.13. *Let ω be a volume 2-form on $\mathring{\mathbb{D}}$ with $\int_{\mathring{\mathbb{D}}} \omega < +\infty$. Let*

f be a non-degenerate diffeomorphism on $\mathbb{D}$. If f satisfies $f^*\omega = \omega$ and has at least two periodic points on $\mathring{\mathbb{D}}$, then the number of positive hyperbolic periodic points on $\mathring{\mathbb{D}}$ is infinite.

As will be seen, the diffeomorphisms in Theorem 3.2.13 have highly compatibility with the return maps of Birkhoff sections of disk type near J-holomorphic curves in 3-dimensional Reeb flows.

Next, we observe how Theorem 3.2.13 is applied to 3-dimensional Reeb flows.

As stated in §2.2 D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano [CHP] showed that a non-degenerate (Y, λ) with $b_1(Y) > 0$ has as least one positive hyperbolic simple orbit by using Embedded contact homology and Monopole floer homology, and asked the following question.

Question 3.2.14. *[CHP] Suppose that a non-degenerate (Y, λ) is not a lens space with exactly two simple elliptic orbit. Does (Y, λ) has as least one positive hyperbolic simple orbit?*

As we will see later, our results support an affirmative answer to the question.

Before proceeding, we need to recall some notions. Let (Y, λ) be a contact 3-manifold. Consider a simple periodic orbit $\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y$ and $\gamma^*\xi \to Y$. Then the linearized flow $d\phi^t|_\xi$ on the periodic orbit induces a flow on $\gamma^*\xi$ and hence on $(\gamma^*\xi\backslash 0)/\mathbb{R}_+$. we write $(\gamma^*\xi\backslash 0)/\mathbb{R}_+$ as $\mathbb{T}_\gamma$ and refer to the vector field induced by $d\phi^t$ on $\mathbb{T}_\gamma$ as linearized polar dynamics along γ. As a set, the blown-up manifold is defined as $Y_\gamma := (Y\backslash\gamma) \bigsqcup \mathbb{T}_\gamma$. Y_γ has a smooth structure of a manifold such that the Reeb vector field X_λ extend smoothly to the linearized polar dynamics on $\mathbb{T}_\gamma$ (see [FHr, v1 Lemma A.1]). It is easy to see that if (Y, λ) is non-degenerate, any periodic orbit of Y_γ is non-degenerate.

Let $u : \mathbb{D} \to Y$ be a Birkhoff section such that $u(\partial\mathbb{D})$ is tangent to γ. Then we can lift the map to $\tilde{u} : \mathbb{D} \to Y_\gamma = (Y\backslash\gamma) \bigsqcup \mathbb{T}_\gamma$ smoothly as follows. If $x \in \mathring{\mathbb{D}}$, then $\tilde{u}(x) = u(x)$. If $x \in \partial\mathbb{D}$, then $\tilde{u}(x) := \mathrm{pr}\circ du(\mathbb{R}_+ v)$ where v is the outward unit vector at x and pr is the projection $TY = \mathbb{R}X_\lambda \oplus \xi \to (\xi\backslash 0)/\mathbb{R}_+$.

Definition 3.2.15. *[FHr, c.f. Definition 1.6] A Birkhoff section $u : \mathbb{D} \to Y$ is ∂-strong if for the lift $\tilde{u} : \mathbb{D} \to Y_\gamma$, $\tilde{u}(\partial\mathbb{D})$ is transverse to the linearized polar dynamics on $\mathbb{T}_\gamma$ and any trajectory on $\mathbb{T}_\gamma$ intersects $\tilde{u}(\partial\mathbb{D})$ infinitely many times in the future and in the past. Here γ is the simple periodic orbit to which $u(\partial\mathbb{D})$ is tangent.*

The following is an application of Theorem 3.2.13.

Theorem 3.2.16. *If a non-degenerate contact 3-manifold (Y, λ) admits a ∂-strong Birkhoff section of disk type and has at least 3 simple periodic orbits, then there exists infinitely many simple positive hyperbolic orbits.*

Proof of Theorem 3.2.16. Since (Y, λ) is non-degenerate and the Birkhoff section is ∂-strong, the return map on $u(\mathring{\mathbb{D}})$ is non-degenerate, area-preserving and orientation preserving map with respect to $d\lambda$. In addition, it extends smoothly to the boundary of the disk with non-degeneracy and $\int_{u(\mathring{\mathbb{D}})} d\lambda < +\infty$ because of the Stokes' theorem. This implies that we can apply Theorem 3.2.13 to this map. □

If a contact manifold is contactomorphic to a universally tight lens space, we may simplify the assumption as follows.

Theorem 3.2.17. *Let λ be a non-degenerate contact form on $(L(p, q), \xi_{\text{std}})$. If $(L(p, q), \lambda)$ admits a Birkhoff section of disk type and has at least 3 simple periodic orbits, then there are infinitely many simple positive hyperbolic orbits.*

Remark 3.2.18. In [HrS1, HrLS], necessary and sufficient conditions for $(L(p, q), \lambda)$ with $\text{Ker}\lambda = \xi_{\text{std}}$ to admit a Birkhoff section of disk type are given.

The next proposition allows us to apply Theorem 3.2.16 to Theorem 3.2.17 and hence Theorem 3.2.17 follows immediately.

Proposition 3.2.19. *Let λ be a non-degenerate contact form on $(L(p, q), \xi_{\text{std}})$. If $(L(p, q), \lambda)$ admits a Birkhoff section of disk type, then there is a ∂-strong Birkhoff section of disk type with the same binding.*

We assume that lens spaces $L(p, q)$ contain S^3 as a lens space with $p = 1$.

As an immediate corollary of Theorem 3.2.17 and Theorem 3.1.11, we have

Corollary 3.2.20. *Let $(L(p, q), \lambda)$ be a non-degenerate dynamically convex. If there is a p-unknotted simple orbit γ with $sl_{\xi}^{\mathbb{Q}}(\gamma) = -\frac{1}{p}$, then there are either infinitely many simple positive hyperbolic orbits or exactly two simple elliptic orbits.*

Corollary 3.2.21. *Assume (Y, λ) be a dynamically convex non-degenerate contact 3-manifold such that Y is diffeomorphic to $L(p, p-1)$ for some p. Then then there are either infinitely many simple positive hyperbolic orbits or exactly two simple elliptic orbits.*

Remark 3.2.22. It follows from [HrS2] that any dynamically convex $(L(p,q), \lambda)$ with even p has an elliptic orbit. Combining with [Shi1, Shi2], we have that any non-degenerate dynamically convex $(L(p,q), \lambda)$ with at least 3 simple periodic orbits has a simple positive hyperbolic orbit.

We end this section with the following question.

Question 3.2.23. *Let (Y, λ) be a non-degenerate contact 3-manifold. Assume (Y, λ) is not a lens space with exactly two simple elliptic orbits. Does (Y, λ) have infinitely many simple positive hyperbolic orbits?*

3.3 Proof of Theorem 3.2.2

3.3.1 Behaviors of Conley-Zehnder index and J-holomorphic curves

In §3.3.1, we observe behaviors of Conley-Zehnder index and J-holomorphic curves under non-degeneracy.

Lemma 3.3.1. *Assume that $(L(p, p-1), \lambda)$ is non-degenerate dynamically convex contact manifold with* $\mathrm{Ker}\lambda = \xi_{\mathrm{std}}$.

(1). For any orbit γ, $\mu_{\mathrm{glob}}(\gamma) \geq 1$.

(2). Suppose that $p = 2$. If $\mu_{\mathrm{glob}}(\gamma) = 1$, γ is a simple non-contractible elliptic orbit and if $\mu_{\mathrm{glob}}(\gamma) = 2$, γ is a simple non-contractible positive hyperbolic orbit.

(3). Suppose that $p > 2$. If $\mu_{\mathrm{glob}}(\gamma) = 1$, γ is a simple non-contractible negative hyperbolic orbit otherwise an elliptic orbit which may not be simple. If $\mu_{\mathrm{glob}}(\gamma) = 2$, γ is a non-contractible positive hyperbolic orbit.

Proof of Lemma 3.3.1. Suppose $\mu_{\mathrm{glob}}(\gamma) < 3$. Since λ is dynamically convex, γ is non-contractible. Note that γ^p is contractible. Suppose that γ is hyperbolic. Then $\mu_{\mathrm{glob}}(\gamma^p) = p\mu_{\mathrm{glob}}(\gamma) \geq 3$. If $p = 2$, this implies that $\mu_{\mathrm{glob}}(\gamma) = 2$ and thus γ is simple positive hyperbolic. If $p > 2$, $\mu_{\mathrm{glob}}(\gamma^p) =$

$p\mu_{\text{glob}}(\gamma) \geq 3$ implies that $\mu_{\text{glob}}(\gamma) = 1$ or 2. If $\mu_{\text{glob}}(\gamma) = 2$, γ is a positive hyperbolic orbit (may not be simple). If $\mu_{\text{glob}}(\gamma) = 1$, γ is a simple negative hyperbolic orbit.

Next suppose that γ with $\mu_{\text{glob}}(\gamma) < 3$ is elliptic. Then there is $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_{\text{glob}}(\gamma^n) = 2\lfloor n\theta \rfloor + 1$ for every $n \in \mathbb{Z}_{>0}$. Since $\mu_{\text{glob}}(\gamma^p) = 2\lfloor p\theta \rfloor + 1 \geq 3$, we have $\lfloor p\theta \rfloor \geq 1$ and hence $\lfloor \theta \rfloor \geq 0$. In particular, since $\mu_{\text{glob}}(\gamma) < 3$, we have $\lfloor \theta \rfloor = 0$. Moreover in the case of $p = 2$, $\lfloor 2\theta \rfloor = 1$. This implies that if $p = 2$, γ is simple and $\mu_{\text{glob}}(\gamma) = 1$. We complete the proof. □

Lemma 3.3.2. *Assume that $(L(p, p-1), \lambda)$ is non-degenerate dynamically convex contact manifold with $\text{Ker}\lambda = \xi_{\text{std}}$. Let α be an orbit set with $[\alpha] = 0 \in H_1(Y)$. If $I(\alpha, \emptyset) = 1$, then $\mathcal{M}^J(\alpha, \emptyset) = \emptyset$.*

Proof of Lemma 3.3.2. Suppose that $\mathcal{M}^J(\alpha, \emptyset) \neq \emptyset$. Choose $u \in \mathcal{M}^J(\alpha, \emptyset)$ and let $\{\gamma_i\}_{i=1,\ldots k}$ denote the set consisting of all orbits to which the positive ends of u are asymptotic. Note that γ_i may not be simple. Since u is embedded and of Fredholm index 1, we have

$$\text{ind}(u) = 1 = -\chi(u) + \sum_{1 \leq i \leq k} \mu_{\text{glob}}(\gamma_i) = 2g - 2 + \sum_{1 \leq i \leq k} (\mu_{\text{glob}}(\gamma_i) + 1). \quad (3.3)$$

Since $\mu_{\text{glob}}(\gamma_i) + 1 \geq 2$, we have $k = 1$. Therefore we have

$$1 = 2g + \mu_{\text{glob}}(\gamma_1) - 1. \quad (3.4)$$

This implies that $\mu_{\text{glob}}(\gamma_1) = 2$ and $g = 0$. It follows from Lemma 3.3.1 that γ_1 is non-contractible and hence $[\alpha] \neq 0$. This contradicts the assumption. We complete the proof. □

For $u \in \mathcal{M}^J(\alpha, \emptyset)$, let $\mathcal{M}^J_u$ denote the connected component of $\mathcal{M}^J(\alpha, \emptyset)$ containing u. we introduce the following proposition given in [CHP] which plays an important role in what follows.

Proposition 3.3.3 ([CHP])**.** *Let (Y, λ) be a non-degenerate contact three-manifold, and let J be a compatible almost complex structure on $\mathbb{R} \times Y$. Let C be an irreducible J-holomorphic curve in $\mathbb{R} \times Y$ such that:*

(1). Every $C \in \mathcal{M}^J_C$ is embedded in $\mathbb{R} \times Y$.

(2). C is of genus 0, has no end asymptotic to a positive hyperbolic orbit and $\text{ind}(C) = 2$.

(3). C does not have two positive ends, or two negative ends, at covers of the same simple Reeb orbit.

(4). Let γ be a simple orbit with rotation nnumber $\theta \in \mathbb{R}\backslash\mathbb{Q}$. If C has a positive end at an m-fold cover of γ, then $\gcd(m, \lfloor m\theta \rfloor) = 1$. If C has a negative end at an m-fold cover of γ, then $\gcd(m, \lceil m\theta \rceil) = 1$.

(5). $\mathcal{M}^J_C/\mathbb{R}$ is compact.

Then $\pi(C) \subset Y$ is a global surface of section for the Reeb flow. In addition for any section $s : \mathcal{M}^J_C/\mathbb{R} \to \mathcal{M}^J_C$, $\bigcup_{t\in S^1} \pi(s(t))$ gives a rational open book decomposition supporting the contact structure $\mathrm{Ker}\lambda = \xi$.

Lemma 3.3.4. *Assume that $(L(p, p-1), \lambda)$ is non-degenerate dynamically convex contact manifold with $\mathrm{Ker}\lambda = \xi_{\mathrm{std}}$. Let α be an ECH generator with $I(\alpha, \emptyset) = 2$ and $[\alpha] = 0$. If $\langle U_{J,z}\alpha, \emptyset\rangle \neq 0$, then there exists an simple orbit γ with $\mu_{\mathrm{glob}}(\gamma) = 1$ and $\mu_{\mathrm{glob}}(\gamma^p) = \mu_{\mathrm{disk}}(\gamma^p) = 3$ for which $\alpha = (\gamma, p)$, otherwise there exist simple orbits γ_1, γ_2 with $\mu_{\mathrm{glob}}(\gamma_1) = \mu_{\mathrm{glob}}(\gamma_2) = 1$ for which $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$. Moreover any element $u \in \mathcal{M}^J(\alpha, \emptyset)$ is of genus 0 and $\pi(u)$ is a global surface of section for X_λ. In addition, $s : \mathcal{M}^J_C/\mathbb{R} \to \mathcal{M}^J_C$, $\bigcup_{t\in S^1} \pi(s(t))$ gives a (rational) open book decomposition supporting the contact structure $\mathrm{Ker}\lambda = \xi$.*

Proof of Lemma 3.3.4. Write $\alpha = \{(\gamma_i, m_i)\}_{i=1,\ldots k}$. Let $u \in \mathcal{M}^J(\alpha, \emptyset)$ be a J-holomorphic curve counted by $U_{z,J}$. Then we have

$$I(u) - J_0(u) = \sum_{i=1,\ldots k} \mu_{\mathrm{glob}}(\gamma_i^{m_i}). \tag{3.5}$$

Therefore

$$2 = -\chi(u) + \sum_{i=1,\ldots k} (n_i - 1) + \sum_{i=1,\ldots k} \mu_{\mathrm{glob}}(\gamma_i^{m_i}) = 2g - 2 + \sum_{i=1,\ldots k} (\mu_{\mathrm{glob}}(\gamma_i^{m_i}) + 2n_i - 1) \tag{3.6}$$

where n_i is the number of positive ends of u asymptotic to γ_i with some multiplicities and g is the genus of u. Since $\mu_{\mathrm{glob}}(\gamma_i^{m_i}) + 2n_i - 1 \geq 2$ for $i = 1, \ldots k$, we have $k = 1$ or $k = 2$.

Suppose that $k = 1$. Then we have $5 = 2g + \mu_{\mathrm{glob}}(\gamma_1^{m_1}) + 2n_1$. Note that $g \geq 0$, $m_1, n_1 \geq 1$ and γ_1 is simple. If $n_1 \geq 2$, we have $g = 0$ and $1 = \mu_{\mathrm{glob}}(\gamma_1^{m_1})$. Since λ is dynamically convex, $\gamma_1^{m_1}$ is a non-contractible elliptic orbit and hence $[\alpha] \neq 0$. This contradicts the assumption. Therefore $n_1 = 1$ and hence $3 = 2g + \mu_{\mathrm{glob}}(\gamma_1^{m_1})$. In the same way, we have $g = 0$ and so $3 = \mu_{\text{glob}}(\gamma^m)$. In order to complete the proof of Lemma 3.3.4 in the case of $k = 1$, we have to show $m = p$, but since additional discussion is needed, we would leave it for later.

Suppose that $k = 2$. Then $6 = 2g + \mu_{\text{glob}}(\gamma_1^{m_1}) + \mu_{\text{glob}}(\gamma_2^{m_2}) + 2n_1 + 2n_2$. Since $g \geq 0$ and $m_i, n_i \geq 1$, we have $n_1 = n_2 = 1$, $g = 0$ and $\mu_{\text{glob}}(\gamma_1^{m_1}) = \mu_{\text{glob}}(\gamma_2^{m_2}) = 1$. We have to show $m_1 = m_2 = 1$. From Lemma 3.3.1, if $p = 2$, $\gamma_1^{m_1}$ and $\gamma_2^{m_2}$ are simple elliptic orbits and hence we have $m_1 = m_2 = 1$. Therefore we may assume that $p > 2$. Without loss of generality, we consider γ_1 and m_1. Suppose that $m_1 \geq 2$. By the definition of ECH generator, γ_1 is elliptic. Therefore there is $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_{\text{glob}}(\gamma_1^m) = 2\lfloor m\theta \rfloor + 1$ for every $m \in \mathbb{Z}_{>0}$. Since $\mu_{\text{glob}}(\gamma_1^{m_1}) = 2\lfloor m_1\theta \rfloor + 1 = 1$, we have $2\lfloor i\theta \rfloor + 1 = 1$ for every $1 \leq i \leq m_1$. Hence $\lfloor m_1\theta \rfloor = m_1\lfloor \theta \rfloor$. It follows from the partition condition (Proposition 1.3.8) that each multiplicity of positive end of u asymptotic to γ_1 is 1 and hence the number of ends asymptotic to γ_1 is m_1. This implies that $n_1 = m_1 \geq 2$. This contradicts $n_1 = 1$. Therefore we have $m_1 = 1$. In the same way, we have $m_2 = 1$.

Next, we show that for any element $u \in \mathcal{M}^J(\alpha, \emptyset)$, $\pi(u)$ is a global surface of section for X_λ. Since $I(u) = 2$, $u \in \mathcal{M}^J(\alpha, \emptyset)$ is embedded.

According to Proposition3.3.3, in order to complete the proof, it is sufficient to show that $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ is compact. Indeed, if $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ is compact, $u \in \mathcal{M}^J(\alpha, \emptyset)$ satisfies all of the assumptions of Proposition3.3.3

Suppose that $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ is not compact. Let $\overline{\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}}$ denote the compactified space of $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ in the sense of SFT compactness. Choose $\bar{u} \in \overline{\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}}\backslash\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$. $\bar{u}$ consists of some J-holomorphic curves in several floors. Let u_0 be the component of $\bar{u}$ in the lowest floor. Then there is an orbit set β such that $u_0 \in \mathcal{M}^J(\beta, \emptyset)/\mathbb{R}$. By the additivity of ECH index, we have $I(\beta, \emptyset) = 1$. This contradicts Lemma 3.3.2. So $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ is compact.

Summarizing the discussion so far, under the assumption of Lemma 3.3.4, we have

(1). There is a contractible simple orbit γ such that $\alpha = (\gamma, m)$ and $3 = \mu_{\text{glob}}(\gamma^m)$. Moreover any $u \in \mathcal{M}^J(\alpha, \emptyset)$ is of genus 0 and $\pi(u)$ is a global surface of section whose boundary is γ^m.

(2). There is non-contractible simple orbits γ_1, γ_2 such that $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ and $1 = \mu_{\text{glob}}(\gamma_1) = \mu_{\text{glob}}(\gamma_2)$. Moreover any $u \in \mathcal{M}^J(\alpha, \emptyset)$ is of genus 0 and $\pi(u)$ is a global surface of section whose boundary is $\gamma_1 \cup \gamma_2$.

Now, it is sufficient to show that $m = p$ in (1) to complete the proof. Take a section $s : \mathcal{M}^J(\alpha,\emptyset)/\mathbb{R} \to \mathcal{M}^J(\alpha,\emptyset)$. Then $\bigcup_{\tau\in\mathcal{M}^J(\alpha,\emptyset)/\mathbb{R}} \overline{\pi(s(\tau))} \to \mathcal{M}^J(\alpha,\emptyset)/\mathbb{R}$ is an (rational) open book decomposition of $L(p,p-1)$ (see Proposition3.3.3). Note that $\mathcal{M}^J(\alpha,\emptyset)/\mathbb{R} \cong S^1$. It is easy to compute that the fundamental group of $\bigcup_{\tau\in\mathcal{M}^J(\alpha,\emptyset)/\mathbb{R}} \overline{\pi(s(\tau))}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$, which means $m = p$. This completes the proof. □

Lemma 3.3.5. *Suppose that α in Lemma 1.21 can be written as $\alpha = (\gamma, p)$ for a simple orbit γ. Then $\gamma \in \mathcal{S}_p$.*

Proof of Lemma 3.3.5. This follows immediately from Lemma 3.3.4 and Theorem 3.1.11. □

Lemma 3.3.6. *Suppose that α in Lemma 1.21 can be written as $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ for simple orbits γ_1, γ_2. Then $\gamma_1, \gamma_2 \in \mathcal{S}_p$*

Lemma 3.3.6 follows from the properties of open book decomposition supporting $(L(p,p-1),\xi_{\mathrm{std}})$. But here, we introduce a proof of Lemma 3.3.6 by constructing a Seifert surface from J-holomorphic curves.

In order to prove Lemma 3.3.6, we recall the existence of the following local coordinate called Martinet tube which is useful for our arguments in what follows.

Proposition 3.3.7. *[HWZ1] Let (Y^3,λ) be a contact three manifold with* $\mathrm{Ker}\lambda = \xi$. *For a simple orbit γ, there is a diffeomorphism called Martinet tube $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\delta \to \bar{U}$ for a sufficiently small $\delta > 0$ such that $F(t,0) = \gamma(t)$ and there exists a smooth function $f : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to (0,+\infty)$ satisfying $f(\theta,0) = T_\gamma$, $df(\theta,0) = 0$ and $F^*\lambda = f(\theta, x+iy)(d\theta + xdy)$. Here $\mathbb{D}_\delta$ is the disk with radius δ. Note that* $\mathrm{Ker}F^*\lambda|_{\mathbb{R}/\mathbb{Z}\times\{0\}} = \mathrm{span}(\partial_x,\partial_y)$. *Let $\tau_F : \gamma^*\xi \to \mathbb{R}/\mathbb{Z}\times\mathbb{R}^2$ denote the induced trivialization by F which maps $a\partial_x + b\partial_y$ to $(a,b) \in \mathbb{R}^2$ on each fiber.*

Remark that we may take F so that the trivialization τ_F realizes a given homotopy class of trivializations over γ.

