Category-theoretic Reconstruction of Log Schemes from Categories of Reduced fs Log Schemes

Tomoki Yuji

Throughout the present paper, we **fix** a Grothendieck universe $\mathbf{U}$. Let

$$S^{\log}$$

be a ($\mathbf{U}$-small) fs log scheme. Write S for the underlying scheme of $S^{\log}$. Let

$$\blacklozenge$$

be a set of properties of morphisms of ($\mathbf{U}$-small) schemes (where we *identify* properties of morphisms of schemes with certain full subcategories of the category of morphisms of schemes). We shall write $\mathsf{Sch}_{/S}$ for the category of ($\mathbf{U}$-small) S-schemes, $\mathsf{Sch}_{\blacklozenge/S} \subset \mathsf{Sch}_{/S}$ for the full subcategory of objects of $\mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property contained in $\blacklozenge/S$, $\mathsf{Sch}^{\log}_{/S^{\log}}$ for the category of ($\mathbf{U}$-small) fs log schemes over $S^{\log}$, and

$$\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$$

for the full subcategory determined by the fs log schemes over $S^{\log}$ whose underlying S-scheme is contained in $\mathsf{Sch}_{\blacklozenge/S}$. In the present paper, we shall mainly be concerned with the situation where $\blacklozenge/S^{\log}$ is contained in the following set of properties of log schemes over $S^{\log}$:

"red", "qcpt", "qsep", "sep", "ft",

i.e., (the source scheme is) "reduced", "quasi-compact over $S^{\log}$", "quasi-separated over $S^{\log}$", "separated over $S^{\log}$", and "of finite type over $S^{\log}$".

In the present paper, we consider the problem of reconstructing the log scheme $S^{\log}$ from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. The problem of reconstructing the scheme S from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$ in the case where the elements of $\blacklozenge/S$ amount essentially to the property of being "finite étale over S" is closely related to Grothendieck's anabelian conjectures and has been investigated by many mathematicians. By contrast, the case where the elements of $\blacklozenge/S$ differ substantially from the property of being "finite étale over S" has only been investigated to a limited degree. In this case, there are some known results, mainly as follows: In [Mzk04, Section 1], Mochizuki gave a solution to this reconstruction problem in the case where S is locally Noetherian, and $\blacklozenge = \{\mathrm{ft}\}$. In [vDdB19], van Dobben de Bruyn gave a solution to this reconstruction problem in the case where S is an arbitrary scheme, and $\blacklozenge = \varnothing$. The arguments in [Mzk04, Section 1] and [vDdB19] make essential use of the existence of non-reduced schemes. On the other hand, in [YJ], the author gave a solution to this reconstruction problem in the case where S is a locally Noetherian normal scheme, and one allows an arbitrary subset $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$.

There are even fewer known results concerning the problem of reconstructing a log scheme $S^{\log}$ from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. In [Mzk15] (and [Mzk04, Section 2]), S. Mochizuki proved that if $\blacklozenge = \{\mathrm{ft}\}$, then a locally Noetherian fs log scheme $S^{\log}$ may be reconstructed category-theoretically from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. In [HoNa], Y. Hoshi and C. Nakayama gave a category-theoretic characterization of strict morphisms in the case where $S^{\log}$ is locally Noetherian, and $\blacklozenge = \{\mathrm{ft}\}$. As discussed in [HoNa, Introduction], the arguments

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of [Mzk15] can be applied in more general situations where $\blacklozenge = \{\mathrm{ft}\}$. For instance, the condition assumed in [Mzk15] that $\blacklozenge = \{\mathrm{ft}\}$ may be replaced by the assumption that $\blacklozenge = \{\mathrm{sep}, \mathrm{ft}\}$ (for a detailed discussion, cf. [HoNa, Introduction]). On the other hand, the proof given in [Mzk15] is based on somewhat complicated combinatorial properties of monoids. By constrast, while the arguments in [HoNa] are somewhat more straightforward than the arguments of [Mzk15], the result of Hoshi and Nakayama depends essentially on the existence of non-separated log schemes in $\mathsf{Sch}^{\log}_{\{\mathrm{ft}\}/S^{\log}}$ (for a detailed discussion, cf. [HoNa, Introduction]). In particular, the arguments in [HoNa] cannot be applied in the situation, for instance, where $\blacklozenge = \{\mathrm{ft}, \mathrm{sep}\}$. Here we note that the arguments in [Mzk15] also make essential use of the existence of non-reduced schemes to give a characterization of "SLEM" morphisms (cf. [Mzk15, Definition 2.1, Proposition 2.2]). Hence, the arguments in [Mzk15] cannot be applied in the case, for instance, where $\blacklozenge = \{\mathrm{ft}, \mathrm{sep}, \mathrm{red}\}$.

In the present paper, we give a **relatively simple** solution to this problem of reconstructing log structures in a situation that **generalizes** the situations discussed in [Mzk04], [Mzk15], and [HoNa] to include the log scheme version of the situation discussed in [YJ]. Our main result is the following:

Theorem A. *Let $S^{\log}, T^{\log}$ be locally Noetherian normal fs log schemes,*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

[possibly empty] subsets, and $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ an equivalence of categories. Assume that one of the following conditions (A), (B) holds:

(A) $\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$, and the underlying schemes of $S^{\log}$ and $T^{\log}$ are normal.
(B) $\blacklozenge = \lozenge = \{\mathrm{ft}\}$.

Then the following assertions hold:

(i) Let $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ be an object. Then there exists an isomorphism of log schemes $X^{\log} \xrightarrow{\sim} F(X^{\log})$ that is functorial with respect to $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(ii) Assume that $\blacklozenge = \lozenge$. Then there exists a unique isomorphism of log schemes $S^{\log} \xrightarrow{\sim} T^{\log}$ such that F is isomorphic to the equivalence of categories $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\blacklozenge/T^{\log}}$ induced by composing with this isomorphism of log schemes $S^{\log} \xrightarrow{\sim} T^{\log}$.

By combining the theory of [YJ] with the above Theorem A (ii), we conclude the following corollary:

Corollary B. *Let $S^{\log}, T^{\log}$ be locally Noetherian normal fs log schemes and*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

subsets such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$, and $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$. If the categories $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ and $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ are equivalent, then $\blacklozenge = \lozenge$, and $S^{\log} \cong T^{\log}$.

References

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