It is convenient to see J-holomorphic curves in this coordinate. Recall the following property.

Proposition 3.3.8. *[HWZ1, cf.Theorem 1.3] Let (Y,λ) be a non-degenerate contact manifold (not necessarily $L(p,p-1)$). Let J be an admissible almost complex structure on $\mathbb{R}\times Y$. Consider a J-holomorphic curve $g = (a,u)$:*

$([0,\infty)\times S^1, j_0)\to(\mathbb{R}\times Y, J)$ *such that* $u(s,t)\to\gamma(mT_\gamma t)$ *for a simple periodic orbit* $\gamma:\mathbb{R}/T_\gamma\mathbb{Z}\to Y$ *and* $m>0$. *Here* $j_0(\partial_s)=\partial_t$. *Take a Martinet coordinate* $F:S^1\times\mathbb{D}_\delta\to U_\gamma$ *and write* $F^{-1}(u(s,t))=(\theta(s,t),z(s,t))$ *Then there is constant* c *and* $d>0$ *such that*

$$|\partial^\beta(a(s,t)-T_\gamma s-c)|\le Me^{-ds},\ |\partial^\beta(\theta(s,t)-mt)|\le Me^{-ds}.$$

for all multi-indices β *with constants* $M=M_\beta$.

In addition, if g *does not map onto a trivial cylinder, there is negative eigenvalue* $\nu\in\sigma(L_{S_{\phi_{\gamma,\tau_F}\circ\rho_m}})$, *eigenfunction* e_ν *of* $L_{S_{\phi_{\gamma,\tau_F}\circ\rho_m}}$ *with eigenvalue* ν, *and a smooth map* $w:[0,\infty)$ *with* $w(s)\to\nu$ *as* $s\to\infty$ *such that*

$$z(s,t)=e^{\int_0^s w(v)dv}(e_\nu(t)+r(s,t))$$

Here $L_{S_{\phi_{\gamma,\tau_F}\circ\rho_m}}$ *is the operator defined in §1.2. Moreover,* $r(s,t)$ *satisfies*

$$\partial^\alpha r(s,t)\to 0$$

for all derivatives $\alpha=(\alpha_1,\alpha_2)$, *uniformly in* $t\in S^1$.

As a corollary, the next follows.

Proposition 3.3.9. *Consider a* J*-holomorphic curve* $g=(a,u):([0,\infty)\times S^1,j_0)\to(\mathbb{R}\times Y,J)$ *such that* $u(s,t)\to\gamma(mT_\gamma t)$ *for a simple periodic orbit* $\gamma:\mathbb{R}/T_\gamma\mathbb{Z}\to Y$ *and* $m>0$. *Suppose that* g *does not map onto a trivial cylinder.*

For large s, *let* $\mathrm{wind}(g)$ *denote the winding number of* $t\to z(s,t)\in\mathbb{D}_\delta$. *Then*

$$\mathrm{wind}(g)\le\lfloor\frac{\mu_{\tau_F}(\gamma^m)}{2}\rfloor.$$

Note that this is independent of large s.

In the same way, consider a J*-holomorphic curve* $g=(a,u):((-\infty,0]\times S^1,j_0)\to(\mathbb{R}\times Y,J)$ *such that* $u(s,t)\to\gamma(mT_\gamma t)$ *for a simple periodic orbit* $\gamma:\mathbb{R}/T_\gamma\mathbb{Z}\to Y$ *and* $m>0$ *as* $s\to-\infty$. *suppose that* g *does not map onto a trivial cylinder. Then The similar result holds. That is, take a Martinet coordinate* $F:S^1\times\mathbb{D}_\delta\to U_\gamma$ *and write* $F^{-1}(u(s,t))=(\theta(s,t),z(s,t))$. *For small* s, *let* $\mathrm{wind}(g)$ *denote the winding number of* $t\to z(s,t)\in\mathbb{D}_\delta$. *Then*

$$\mathrm{wind}(g)\ge\lceil\frac{\mu_{\tau_F}(\gamma^m)}{2}\rceil.$$

Let $Z : L(p, p-1) \to \xi_{\text{std}}$ denote a non-vanishing global section satisfying $\tau_{\text{glob}}(Z(x)) = (x, 1) \in L(p, p-1) \times \mathbb{C}$ where τ_{glob} is the fixed global trivialization.

We fix the parameters $\gamma_1 : \mathbb{R}/T_1\mathbb{Z} \to Y$, $\gamma_2 : \mathbb{R}/T_2\mathbb{Z} \to Y$. Here we write the periods of γ_1, γ_2 as T_1, T_2 for simplicity.

The next theorem is used to construct Seifert surface to compute their self-linking number and will be proved in §3.3.4.

Theorem 3.3.10. *Assume that $(L(p, p-1), \lambda)$ is non-degenerate dynamically convex contact manifold with $\text{Ker}\lambda = \xi_{\text{std}}$. Let α be an ECH generator with $I(\alpha, \emptyset) = 2$ and $[\alpha] = 0$. Suppose that $\langle U_{J,z}\alpha, \emptyset \rangle \neq 0$ and α can be written as $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ for simple orbits γ_1, γ_2. Then there is a smooth map $\tilde{u} = (a, u) : S^1 \times \mathbb{R} \times S^1 \to \mathbb{R} \times Y$ satisfying the following properties.*

1. *$u : S^1 \times \mathbb{R} \times S^1 \to Y$ is embedding.*
2. *For any $\sigma \in S^1$, $\tilde{u}(\sigma, \cdot, \cdot) = \tilde{u}_\sigma = (a_\sigma, u_\sigma) \in \mathcal{P}^J$ with $u(s,t) \to \gamma_1(T_1(t + p\sigma))$ as $s \to +\infty$ and $u(s,t) \to \gamma_1(-T_2 t)$ as $s \to -\infty$*
3. *$S^1 \ni \sigma \mapsto [\tilde{u}_\sigma] \in \mathcal{M}^J(\alpha, \emptyset)/\mathbb{R}$ is a diffeomoprhism map where $[\tilde{u}_\sigma]$ is the equivalent class containing $\tilde{u}_\sigma$.*

Proof of Lemma 3.3.6. Without loss of generality, we may consider γ_1. Let $\tilde{u} = (a, u) : S^1 \times \mathbb{R} \times S^1 \to \mathbb{R} \times Y$ be as in Theorem 3.3.10. We define a continuous map $v : \mathbb{C} \to L(p, p-1)$ by $v(re^{2\pi\tau}) = u_\tau(\text{log}r, 0)$ and $v(0) = \gamma_2(0)$. Then by the construction, we have $v(re^{2\pi\tau}) \to \gamma_1(pT_1\tau)$ as $r \to +\infty$ and moreover this map is immersion and injective other than 0 and γ_1. In order to obtain Seifert surface to compute the self-linking number, we change v near $\gamma_2(0)$ and γ_1

Take Martinet tubes of γ_1, $F : S^1 \times \mathbb{D}_\delta \to L(p, p-1)$ so that τ_F and τ_{glob} are in the same homotopy class. Fix $\sigma \in S^1$. According to [CHP, Proof of Lemma 3.3], the winding number $\text{wind}(u_\tau)$ as in Proposition 3.3.8 is 0. Therefore, by moving $\sigma \in S^1$, it follows that for small δ, $F^{-1}(F(S^1 \times \partial\mathbb{D}_\delta) \cap \pi(v(\mathbb{C})))$ is $(p, 1)$-cabling or $(p, -1)$-cabling.

For small $\delta > 0$, we consider a surface constructed by connecting $F(S^1 \times \partial\mathbb{D}_\delta) \cap \pi(v(\mathbb{C}))$ with γ_1 by an isotopy. And we connect it smoothly with the outer part of $v(\mathbb{C})$ together and change small neighborhood of $v(0)$ to be embedded. We write the surface as $D \subset L(p, p-1)$. By the construction, this is a (rational) Seifert surface of disk type binding γ_1 with multiplicity p. Moreover by the construction $sl_\xi^{\mathbb{Q}}(\gamma_1) = -\frac{1}{p}$ or $\frac{1}{p}$ which depends on whether

$F^{-1}(F(S^1 \times \partial\mathbb{D}_\epsilon) \cap \pi(v(\mathbb{C})))$ is $(p,1)$-cabling or $(p,-1)$-cabling. On the other hand, since $\xi_{\rm std}$ is universally tight and γ_1 is transversal knot, we may apply (rational) Bennequin inequality to this (see [BE]). Therefore we have $sl_\xi^{\mathbb{Q}}(\gamma_1) \leq -\frac{1}{p}\chi(F) = -\frac{1}{p}$. This implies $sl_\xi^{\mathbb{Q}}(\gamma_1) = -\frac{1}{p}$. We complete the proof. □

3.3.2 Proof of Theorem 3.2.2 in non-degenerate cases

The purpose of this section is to prove Theorem 3.2.2 under non-degeneracy. It is easy to check that Theorem 3.2.2 under non-degeneracy follows from the next propositions. We prove them respectively.

Proposition 3.3.11. *Assume that $(L(p,p-1),\lambda)$ is non-degenerate and dynamically convex with* Ker$\lambda = \xi_{\rm std}$. *Then there exists $\gamma \in \mathcal{S}_p$ satisfying $\mu_{\rm glob}(\gamma) = 1$ and*

$$\int_\gamma \lambda \leq \frac{1}{2}\, c_1^{\rm ECH}(L(p,p-1),\lambda). \tag{3.7}$$

Proposition 3.3.12. *Assume that $(L(2,1),\lambda)$ is non-degenerate and dynamically convex. Then*

$$\frac{1}{2}\, c_1^{\rm ECH}(L(2,1),\lambda) \leq \inf_{\gamma\in\mathcal{S}_2, \mu_{\rm glob}(\gamma)=1} \int_\gamma \lambda \tag{3.8}$$

Proof of Proposition 3.3.11

Here, we prove For a sum of ECH generators $\alpha_1 + ... + \alpha_k$ with $\partial_J(\alpha_1 + ... + \alpha_k) = 0$, we write $\langle \alpha_1 + ... + \alpha_k\rangle$ as the equivalence class in $ECH(Y,\lambda)$.

Lemma 3.3.13. *Let $\alpha_1, ..., \alpha_k$ be ECH generators with $[\alpha_i] = 0$ and $I(\alpha_i, \emptyset) = 2$ for $i = 1, ...k$. Suppose that $\partial_J(\alpha_1 + ... + \alpha_k) = 0$ and $0 \neq \langle\alpha_1 + ... + \alpha_k\rangle \in ECH_2(Y,\lambda,0)$. Then there exists i such that $\langle U_{J,z}\alpha_i, \emptyset\rangle \neq 0$.*

Proof of Lemma 3.3.13. Since Ker$\lambda = \xi_{\rm std}$, $\emptyset + {\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ is not zero in $ECH_0(Y,\lambda,0)$. It follows from Lemma 3.3.2 that if non zero element in $ECH_0(Y,\lambda,0)$ is represented as $\beta_1 + ... + \beta_j + {\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ for some ECH generators $\beta_1, ..., \beta_j$, there is k such that $\beta_k = \emptyset$ and $\sum_{i\neq k}\beta_i$ is in ${\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$. Indeed, if $\beta_k \neq \emptyset$ for every $1 \leq k \leq j$, $\emptyset + {\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ and $\beta_1 + ... + \beta_j + {\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ are linearly independent in $ECH_0(Y,\lambda,0)$. Otherwise since $ECH_2(Y,\lambda,0) \cong \mathbb{F}$, $\emptyset + \beta_1 + ... + \beta_j + {\rm Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ is zero in $ECH_0(Y,\lambda,0)$ and so $\emptyset + \beta_1 + ... + \beta_j \in$

$\mathrm{Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$. This implies that there is an ECH generator σ with $I(\sigma,\emptyset)=1$ such that $\mathcal{M}^J(\sigma,\emptyset)\neq\emptyset$, but this contradicts Lemma 3.3.2.

Since $U: ECH_2(Y,\lambda,0)\to ECH_0(Y,\lambda,0)$ is isomorphism, it follows that $U_{J,z}(\alpha_1+...+\alpha_k)+\mathrm{Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$ is not zero in $ECH_0(Y,\lambda,0)$. Summarizing these arguments, if $\langle U_{J,z}\alpha_i,\emptyset\rangle=0$ for any i, we have $U_{J,z}(\alpha_1+...+\alpha_k)\in\mathrm{Im}(\partial_J|_{ECC_1(Y,\lambda,0)})$. Therefore there is i such that $\langle U_{J,z}\alpha_i,\emptyset\rangle\neq 0$. We complete the proof. □

Proof. Now we complete the proof of Proposition 3.3.11. From the definition of c_1^{ECH}, there are ECH generators $\alpha_1,...,\alpha_k$ with $[\alpha_i]=0$, $I(\alpha_i,\emptyset)=2$ and $A(\alpha_i)\leq c_1^{\mathrm{ECH}}(L(p.p-1),\lambda)$ for $i=1,...k$ such that $\partial_J(\alpha_1+...+\alpha_k)=0$ and $0\neq\langle\alpha_1+...+\alpha_k\rangle\in ECH_2(Y,\lambda,0)$. By Lemma 3.3.13, we can find i such that $\langle U_{J,z}\alpha_i,\emptyset\rangle\neq 0$. By combining with Lemma 3.3.4 and Lemma 3.3.6, there is either $\gamma\in\mathcal{S}_p$ with $\mu_{\mathrm{glob}}(\gamma)=1$ for which $\alpha_i=(\gamma,p)$ or $\gamma_1,\gamma_2\in\mathcal{S}_p$ with $\mu_{\mathrm{glob}}(\gamma_1)=\mu_{\mathrm{glob}}(\gamma_2)=1$ for which $\alpha_i=(\gamma_1,1)\cup(\gamma_2,1)$. In any cases, this implies that there is $\gamma\in\mathcal{S}_p$ with $\mu(\gamma)=1$ such that $c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)\geq 2\int_\gamma\lambda$. Thus we have $\frac{1}{2}c_1^{\mathrm{ECH}}(L(p,p-1),\lambda)\geq\int_\gamma\lambda$. This completes the proof. □

Proof of Proposition 3.3.12

Here, we focus on the case of $p=2$.

Lemma 3.3.14. *For any $\gamma\in\mathcal{S}_2$ with $\mu_{\mathrm{glob}}(\gamma)=1$ (that is, $\mu_{\mathrm{disk}}(\gamma^2)=\mu_{\mathrm{glob}}(\gamma^2)=3$). Then $(\gamma,2)$ is an ECH generator and $I((\gamma,2),\emptyset)=2$.*

Proof of Lemma 3.3.14. We will give the proof more generally in Lemma 3.4.18. □

Lemma 3.3.15. *Let $(L(2,1),\lambda)$ be a non-degenerate dynamically convex contact manifold. Let $\alpha_\gamma=(\gamma,2)$ for $\gamma\in\mathcal{S}_p$. If $\mu_{\mathrm{disk}}(\gamma^2)=3$, then there is no somewhere injective J-holomorphic curve satisfying the following;*

(1). There is only one positive end. In addition, the positive end is asymptotic to γ with multiplicity p.

(2). There is at least one negative end.

(3). Any puncture on the domain is either positive or negative end.

Proof of Lemma 3.3.15. We will give the proof more generally in Lemma 3.4.19. □

Lemma 3.3.16. *For any* $\gamma \in \mathcal{S}_2$ *with* $\mu_{\text{disk}}(\gamma^2) = 3$, $\partial_J \alpha_\gamma = 0$.

Proof of Lemma 3.3.16. We will give the proof more generally in Lemma 3.4.21. □

Define a set $\mathcal{G}$ consisting of ECH generators as

$$\mathcal{G} := \{\alpha | \ \langle U_{J,z}\alpha, \emptyset\rangle \neq 0 \ \}. \tag{3.9}$$

Lemma 3.3.17. *For* $\gamma \in \mathcal{S}_2$ *with* $\mu(\gamma) = 1$, *let* $\alpha_\gamma = (\gamma, 2)$. *Then* $\langle U_{J,z}\alpha_\gamma, \emptyset\rangle \neq 0$ *for generic* $z \in L(2,1)$. *That is* $\alpha_\gamma \in \mathcal{G}$.

Proof of Lemma 3.3.17. Recall that each page of the rational open book decomposition constructed in [HrS2, Theorem 1.7, Corollary 1.8] is the projection of J-holomorphic curve from $(\mathbb{C}, i)$ to $L(2,1)$. Moreover in this case, $\mathcal{M}^J(\alpha_\gamma, \emptyset)/\mathbb{R}$ is compact and any two distinct elements $u_1, u_2 \in \mathcal{M}^J(\alpha_\gamma, \emptyset)$ has no intersection point. Hence : $\mathcal{M}^J(\alpha_\gamma, \emptyset)/\mathbb{R} \cong S^1$ and for a section $s : \mathcal{M}^J(\alpha_\gamma, \emptyset)/\mathbb{R} \to \mathcal{M}^J(\alpha_\gamma, \emptyset)$, $\bigcup_{\tau \in \mathcal{M}^J(\alpha_\gamma, \emptyset)/\mathbb{R}} \overline{\pi(s(\tau))} \to \mathcal{M}^J(\alpha_\gamma, \emptyset)/\mathbb{R}$ is an (rational) open book decomposition of $L(2,1)$. This implies that for $z \in L(2,1)$ not on γ, there is exactly one J-holomorphic curve in $\mathcal{M}^J(\alpha_\gamma, \emptyset)$ through $(0, z) \in \mathbb{R} \times L(2,1)$. Therefore we have $\langle U_{J,z}\alpha_\gamma, \emptyset\rangle \neq 0$. □

Lemma 3.3.18. *Suppose that* β *is an ECH generator with* $I(\beta, \alpha) = 1$. *Then*

$$\sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha\rangle = 0 \tag{3.10}$$

Proof of Lemma 3.3.18. Write

$$\partial_J \beta = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha\rangle \alpha + \sum_{I(\beta,\sigma)=1, \sigma \notin \mathcal{G}} \langle \partial_J \beta, \sigma\rangle \sigma. \tag{3.11}$$

Then we have

$$\langle U_{J,z}\partial_J \beta, \emptyset\rangle = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha\rangle \langle U_{J,z}\alpha, \emptyset\rangle + \sum_{I(\beta,\sigma)=1, \sigma \notin \mathcal{G}} \langle \partial_J \beta, \sigma\rangle \langle \sigma, \emptyset\rangle = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha\rangle \tag{3.12}$$

Here we use that for $\alpha \in \mathcal{G}$, $\langle U_{J,z}\alpha, \emptyset\rangle = 1$ and for σ with $\sigma \notin \mathcal{G}$, $\langle U_{J,z}\sigma, \emptyset\rangle = 0$.

Since $U_{J,z}\partial_J = \partial_J U_{J,z}$, we have $\langle U_{J,z}\partial_J\beta, \emptyset\rangle = \langle \partial_J U_{J,z}\beta, \emptyset\rangle = 0$ (Here we use Lemma 3.3.2). This completes the proof. □

Lemma 3.3.19. *For any $\gamma \in \mathcal{S}_2$ with $\mu_{\mathrm{glob}}(\gamma) = 1$, $0 \neq \langle \alpha_\gamma \rangle = \langle (\gamma, 2) \rangle \in ECH_2(Y, \lambda, 0)$.*

Proof of Lemma 3.3.19. Suppose that $0 = \langle \alpha_\gamma \rangle \in ECH_2(Y, \lambda, 0)$. Then there are ECH generators $\beta_1, ...\beta_j$ with $I(\beta_i, \alpha_\gamma) = 1$ for any i such that $\partial_J(\beta_1 + ... + \beta_j) = \alpha_\gamma$. From Lemma 3.3.18, we have

$$\sum_{1\leq i\leq j}\sum_{\alpha\in\mathcal{G}}\langle \partial_J\beta_i, \alpha\rangle = \sum_{\alpha\in\mathcal{G}}\langle \alpha_\gamma, \alpha\rangle = 0. \tag{3.13}$$

But since $\alpha_\gamma \in \mathcal{G}$, $\sum_{\alpha\in\mathcal{G}}\langle \alpha_\gamma, \alpha\rangle = 1$. This is a contradiction. We complete the proof. □

Proof. Now, we complete the proof of Proposition 3.3.12. From Lemma 3.3.19 and the definition of $c_1^{\mathrm{ECH}}(L(2,1), \lambda)$, we have $2\int_\gamma \lambda \geq c_1^{\mathrm{ECH}}(L(2,1), \lambda)$ for any $\gamma \in \mathcal{S}_2$ with $\mu_{\mathrm{glob}}(\gamma) = 1$. This implies $\inf_{\gamma\in\mathcal{S}_2, \mu_{\mathrm{glob}}(\gamma)=1}\int_\gamma \lambda \geq \frac{1}{2}c_1^{\mathrm{ECH}}(L(2,1), \lambda)$. This completes the proof. □

3.3.3 Extend the results to degenerate cases

Case of $p = 2$

Here, we prove Theorem 3.2.2 (1) under degenerate strictly convex as a limiting case of non-degenerate result. At first, we show;

Proposition 3.3.20. *Assume that $(L(2,1), \lambda)$ is strictly convex. Then there exists a simple orbit $\gamma \in \mathcal{S}_2$ such that $\mu_{\mathrm{glob}}(\gamma) = 1$ and $\int_\gamma \lambda = \frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1), \lambda)$. In particular,*

$$\inf_{\gamma\in\mathcal{S}_2, \mu_{\mathrm{glob}}(\gamma)=1}\int_\gamma \lambda \leq \frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1), \lambda). \tag{3.14}$$

Proof of Proposition 3.3.20. Let $L = c_1^{\mathrm{ECH}}(L(2,1), \lambda)$. Take a sequence of strictly convex contact forms λ_n such that $\lambda_n \to \lambda$ in C^∞-topology and λ_n is non-degenerate for each n. Therefore we have

$$\inf_{\gamma\in\mathcal{S}_2, \mu(\gamma)=1}\int_\gamma \lambda_n = \frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1), \lambda_n) \tag{3.15}$$

Note that $c_1^{\mathrm{ECH}}(L(2,1),\lambda_n) \to L$ as $n \to +\infty$. This means that there is a sequence of $\gamma_n \in \mathcal{S}_2(L(2,1), f_n\lambda)$ with $\mu_{\mathrm{glob}}(\gamma_n) = 1$ such that $\int_{\gamma_n} \lambda_n \to \frac{1}{2}L$. By Arzelà–Ascoli theorem, we can find a subsequence which converges to a periodic orbit γ of λ in C^∞-topology.

Claim 3.3.21. *γ is simple. In particular, $\gamma \in \mathcal{S}_2(L(2,1),\lambda)$ and $\mu_{\mathrm{glob}}(\gamma) = 1$.*

Proof of Claim 3.3.21. By the argument so far, there is a sequence of $\gamma_n \in \mathcal{S}_2(L(2,1),\lambda_n)$ with $\mu_{\mathrm{glob}}(\gamma_n) = 1$ which converges to γ in C^∞. Note that $\mu_{\mathrm{glob}}(\gamma^2) = 3$. Suppose that γ is not simple, that is, there is a simple orbit γ' and $k \in \mathbb{Z}_{>0}$ with $\gamma'^k = \gamma$. Form the lower semi-continuity of μ, we have $\mu_{\mathrm{glob}}(\gamma_n^2) \to \mu_{\mathrm{glob}}(\gamma'^{2k}) = \mu_{\mathrm{glob}}((\gamma'^2)^k) = 3$. Note that here we use the fact that γ'^{2k} is contractible and $\mu_{\mathrm{glob}}((\gamma'^2)^k) \geq 3$. But since γ'^2 is also contractible, we have $\mu_{\mathrm{glob}}(\gamma'^2) \geq 3$. Hence it follows from Proposition 1.2.4 that $\mu_{\mathrm{glob}}((\gamma'^2)^k) \geq 2k+1 \geq 5$. This is a contradiction. Therefore γ is simple. This implies that for sufficiently large n, γ_n is transversally isotopic to γ. This implies that γ is 2-unknotted and has self-linking number $-\frac{1}{2}$.

At last, we prove $\mu_{\mathrm{glob}}(\gamma) = 1$. Form the lower semi-continuity of μ, we have $\mu_{\mathrm{glob}}(\gamma_n) \to \mu_{\mathrm{glob}}(\gamma) = 1$ or 0. Suppose $\mu_{\mathrm{glob}}(\gamma) = 0$. Then from Proposition 1.2.4, we have $\mu_{\mathrm{glob}}(\gamma^2) = 0$. This contradicts the assumption of dynamical convexity. Hence we have $\mu_{\mathrm{glob}}(\gamma) = 1$. We complete the proof. □

As discussion so far, there is a sequence of $\gamma_n \in \mathcal{S}_2(L(2,1), f_n\lambda)$ with $\mu_{\mathrm{glob}}(\gamma_n) = 1$ and $\gamma \in \mathcal{S}_2(L(2,1),\lambda)$ with $\mu_{\mathrm{glob}}(\gamma) = 1$ such that $\int_{\gamma_n} f_n\lambda \to \frac{1}{2}L$ and γ_n converges to γ of λ in C^∞-topology. Therefore we have $\int_\gamma \lambda = \frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1),\lambda)$ in C^∞-topology. we complete the proof of Proposition 3.3.20. □

Now, we have Proposition 3.3.20. Therefore in order to complete the proof of Theorem 3.2.2 (1), it is sufficient to show the next proposition.

Proposition 3.3.22. *Assume that $(L(2,1),\lambda)$ is strictly convex. Then*

$$\frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1),\lambda) \leq \inf_{\gamma \in \mathcal{S}_2, \mu_{\mathrm{glob}}(\gamma)=1} \int_\gamma \lambda. \tag{3.16}$$

Proof of Proposition 3.3.22. We prove this by contradiction. Suppose that there exists $\gamma_\lambda \in \mathcal{S}_2(L(2,1),\lambda)$ with $\mu_{\mathrm{glob}}(\gamma_\lambda) = 1$ such that $\frac{1}{2}\, c_1^{\mathrm{ECH}}(L(2,1),\lambda) > \int_{\gamma_\lambda} \lambda$.

Lemma 3.3.23. *There exists a sequence of smooth functions $f_n : L(2,1) \to \mathbb{R}_{>0}$ such that $f_n \to 1$ in C^∞-topology and satisfying $f_n|_{\gamma_\lambda} = 1$ and $df_n|_{\gamma_\lambda} = 0$. Moreover, all periodic orbits of $X_{f_n\lambda}$ of periods $< n$ are non-degenerate and all contractible orbits of periods $< n$ have Conley-Zehnder index ≥ 3. In addition, γ_λ is a non-degenerate periodic orbit of $X_{f_n\lambda}$ with $\mu_{\text{glob}}(\gamma_\lambda) = 1$ for every n.*

Proof of Lemma 3.3.23. See [HWZ4, Lemma 6.8, 6.9] □

For a sequence of smooth functions $f_n : L(2,1) \to \mathbb{R}_{>0}$ in Lemma 3.3.23, fix $N >> 0$ sufficient large so that $c_1^{\text{ECH}}(L(2,1), f_N\lambda) > \int_{\gamma_\lambda} \lambda$ and $N > 2c_1^{\text{ECH}}(L(2,1), f_N\lambda)$. We can take such f_N because c_1^{ECH} is continuous in C^0-topology.

Lemma 3.3.24. *Let $f : L(2,1) \to \mathbb{R}_{>0}$ be a smooth function such that $f(x) < f_N(x)$ for any $x \in L(2,1)$. Suppose that $f\lambda$ is non-degenerate dynamically convex. Then there exists a simple periodic orbit $\gamma \in \mathcal{S}_2(L(2,1), f\lambda)$ with $\mu_{\text{glob}}(\gamma) = 1$ such that $\int_\gamma f\lambda < \int_{\gamma_\lambda} \lambda$.*

Outline of the proof of Lemma 3.3.24. See [HrS2, Proposition 3.1]. In the proof and statement of [HrS2, Proposition 3.1], ellipsoids are used instead of $(L(2,1), f_N\lambda)$, but the important point in the proof is to find 2-unknotted self-linking number $-\frac{1}{2}$ orbit with Conley-Zehnder index 1 and construct a suitable J-holomorphic curve from [HrLS, Proposition 6.8]. Now, we have $\gamma_\lambda \in \mathcal{S}_2(L(2,1), f_N\lambda)$ with $\mu_{\text{glob}}(\gamma_\lambda) = 1$ and hence by applying [HrLS, Proposition 6.8], we can construct a suitable J-holomorphic curve. By using this curves instead of ones in the original proof, we can show Proposition 3.3.24. Here we note that Lemma 3.3.15 is needed to prove the same result of [HrS2, Theorem 3.15] □

Now, we would complete the proof. Let $f : L(2,1) \to \mathbb{R}_{>0}$ be a smooth function such that $f(x) < f_N(x)$ for any $x \in L(2,1)$, $f\lambda$ be non-degenerate strictly convex and $\int_{\gamma_\lambda} \lambda < \frac{1}{2}c_1^{\text{ECH}}(L(2,1), f\lambda) < \frac{1}{2}c_1^{\text{ECH}}(L(2,1), f_N\lambda)$. We can check easily that it is possible to take such f. Due to Lemma 3.3.24, there exists a simple periodic orbit $\gamma \in \mathcal{S}_2(L(2,1), f\lambda)$ with $\mu_{\text{glob}}(\gamma) = 1$ such that $\int_\gamma f\lambda < \int_{\gamma_\lambda} \lambda$. Since $\inf_{\gamma \in \mathcal{S}_2, \mu_{\text{glob}}(\gamma)=1} \int_\gamma f\lambda = \frac{1}{2}\, c_1^{\text{ECH}}(L(2,1), f\lambda)$, we have $\int_{\gamma_\lambda} \lambda < \frac{1}{2}c_1^{\text{ECH}}(L(2,1), f\lambda) \leq \int_\gamma f\lambda$. This is a contradiction. We complete the proof. □

Case of $p = 3, 4, 6$

At first, we consider general p and after that, we focus on $p = 3, 4, 6$.

Suppose that $(L(p, p-1), \lambda)$ is strictly convex. Then we can take a sequence of strictly convex non-degenerate contact form with $\lambda_n \to \lambda$. Hence from Proposition 3.3.11, we can find $\gamma_n \in \mathcal{S}_p(L(p, p-1), \lambda_n)$ with $\mu_{\text{glob}}(\gamma_n) = 1$ such that $\int_{\gamma_n} \lambda_n \leq \frac{1}{2} c_1^{\text{ECH}}(L(p, p-1), \lambda_n)$. If γ_n converges to a simple orbit γ, It follows from the same mathod of Proposition 3.3.20 and Claim 3.3.21 that $\gamma \in \mathcal{S}_p(L(p, p-1), \lambda)$ and $\mu_{\text{glob}}(\gamma) = 1$. But since $\mu_{\text{glob}}(\gamma^p)$ may be larger than 3 for $p \geq 3$, the limiting orbit γ of γ_n may not be simple and hence we can't say $\gamma \in \mathcal{S}_p(L(p, p-1), \lambda)$ directly. In this subsection, we observe the limiting behavior of γ_n and find a subsequence converging to a simple orbit γ in the case of $p = 3, 4, 6$.

At first, we recall how to find $\gamma_n \in \mathcal{S}_p$ in $(L(p, p-1), \lambda_n)$. Due to Section 3,4, from the algenbraic structure of ECH and the behaviors of J-holomorphic curves, we can find an ECH generator α_n satisfying $\langle U_{J_n,z}\alpha_n, \emptyset \rangle \neq 0$ and $A(\alpha_n) \leq c_1^{\text{ECH}}(L(p, p-1), \lambda_n)$. In addition, we can see that if $\langle U_{J_{n,z}}\alpha_n, \emptyset \rangle \neq 0$, α_n is described as either

(1). $\alpha_n = (\gamma_n, p)$ with $\gamma_n \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_n^p) = 3$ and $\mu_{\text{glob}}(\gamma_n) = 1$, or

(2). $\alpha_n = (\gamma_{n,1}, 1) \cup (\gamma_{n,2}, 1)$ with $\gamma_{n,1}, \gamma_{n,2} \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_{n,1}) = \mu_{\text{glob}}(\gamma_{n,2}) = 1$.

In any case, we obtain exactly what we want.

Now, for $\lambda_n \to \lambda$, we pick and fix a sequence of ECH generators α_n satisfying $\langle U_{J_n,z}\alpha_n, \emptyset \rangle \neq 0$ and $A(\alpha_n) \leq c_1^{\text{ECH}}(L(p, p-1), \lambda_n)$.

Lemma 3.3.25. *Suppose that* $\{\alpha_n\}$ *consists of* $\alpha_n = (\gamma_n, p)$ *with* $\gamma_n \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_n^p) = 3$ *and* $\mu_{\text{glob}}(\gamma_n) = 1$. *Then there exist a simple orbit* γ *of* $(L(p, p-1), \lambda)$ *and a subsequence* $\{\gamma_{n_k}\}$ *such that* $\gamma_{n_k} \to \gamma$. *In particular,* $\gamma \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma^p) = 3$ *and* $\mu_{\text{glob}}(\gamma) = 1$.

Proof of Lemma 3.3.25. The proof is the same as the one of Claim 3.3.21. □

Next, suppose that $\{\alpha_n\}$ consists of $\alpha_n = (\gamma_{n,1}, 1) \cup (\gamma_{n,2}, 1)$ with $\gamma_{n,1}, \gamma_{n,2} \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_{n,1}) = \mu_{\text{glob}}(\gamma_{n,2}) = 1$. By Arzel'a–Ascoli theorem, we may assume that there are simple orbits $\gamma_{\infty,1}$ and $\gamma_{\infty,2}$ such that $\gamma_{n,1} \to \gamma_{\infty,1}^{k_1}$ and $\gamma_{n,2} \to \gamma_{\infty,2}^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}_{>0}$.

Lemma 3.3.26. *If $\gamma_{\infty,1} \neq \gamma_{\infty,2}$, then $k_1 = k_2 = 1$. In particular, $\gamma_{\infty,1}, \gamma_{\infty,2} \in \mathcal{S}_p$ and $\mu_{\mathrm{glob}}(\gamma_{\infty,1}) = \mu_{\mathrm{glob}}(\gamma_{\infty,2}) = 1$.*

Proof of Lemma 3.3.26. Without loss of generality, we focus on $i = 2$. Note that $H_1(L(p,p-1)\backslash\gamma_{n,1}) \cong \mathbb{Z}$ and $[\gamma_{n,2}]$ generates $H_1(L(p,p-1)\backslash\gamma_{n,1})$. Take small neighborhoods $\gamma_{\infty,i} \subset V_i$ for $i = 1, 2$ satisfying $V_1 \cap V_2 = \emptyset$. Then, $\gamma_{n,i} \subset V_i$ for sufficiently large n. This means that when we fix large n_0, $\gamma_{n,2}$ is isotopic to $\gamma^k_{\infty,2}$ in $L(p,p-1)\backslash\gamma_{n_0,1}$ for any $n \geq n_0$. Hence $[\gamma_{n,2}]$ generates $H_1(L(p,p-1)\backslash\gamma_{n_0,1})$ and $[\gamma_{n,2}] = k_2[\gamma_{\infty,2}]$. Therefore we have $k_2 = 1$. In particular, by the same method with Claim 3.3.21, we have $\gamma_{\infty,2} \in \mathcal{S}_p$ and $\mu(\gamma_{\infty,2}) = 1$. We complete the proof. □

Lemma 3.3.27. *If $\gamma_{\infty,1} = \gamma_{\infty,2}$, then $k_1 + k_2 = p$. In addition, both k_1 and k_2 are mutually prime with p.*

Proof of Lemma 3.3.27. For simplicity, write $\gamma_\infty := \gamma_{\infty,1} = \gamma_{\infty,2}$. Since $\gamma_{n,1} \to \gamma^{k_1}_\infty$ and $\gamma_{n,2} \to \gamma^{k_2}_\infty$, we have $[\gamma_{n,1}] = k_1[\gamma_\infty]$ and $[\gamma_{n,2}] = k_2[\gamma_\infty]$ in $H_1(L(p,p-1)) \cong \mathbb{Z}/p\mathbb{Z}$. Since $[\gamma_{n,1}] + [\gamma_{n,2}] = 0$ and $[\gamma_{n,1}]$ generates $H_1(L(p,p-1))$, $[\gamma_\infty]$ also generates $H_1(L(p,p-1))$ and k_1, k_2 are mutually prime with p. Moreover, $k_1 + k_2 = kp$ for some $k \in \mathbb{Z}_{>0}$.

Suppose that $k \geq 2$. Then $k_1 \geq p+1$ or $k_2 \geq p+1$. Without loss of generality, we may assume $k_1 \geq p+1$.

Since γ^p_∞ is contractible, we have $\mu_{\mathrm{glob}}(\gamma^p_\infty) \geq 3$ and hence $\mu_{\mathrm{glob}}((\gamma^p_\infty)^{k_1}) \geq 2k_1 + 1 \geq 2p + 3$ (Proposition 1.2.4).

On the other hand, for any n, $2p - 1 \geq \mu_{\mathrm{glob}}(\gamma^p_{n,1})$. Indeed, if $\gamma_{n,1}$ is hyperbolic, $\mu_{\mathrm{glob}}(\gamma^p_{n,1}) = p\mu_{\mathrm{glob}}(\gamma_{n,1}) = p$. If $\gamma_{n,1}$ is elliptic, since $\gamma_{n,1}$ is non-degenerate, there is $\theta_n \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_{\mathrm{glob}}(\gamma^m_{n,1}) = 2\lfloor m\theta_n \rfloor + 1$ for every $m \in \mathbb{Z}_{>0}$. Note that $0 < \theta_n < 1$ because $\mu_{\mathrm{glob}}(\gamma_{n,1}) = 1$. Hence we have $\mu_{\mathrm{glob}}(\gamma^p_{n,1}) = 2\lfloor p\theta_n \rfloor + 1 \leq 2p - 1$.

Now, we have $2p - 1 \geq \mu_{\mathrm{glob}}(\gamma^p_{n,1})$. Since $\gamma^p_{n,1} \to \gamma^{pk_1}_\infty$ and μ is lower semi-continuous, we have $2p - 1 \geq \mu_{\mathrm{glob}}(\gamma^{pk_1}_\infty) = \mu_{\mathrm{glob}}((\gamma^p_\infty)^{k_1})$, but this contradicts $\mu_{\mathrm{glob}}((\gamma^p_\infty)^{k_1}) \geq 2k_1 + 1 \geq 2p + 3$. Therefore, we have $k = 1$ and complete the proof. □

Complete the proof in the case of $p = 3, 4, 6$. For $\lambda_n \to \lambda$, consider a sequence of ECH generators α_n satisfying $\langle U_{J_n,z}\alpha_n, \emptyset \rangle \neq 0$ and $A(\alpha_n) \leq c^{\mathrm{ECH}}_1(L(p,p-1), \lambda_n)$.

If $\{\alpha_n\}$ contains an infinity subsequence consisting of $\alpha_n = (\gamma_{n,1}, p)$ with $\gamma_n \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_n^p) = 3$, we can apply Lemma 3.3.25 and hence we obtain $\gamma \in \mathcal{S}_p$ satisfying $\mu_{\text{glob}}(\gamma) = 1$ and $p\int_\gamma \lambda \leq c_1^{\text{ECH}}(L(p,p-1),\lambda)$.

If $\{\alpha_n\}$ contains an infinity subsequence consisting of $\alpha_n = (\gamma_{n,1}, 1) \cup (\gamma_{n,2}, 1)$ with $\gamma_{n,1}, \gamma_{n,2} \in \mathcal{S}_p$, $\mu_{\text{glob}}(\gamma_{n,1}) = \mu_{\text{glob}}(\gamma_{n,2}) = 1$. In addition, suppose that $\gamma_{n,1}$ and $\gamma_{n,2}$ converge to different orbits, then we may apply Lemma 3.3.26 and hence we obtain two simple orbits $\gamma_{\infty,1}\gamma_{\infty,2} \in \mathcal{S}_p$ with $\mu_{\text{glob}}(\gamma_{\infty,1}) = \mu_{\text{glob}}(\gamma_{\infty,2}) = 1$ satisfying $\gamma_{n,1} \to \gamma_{\infty,1}$, $\gamma_{n,2} \to \gamma_{\infty,2}$. Hence we have $\int_{\gamma_{\infty,1}} \lambda + \int_{\gamma_{\infty,2}} \lambda \leq c_1^{\text{ECH}}(L(p,p-1),\lambda)$.

If $\{\alpha_n\}$ contains an infinity subsequence consisting of $\alpha_n = (\gamma_{n,1}, 1) \cup (\gamma_{n,2}, 1)$ with $\gamma_{n,1}, \gamma_{n,2} \in \mathcal{S}_p$ and $\mu_{\text{glob}}(\gamma_{n,1}) = \mu_{\text{glob}}(\gamma_{n,2}) = 1$. In addition, suppose that $\gamma_{n,1}$ and $\gamma_{n,2}$ converge to the same orbit γ_∞ with some multiplicities k_1, k_2. Then we may apply Lemma 3.3.27. If $p = 3, 4, 6$, any pairs (k_1, k_2) satisfying Lemma 3.3.27 contain 1. Therefore $(k_1, k_2) = (1, p-1)$ or $(p-1, 1)$. This implies that $\gamma_{n,1} \to \gamma_\infty$, $\gamma_{n,2} \to \gamma_\infty^{p-1}$ or $\gamma_{n,1} \to \gamma_\infty^{p-1}$, $\gamma_{n,2} \to \gamma_\infty$ and hence we have $\gamma_\infty \in \mathcal{S}_p$ and $\mu_{\text{glob}}(\gamma_\infty) = 1$ by the same way as Claim 3.3.21. Moreover, we have $p\int_{\gamma_\infty} \lambda \leq c_1^{\text{ECH}}(L(p,p-1),\lambda)$.

In any case, we complete the proof. □

3.3.4 Construction of a family of J-holomorphic curves

The purpose of this section is to prove Theorem 3.3.10.

Recall the conditions.

Let $\gamma_1 : \mathbb{R}/T_1\mathbb{Z} \to Y$ and $\gamma_2 : \mathbb{R}/T_2\mathbb{Z} \to Y$ be simple periodic orbits with $\mu_{\text{glob}}(\gamma_1) = \mu_{\text{glob}}(\gamma_2) = 1$. Here we fix their parametrizations. Let $\mathcal{P}^J$ denote the set of J-holomorphic curves $(a, u) : (\mathbb{R} \times S^1, j_0) \to (\mathbb{R} \times Y, J)$ with $u(s,t) \to \gamma_1(T_1 t + e_1)$ as $s \to +\infty$ and $u(s,t) \to \gamma_1(-T_2 t + e_2)$ as $s \to -\infty$ for some $e_i \in \mathbb{R}$ where $j_0(\partial_s) = \partial_t$. Note that $a(s,t) \to +\infty$ as $s \to \pm\infty$.

In order to prove the theorem, we recall the description of P^J as a finite dimensional submanifold in a suitable infinite dimensional Banach manifold according to [Dr].

For γ_i $i = 1, 2$, there is an open neighborhood $\gamma_i(\mathbb{R}/T_i\mathbb{Z}) \subset U_i \subset Y$ and a Martinet tube $F_i : S^1 \times \mathbb{D}_{\delta_0} \to U_i$ such that the trivialization τ_F is in the same fomotopy class with τ_{glob}.

Definition 3.3.28. *[Dr, cf. Definiton 4] Let $\delta > 0$ and $q > 2$. $(a, u) \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times Y)$ is $(\delta, 1, q)$–convergence to γ_1, $-\gamma_2$ if it satisfies the* *following properties.*

1. *there is a sufficient large number $R >>$ such that $u([R, +\infty) \times S^1) \subset U_1$ and $u((-\infty, -R] \times S^1) \subset U_2$.*

2. *Let $F_1^{-1}(u(s,t)) = (\theta_1(s,t), z_1(s,t))$ and $F_2^{-1}(u(s,t)) = (\theta_2(s,t), z_2(s,t))$. Then there are $d_i \in \mathbb{R}$ and $c_i \in \mathbb{R}$ such that*

$$e^{\delta s}(a_i(\epsilon_i s, t) - \epsilon_i T_i s - d_i),\ e^{\delta s}(\theta_i(\epsilon_i s, t) - \epsilon_i t - c_i) \in W^{1,q}([R, +\infty) \times S^1, \mathbb{R}) \tag{3.17}$$

and

$$e^{\delta s} z_i(\epsilon_i s, t) \in W^{1,q}([R, +\infty) \times S^1, \mathbb{R}^2) \tag{3.18}$$

where $\epsilon_1 = +1$, $\epsilon_2 = -1$.

Define $\mathcal{B}_\delta^\infty \subset C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times Y)$ as the set consisting of $(\delta, 1, q)$–convergence elements to γ_1, $-\gamma_2$ of $C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times Y)$.

Note that for sufficiently small δ, any element in P^J is $(\delta, 1, q)$–convergence and hence $P^J \subset \mathcal{B}_\delta^\infty$ [HWZ1, cf. Theorem 1.3].

Next we complete $\mathcal{B}_\delta^\infty$ to a suitable Banach manifold. Fix a Riemannian metric g_J on $\mathbb{R} \times Y$ where

$$g_J(a\partial_s + h, b\partial_s + k) = ab + \lambda(h)\lambda(k) + d\lambda(h, Jk). \tag{3.19}$$

Let $(a, u) \in \mathcal{B}^J$ and $\tilde{\gamma} : \mathbb{R} \times S^1 \to T(\mathbb{R} \times Y)$ such that $\tilde{\gamma}(s,t) \in T_{(a(s,t), u(s,t))}(\mathbb{R} \times Y)$.

Definition 3.3.29. *[Dr, cf. Definition 5.] For $\tilde{\gamma} \in W^{1,q}_{\text{loc}}((a,u)^* T(\mathbb{R} \times Y))$, write*

$$\tilde{\gamma}(s,t) = (b(s,t)\partial_t, h(s,t)X_\lambda(u(s,t)) + Q(u(s,t)))$$

where $Q(u(s,t)) \in \xi_{u(s,t)}$. If b, h, Q satisfies,

$$e^{\delta s}(b, h) \in W^{1,q}([R, \infty)] \times S^1, \mathbb{R}^2),\ \ e^{-\delta s}(b, h) \in W^{1,q}((-\infty, -R] \times S^1, \mathbb{R}^2)$$

and

$$e^{\delta s} Q \in W^{1,q}((u|_{[R,\infty)] \times S^1})^* \xi),\ e^{-\delta s} Q \in W^{1,q}((u|_{(-\infty,-R] \times S^1})^* \xi)$$

for a sufficiently large $R >> 0$. Then we say $\tilde{\gamma} \in W^{1,q}_\delta((a,u)^* T(\mathbb{R} \times Y))$.

Let $\tilde{h} = (a, u) \in \mathcal{B}^\infty$ and suppose that $\tilde{h}([R, +\infty) \times S^1) \subset U_1$, $\tilde{h}((-\infty, -R] \times S^1) \subset U_2$ for a sufficient large $R >> 0$. Let $F_1^{-1}(u(s,t)) = (\theta_1(s,t), z_1(s,t))$ and $F_2^{-1}(u(s,t)) = (\theta_2(s,t), z_2(s,t))$. Consider a smooth function $\kappa : \mathbb{R} \to [0,1]$ such that $\kappa(s) = 0$ for $|s| < R + \frac{1}{2}$ and $\kappa(s) = 1$ for $|s| > R + 1$. For $d = (d_1, d_2), c = (c_1, c_2) \in \mathbb{R}^2$, define $\tilde{h}_{(c,d)}$ as $\tilde{h}_{(c,d)} = h$ on $[-R, R] \times S^1$ and

$$\tilde{h}_{(c,d)}(s,t) = (a(s,t) + \kappa(s) d_i, \phi_i(\theta_i(s,t) + \kappa(s) c_i, z_i(s,t))) \tag{3.20}$$

on $\mathbb{R} \backslash [-R, R] \times S^1$

Definition 3.3.30. *[Dr, cf. Definition 6] Fix small $\epsilon > 0$ so that 2ϵ is smaller than the injective radius with respect to g_J. We define*

$$\mathcal{B}_\delta^{1,q} := \{\exp_{\tilde{h}_{(c,d)}} \circ \tilde{\gamma} |\ \tilde{\gamma} \in W_\delta^{1,q}(\tilde{h}_{(c,d)}^* T(\mathbb{R} \times Y)),\ (c,d) \in \mathbb{R}^4,\ |\tilde{\gamma}|_{C^0} < \epsilon,\ |c_i|, |d_i| < \epsilon,\ i = 1, 2\} \tag{3.21}$$

where $\tilde{h} \in \mathcal{B}_\delta^\infty$.

Theorem 3.3.31. *$\mathcal{B}_\delta^{1,q}$ is endowed with the differentiable structure of an infinite-dimensional, separable Banach manifold.*

For a map $\tilde{h} \in \mathcal{B}_\delta^{1,q}$, let

$$U = \{(\tilde{\gamma}, (c,d)) \in W_\delta^{1,q}(\tilde{h}^* T(\mathbb{R} \times Y)) \times \mathbb{R}^4 |\ |\tilde{\gamma}|_{C^0} < \epsilon,\ |c_i|, |d_i| < \epsilon,\ i = 1, 2\}. \tag{3.22}$$

Then by the construction, we can describe a local chart around $\tilde{h} \in \mathcal{B}_\delta^{1,q}$ as

$$E_{\tilde{h}} : U \to \{\exp_{\tilde{h}_{(c,d)}} \circ \Pi_{(c,d)} \tilde{\gamma} |\ (\tilde{\gamma}, (c,d)) \in U\} \subset \mathcal{B}_\delta^{1,q} \tag{3.23}$$

where $E_{\tilde{h}}(\tilde{\gamma}, (c,d)) = \exp_{\tilde{h}_{(c,d)}} \circ \Pi_{(c,d)} \tilde{\gamma}$ and $\Pi_{(c,d)} : \tilde{h}^* T(\mathbb{R} \times Y) \to \tilde{h}_{(c,d)}^* T(\mathbb{R} \times Y)$ is the parallel transport along the shortest geodesic from a point of $\tilde{h}$ to a point of $\tilde{h}_{(c,d)}$. This implies that there is a natural identification

$$T_{\tilde{h}} \mathcal{B}_\delta^{1,q} \cong W_\delta^{1,q}(\tilde{h}^* T(\mathbb{R} \times Y)) \oplus \mathbb{R}^4 \tag{3.24}$$

where $T_{\tilde{h}} \mathcal{B}_\delta^{1,q}$ is the tangent space at $\tilde{h} \in \mathcal{B}_\delta^{1,q}$.

Next, we consider the tangent space at $\tilde{h} \in P^J$ as a subspace of $T_{\tilde{h}} \mathcal{B}_\delta^{1,q}$.

From the standard argument, $T_{\tilde{h}} P^J$ can be identified with the kernel of

$$F_{\tilde{h}} : T_{\tilde{h}} \mathcal{B}_\delta^{1,q} \cong W_\delta^{1,q}(\tilde{h}^* T(\mathbb{R} \times Y)) \oplus \mathbb{R}^4 \to L_\delta^q(\wedge^{0,1} T^*(\mathbb{R} \times S^1) \otimes \tilde{h}^* T(\mathbb{R} \times Y)). \tag{3.25}$$

Here, $F_{\tilde{h}}$ is the linearization of the Cauchy-Riemann operator at $\tilde{h}$ and described as follows.

Define

$$P_{\tilde{h}}(\tilde{\gamma},(c,d)) = \Pi_{(c,d)} \circ \Phi_{\tilde{h}}^{(c,d)}(\Pi_{(c,d)}\tilde{\gamma})^{-1} \circ \bar{\partial}_J \exp_{\tilde{h}_{(c,d)}} \Pi_{(c,d)}\tilde{\gamma} \tag{3.26}$$

where $\Phi_{\tilde{h}}^{(c,d)}(\tilde{\xi}) : T_{\tilde{h}_{(c,d)}}(\mathbb{R}\times Y) \to T_{\exp_{\tilde{h}_{(c,d)}}(\tilde{\xi})}(\mathbb{R}\times Y)$ is the parallel transport for $\tilde{\xi} \in T_{\tilde{h}_{(c,d)}}(\mathbb{R}\times Y)$.

Then $F_{\tilde{h}}$ is given by

$$F_{\tilde{h}}(\tilde{\gamma},(c,d)) = \frac{d}{d\lambda} P_{\tilde{h}}(\lambda\tilde{\gamma},\lambda(c,d))|_{\lambda=0}. \tag{3.27}$$

Moreover, there is a natural identification

$$T_{\tilde{h}}P^J \cong \mathrm{Ker} F_{\tilde{h}} \tag{3.28}$$

From this construction, it can be described as

$$F_{\tilde{h}}(\tilde{\gamma},(c,d)) = D_{\tilde{h}}(\tilde{\gamma}) + K(c,d) \tag{3.29}$$

where $D_{\tilde{h}} : W_\delta^{1,q}(\tilde{h}^*T(\mathbb{R}\times Y)) \to L_\delta^q(\wedge^{0,1}T^*(\mathbb{R}\times S^1)\otimes_J \tilde{h}^*T(\mathbb{R}\times Y))$ and $K : \mathbb{R}^4 \to L_\delta^q(\wedge^{0,1}T^*(\mathbb{R}\times S^1)\otimes_J \tilde{h}^*T(\mathbb{R}\times Y))$.

Note that $F_{\tilde{h}}$ and $D_{\tilde{h}}$ are Fredholm operators and we fix a generic J so that $F_{\tilde{h}}$ is surjective for any $\tilde{h} \in P^J$.

Proposition 3.3.32. *[Dr, Theorem 9]*

$$\mathrm{Ind} F_{\tilde{h}} = \mathrm{Ind} D_{\tilde{h}} + 4 = 4. \tag{3.30}$$

In particular, $\mathrm{Ind} D_{\tilde{h}} = 0$.

From Proposition 3.3.32, we can see that $\mathrm{pr} : T_{\tilde{h}}P^J \cong \mathrm{Ker} F_{\tilde{h}} \to \mathbb{R}^4$ is an isomorphism where pr is the restriction of the natural projection $\mathrm{pr}_2 : T_{\tilde{h}}\mathcal{B}_\delta^{1,q} \cong W_\delta^{1,q}(\tilde{h}^*T(\mathbb{R}\times Y)) \oplus \mathbb{R}^4 \to \mathbb{R}^4$ to $T_{\tilde{h}}P^J \cong \mathrm{Ker} F_{\tilde{h}}$.

Recall the moduli space $\mathcal{M}^J(\alpha,\emptyset)/\mathbb{R} \cong S^1$. This is a quotient space of P^J. Consider the projection $\pi : P^J \to \mathcal{M}^J(\alpha,\emptyset)/\mathbb{R} \cong S^1$. For $\tilde{h} = (a,h) \in P^J$,

$$\pi(\pi(\tilde{h}))^{-1} = \{(a(s+d,t+c)+e, h(s+d,t+c)) \in P^J | \, (c,d,e) \in S^1 \times \mathbb{R} \times \mathbb{R}\} \tag{3.31}$$

So $\mathcal{M}^J(\alpha, \emptyset)/\mathbb{R} \cong S^1$ is a fiber bundle whose fiber is isomorphic to $S^1 \times \mathbb{R} \times \mathbb{R}$.

Take a section $\tilde{v} : \mathcal{M}^J(\alpha, \emptyset)/\mathbb{R} \cong S^1 \to P^J$. For $\sigma \in S^1$, write $\tilde{v}(\sigma) = (a_\sigma, v_\sigma) \in P^J$. Let p_1, $p_2 : S^1 \to S^1$ denote the functions defined by

$$\lim_{s \to +\infty} v_\sigma(s,t) = \gamma_1(T_1(t + p_1(\sigma))) \tag{3.32}$$

$$\lim_{s \to -\infty} v_\sigma(s,t) = \gamma_2(-T_2(t + p_2(\sigma))) \tag{3.33}$$

Proposition 3.3.33. *p_1, $p_2 : S^1 \to S^1$ are smooth functions such that $\iota = p_1 - p_2 : S^1 \to S^1$ is a locally diffeomorphism.*

Proof of Proposition 3.3.33. In order to prove Proposition 3.3.33, we have to observe the tangent space along the fiber.

Fix $\sigma_0 \in S^1$. Take $e_i \in S^1$ and $f_i \in \mathbb{R}$ so that

$$\lim_{\epsilon_i s \to +\infty} v_{\sigma_0}(s,t) = \gamma_i(\epsilon_i T_i(t + e_i)), \quad \lim_{\epsilon_i s \to +\infty} a_{\tau_0}(s,t) - \epsilon_i T_i s = f_i. \tag{3.34}$$

Let $\tilde{w} : (-\epsilon, \epsilon) \to P^J$ be a smooth path with $\tilde{w}(0) = \tilde{v}(\sigma_0)$ where $\epsilon > 0$ is sufficiently small. Write $\tilde{w}(\lambda) = (b_\sigma, w_\sigma) \in P^J$ and let $c_i : (-\epsilon, \epsilon) \to S^1$, $d_i : (-\epsilon, \epsilon) \to \mathbb{R}$ $i = 1, 2$ denote the functions defined by

$$\lim_{\epsilon_i s \to +\infty} w_\sigma(s,t) = \gamma_i(\epsilon_i T_i(t + c_i(\sigma))), \quad \lim_{\epsilon_i s \to +\infty} b_\sigma(s,t) - \epsilon_i T_i s = d_i(\sigma). \tag{3.35}$$

From the construction we can easily check that

$$\mathrm{pr}_2 \circ E^{-1}_{\tilde{v}(\sigma_0)}(\tilde{w}(\sigma)) = (c_1(\sigma) - e_1, c_2(\sigma) - e_2, d_1(\sigma) - f_1, d_2(\sigma) - f_2) \tag{3.36}$$

where $E_{\tilde{v}_{\sigma_0}}$ is the map defined by (3.23).

This implies that for $\tilde{w}'(0) \in T_{\tilde{v}(\sigma_0)} P^J \cong \mathrm{Ker} F_{\tilde{v}(\sigma_0)}$,

$$\mathrm{pr}(\tilde{w}'(0)) = (c_1'(0), c_2'(0), d_1'(0), d_2'(0)). \tag{3.37}$$

Under these understandings, we observe the tangent space along the fiber at $\tilde{v}(\sigma_0)$. For $i = 1, 2, 3$, define $\tilde{w}_i : (-\epsilon, \epsilon) \to P^J$ as

$$\tilde{w}_1(\sigma) = (a_{\sigma_0}(s,t) + \sigma, v_{\sigma_0}(s,t)), \tag{3.38}$$

$$\tilde{w}_2(\sigma) = (a_{\sigma_0}(s+\sigma, t), v_{\tau_0}(s+\sigma, t)), \tag{3.39}$$

$$\tilde{w}_3(\sigma) = (a_{\sigma_0}(s, t+\sigma), v_{\tau_0}(s, t+\sigma)). \tag{3.40}$$

Since (3.31), the tangent space along the fiber at $\tilde{v}(\sigma_0)$ is spaned by $\tilde{w}_1'(0)$, $\tilde{w}_2'(0)$ and $\tilde{w}_3'(0)$. From the definition, we have

$$\mathrm{pr}(\tilde{w}_1'(0)) = (0,0,1,1),\ \mathrm{pr}(\tilde{w}_2'(0)) = (0,0,T_1,-T_2),\ \mathrm{pr}(\tilde{w}_3'(0)) = (1,1,0,0). \tag{3.41}$$

Since $\tilde{v} : \mathcal{M}^J(\alpha, \emptyset)/\mathbb{R} \cong S^1 \to P^J$ is a section, $\tilde{v}'(\tau_0)$, $\tilde{w}_1'(0)$, $\tilde{w}_2'(0)$ and $\tilde{w}_3'(0)$ span $T_{\tilde{v}(\sigma_0)}P^J \cong \mathrm{Ker}F_{\tilde{v}(\sigma_0)}$. From the definition, the first two coordinate of $\mathrm{pr}(\tilde{v}'(\sigma_0))$ is $(p_1'(\sigma_0), p_2'(\sigma_0))$. Since $\mathrm{pr} : T_{\tilde{h}}P^J \cong \mathrm{Ker}F_{\tilde{h}} \to \mathbb{R}^4$ is an isomorphism, $\mathrm{pr}(\tilde{w}_1'(0))$, $\mathrm{pr}(\tilde{w}_2'(0))$, $\mathrm{pr}(\tilde{w}_3'(0))$ and $\mathrm{pr}(\tilde{v}'(\sigma_0))$ span $\mathbb{R}^4$ and so we have $p_1'(\sigma_0) - p_2'(\sigma_0) \neq 0$. This implies that $\iota = p_1 - p_2 : S^1 \to S^1$ is a local diffeomorphism and hence we complete the proof. □

Proposition 3.3.34. *$\iota = p_1 - p_2 : S^1 \to S^1$ is a p-fold cover.*

Proof of Proposition 3.3.34. By considering a composition function of $t \to t - p_2(\sigma)$ with $\tilde{v}(\sigma)$, we may assume that for $\tilde{v}(\sigma) = (a_\sigma, v_\sigma) \in P^J$,

$$\lim_{s\to+\infty} v_\sigma(s,t) = \gamma_1(T_1(t + p(\sigma))) \tag{3.42}$$

$$\lim_{s\to-\infty} v_\sigma(s,t) = \gamma_2(-T_2 t) \tag{3.43}$$

Moreover by considering $\sigma \to -\sigma$, we may assume that $\iota : S^1 \to S^1$ is an orientation preserving map.

For $R >> 0$ Consider two solid torus $\bigcup_{\sigma\in S^1} v_\sigma((-\infty, -R] \times S^1) \cup \mathrm{Im}\gamma_2$ and $\bigcup_{\sigma\in S^1} v_\sigma([R, +\infty) \times S^1) \cup \mathrm{Im}\gamma_1$ whose boundaries have coordinates

$$S^1 \times S^1 \ni (\sigma, t) \mapsto v_\sigma(-R, t),\ v_\sigma(R, t). \tag{3.44}$$

From the construction, we can see that the map on the boundaries

$$\psi : \bigcup_{\sigma\in S^1} v_\sigma(\{-R\} \times S^1) \to \bigcup_{\sigma\in S^1} v_\sigma(\{R\} \times S^1),\ v_\sigma(-R,t) \mapsto v_\sigma(R,t) \tag{3.45}$$

is a diffeomorphism and the lens space constructed by gluing the two solid torus by this map is diffeomorphic to $L(p, p-1)$. This implies that for fixed $\sigma_0 \in S^1$, $\psi_*([\bigcup_{\sigma\in S^1} v_\sigma(-R, 0)]) = p[v_{\sigma_0}(\{-R\} \times S^1)]$ where

$$\psi_* : H_1(\bigcup_{\sigma\in S^1} v_\sigma(\{-R\} \times S^1)) \to H_1(\bigcup_{\sigma\in S^1} v_\sigma(\{R\} \times S^1)) \tag{3.46}$$

is the induced map by ψ. Consider the inclusion $i : \bigcup_{\sigma\in S^1} v_\sigma(\{R\}\times S^1) \to \bigcup_{\sigma\in S^1} v_\sigma([R,+\infty)\times S^1)\cup \mathrm{Im}\gamma_1$. Then $i_*\circ\psi_*([\bigcup_{\sigma\in S^1} v_\sigma(-R,0)]) = p[\gamma_1]$. This implies that the multiplicity of $\iota : S^1 \to S^1$ is p. We complete the proof of Proposition 3.3.34. □

Proof of Theorem 3.3.10. Since $\iota : S^1 \to S^1$ is a p-fold cover, by considering a composite function $\sigma \mapsto \sigma - \sigma_0$, we may assume that $\iota(0) = 0$. Let $\tilde{S}^1 = \mathbb{R}/p\mathbb{Z}$ and $\pi : \tilde{S}^1 \to S^1$ be the natural projection. Then there is a lift $\tilde{p} : (S^1, 0) \to (\tilde{S}^1, 0)$. Since $\tilde{p}$ is a diffeomorphism, the map $\tilde{p}^{-1} : \tilde{S}^1 \to S^1$ is defined and $p\circ\tilde{p}^{-1} = \pi$. Define $j : S^1 \to S^1$ as $j(\sigma) = p\sigma$. then there is a lift $\tilde{j} : (S^1, 0) \to (\tilde{S}^1, 0)$ with $\pi\circ\tilde{j} = j$. From the construction, we have $\iota\circ\tilde{p}^{-1}\circ\tilde{j} = j$.

Define $\tilde{u}(\sigma) = \tilde{v}(\tilde{p}^{-1}\circ\tilde{j}(\sigma))$. Then this is exactly what we want. Indeed

$$\lim_{s\to+\infty} u_\sigma(s,t) = \gamma_1(T_1(t + p\circ\tilde{p}^{-1}\circ\tilde{j}(\sigma)) = \gamma_1(T_1(t + j(\sigma))) = \gamma_1(T_1(t + p\sigma)) \tag{3.47}$$

$$\lim_{s\to-\infty} u_\sigma(s,t) = \gamma_2(-T_2 t). \tag{3.48}$$

We complete the proof of Theorem 3.3.10. □

3.4 Proof of Theorem 3.2.7 and Theorem 3.2.11

At first, we prove Theorem 3.2.7.

Consider $V_1 = S^1\times\mathbb{D}$, $V_2 = S^1\times\mathbb{D}$ and a gluing map $g : \partial V_1 = S^1\times\partial\mathbb{D} \to S^1\times\partial\mathbb{D} = \partial V_2$ which is described as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in standard longitude-meridian coordinates on the torus where $a, b, c, d \in \mathbb{Z}$, $b > 0$ and $ad - bc = 1$. Then, there is an orientation-preserving diffeomorphism from the glued manifold $V_1 \cup_g V_2$ to $L(p,q)$ if and only if $b = p$ and in addition either $d = -q \mod p$ or $dq = -1 \mod p$ (as remarked, the orientation of $L(p,q)$ is induced by the 4-dimensional ball). Note that the boundary of a meridian disk of V_1 is glued by g along a (p,d)-cable curve on $\partial V_2 = S^1\times\partial\mathbb{D}$.

Let $\gamma \subset (L(p,q),\lambda)$ be a p-unknotted Reeb orbits and $u : \mathbb{D} \to L(p,q)$ be a rational Seifert surface of γ^p. Take a Martinet tube $F : \mathbb{R}/T_\gamma\mathbb{Z}\times\mathbb{D}_\delta \to \bar{U}$ for a sufficiently small $\delta > 0$ onto a small open neighbourhood $\gamma \subset \bar{U}$.

As Remark 3.1.8, $\bar{U}$ is a solid torus such that $L(p,q)\backslash U$ is also a solid torus, which gives a Heegaard decomposition of genus 1. In addition, $u(\mathbb{D})\cap(L(p,q)\backslash U)$ is a meridian disk of $L(p,q)\backslash U$. Therefore, $F^{-1}(u(\mathbb{D})\cap\partial\bar{U})$ is a (p,r) cable such that either $-r=q \bmod p$ or $-rq=1 \bmod p$ with respect to the coordinate of $\mathbb{R}/T_\gamma\mathbb{Z}\times\mathbb{D}$. Therefore, Theorem 3.2.7 follows directly from the next proposition.

Proposition 3.4.1. *Suppose that the above $F^{-1}(u(\mathbb{D})\cap\partial\bar{U})$ is a (p,r) cable. If $-2r-2p\cdot sl_\xi^{\mathbb{Q}}(\gamma)-\mu_{\text{disk}}(\gamma^p)$ is not divisible by p, then γ is elliptic.*

To prove the above proposition, we take sections $Z_{\text{disk}}:\mathbb{R}/pT_\gamma\mathbb{Z}\to(\gamma^p)^*\xi$ and $Z_F:\mathbb{R}/pT_\gamma\mathbb{Z}\to(\gamma^p)^*\xi$ so that Z_{disk} extends to a non-vanishing section on $u^*\xi$ and Z_F corresponds to ∂_x on the coordinate induced by F. Let $Z_{\text{disk}}^\epsilon:\mathbb{R}/pT_\gamma\mathbb{Z}\to L(p,q)$ and $Z_F^\epsilon:\mathbb{R}/pT_\gamma\mathbb{Z}\to L(p,q)$ denote the curves $Z_{\text{disk}}^\epsilon(t)=\exp_{\gamma(t)}(\epsilon Z_{\text{disk}}(t))$ and $Z_F^\epsilon(t)=\exp_{\gamma(t)}(\epsilon Z_F(t))$ for small $\epsilon>0$ respectively. Then, it follows from a direct observation and the definition that $\#(u(\mathbb{D})\cap Z_0^\epsilon)=-rp$ and $\#(u(\mathbb{D})\cap Z_{\text{disk}}^\epsilon)=p^2sl_\xi^{\mathbb{Q}}(\gamma)$ (note the orientation and sign).

Let $\rho:S^3\to L(p,q)$ be the covering map. We can take lifts of $\gamma^p:\mathbb{R}/pT_\gamma\mathbb{Z}\to L(p,q)$ and the rational Seifert surface $u:\mathbb{D}\to L(p,q)$ to S^3, and write $\tilde{\gamma}:\mathbb{R}/pT_\gamma\mathbb{Z}\to S^3$, $\tilde{u}:\mathbb{D}\to S^3$ respectively. We may assume that $\tilde{\gamma}(pT_\gamma t)=\tilde{u}(e^{2\pi t})$. In the same way, we take lifts of Z_{disk}, $Z_F:\mathbb{R}/pT_\gamma\mathbb{Z}\to(\gamma^p)^*\xi$ and write $\tilde{Z}_{\text{disk}}$, $\tilde{Z}_F:\mathbb{R}/pT_\gamma\mathbb{Z}\to\tilde{\gamma}^*\xi$ respectively. Let $\tilde{Z}_{\text{disk}}^\epsilon(t)=\exp_{\tilde{\gamma}(t)}(\epsilon\tilde{Z}_{\text{disk}}(t))$ and $\tilde{Z}_F^\epsilon(t)=\exp_{\tilde{\gamma}(t)}(\epsilon\tilde{Z}_F(t))$. Then it follows from the construction that $\tilde{Z}_{\text{disk}}^\epsilon$ and $\tilde{Z}_F^\epsilon$ are lifts of Z_{disk}^ϵ and Z_F^ϵ respectively. Since there are p ways of lifting Z_{disk}^ϵ, Z_F^ϵ and each intersection number of a lift with $\tilde{u}(\mathbb{D})$ is equal to each other, we have

$$\#(u(\mathbb{D})\cap Z_F^\epsilon)=p\#(\tilde{u}(\mathbb{D})\cap\tilde{Z}_F^\epsilon),\ \#(u(\mathbb{D})\cap Z_{\text{disk}}^\epsilon)=p\#(\tilde{u}(\mathbb{D})\cap\tilde{Z}_{\text{disk}}^\epsilon)$$

and hence $\#(\tilde{u}(\mathbb{D})\cap\tilde{Z}_F^\epsilon)=-r,\ \#(\tilde{u}(\mathbb{D})\cap\tilde{Z}_{\text{disk}}^\epsilon)=psl_\xi^{\mathbb{Q}}(\gamma)$.

Recall that $\tau_F:\gamma^*\xi\to\mathbb{R}/T_\gamma\times\mathbb{R}^2$ denotes the trivialization induced by F. As mentioned, we use the same notation τ_F for a trivialization $(\gamma^p)^*\xi\to\mathbb{R}/T_\gamma\times\mathbb{R}^2$ induced by $\tau_F:\gamma^*\xi\to\mathbb{R}/T_\gamma\times\mathbb{R}^2$.

Lemma 3.4.2. $\mu_{\tau_F}(\gamma^p)-2psl_\xi^{\mathbb{Q}}(\gamma)-2r=\mu_{\text{disk}}(\gamma^p)$

Proof of Lemma 3.4.2. Take a section $\tilde{W}_{\text{disk}}:\mathbb{R}/pT_\gamma\mathbb{Z}\to(\tilde{\gamma})^*\xi$ so that the map $\tilde{\gamma}^*\xi\ni a\tilde{Z}_{\text{disk}}+b\tilde{W}_{\text{disk}}\mapsto a+ib\in\mathbb{R}^2$ gives a symplectic trivialization. Since $\tilde{Z}_{\text{disk}}:\mathbb{R}/pT_\gamma\mathbb{Z}\to(\tilde{\gamma})^*\xi$ extends globally to a non-vanishing section on

$\tilde{u}^*\xi$, the homotopy class of this trivialization is τ_{disk}. In the same way, we take a section $\tilde{W}_F : \mathbb{R}/pT_\gamma\mathbb{Z} \to (\tilde{\gamma})^*\xi$ so that the map $\tilde{\gamma}^*\xi \ni a\tilde{Z}_F + b\tilde{W}_F \mapsto a + ib \in \mathbb{R}^2$ gives a symplectic trivialization. Since $\tilde{Z}_F : \mathbb{R}/pT_\gamma\mathbb{Z} \to (\tilde{\gamma})^*\xi$ corresponds to ∂_x with respect to the coordinate induced by F, the homotopy class of this trivialization is τ_F. Therefore it follows directly that

$$\text{wind}(\tau_F, \tau_{\text{disk}}) = \#(\tilde{u}(\mathbb{D}) \cap \tilde{Z}_F^\epsilon) - \#(\tilde{u}(\mathbb{D}) \cap \tilde{Z}_{\text{disk}}^\epsilon) = -r - psl_\xi^{\mathbb{Q}}(\gamma). \quad (3.49)$$

It follows from Proposition 1.2.3 that

$$\mu_{\tau_F}(\gamma^p) + 2\text{wind}(\tau_F, \tau_{\text{disk}}) = \mu_{\tau_F}(\gamma^p) - 2r - 2psl_\xi^{\mathbb{Q}}(\gamma) = \mu_{\text{disk}}(\gamma^p). \quad (3.50)$$

This completes the proof. □

Now, we shall complete the proof of Proposition 3.4.1. Suppose that $\gamma \subset L(p,q)$ is hyperbolic. Then according to Proposition 1.2.4, $\mu_{\tau_F}(\gamma^p) = p\mu_{\tau_F}(\gamma)$. Therefore it follows from Lemma 3.4.2 that $-p\mu_{\tau_F}(\gamma) = -2psl_\xi^{\mathbb{Q}}(\gamma) - 2r - \mu_{\text{disk}}(\gamma^p)$. This means that if $-2psl_\xi^{\mathbb{Q}}(\gamma) - 2r - \mu_{\text{disk}}(\gamma^p)$ is not divisible by p, then γ must be elliptic. This completes the proof.

3.4.1 Immersed J-holomorphic curves

In this sectin, we assume that $Y \cong L(3,1)$ and (Y, λ) is non-degenerate dynamically convex. From now, we fix a generic admissible almost complex structure J and trivialization $\tau_\gamma \in \mathcal{P}(\gamma)$ of the contact surface ξ on each simple orbit γ. Let $\tau := \{\tau_\gamma\}_\gamma$.

Proposition 3.4.3. *Let $u : (\Sigma, j) \to (\mathbb{R}\times Y, J)$ be an immersed J-holomorphic curve with no negative end.*

(1). $\text{ind}(u)$ *is not equal to* 1.

(2). If $\text{ind}(u) = 2$, *then* u *is of genus* 0.

Proof of Proposition 3.4.3. Let $g(u)$ denote the genus of u. Since u is immersion, we have

$$\begin{aligned} \text{ind}(u) &= -(2 - 2g(u) - k) + 2c_\tau(\xi|_{[u]}) + \sum_{1\leq i\leq k} \mu_\tau(\gamma_i) \\ &= -(2 - 2g(u)) + 2c_\tau(\xi|_{[u]}) + \sum_{1\leq i\leq k} (\mu_\tau(\gamma_i) + 1) \end{aligned} \quad (3.51)$$

where $\{\gamma_i\}_i$ is the set of periodic orbits to which the ends of u are asymptotic. We may assume that γ_i is hyperbolic if $1 \leq i \leq k'$ and elliptic if $k'+1 \leq i \leq k$.

Lemma 3.4.4. *Let a periodic orbit γ be elliptic. For any symplectic trivialization $\tau : \gamma^*\xi \to \gamma \times \mathbb{R}^2$ and $p \in \mathbb{Z}_{>0}$, we have*

$$p\mu_\tau(\gamma) - \mu_\tau(\gamma^p) + p \geq 1. \tag{3.52}$$

Proof of Lemma 3.4.4. Let θ denote the monodolomy angle with respect to τ. Then $\mu_\tau(\gamma^p) = 2\lfloor p\theta \rfloor + 1$ for any $p \in \mathbb{Z}_{>0}$. Since either $\lfloor 2\theta \rfloor = 2\lfloor \theta \rfloor + 1$ or $\lfloor 2\theta \rfloor = 2\lfloor \theta \rfloor$, we have $2\lfloor \theta \rfloor + 1 \geq \lfloor 2\theta \rfloor$. It is easy to check by inducting that $p\lfloor \theta \rfloor + p - 1 \geq \lfloor p\theta \rfloor$. This completes the proof. □

Recall $Y \cong L(3,1)$. Hence for any periodic orbit γ, γ^3 is contractible. Let α denote the orbit set to which the positive ends of u are asymptotic. Then $\{[u]\} = H_2(Y, \alpha, \emptyset)$. Now, we note some obvious facts. First, $c_\tau(\xi|_{3[u]}) = 3c_\tau(\xi|_{[u]})$ where c_τ is the first relative Chern number and $\{3[u]\} = H_2(Y, 3\alpha, \emptyset)$. Next, since any γ_i^3 is contractible, $2c_\tau(\xi|_{3[u]}) + \sum_{1\leq i\leq k} \mu_\tau(\gamma_i^3) = \sum_{1\leq i\leq k} \mu_{\mathrm{disk}}(\gamma_i^3)$.

Based on this understanding, we multiply both sides of (3.51) by 3. Then we have

$$\begin{aligned}
3\mathrm{ind}(u) =& -3(2-2g(u)) + 2c_\tau(\xi|_{3[u]}) + \sum_{1\leq i\leq k'} (\mu_\tau(\gamma_i^3) + 3) \\
&+ \sum_{k'+1\leq i\leq k} \mu_\tau(\gamma_i^3) + \sum_{k'+1\leq i\leq k} (p\mu_\tau(\gamma_i) - \mu_\tau(\gamma_i^3) + 3) \\
=& -3(2-2g(u)) + \sum_{1\leq i\leq k'} (\mu_{\mathrm{disk}}(\gamma_i^3) + 3) \\
&+ \sum_{k'+1\leq i\leq k} \mu_{\mathrm{disk}}(\gamma_i^3) + \sum_{k'+1\leq i\leq k} (3\mu_\tau(\gamma_i) - \mu_\tau(\gamma_i^3) + 3) \\
\geq& 6g(u) - 6 + \sum_{1\leq i\leq k'} (\mu_{\mathrm{disk}}(\gamma_i^3) + 3) + \sum_{k'+1\leq i\leq k} (\mu_{\mathrm{disk}}(\gamma_i^3) + 1)
\end{aligned} \tag{3.53}$$

Here Lemma 3.4.4 is used.

Suppose that $\mathrm{ind}(u) = 1$. From (3.53), we have

$$9 \geq 6g(u) + \sum_{1\leq i\leq k'} (\mu_{\mathrm{disk}}(\gamma_i^3) + 3) + \sum_{k'+1\leq i\leq k} (\mu_{\mathrm{disk}}(\gamma_i^3) + 1) \tag{3.54}$$

We note that $\mu_{\mathrm{disk}}(\gamma_i^3) + 3$ is at least 6 and $\sum_{k'+1\leq i\leq k}(\mu_{\mathrm{disk}}(\gamma_i^3) + 1)$ is at least 4 because of dynamical convexity. Since $\mathrm{ind}(u) = 1$ is odd, it follows from (3.51) that at least one γ_i must be positive hyperbolic. Therefore $k' \geq 1$ and thus $k' = k = 1$. This means that the only one orbit γ_1 is contractible. Now we go back to (3.51). We have $1 = 2g(u) - 1 + \mu_{\text{disk}}(\gamma_1)$. This does not happen since $\mu_{\text{disk}}(\gamma_1) \geq 3$ and $g(u) \geq 0$. This proves Proposition 3.4.3(1).

Next we suppose that $\text{ind}(u) = 2$ and $g(u) \geq 1$. From (3.53), we have

$$6 \geq \sum_{1 \leq i \leq k'} (\mu_{\text{disk}}(\gamma_i^3) + 3) + \sum_{k'+1 \leq i \leq k} (\mu_{\text{disk}}(\gamma_i^3) + 1) \tag{3.55}$$

In the same way as above, we have $k = 1$ and so from (3.51) $2 = \text{ind}(u) = 2g(u) - 1 + \mu_{\text{disk}}(\gamma_1)$. But this does not happen since $\mu_{\text{disk}}(\gamma_1) \geq 3$ and $g(u) \geq 1$. This proves Proposition 3.4.3(2). □

Remark 3.4.5. In general, the above argument does not work for any $L(p, q)$.

Proposition 3.4.6. *Let α be an ECH generator with $\langle U_{J,z}\alpha, \emptyset\rangle \neq 0$. Then α satisfies one of the following:*

(1). There are simple elliptic orbits γ_1, γ_2, $\gamma_3 \in \mathcal{S}_3$ with $\mu_{\text{disk}}(\gamma_i^3) = 3$ for $i = 1, 2, 3$ such that $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1) \cup (\gamma_3, 1)$.

(2). There is a simple elliptic orbit $\gamma \in \mathcal{S}_3$ with $\mu_{\text{disk}}(\gamma^3) = 3$ such that $\alpha = (\gamma, 3)$.

(3). There are simple elliptic orbits γ_1, $\gamma_2 \in \mathcal{S}_3$ with $\mu_{\text{disk}}(\gamma_1^3) = \mu_{\text{disk}}(\gamma_2^3) = 3$ such that $\alpha = (\gamma_1, 2) \cup (\gamma_2, 1)$.

There are many steps to prove Proposition 3.4.6. At first, we compute some indices and list the properties of ECH generators α which satisfy $\langle U_{z,J}\alpha, \emptyset\rangle \neq 0$.

Lemma 3.4.7. *Let α be an ECH generator with $\langle U_{J,z}\alpha, \emptyset\rangle \neq 0$. Then by conducting some computations regarding indices, we have that any $u \in \mathcal{M}^J(\alpha, \emptyset)$ has no two ends asymptotic to the same orbit. In addition, α satisfies one of the following:*

(1). There are simple elliptic orbits γ_1, γ_2, γ_3 with $\mu_{\text{disk}}(\gamma_i^3) = 3$ for $i = 1, 2, 3$ such that $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1) \cup (\gamma_3, 1)$.

(2). There is a simple elliptic orbit γ with $\mu_{\text{disk}}(\gamma^3) = 3$ such that $\alpha = (\gamma, 3)$.

(3). There are simple elliptic orbits γ_1 and γ_2 with $\mu_{\text{disk}}(\gamma_1^3) = \mu_{\text{disk}}(\gamma_2^3) = 3$ such that $\alpha = (\gamma_1, 2) \cup (\gamma_2, 1)$.

(4). There are simple elliptic orbits γ_1 γ_2 with $\mu_{\text{disk}}(\gamma_1^6) = 5$ and $\mu_{\text{disk}}(\gamma_2^3) = 5$ such that $\alpha = (\gamma_1, 2) \cup (\gamma_2, 1)$.

(5). There are simple orbits γ_1 and γ_2 such that $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ and each of them is not positive hyperbolic.

(6). There are simple elliptic orbits γ_1 and γ_2 such that $\alpha = (\gamma_1, 2) \cup (\gamma_2, 2)$.

(7). There is a simple orbit γ such that $\alpha = (\gamma, 1)$ and γ is not positive hyperbolic.

Proof of Lemma 3.4.7. Set $\alpha = \{(\gamma_i, m_i)\}_{1 \leq i \leq k}$. We may assume that γ_i is hyperbolic for $1 \leq i \leq k'$ and elliptic for $k'+1 \leq i \leq k$. Of course, $m_i = 1$ for $1 \leq i \leq k'$ since α is an ECH generator. Let $u \in \mathcal{M}^J(\alpha, \emptyset)$ be a J-holomorphic curve counted by $\langle U_{z,J}\alpha, \emptyset \rangle \neq 0$. Then we have

$$\begin{aligned} 2c_\tau(\xi|_{[u]}) + \sum_{1 \leq i \leq k} \mu_\tau(\gamma_i^{m_i}) =& I(u) - J_0(u) \\ =& 2 - (-\chi(u) + \sum_{1 \leq i \leq k} (n_i^+ - 1)) \\ =& 2 - (2g(u) - 2 + h + \sum_{1 \leq i \leq k} (n_i^+ - 1)). \end{aligned} \tag{3.56}$$

Here h is the number of the positive ends of u and n_i^+ is the number of the positive ends asymptotic to γ_i. It follows from Proposition 3.4.3 (2) that $g(u) = 0$. Hence we have

$$4 = h + \sum_{1 \leq i \leq k} (n_i^+ - 1) + 2c_\tau(\xi|_{[u]}) + \sum_{1 \leq i \leq k} \mu_\tau(\gamma_i^{m_i}) \tag{3.57}$$

Now, we conduct similar calculations with the proof of Proposition 3.4.3.

Multiplying both side by 3, it follows from Lemma 3.4.4 that

$$\begin{aligned}
12 =&3h + 3\sum_{1\leq i\leq k}(n_i^+ - 1) + 2c_\tau(\xi|_{3[u]}) + \sum_{1\leq i\leq k'}\mu_\tau(\gamma_i^3)\\
&+\sum_{k'+1\leq i\leq k}\mu_\tau(\gamma_i^{3m_i}) + \sum_{k'+1\leq i\leq k}(3\mu_\tau(\gamma_i^{m_i}) - \mu_\tau(\gamma_i^{3m_i}))\\
=&3h + 3\sum_{1\leq i\leq k}(n_i^+ - 1) + \sum_{1\leq i\leq k'}(\mu_{\text{disk}}(\gamma_i^3) + 3)\\
&+\sum_{k'+1\leq i\leq k}\mu_{\text{disk}}(\gamma_i^{3m_i}) + \sum_{k'+1\leq i\leq k}(3\mu_\tau(\gamma_i^{m_i}) - \mu_\tau(\gamma_i^{3m_i}) + 3) - 3k\\
\geq&3(h-k) + 3\sum_{1\leq i\leq k}(n_i^+ - 1)\\
&+\sum_{1\leq i\leq k'}(\mu_{\text{disk}}(\gamma_i^3) + 3) + \sum_{k'+1\leq i\leq k}(\mu_{\text{disk}}(\gamma_i^{3m_i}) + 1).
\end{aligned} \tag{3.58}$$

Claim 3.4.8. *$n_i^+ = 1$ for any $1 \leq i \leq k$. That is, the number of the ends of u asymptotic to γ_i is 1 for any $1 \leq i \leq k$. This means $h = k$ and $\sum_{1\leq i\leq k}(n_i^+ - 1) = 0$ in (3.58).*

Proof of Claim 3.4.8. We prove this by contradiction. Assume that $n_j^+ > 1$ for some $k'+1 \leq j \leq k$. Then $3(h-k) \geq 3$ and $3\sum_{1\leq i\leq k}(n_i^+ - 1) \geq 3$. Therefore we have

$$6 \geq \sum_{1\leq i\leq k'}(\mu_{\text{disk}}(\gamma_i^3) + 3) + \sum_{k'+1\leq i\leq k}(\mu_{\text{disk}}(\gamma_i^{3m_i}) + 1). \tag{3.59}$$

Moreover $m_j > 1$. It follows easily from (3.59) that $k = 1$. Indeed If $k > 1$, the right hand side of (3.59) is at least 8. Hence $k = j = 1$. If γ_1 is contractible, $\mu_{\text{disk}}(\gamma^{3m_1}) \geq 6m_1+1 \geq 13$. This is a contradiction. So γ_1 must be non-contractible. Suppose that γ is non-contractible. Since $[\alpha] = m_1[\gamma_1]$ must be zero in $H_1(L(3,1))$, m_1 is divisible by 3. Write $m_1 = 3m_1'$. We have $\mu_{\text{disk}}(\gamma^{3m_1}) + 1 = \mu_{\text{disk}}((\gamma^3)^{3m_1'}) + 1 \geq 6m_1' + 1 + 1 \geq 8$ (Proposition 1.2.2). This contradicts (3.59). In summary, we have $n_i^+ = 1$ for any i. This completes the proof. ☐

Having Claim 3.4.8, it follows from (3.58) that

$$\begin{aligned}
12 \geq& \sum_{1\leq i\leq k'}(\mu_{\text{disk}}(\gamma_i^3) + 3) + \sum_{k'+1\leq i\leq k}(\mu_{\text{disk}}(\gamma_i^{3m_i}) + 1)\\
\geq&6k' + 4(k-k').
\end{aligned} \tag{3.60}$$

The pair $(k-k',k')$ must satisfy (3.60). Thus $(k-k',k')$ is one of the following. $(k-k',k') = (3,0),(0,2),(1,1),(2,0),(0,1),(1,0)$. We check their properties one by one.

The next claim is obvious but it is worth to be mentioned explicitly for further arguments.

Claim 3.4.9. *Suppose that $\mu_\tau(\gamma) \geq 3$. If $\mu_\tau(\gamma^k) = 5$ or 7 for $k > 1$, then $\mu_\tau(\gamma) = 3$. In addition if $\mu_\tau(\gamma^k) = 5$ for $k > 1$, then $k = 2$ and if $\mu_\tau(\gamma^k) = 7$ for $k > 1$, then either $k = 2$ or $k = 3$.*

Proof of Claim 3.4.9. Note that γ is elliptic if $\mu_\tau(\gamma^k) = 5$ or 7 for $k > 1$. Indeed if γ is hyperbolic, then $\mu_\tau(\gamma^k) = k\mu_\tau(\gamma)$ (c.f. Proposition 1.2.2), but since 5 and 7 are prime, this is a contradiction. Since γ is elliptic, there is $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_\tau(\gamma^k) = 2\lfloor k\theta \rfloor + 1$ for any $k \geq 1$ (Proposition 1.2.2). If $\mu_\tau(\gamma) \geq 5$, then $\theta \geq 2$ because $\mu_\tau(\gamma) = 2\lfloor \theta \rfloor + 1 \geq 5$. Therefore $\mu_\tau(\gamma^k) = 2\lfloor k\theta \rfloor + 1 \geq 4k+1$ (c.f. Proposition 1.2.4). But any $k > 1$ does not satisfy $\mu_\tau(\gamma^k) = 5$ and 7. This is a contradiction. Thus we have $\mu_\tau(\gamma) = 3$.

If $\mu_\tau(\gamma^k) = 5$ for $k > 1$, $k = 2$ follows directly from Proposition 1.2.4. In the same way, we have if $\mu_\tau(\gamma^k) = 7$ for $k > 1$, then either $k = 2$ or $k = 3$. □

(a) If $(k-k',k') = (3,0)$. In this case, we have $\alpha = (\gamma_1, m_1) \cup (\gamma_2, m_2) \cup (\gamma_3, m_3)$ for some elliptic orbits γ_i. Moreover from (3.60), $\mu_{\text{disk}}(\gamma_i^{3m_i}) = 3$ for $i = 1,2,3$. This implies that $m_i = 1$ (Proposition 1.2.4) and thus $\mu_{\text{disk}}(\gamma_i^3) = 3$ for $i = 1,2,3$. Thus α satisfies (1) in Lemma 3.4.7.

(b) If $(k-k',k') = (0,2)$. In this case, we have $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ with $\mu_{\text{disk}}(\gamma_1^3) = \mu_{\text{disk}}(\gamma_2^3) = 3$. Thus γ_1 and γ_2 are negative hyperbolic, and α satisfies (4) in Lemma 3.4.7.

(c) If $(k-k',k') = (1,1)$. In this case, $\alpha = (\gamma_1, 1) \cup (\gamma_2, m_2)$ for elliptic γ_2 and negative hyperbolic γ_1. Since $12 \geq (\mu_{\text{disk}}(\gamma_1^3)+3) + (\mu_{\text{disk}}(\gamma_2^{3m_2})+1)$, it follows from dynamical convexity that either $(\mu_{\text{disk}}(\gamma_1^3), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (3,5)$ or $(5,3)$ or $(3,3)$.

Suppose that $(\mu_{\text{disk}}(\gamma_1^3), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (3,5)$. Then $m_2 = 1$ or 2 (Clam 3.4.9). If $m_2 = 1$, α satisfies Lemma 3.4.7 (5). If $m_2 = 2$, $\mu_{\text{disk}}(\gamma_2^{3m_2}) = \mu_{\text{disk}}((\gamma_2^3)^2)) = 5$ implies that $\mu_{\text{disk}}(\gamma_3) = 3$ (Claim 3.4.9) and thus $\alpha = (\gamma_1, 1) \cup (\gamma_2, 2)$ satisfies Lemma 3.4.7 (3).

Suppose that $(\mu_{\text{disk}}(\gamma_1^3), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (5,3)$ or (3.3). $\mu_{\text{disk}}(\gamma_2^{3m_2}) = 3$

implies that $m_1 = 1$ (Proposition 1.2.4) and thus $\alpha = (\gamma_1, 1) \cup (\gamma_2, 1)$ satisfies (5) in Lemma 3.4.7.

(d) If $(k - k', k') = (2, 0)$. In this case, we have $\alpha = (\gamma_1, m_1) \cup (\gamma_2, m_2)$ for elliptic orbits γ_1 and γ_2. It follows from (3.60) that $10 \geq \mu_{\text{disk}}(\gamma_1^{3m_1}) + \mu_{\text{disk}}(\gamma_2^{3m_2})$. Without of loss generality, we may assume that $\mu_{\text{disk}}(\gamma_1) \geq \mu_{\text{disk}}(\gamma_2)$. Then we have $(\mu_{\text{disk}}(\gamma_1^{3m_1}), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (7, 3), (5, 5), (5, 3), (3, 3)$.

Suppose that $(\mu_{\text{disk}}(\gamma_1^{3m_1}), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (7, 3)$. Then $m_2 = 1$ and $\mu_{\text{disk}}(\gamma_2) = 3$ (Proposition 1.2.4) and in addition either $m_1 = 1$ or $m_1 = 2$ or $m_1 = 3$ (Claim 3.4.9). If $m_1 = 1$, then α satisfies (5) in Lemma 3.4.7. If $m_1 = 2$, then $\mu_{\text{disk}}(\gamma_2^3) = 3$ (Claim 3.4.9) and thus α satisfies (3) in Lemma 3.4.7. If $m_1 = 3$, then since $[\alpha] = m_1[\gamma_1] + [\gamma_2] = 0$, $[\gamma_2] = 0$ and thus γ_2 is contractible. But this is a contradiction because $\mu_{\text{disk}}(\gamma_2^3) \geq 2 \times 3 + 1 = 7$ (Proposition 1.2.4).

Suppose that $(\mu_{\text{disk}}(\gamma_1^{3m_1}), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (5, 5)$. Then it follows from Claim 3.4.9 that either $m_i = 1$ or 2 for each $i = 1, 2$ and in addition if $m_i = 2$, then $\mu_{\text{disk}}(\gamma_i^3) = 3$. This means that α satisfies either (4) or (5) or (6) in Lemma 3.4.7.

Suppose that $(\mu_{\text{disk}}(\gamma_1^{3m_1}), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (5, 3)$. Then $m_2 = 1$ and $\mu_{\text{disk}}(\gamma_2) = 3$ (Proposition 1.2.4) and in addition either $m_1 = 1$ or $m_1 = 2$. If $m_1 = 1$, then α satisfies (5) in Lemma 3.4.7. If $m_1 = 2$, then $\mu_{\text{disk}}(\gamma_2^3) = 3$ (Claim 3.4.9) and thus α satisfies (3) in Lemma 3.4.7.

Suppose that $(\mu_{\text{disk}}(\gamma_1^{3m_1}), \mu_{\text{disk}}(\gamma_2^{3m_2})) = (3, 3)$. Then $m_1 = m_2 = 1$. Thus α satisfies (5) in Lemma 3.4.7.

(e) If $(k - k', k') = (0, 1)$. In this case, we have $\alpha = (\gamma_1, 1)$ for a negative hyperbolic γ_1 and thus α satisfies (7) in Lemma 3.4.7.

(f) If $(k - k', k') = (1, 0)$. In this case, $\alpha = (\gamma_1, m_1)$ for some m_1 and elliptic γ. Suppose that γ_1 is contractible. Then $\mu_{\text{disk}}(\gamma_1^{m_1}) \geq 2m_1 + 1$ (Proposition 1.2.4). If $m_1 \geq 2$, we have $\text{ind}(u) = \mu_{\text{disk}}(\gamma_1^{m_1}) - 1 \geq 4$. This contradicts $\text{ind}(u) = 2$. Therefore $m_1 = 1$ and α satisfies (7).

Next, suppose that γ_1 is not contractible. Since $[\alpha] = 0$, $m_1 = 3m_1'$ for some $m_1' \in \mathbb{Z}_{>0}$. Therefore, we have $\text{ind}(u) = \mu_{\text{disk}}((\gamma_1^3)^{m_1'}) - 1 \geq 2m_1'$. This implies that $m_1' = 1$ and $\mu_{\text{disk}}(\gamma_1^3) = 3$. Hence we have $\alpha = (\gamma_1, 3)$, which satisfies (2).

In summary. we complete the proof of Lemma 3.4.7. □

3.4.2 Rational open book decompositions and binding orbits

In this section, we narrow down the list in Lemma 3.4.7. For the purpose, we observe topological properties of the moduli space of holomorphic curves. The next lemma plays important roles in what follows.

Lemma 3.4.10. *Let α be an ECH generator with $\langle U_{J,z}\alpha, \emptyset\rangle \neq 0$. Suppose that $u \in \mathcal{M}^J(\alpha, \emptyset)$ is a J-holmorphic curve counted by $U_{J,z}$. Then the quotient space $\mathcal{M}^J_u/\mathbb{R}$ is compact and thus diffeomorphic to S^1. In addition, for any section $s : \mathcal{M}^J_u/\mathbb{R} \to \mathcal{M}^J_u$, $\bigcup_{t\in S^1} \pi(s(t))$ gives a rational open book decomposition on $Y \cong L(3,1)$ where $\pi : \mathbb{R} \times Y \to Y$ is the projection.*

Proof of Lemma 3.4.10. It follows from Lemma 3.4.7 that any $u \in \mathcal{M}^J(\alpha, \emptyset)$ counted by $U_{J,z}$ satisfies (1), (2), (3) in Proposition 3.3.3. To prove that $u \in \mathcal{M}^J(\alpha, \emptyset)$ satisfies (4) in Proposition 3.3.3, we recall the following property.

Claim 3.4.11. *Let $\theta \in \mathbb{R}\backslash\mathbb{Z}$. For any $q \in S_{-\theta}$, $\gcd(q, \lfloor q\theta \rfloor) = 1$.*

Proof of Claim 3.4.11. Claim 3.4.11 follows directly from the definition of S_θ. Here, note that $-\lfloor \theta q \rfloor = \lceil -q\theta \rceil$. □

According to Proposition 1.3.8, if an end of $u \in \mathcal{M}^J(\alpha, \emptyset)$ is asymptotic to simple orbit γ with some multiplicity, the multiplicity is in $S_{-\theta}$ where θ is the monodromy angle of γ. Therefore it follows that$u \in \mathcal{M}^J(\alpha, \emptyset)$ satisfies (4) in Proposition 3.3.3.

At last, we check that $u \in \mathcal{M}^J(\alpha, \emptyset)$ satisfies (5) in Proposition 3.3.3. Suppose that $\mathcal{M}^J_u/\mathbb{R}$ is not compact. Let $\overline{\mathcal{M}^J_u/\mathbb{R}}$ denote the compactified space of $\mathcal{M}^J_u/\mathbb{R}$ in the sense of SFT compactness. Choose $\bar{u} \in \overline{\mathcal{M}^J_u/\mathbb{R}}\backslash(\mathcal{M}^J_u/\mathbb{R})$. $\bar{u}$ consists of some J-holomorphic curves in several floors. Let u' be the component of $\bar{u}$ in the lowest floor. Then there is an orbit set β such that $u' \in \mathcal{M}^J(\beta, \emptyset)/\mathbb{R}$. By the additivity of ECH index, we have $I(\beta, \emptyset) = 1$. This contradicts Lemma 3.4.3. Thus $\mathcal{M}^J_u/\mathbb{R}$ is compact. □

Lemma 3.4.12. $L(3,1)$ *does not admit rational open book decompositions coming from (6), (7) in Lemma 3.4.7.*

Proof of Lemma 3.4.12. It is obvious that if a 3-manifol Y has a open book decomposition such that each page is embedded disk, then Y is diffeomorphic to S^3. Therefore (7) is excluded.

Next, we consider (6) in Lemma 3.4.7. For γ_1 and γ_2, we take Martinet tubes $F_i : \mathbb{R}/T_{\gamma_i}\mathbb{Z} \times \mathbb{D}_\delta \to \bar{U}_i$ for a sufficiently small $\delta > 0$ where $\gamma_i \subset U_i$. Since $\pi(s(t))$ is embedded and connected on Y, $F_i^{-1}(\mathbb{R}/T_{\gamma_i}\mathbb{Z} \times \partial\mathbb{D}_\delta \cap \pi(s(t)))$ is $(2, p_i)$-cable for odd integers $p_i \in \mathbb{Z}$ for any $t \in S^1$. In addition, the gluing map from $F_1(\mathbb{R}/T_{\gamma_1}\mathbb{Z} \times \partial\mathbb{D}_\delta)$ to $F_2(\mathbb{R}/T_{\gamma_2}\mathbb{Z} \times \partial\mathbb{D}_\delta)$ which maps the $(2, p_1)$-cabling curve to the $(-2, -p_2)$-cabling curve along $\mathrm{pr}_2(s(t))$ for each $t \in S^1$ recovers $Y \cong L(3,1)$ (note the sign). We note that the gluing map is described as

$$\begin{pmatrix} a & -3 \\ c & d \end{pmatrix}$$

in standard longitude-meridian coordinates on the torus. Since the matrix send a $(2, p_1)$-cabling curve to a $(-2, -p_2)$-cabling curve, it follows from the first line of the matrix that $-2 = 2a - 3p_1$. Since p_1 is odd, this can not occur. Thus (7) is excluded. □

Lemma 3.4.13. *Any open book decomposition coming from (5) in Lemma 3.4.7 does not support* $(L(3,1), \xi_{\mathrm{std}})$.

Proof of Claim 3.4.13. As mentioned, for any section $s : \mathcal{M}_u^J/\mathbb{R} \to \mathcal{M}_u^J$, $\bigcup_{t\in S^1} \pi(s(t)))$ gives a rational open book decomposition of $L(3,1)$ supporting ξ_{std}. But it is well-known that the contact structure on $L(3,1)$ supported by an open book decomposition such that each page is an embedded annulus is overtwisted. This completes the proof. □

Lemma 3.4.14. *Suppose that α satisfies either (1) or (2) or (3) or (4) in Lemma 3.4.7. Then any simple orbit γ_i in α is in $\mathcal{S}_3$.*

Proof of Lemma 3.4.14. It follows from Theorem 3.1.11 that if α satisfies (2) in Lemma 3.4.7, γ in α is in $\mathcal{S}_3$.

Next, we consider (1), (3), (4) in Lemma 3.4.7.

We note that each monodromy of (rational) open book decomposition coming from (1), (3), (4) is unique up to isotopy. Indeed, in the case of (3) or (4), it follows from straightforward arguments since each page is annulus type. In the case of (1), it is complicated but follows from the classification of contact structures supported by planer open book decompositions with three boundaries given in [Ar]. Therefore, it suffices to check the statement specifically. For the purpose, we give the specific (rational) open book decompositions as Milnor fibrations.

Let $S^3 := \{(z_1, z_2) |\, |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ denote the unit sphere. Recall that $(S^3, \lambda_0|_{S^3})$ is a contact 3-sphere with tight contact structure whose Reeb vector field is given by the derivative of the action of multiplying by $e^{2\pi t}$. In this case, any flow is periodic and any simple periodic orbit is a Hopf fiber. In addition, it is obvious that any Hopf fiber is in $\mathcal{S}_1$.

Let $(L(3,1), \lambda_0|_{L(3,1)})$ denote the contact manifold obtained by taking the quotient of $(S^3, \lambda_0|_{S^3})$ under the action $(z_1, z_2) \to e^{\frac{2\pi i}{3}}(z_1, z_2)$. Under the action, any Hopf fiber is preserved. Therefore any flow on $(L(3,1), \lambda_0|_{L(3,1)})$ is periodic. In addition, any simple periodic orbit is tangent to a image of a Hopf fiber of $S^3 \to L(3,1)$ and thus obviously in $\mathcal{S}_3$.

At first, we describe (1) as a Milnor fibration. Consider a map $f : \mathbb{C}^2 \to S^1$ with $f(z_1, z_2) = z_1^3 + z_2^3$. It is easy to check that $g = f/|f| : S^3 \backslash f^{-1}(0) \to S^1$ gives a open book decomposition supporting the tight contact structure and the binding consists of periodic orbits of $(S^3, \lambda_0|_{S^3})$. In addition, for each $\theta \in S^1$, $g^{-1}(\theta)$ is connected and of genus 1 with three boundary components (cf. [M]). Since $f(e^{\frac{2\pi}{3}}z_1, e^{\frac{2\pi}{3}}z_2) = f(z_1, z_2)$, g induces, by taking a quotient space, an open book decomposition on $(L(3,1), \lambda_0|_{L(3,1)})$. It follows that each fiber of the open book decomposition is of genus 0 and three boundary components consisting of periodic orbits on $(L(3,1), \lambda_0|_{L(3,1)})$, which gives a description of (1). Therefore any simple orbit γ_i in α is in $\mathcal{S}_3$.

Second we describe (3) and (4). Consider a map $f : \mathbb{C}^2 \to S^1$ with $f(z_1, z_2) = z_1 z_2^2$. $g = f/|f| : S^3 \backslash f^{-1}(0) \to S^1$ gives a open book decomposition supporting the tight contact structure and the binding link consists of periodic orbits of $(S^3, \lambda_0|_{S^3})$ (see the remark below). In addition, since $f(e^{\frac{2\pi i}{3}}z_1, e^{\frac{2\pi i}{3}}z_2) = f(z_1, z_2)$, g induces, by taking a quotient space, an open book decomposition on $(L(3,1), \lambda_0|_{L(3,1)})$ whose boundary consists of periodic orbits on $(L(3,1), \lambda_0|_{L(3,1)})$. Moreover, for any $\theta \in S^1$, $g^{-1}(\theta)$ is of genus 0 with two boundary components and thus each page of the induced open book decomposition of $(L(3,1), \xi_{\mathrm{std}})$ is of genus 0 with two boundary components which are periodic orbits of $(L(3,1), \lambda_0|_{L(3,1)})$, which gives a description of (3) and (4). This means that if α satisfies (3) or (4) in Lemma 3.4.7, then any simple orbit γ_i in α is in $\mathcal{S}_3$. □

Remark 3.4.15. Consider a map $g = f/|f| : S^3 \to S^1$ with $f(z_1, z_2) = \frac{z_1 z_2^2}{|z_1 z_2^2|}$. Then for $\theta \in S^1$, the fiber $g^{-1}(\theta)$ is the set $\{(re^{2\pi i\theta_1}, \sqrt{1-r^2}e^{2\pi i\theta_1}) |\, \theta_1 + 2\theta_2 = \theta\}$.

Remark 3.4.16. Let $p_1 : M \backslash B_1 \to S^1$ and $p_2 : M \backslash B_2 \to S^1$ be rational open book decompositions supporting a contact manifold (M, ξ). Suppose that each page are diffeomorphic to each other and the monodromies are the same up to isotopy. Then the binding links B_1 and B_2 are the same as transversal links. Indeed, Let $B_i \subset U_i$ be a sufficiently small tubular neighborhood. Then we can construct a diffeomorphism $f : M \to M$ such that $f(B(1)) = B(2)$ and f maps each page of $p_1 : M \backslash U_1 \to S^1$ to $p_2 : M \backslash U_2 \to S^1$. It follows from standard arguments that we can construct an isotopy of contact structures from $f_*(\xi)$ to ξ so that the binding link $f(B(1)) = B(2)$ is preserved with transversal to the isotopic contact structures.

Proof of Proposition 3.4.1. Having Lemma 3.4.12, Lemma 3.4.13 and Lemma 3.4.14, it suffices to exclude (4) in Lemma 3.4.7. Suppose that α satisfies (4) in Lemma 3.4.14. Then there are two elliptic orbits γ_1 and γ_2 such that $\alpha = (\gamma_1, 2) \cup (\gamma_2, 1)$, $\mu_{\mathrm{disk}}(\gamma_1^6) = 5$ and $\mu_{\mathrm{disk}}(\gamma_2^3) = 5$. Note that $\mu_{\mathrm{disk}}(\gamma_1^6) = 5$ means $\mu_{\mathrm{disk}}(\gamma_1^3) = 3$

In addition, according to Lemma 3.4.14, we have $\gamma_2 \in \mathcal{S}_3$. Recall that $\gamma_2 \in \mathcal{S}_3$ means that there is a rational Seifert surface $u : \mathbb{D} \to L(3,1)$ with $u(e^{2\pi t}) = \gamma_2(3T_{\gamma_2} t)$ and $sl_\xi^{\mathbb{Q}}(\gamma_2) = -\frac{1}{3}$. Take a Martinet tube $F : \mathbb{R}/T_{\gamma_2}\mathbb{Z} \times \mathbb{D}_\delta \to \bar{U}$ for a sufficiently small $\delta > 0$ onto a small open neighbourhood $\gamma_2 \subset \bar{U}$. Let $\tau_F : \gamma_i^* \xi \to \mathbb{R}/T_{\gamma_2}\mathbb{Z} \times \mathbb{R}^2$ be a trivialization induced by F. We may choose F so that $\mu_{\tau_F}(\gamma_2) = 1$.

Note that $F^{-1}(u(\mathbb{D}) \cap \partial \bar{U})$ is a $(3, r)$ cable such that $r = 3k - 1$ for some $k \in \mathbb{Z}$ with respect to the coordinate of $\mathbb{R}/T_{\gamma_2}\mathbb{Z} \times \mathbb{D}$. It follows from Lemma 3.4.2 that

$$\mu_{\tau_F}(\gamma_2^3) - 2r - 6sl_\xi^{\mathbb{Q}}(\gamma_2) = \mu_{\mathrm{disk}}(\gamma_2^3)$$

In this case, it follows from $r = 3k - 1$ and $sl_\xi^{\mathbb{Q}}(\gamma_2) = -\frac{1}{3}$ that $\mu_{\tau_F}(\gamma_2^3) - 2r - 6sl_\xi^{\mathbb{Q}}(\gamma_2) = \mu_{\tau_F}(\gamma_2^3) + 4 - 6k$. Since γ_2 is elliptic, there is $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that $\mu_{\tau_F}(\gamma_2^m) = 2\lfloor m\theta \rfloor + 1$ for every $m \in \mathbb{Z}_{>0}$. Recall that we take τ_F so that $\mu_{\tau_F}(\gamma_2) = 1$. Hence $0 < \theta < 1$ and so we have $\mu_{\tau_F}(\gamma_2^3) = 1$ or 3 or 5.

Since $\mu_{\mathrm{disk}}(\gamma_2^3) = 5$. $\mu_{\tau_F}(\gamma_2^3) + 4 - 6k = \mu_{\mathrm{disk}}(\gamma^3)$ holds only if $k = 0$ and $\mu_{\tau_F}(\gamma_2^3) = 1$. Let $u \in \mathcal{M}^J(\alpha, \emptyset)$. From (3.53), it follows that

$$\begin{aligned}
3\mathrm{ind}(u) =& -3(2 - 2g(u) - 2) + 2c_\tau(\xi|_{3[u]}) + 3\mu_{\tau_0}(\gamma_2) + 3\mu_\tau(\gamma_1^2) \\
=& -6 + \mu_{\mathrm{disk}}(\gamma_1^3) + \mu_{\mathrm{disk}}(\gamma_2^6) \\
&+ (3\mu_{\tau_F}(\gamma_2) - \mu_{\tau_F}(\gamma_2^3) + 3) + (3\mu_\tau(\gamma_1^2) - \mu_\tau((\gamma_1^2)^3) + 3) \\
\geq& -6 + \mu_{\mathrm{disk}}(\gamma_1^3) + \mu_{\mathrm{disk}}(\gamma_2^6) + (3\mu_{\tau_0}(\gamma_2) - \mu_{\tau_F}(\gamma_2^3) + 3) + 1
\end{aligned}$$

Here τ is a trivialization on γ_1 and we use Lemma 3.4.4. Since $\mu_{\tau_F}(\gamma_2) = \mu_{\tau_F}(\gamma_2^3) = 1$, we have $3\mu_{\tau_F}(\gamma_2) - \mu_{\tau_F}(\gamma_2^3) + 3 = 5$. In addition since $\mu_{\text{disk}}(\gamma_1^3) = \mu_{\text{disk}}(\gamma_2^6) = 5$, we have $\mu_{\text{disk}}(\gamma_1^3) + \mu_{\text{disk}}(\gamma_2^6) = 10$. In summary we have

$$-6 + \mu_{\text{disk}}(\gamma_1^3) + \mu_{\text{disk}}(\gamma_2^6) + (3\mu_{\tau_F}(\gamma_2) - \mu_{\tau_F}(\gamma_2^3) + 3) + 1 = 10.$$

But $3\text{ind}(u) = 6$. This contradicts the inequality. This means that (4) in Lemma 3.4.7 is impossible. This completes the proof. □

3.4.3 Proof of the main theorem under non-degeneracy

What follows in this section is written in almost the same way with §3.3.2.

Let $\langle \alpha_1 + ... + \alpha_k \rangle$ denotes the element in $ECH(Y, \lambda)$ for a sum of ECH generators $\alpha_1 + ... + \alpha_k$ with $\partial_J(\alpha_1 + ... + \alpha_k) = 0$.

Lemma 3.4.17. *Let $(L(3,1), \lambda)$ be a dynamically convex non-degenerate contact manifold with $\lambda \wedge d\lambda > 0$. Let $\alpha_1, ..., \alpha_k$ be ECH generators with $[\alpha_i] = 0$ and $I(\alpha_i, \emptyset) = 2$ for $i = 1, ...k$. Suppose that $\partial_J(\alpha_1 + ... + \alpha_k) = 0$ and $0 \neq \langle \alpha_1 + ... + \alpha_k \rangle \in ECH_2((3,1), \lambda, 0)$. Then there exists i such that $\langle U_{J,z}\alpha_i, \emptyset \rangle \neq 0$.*

Proof of Lemma 3.4.17. The proof is the same with Lemma 3.3.13. □

Lemma 3.4.18. *Let $(L(p,1), \lambda)$ be a (not necessarily dynamically convex) non-degenerate contact manifold with $\lambda \wedge d\lambda > 0$. Let $\alpha_\gamma = (\gamma, p)$ for $\gamma \in \mathcal{S}_p$. If $\mu_{\text{disk}}(\gamma^p) = 3$, then α_γ is an ECH generator. In addition $I(\alpha_\gamma, \emptyset) = 2$.*

Proof of Lemma 3.4.18. According to Corollary 3.2.8, γ is elliptic and hence by definition $\alpha_\gamma = (\gamma, p)$ is an ECH generator.

Take a Martinet tube $F : \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{D}_\delta \to \bar{U}$ for a sufficiently small $\delta > 0$ onto a small open neighbourhood $\gamma \subset \bar{U}$.

Recall that $\gamma \in \mathcal{S}_p$ means that there is a rational Seifert surface $u : \mathbb{D} \to L(p,1)$ with $u(e^{2\pi t}) = \gamma(pT_{\gamma_2}t)$ and $sl_\xi^{\mathbb{Q}}(\gamma) = -\frac{1}{p}$.

We may take F so that $F^{-1}(u(\mathbb{D}) \cap \partial\bar{U})$ is $(p, p-1)$-cabling such that $r = 3k - 1$ for some $k \in \mathbb{Z}$ with respect to the coordinate of $\mathbb{R}/T_{\gamma_2}\mathbb{Z} \times \mathbb{D}_\delta$. Let $\tau_F : \gamma^*\xi \to \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}^2$ be the trivialization induced by F. Let $\theta \in$

$\mathbb{R}\backslash\mathbb{Q}$ denote the monodoromy angle of γ with respect to τ_F. In this case, $\mu_{\tau_F}(\gamma^k) = 2\lfloor k\theta \rfloor + 1$ for any $k \in \mathbb{Z}_{>0}$. It follows from Lemma 3.4.2 that

$$\mu_{\tau_F}(\gamma^p) + 4 - 2p = 2\lfloor p\theta \rfloor + 5 - 2p = \mu_{\text{disk}}(\gamma^p) = 3.$$

and hence

$$\lfloor p\theta \rfloor = p - 1.$$

This implies that $1 - \frac{1}{p} < \theta < 1$. In particular, this means that $\mu_{\tau_F}(\gamma^k) = 2(k-1) + 1$ for any $1 \leq k \leq p$. Therefore we have

$$\sum_{1 \leq k \leq p} \mu_{\tau_F}(\gamma^k) = p^2.$$

To compute the relative self intersection number Q_{τ_0}, take another Seifert surface $u' : \mathbb{D} \to Y$ of γ so that $u(\mathring{\mathbb{D}}) \cap u'(\mathring{\mathbb{D}}) = \emptyset$. This is possible because $u(\mathbb{D})$ is a page of a rational open book decomposition. Now, we take immersed $S_1, S_2 \subset [0,1] \times Y$ which are transverse to $\{0,1\} \times Y$ so that $\mathrm{pr}_2(S_1) = u(\mathbb{D})$ and $\mathrm{pr}_2(S_2) = u'(\mathbb{D})$ where $\mathrm{pr}_2 : [0,1] \times Y \to Y$ is the projection. Set $\{Z\} := H_2(Y, \alpha_\gamma, \emptyset)$. It follows from the definition that we have $Q_{\tau_F}(Z) = -l_{\tau_F}(S_1, S_2) + \#(S_1 \cap S_2)$ where $l_{\tau_F}(S_1, S_2)$ is the linking number. More precisely, $l_{\tau_F}(S_1, S_2)$ is one half the signed number of crossings of $F^{-1}(\mathrm{pr}_2(S_1 \cap \{1-\epsilon\} \times Y))$ with $F^{-1}(\mathrm{pr}_2(S_2 \cap \{1-\epsilon\} \times Y))$ for a small $\epsilon > 0$ in the projection $(\mathrm{id}, \mathrm{pr}_1) : \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}$ where we naturally assume that $\mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{D}_\delta \subset \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}^2$ and $\mathrm{pr}_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection to the first coordinate. For more details, see [H1].

In this situation, it follows from the choice of S_1 and S_2 that $\#(S_1 \cap S_2) = 0$. In addition, since we take F so that $F^{-1}(u(\mathbb{D}) \cap \partial \bar{U})$ is a $(p, p-1)$ cabling, we have $l_{\tau_F}(S_1, S_2) = p(p-1)$. In summary, we have $Q_{\tau_F}(Z) = -p(p-1)$.

At last, we compute the relative first chern number c_{τ_F}. Since $\tau_{\text{disk}} : (\gamma^p)^*\xi \to \mathbb{R}/pT_\gamma\mathbb{Z} \times \mathbb{R}^2$ extends to a trivialization by definition, we have $c_{\tau_F} = \mathrm{wind}(\tau_F, \tau_{\text{disk}})$. It follows from (3.49) that $\mathrm{wind}(\tau_F, \tau_{\text{disk}}) = 2 - p$ and hence $c_{\tau_F} = \mathrm{wind}(\tau_F, \tau_{\text{disk}}) = 2 - p$. By summarizing above results, we have $I(\alpha_\gamma, \emptyset) = (2-p) + (-p(p-1)) + p^2 = 2$. This completes the proof. □

Lemma 3.4.19. *Let $(L(p,1), \lambda)$ be a non-degenerate dynamically convex contact manifold with $\lambda \wedge d\lambda > 0$. Let $\alpha_\gamma = (\gamma, p)$ for $\gamma \in \mathcal{S}_p$. If $\mu_{\text{disk}}(\gamma^p) = 3$, then there is no somewhere injective J-holomorphic curve satisfying the following;*

(1). There is only one positive end. In addition, the positive end is asymptotic to γ with multiplicity p.

(2). There is at least one negative end.

(3). Any puncture on the domain is either positive or negative end.

Proof of Lemma 3.4.19. In this proof, we set $Y = L(p,1)$. Suppose that $h : (\Sigma, j) \to (\mathbb{R} \times Y, J)$ is a somewhere injective curve satisfying the properties.

Recall that $\gamma \in \mathcal{S}_p$ means that there is a rational Seifert surface $u : \mathbb{D} \to L(3,1)$ with $u(e^{2\pi t}) = \gamma(pT_{\gamma_2}t)$ and $sl_{\xi}^{\mathbb{Q}}(\gamma) = -\frac{1}{p}$ such that $u(\mathbb{D})$ is a Birkhoff section for X_λ of disk type. We note that $H_1(Y \backslash \gamma) \cong \mathbb{Z}$.

For a sufficiently large $s >> 0$, consider $\pi(h(\Sigma) \cap ([-s,s] \times Y)) \subset Y$. Since $\#(\gamma \cap \pi(h(\Sigma) \cap ([-s,s] \times Y))) \geq 0$ because of positivity of intersection, it follows topologically that $\#(u(\mathbb{D}) \cap \pi(h(\Sigma) \cap (\{s\} \times Y))) \geq \#(u(\mathbb{D}) \cap \pi(h(\Sigma) \cap (\{-s\} \times Y)))$. This contradicts the next claim.

Claim 3.4.20.

(1). $0 \geq \#(u(\mathbb{D}) \cap \pi(h(\Sigma) \cap (\{s\} \times Y)))$

(2). $\#(u(\mathbb{D}) \cap \pi(h(\Sigma) \cap (\{-s\} \times Y))) \geq 1$

Proof of Claim 3.4.20. Take a Martinet tube $F : \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{D}_\delta \to \bar{U}$ for a sufficiently small $\delta > 0$ onto a small open neighbourhood $\gamma \subset \bar{U}$.

We may take F so that $F^{-1}(u(\mathbb{D}) \cap \partial\bar{U})$ is a $(p, p-1)$ cable. Let $\tau_F : \gamma^*\xi \to \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{R}^2$ be a trivialization induced by F. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ denote the monodromy angle of γ with respect to τ_0. That is $\mu_{\tau_0}(\gamma^k) = 2\lfloor k\theta \rfloor + 1$ for any $k \in \mathbb{Z}_{>0}$. It follows from the same argument that $1 - \frac{1}{p} < \theta < 1$. In particular, this means that $\lfloor k\theta \rfloor = k-1$ for any $1 \leq k \leq p$.

At first, we consider $F^{-1}(\pi(h(\Sigma) \cap (\{s\} \times Y)) \cap \partial\bar{U})$. Let $\mathrm{pr}_{\mathbb{D}_\delta} : \mathbb{R}/T_\gamma\mathbb{Z} \times \mathbb{D}_\delta \to \mathbb{D}_\delta$ denote the projection. Then it follows that the winding number of $\mathrm{pr}_{\mathbb{D}_\delta} \circ F^{-1}(\pi(h(\Sigma) \cap (\{s\} \times Y)) \cap \partial\bar{U}) \subset \mathbb{D}_\delta$ is at most $\lfloor \frac{\mu_{\tau_0}(\gamma^p)}{2} \rfloor = p-1$. This means that the slope of $F^{-1}(\pi(h(\Sigma) \cap (\{s\} \times Y)) \cap \partial\bar{U})$ is at most $\frac{p-1}{p}$. Since the multiplicity of the positive end is p, it follows that $0 \geq \#(u(\mathbb{D}) \cap \pi(h(\Sigma) \cap (\{s\} \times Y)))$. This proves (1).

Next, we consider $F^{-1}(\pi(h(\Sigma) \cap (\{-s\} \times Y)) \cap \partial\bar{U})$. We note that the total multiplicity of negative ends asymptotic to γ is at most $p-1$ because of the Stokes' theorem. Suppose that a negative end of h is asymptotic to γ with multiplicity $1 \leq k \leq p-1$. Then it follows that the winding number of $\mathrm{pr}_{\mathbb{D}_\delta} \circ F^{-1}(\pi(h(\Sigma) \cap (\{s\} \times Y)) \cap \partial\bar{U}) \subset \mathbb{D}_\delta$ is at least $\lceil \frac{\mu_{\tau_0}(\gamma^k)}{2} \rceil = k$. This means that the slope of $F^{-1}(\pi(h(\Sigma)\cap(\{s\}\times Y))\cap\partial\bar{U})$ is at least 1. On the other hand, $F^{-1}(u(\mathbb{D})\cap\partial\bar{U})$ is a $(p,p-1)$ cable and hence the slope is $\frac{p-1}{p}$. This means that the negative end intersects $u(\mathbb{D})$ positively. Therefore, if there is a negative end asymptotic to γ, we have $\#(u(\mathbb{D})\cap\pi(h(\Sigma)\cap(\{-s\}\times Y)))\geq 1$. Finally suppose that there is no negative end asymptotic to γ. Since $u(\mathbb{D})$ is a Birkhoff section for X_λ, any periodic orbit other than γ intersects $u(\mathbb{D})$ positively. Therefore it follows from the assumption that $\#(u(\mathbb{D})\cap\pi(h(\Sigma)\cap(\{-s\}\times Y)))\geq 1$. This completes the proof. □

Having Claim 3.4.20, Lemma 3.4.19 follows. □

Lemma 3.4.21. *For any $\gamma\in\mathcal{S}_p$ with $\mu_{\mathrm{disk}}(\gamma^p)=3$, $\partial_J\alpha_\gamma=0$.*

Proof of Lemma 3.4.21. Let β be an ECH generator with $I(\alpha_\gamma,\beta)$. Let $h\in\mathcal{M}^J(\alpha_\gamma,\beta)$. It follows from the partition condition that the number of positive ends of h is one and the positive end is asymptotic to γ with multiplicity p. According to Lemma 3.4.19, such a h does not exist. This means that $\mathcal{M}^J(\alpha_\gamma,\beta)=\emptyset$. Hence we have $\partial_J\alpha_\gamma=0$. □

Now, we focus on $Y\cong L(3,1)$. Since $\partial_J\alpha_\gamma=0$ for any $\gamma\in\mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3)=3$, we may consider α_γ as an element in $ECH_2(Y,\lambda,0)$. The following lemma means that α_γ is not zero in $ECH_2(Y,\lambda,0)$.

Lemma 3.4.22. *For any $\gamma\in\mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3)=3$, $0\neq\langle\alpha_\gamma\rangle=\langle(\gamma,3)\rangle\in ECH_2(Y,\lambda,0)$.*

To prove Lemma 3.4.22, define a set $\mathcal{G}$ consisting of ECH generators as

$$\mathcal{G}:=\{\alpha|\ \langle U_{J,z}\alpha,\emptyset\rangle\neq 0\ \}. \tag{3.61}$$

Note that $\langle U_{J,z}\alpha,\emptyset\rangle\neq 0$ if and only if $\alpha\in\mathcal{G}$.

Claim 3.4.23. *For any $\gamma\in\mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3)=3$, $\alpha_\gamma\in\mathcal{G}$.*

Proof of Claim 3.4.23. Recall that each page of the rational open book decomposition constructed in Theorem 3.1.11([HrS2]) is the projection of J-holomorphic curve from $(\mathbb{C},i)$ to $L(3,1)$. Moreover in this case, : $\mathcal{M}^J(\alpha_\gamma,\emptyset)/\mathbb{R}$ is compact and any two distinct elements $u_1,u_2\in\mathcal{M}^J(\alpha_\gamma,\emptyset)$ has no intersection point. Hence : $\mathcal{M}^J(\alpha_\gamma,\emptyset)/\mathbb{R}\cong S^1$ and for a section $s:\mathcal{M}^J(\alpha_\gamma,\emptyset)/\mathbb{R}\to\mathcal{M}^J(\alpha_\gamma,\emptyset)$, $\bigcup_{\tau\in\mathcal{M}^J(\alpha_\gamma,\emptyset)/\mathbb{R}}\overline{\pi(s(\tau))}\to\mathcal{M}^J(\alpha_\gamma,\emptyset)/\mathbb{R}$ is an (rational) open book decomposition of $L(2,1)$. This implies that for $z\in L(3,1)$ not on γ, there is exactly one J-holomorphic curve in $\mathcal{M}^J(\alpha_\gamma,\emptyset)$ through $(0,z)\in\mathbb{R}\times L(3,1)$. Therefore we have $\langle U_{J,z}\alpha_\gamma,\emptyset\rangle\neq 0$. □

Claim 3.4.24. *Let $\alpha \in \mathcal{G}$. Suppose that β is an ECH generator with $I(\beta, \alpha) = 1$. Then*

$$\sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha \rangle = 0 \tag{3.62}$$

Proof of Claim 3.4.24. This proof is completely the same with Lemma 3.3.18. We give it below.

Write

$$\partial_J \beta = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha \rangle \alpha + \sum_{I(\beta,\sigma)=1, \sigma \notin \mathcal{G}} \langle \partial_J \beta, \sigma \rangle \sigma. \tag{3.63}$$

Then we have

$$\langle U_{J,z} \partial_J \beta, \emptyset \rangle = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha \rangle \langle U_{J,z} \alpha, \emptyset \rangle + \sum_{I(\beta,\sigma)=1, \sigma \notin \mathcal{G}} \langle \partial_J \beta, \sigma \rangle \langle \sigma, \emptyset \rangle = \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta, \alpha \rangle \tag{3.64}$$

Here we use that for $\alpha \in \mathcal{G}$, $\langle U_{J,z} \alpha, \emptyset \rangle = 1$ and for σ with $\sigma \notin \mathcal{G}$, $\langle U_{J,z} \sigma, \emptyset \rangle = 0$. Since $U_{J,z} \partial_J = \partial_J U_{J,z}$, we have $\langle U_{J,z} \partial_J \beta, \emptyset \rangle = \langle \partial_J U_{J,z} \beta, \emptyset \rangle = 0$. This completes the proof. □

Proof of Lemma 3.4.22. Suppose that $0 = \langle \alpha_\gamma \rangle \in ECH_2(Y, \lambda, 0)$. Then there are ECH generators $\beta_1, ... \beta_j$ with $I(\beta_i, \alpha_\gamma) = 1$ for any i such that $\partial_J(\beta_1 + ... + \beta_j) = \alpha_\gamma$. From Lemma 3.4.24, we have

$$\sum_{1 \leq i \leq j} \sum_{\alpha \in \mathcal{G}} \langle \partial_J \beta_i, \alpha \rangle = \sum_{\alpha \in \mathcal{G}} \langle \alpha_\gamma, \alpha \rangle = 0. \tag{3.65}$$

But since $\alpha_\gamma \in \mathcal{G}$, $\sum_{\alpha \in \mathcal{G}} \langle \alpha_\gamma, \alpha \rangle = 1$. This is a contradiction. We complete the proof. □

Proof of Theorem 3.2.11 under non-degeneracy. At first, we estimate $c_1^{\mathrm{Ech}}(L(3,1)\lambda)$ from below. It follows from the definitin of ECH spectrum that we can take an ECH generator α such that $\langle U_{J,z} \alpha, \emptyset \rangle \neq 0$ and $A(\alpha) \leq c_1^{\mathrm{ECH}}(L(3,1), \lambda)$. According to Proposition 3.4.1, α contains $\gamma \in \mathcal{S}_3$ such that $\mu_{\mathrm{disk}}(\gamma^3) = 3$. We may assume that γ has the minimum period in α. Then $\int_\gamma \lambda \leq \frac{1}{3} A(\alpha) \leq \frac{1}{3} c_1^{\mathrm{ECH}}(L(3,1), \lambda)$. This means that there is a $\gamma \in \mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3) = 3$ such that $\int_\gamma \lambda \leq \frac{1}{3} A(\alpha) \leq \frac{1}{3} c_1^{\mathrm{ECH}}(L(3,1), \lambda)$.

At last, we estimate $c_1^{\mathrm{ECH}}(L(3,1)\lambda)$ from above. Since $0 \neq \langle \alpha_\gamma \rangle = \langle (\gamma, 3) \rangle \in ECH_2(Y, \lambda, 0)$ for any $\gamma \in \mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3) = 3$ (Lemma 3.4.22), we have $c_1^{\mathrm{EcH}}(L(3,1)\lambda) \leq A(\alpha_\gamma)$ for any $\gamma \in \mathcal{S}_3$ with $\mu_{\mathrm{disk}}(\gamma^3) = 3$. This means that $\frac{1}{3} c_1^{\mathrm{ECH}}(L(3,1)\lambda) \leq \inf_{\gamma \in \mathcal{S}_3,\, \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda$.

In summary, we have $\frac{1}{3}c_1^{\mathrm{ECH}}(L(3,1)\lambda) = \inf_{\gamma \in \mathcal{S}_3,\, \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda$. $\square$

3.4.4 Extend the results to degenerate cases

In this subsection, we extend the above result to degenerate case as a limit of non-degenerate result. The content in this section is completely the same with §3.3.3. However we provide the details as follows.

At first, we show;

Proposition 3.4.25. *Assume that $(L(3,1),\lambda)$ is strictly convex. Then there exists a simple orbit $\gamma \in \mathcal{S}_3$ such that $\mu_{\mathrm{disk}}(\gamma^3) = 3$ and $\int_\gamma \lambda = \frac{1}{3}\, c_1^{\mathrm{ECH}}(L(3,1),\lambda)$. In particular,*

$$\inf_{\gamma \in \mathcal{S}_3, \mu_{\mathrm{disk}}(\gamma^3)=1} \int_\gamma \lambda \leq \frac{1}{3}\, c_1^{\mathrm{ECH}}(L(3,1),\lambda). \tag{3.66}$$

Proof of Proposition 3.4.25. Let $L = c_1^{\mathrm{ECH}}(L(3,1),\lambda)$. Take a sequence of strictly convex contact forms λ_n such that $\lambda_n \to \lambda$ in C^∞-topology and λ_n is non-degenerate for each n. Therefore we have

$$\inf_{\gamma \in \mathcal{S}_3, \mu_{\mathrm{disk}}(\gamma^3)=3} \int_\gamma \lambda_n = \frac{1}{3}\, c_1^{\mathrm{ECH}}(L(3,1),\lambda_n) \tag{3.67}$$

Note that $c_1^{\mathrm{ECH}}(L(3,1),\lambda_n) \to L$ as $n \to +\infty$. This means that there is a sequence of $\gamma_n \in \mathcal{S}_3(L(3,1), f_n\lambda)$ with $\mu_{\mathrm{disk}}(\gamma_n^3) = 3$ such that $\int_{\gamma_n} \lambda_n \to \frac{1}{3}L$. By Arzelà–Ascoli theorem, we can find a subsequence which converges to a periodic orbit γ of λ in C^∞-topology.

Claim 3.4.26. *γ is simple. In particular, $\gamma \in \mathcal{S}_3(L(3,1),\lambda)$ and $\mu_{\mathrm{disk}}(\gamma^3) = 3$.*

Proof of Claim 3.4.26. By the argument so far, there is a sequence of $\gamma_n \in \mathcal{S}_3(L(3,1),\lambda_n)$ with $\mu_{\mathrm{disk}}(\gamma_n^3) = 3$ which converges to γ in C^∞. Suppose that γ is not simple, that is, there is a simple orbit γ' and $k \in \mathbb{Z}_{>0}$ with $\gamma'^k = \gamma$. From the lower semi-continuity of μ, we have $\mu_{\mathrm{disk}}(\gamma_n^3) \to \mu_{\mathrm{disk}}(\gamma'^{3k}) = \mu_{\mathrm{disk}}((\gamma'^3)^k) = 3$. This contradicts Proposition 1.2.4. Therefore γ is simple. This means that for sufficiently large n, γ_n is transversally isotopic to γ. Therefore, γ is 3-unknotted and has self-linking number $-\frac{1}{3}$.

At last, we prove $\mu_{\mathrm{disk}}(\gamma^3) = 3$. From the lower semi-continuity of μ, we have $\mu_{\mathrm{disk}}(\gamma_n^3) \to \mu_{\mathrm{disk}}(\gamma^3) = 3$ or 2. $\mu_{\mathrm{disk}}(\gamma^3) = 2$ contradicts the assumption of dynamical convexity. Thus we have $\mu_{\mathrm{disk}}(\gamma^3) = 3$. We complete the proof. $\square$

As discussion so far, there is a sequence of $\gamma_n \in \mathcal{S}_3(L(3,1), f_n\lambda)$ with $\mu_{\text{disk}}(\gamma_n^3) = 3$ and $\gamma \in \mathcal{S}_3(L(3,1),\lambda)$ with $\mu_{\text{disk}}(\gamma^3) = 3$ such that $\int_{\gamma_n} f_n\lambda \to \frac{1}{3}L$ and γ_n converges to γ of λ in C^∞-topology. Therefore we have $\int_\gamma \lambda = \frac{1}{3}\, c_1^{\text{ECH}}(L(3,1),\lambda)$ in C^∞-topology. we complete the proof of Proposition 3.4.25. □

Having Proposition 3.4.25, to complete the proof of Theorem 3.2.7, it is sufficient to show the next proposition.

Proposition 3.4.27. *Assume that* $(L(3,1),\lambda)$ *is strictly convex. Then*

$$\frac{1}{3}\, c_1^{\text{ECH}}(L(3,1),\lambda) \leq \inf_{\gamma\in\mathcal{S}_3,\mu_{\text{disk}}(\gamma^3)=3} \int_\gamma \lambda. \tag{3.68}$$

Proof of Proposition 3.4.27. We prove this by contradiction. Suppose that there exists $\gamma_\lambda \in \mathcal{S}_3(L(3,1),\lambda)$ with $\mu_{\text{disk}}(\gamma_\lambda^3) = 3$ such that $\frac{1}{3}\, c_1^{\text{ECH}}(L(3,1),\lambda) > \int_{\gamma_\lambda} \lambda$.

Lemma 3.4.28. *There exists a sequence of smooth functions* $f_n : L(3,1) \to \mathbb{R}_{>0}$ *such that* $f_n \to 1$ *in* C^∞*-topology and satisfying* $f_n|_{\gamma_\lambda} = 1$ *and* $df_n|_{\gamma_\lambda} = 0$. *Moreover, all periodic orbits of* $X_{f_n\lambda}$ *of periods* $< n$ *are non-degenerate and all contractible orbits of periods* $< n$ *have Conley-Zehnder index* ≥ 3. *In addition,* γ_λ *is a non-degenerate periodic orbit of* $X_{f_n\lambda}$ *with* $\mu_{\text{disk}}(\gamma_\lambda^3) = 3$ *for every* n.

Proof of Lemma 3.4.28. See [HWZ4, Lemma 6.8, 6.9] □

For a sequence of smooth functions $f_n : L(3,1) \to \mathbb{R}_{>0}$ in Lemma 3.4.28, fix $N >> 0$ sufficient large so that $c_1^{\text{ECH}}(L(3,1), f_N\lambda) > \int_{\gamma_\lambda} \lambda$ and $N > 2c_1^{\text{ECH}}(L(3,1), f_N\lambda)$. We may take such f_N because c_1^{ECH} is continuous in C^0-topology.

Lemma 3.4.29. *Let* $f : L(3,1) \to \mathbb{R}_{>0}$ *be a smooth function such that* $f(x) < f_N(x)$ *for any* $x \in L(3,1)$. *Suppose that* $f\lambda$ *is non-degenerate dynamically convex. Then there exists a simple periodic orbit* $\gamma \in \mathcal{S}_3(L(3,1), f\lambda)$ *with* $\mu(\gamma) = 1$ *such that* $\int_\gamma f\lambda < \int_{\gamma_\lambda} \lambda$.

Outline of the proof of Lemma 3.4.29. See [Sch, Proposition 4.1]. In the proof and statement of [Sch, Proposition 4.1], ellipsoids are used instead of $(L(3,1), f_N\lambda)$, but the important point in the proof is to find 3-unknotted self-linking number $-\frac{1}{3}$ orbit γ with $\mu_{\text{disk}}(\gamma^3) = 3$ and construct a suitable J-holomorphic curve from [Sch, Proposition 4.1]. Now, we have $\gamma_\lambda \in \mathcal{S}_3(L(3,1), f_N\lambda)$ with $\mu_{\text{disk}}(\gamma_\lambda^3) = 3$ and hence by applying [Sch, Proposition 4.1], we can construct a suitable J-holomorphic curve. By using this curves instead of ones in the original proof, we can show Proposition 3.4.29. Here we note that the discussion in the proof of Lemma 3.4.19 is needed to prove the same result of [Sch]. □

Now, we would complete the proof of Proposition. Let $f : L(3,1) \to \mathbb{R}_{>0}$ be a smooth function such that $f(x) < f_N(x)$ for any $x \in L(3,1)$, $f\lambda$ be non-degenerate strictly convex and $\int_{\gamma_\lambda} \lambda < \frac{1}{3}c_1^{\text{ECH}}(L(3,1), f\lambda) < \frac{1}{3}c_1^{\text{ECH}}(L(3,1), f_N\lambda)$. We can check easily that it is possible to take such f. Due to Lemma 3.4.29, there exists a simple periodic orbit $\gamma \in \mathcal{S}_3(L(3,1), f\lambda)$ with $\mu_{\text{disk}}(\gamma^3) = 3$ such that $\int_\gamma f\lambda < \int_{\gamma_\lambda} \lambda$. Since $\inf_{\gamma \in \mathcal{S}_3, \mu_{\text{disk}}(\gamma^3)=3} \int_\gamma f\lambda = \frac{1}{3}\, c_1^{\text{ECH}}(L(3,1), f\lambda)$, we have $\int_{\gamma_\lambda} \lambda < \frac{1}{3}c_1^{\text{ECH}}(L(3,1), f\lambda) \leq \int_\gamma f\lambda$. This is a contradiction. We complete the proof. □

3.5 Proof of Theorem 3.2.13 and Proposition 3.2.19

3.5.1 Proof of Theorem 3.2.13

Here, we prove Theorem 3.2.13. For the purpose, we start in general situations.

Let Σ be a surface. We denote the set of fixed point of f by $\text{Fix}(f)$, the set $\bigcup_{n:\text{odd}} \text{Fix}(f^n)$ of periodic points with odd period by $\text{Per}^{\text{odd}}(f)$ and the set of positive hyperbolic periodic points by $\text{Per}_{h+}(f)$. For any isolated fixed point p of f, let $\text{ind}(p, f)$ be the fixed point index of f. Notice that the fixed point index of a fixed point in the boundary is defined by the fixed point index for then extension $\tilde{f}$ of f to an open manifold $\tilde{\Sigma}$ such that $\tilde{f}(\tilde{\Sigma}) = \Sigma$. When the diffeomorphism is non-degenerate, any periodic points at the boundary are 'positive half-saddles', whose fixed point index is 0 (contracting along the boundary circle) or -1 (expanding along the boundary circle).

Fix an area preserving diffeomorphism f which is non-degenerate and is isotopic to the identity map. The following lemmas reduce Theorem 3.2.13 to finding infinitely many periodic points with odd period of f or f^2.

Lemma 3.5.1. *Let f be a non-degenerate diffeomorphism of compact surface Σ which is area-preserving on the interior and isotopic to the identity* *map. In addition, we assume that f is area preserving on the interior with respect to a volume form defined on the interior. If* $\mathrm{Per}^{odd}(f)$ *is infinite, then* $\mathrm{Per}^{odd}(f) \cap \mathrm{Per}_{h+}(f)$ *is infinite.*

Proof of Lemma 3.5.1. Suppose that f admits infinite number of periodic points with odd period but only finite number of them are positive hyperbolic. Let K be the number of positive hyperbolic periodic points of f and periodic points in the boundary of Σ with odd period. Put $\Lambda_n = \mathrm{Fix}(f^n) \cap (\mathrm{Per}_{h+}(f) \cup \partial\Sigma)$. Then, we have

$$\sum_{p \in \Lambda_n} \mathrm{ind}(p, f^n) \geq -K$$

for any odd n. Put $L = \sum_{i \geq 0} (-1)^i \dim H_i(S)$. Since f is isotopic to the identity map, the Lefschetz number of f^n equals to L for any $n \geq 1$. There are infinitely many periodic points with odd period which are not positive hyperbolic and the boundary of Σ contains only finitely many periodic points. Hence, we can take periodic points $p_1, \ldots, p_{K+L+1}$ in $\mathrm{Per}^{\mathrm{odd}}(f) \setminus (\mathrm{Per}_{h+}(f) \cup \partial\Sigma)$. Let N be the product of the periods of $p_1, \ldots, p_{K+L+1}$ Then N is odd and any point $p \in \mathrm{Fix}(f^N) \setminus \Lambda_n$ satisfies $\mathrm{ind}(p, f^N) = 1$ We have

$$\begin{aligned}
\sum_{p \in \mathrm{Fix}(f^N)} \mathrm{ind}(p, f^N) &= \sum_{p \in \mathrm{Fix}(f^N) \setminus \Lambda_n} \mathrm{ind}(p, f^N) + \sum_{p \in \Lambda_n} \mathrm{ind}(p, f^N) \\
&\geq \sum_{i=1}^{K+L+1} \mathrm{ind}(p_j, f^N) + \sum_{p \in \Lambda_n} \mathrm{ind}(p, f^N) \\
&\geq K + L + 1 - K \geq L + 1.
\end{aligned}$$

This contradicts to the Lefschetz fixed point theorem since the Lefschetz number of f^N equals to L. □

Lemma 3.5.2. *Let f be a non-degenerate diffeomorphism of compact surface Σ which is area-preserving on the interior and isotopic to the identity map. Suppose that* $\mathrm{Per}^{odd}(f^2)$ *is infinite then* $\mathrm{Per}_{h+}(f)$ *is infinite.*

Proof of Lemma 3.5.2. If $\mathrm{Per}^{\mathrm{odd}}(f)$ is infinite, then $\mathrm{Per}_{h+}(f)$ is infinite by Lemma 3.5.1 again. Suppose that $\mathrm{Per}^{\mathrm{odd}}(f)$ is finite. By Lemma 3.5.1, the set $\mathrm{Per}_{h+}(f^2) \cap \mathrm{Per}^{\mathrm{odd}}(f^2)$ is infinite. This implies that $\mathrm{Per}_{h+}(f^2) \cap (\mathrm{Per}^{\mathrm{odd}}(f^2) \setminus \mathrm{Per}^{\mathrm{odd}}(f))$ is infinite. The period of any point in $\mathrm{Per}_{h+}(f^2) \cap (\mathrm{Per}^{\mathrm{odd}}(f^2) \setminus \mathrm{Per}^{\mathrm{odd}}(f))$ is twice of an odd number, and hence, such a point is positively hyperbolic. Hence, $\mathrm{Per}_{h+}(f)$ is infinite. □

Let A be the annulus $A = S^1 \times [0,1]$ and $\pi : \mathbb{R} \times [0,1] \to A$ the universal covering. For a homeomorphism $\tilde{f}$ of $\mathbb{R} \times [0,1]$ and $\tilde{x} \in \times[0,1]$, we define t*the translation number* $\tau(\tilde{x})$ by

$$\tau(\tilde{x}) = \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{x})_1 - \tilde{x}_1}{n}$$

if the limit exists, where $\tilde{f}^n(\tilde{x})_1$ and $\tilde{x}_1$ are the first coordinates of $\tilde{f}^n(\tilde{x})$ and $\tilde{x}$. For a homeomorphism f of A and $x \in A$, take lifts $\tilde{f}$ of f and $\tilde{x}$ of x to $\mathbb{R} \times [0,1]$. Then, the translation number $\tau(\tilde{x})$ modulo $\mathbb{Z}$ does not depend on the choice of lift if it exists. We define *the rotation number* $\rho(x)$ by $\rho(x) = \tau(\tilde{x}) + \mathbb{Z}$. To finding infinitely many periodic points with odd period, we use the following fixed point theorem by Franks.

Theorem 3.5.3. *[Fr1, Cororraly 2.4][Fr2, Theorem 2.1] Let f be a homeomorphism of A which is isotopic to the identity map such that any point of A is chain recurrent. Suppose that a lift of f to $\mathbb{R} \times [0,1]$ admits points $\tilde{x}, \tilde{y} \in \mathbb{R} \times [0,1]$ such that the translation numbers $\tau(\tilde{x}), \tau(\tilde{y})$ exists and $\tau(\tilde{x}) < \tau(\tilde{y})$. Then for any pair $(m,n)$of co-prime integers with $n \geq 1$ and $\tau(\tilde{x}) < m/n < \tilde{y}$, there exists $\tilde{x}_{m/n} \in \mathbb{R} \times [0,1]$ such that $\tilde{f}^n(\tilde{x}_{m/n}) = T^m(\tilde{x}_{m/n})$, where $T : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ is the translation given by$T(x,y) = (x+1,y)$, In particular, $\pi(\tilde{x}_{m/n})$ is a periodic point of f whose period is n.*

Corollary 3.5.4. *Let f be a homeomorphism of A which is isotopic to the identity map such that any point of A is chain recurrent. If there exists $x, y \in A$ such that $\rho(x) \neq \rho(y)$ then, f has infinitely many periodic points of odd period.*

Remark 3.5.5. See for the definition of chain recurrence [Fr1]. Note that for a diffeomorphism f on $\mathbb{D}$ in Theorem 3.2.13, any point in $\mathbb{D}$ is chain recurrent. This follows immediately from the Poincare recurrence theorem.

Now, we prove Theorem 3.2.13. Let f be an area preserving diffeomorphism of $\mathbb{D}$ on the interior which is non-degenerate and is orientation preserving. We show that f or f^2 has infinitely many periodic points with odd period. Then, Lemmas 3.5.1 and 3.5.2 imply that f admits infinitely many positive hyperbolic periodic points.

Recall that the fixed point index of any possible fixed point on the boundday of $\mathbb{D}$ is 0 or -1. By the Lefschetz fixed point theorem, f admits a fixed point p_* in the interior of $\mathbb{D}$ with $\mathrm{ind}(p_*, f) = 1$. Take the blow-up annulus A_{p_*}at p_* and lift the diffeomorphism f to a diffeomorphism $\hat{f}$ on A_{p_*}. Let

ρ_D be the rotation number of $\hat{f}$ along the boundary component of A_{p_*} which corresponds to the boundary of $\mathbb{D}$ and ρ_{p_*} the rotation number of $\hat{f}$ along the boundary component of A_{p_*} which corresponds to p_*. Since the fixed point index of p_* is one, p_* is either negative hyperbolic or elliptic. We have $\rho_{p_*} = 1/2$ in the former case and ρ_{p_*} is irrational in the latter case.

The easiest case is that $\rho_D \neq \rho_{p_*}$. In this case, Corollary 3.5.4 implies that $\hat{f}$, and hence, f has infinitely many periodic points of odd period.

The second case is that $\rho_D = \rho_{p_*}$ and they are irrational. By the assumption, f has at least two periodic points. Hence, there exists a periodic point q_* of f different from p_*. In the blow-up annulus A_{p_*}, the periodic point q_* has rational rotation number for $\hat{f}$. Since $\rho_D = \rho_{p_*}$ is irrational, we can apply Corollary 3.5.4 and obtain infinitely many periodic points of odd period.

The last case is that $\rho_D = \rho_{p_*} = 1/2$. In this case, p_* is negative hyperbolic. If f has a fixed point q_* different from p_*, then the lift to the blow up annulus at p_* has a fixed point q_*, whose rotation number is zero by definition, and the boundary components whose rotation number is $1/2$. By Corollary 3.5.4, $\hat{f}$, and hence, f has infinitely many periodic points of odd period. Suppose that f has no fixed point other than p_*. Since p_* is a positive hyperbolic fixed point of f^2, we have $\mathrm{ind}(p_*, f^2) = -1$. Recall that the fixed point index of any fixed point in the boundary is non-positive. By the Lefschetz fixed point theorem, f must have a 2-periodic point r_* with $\mathrm{ind}(r_*, f^2) = 1$. The rotation number of f^2 along the boundary is 0 and the rotation number of the blow up of f^2 at r_* is $1/2$ or irrational. Therefore, f^2 has infinitely many periodic points with odd period by Corollary 3.5.4. Now, Lemma 3.5.2 completes the proof.

3.5.2 Proof of Proposition 3.2.19

In the assumptions, we may replace the given Birkhoff section of disk type to one coming from a J-holomrphic plane as follows. Let γ be the simple orbit to which the given Birkhoff section is tangent. Note that by the assumption, any periodic orbit in $L(p,q))\backslash\gamma$ is not contractible in $L(p,q))\backslash\gamma$. According to [HrLS, Theorem 1.12, i) $\to$ iii)], γ is p-unknotted and $sl^{\mathbb{Q}}_{\xi}(\gamma) = -\frac{1}{p}$. In addition, the Conley-Zehnder index of γ^p with respect to a trivialization induced by a binding disk is at least 3. Now, we recall the proof of [HrLS, Theorem 1.12, iii) $\to$ i)]. To explain it, we consider an almost complex structure J on $\mathbb{R} \times L(p,q)$ which satisfies $J\xi_{\mathrm{std}} = \xi_{\mathrm{std}}$, $J(\partial_t) = X_\lambda$, $d\lambda$-

compatible and $\mathbb{R}$-invariant, where t is the coordinate of $\mathbb{R}$. Let $\mathrm{pr} : \mathbb{R} \times L(p,q) \to L(p,q)$ be the projection In the proof, they find an almost complex structure J as above and a J-holomorphic plane $h : (\mathbb{C}, j) \to (\mathbb{R} \times Y, J)$ such that $\mathrm{pr} \circ h(re^{2\pi t}) \to \gamma(pT_\gamma t)$ as $r \to +\infty$ and in addition $\mathrm{pr}(\overline{h(\mathbb{C})})$ becomes a Birkhoff section. More precisely, there is a C^1 Birkhoff section of disk type $u : \mathbb{D} \to L(p,q)$ such that $\mathrm{pr}(\overline{h(\mathbb{C})}) = u(\mathbb{D})$ as sets (see [FHr, v1, Lemma C.3]).

Having a C^1 Birkhoff section of disk type $u : \mathbb{D} \to L(p,q)$ coming from a J-holomorphic plane, it follows from [FHr, v1, Lemma C.6.] that we can find a C^∞ ∂-strong Birkhoff section $u' : \mathbb{D} \to L(p,q)$ which is arbitrary close to u in C^1-topology. Which completes the proof of Proposition 3.2.19. We note that although originally [FHr, Lemma C.3] and [FHr, Lemma C.6.] are discussed on S^3, we may apply the proofs to $L(p,q)$ in exactly the same way.

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