Ph.D. thesis

Anomaly and Superconnection

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Abstract

The anomaly is a very important tool to understand the nature of the Quantum Field Theory (QFT). To understand QFT, its infra-red (IR) phase is important. The anomaly is useful because it is invariant under the renormalization group flow.

In this thesis, the anomalies of fermions with spacetime dependent mass are studied. Using Fujikawa's method, it is found that the anomalies associated with the $U(N)_+ \times U(N)_-$ chiral symmetry and $U(N)$ flavor symmetry for even and odd dimensions, respectively, can be written in terms of superconnections. This anomaly is characterized by a $(D+2)$-form part of the Chern character of the superconnection. It is a generalization of the usual anomaly polynomial for the massless fermions. These results enable us to analyze anomalies in the systems with interfaces and spacetime boundaries in a unified way. Applications to index theorems, including Atiyah-Patodi-Singer index theorem and Callias-type index theorem, are also discussed.

This thesis is based on the paper [1].

Contents

Chapter 1

Introduction

Quantum many body systems are described by Quantum Field Theory (QFT). However, in many cases, to solve QFT is very difficult. For example, Quantum ChromoDynamics (QCD) is a well-known that it is very difficult to solve, because of its nature of confinement. QFT is often defined as an Ultra-Violet (UV) action. In this case, if the β-function of the UV action is negative, this QFT is strongly coupled in Infra-Red (IR) and it is often difficult to solve. We want to understand the nature of such strongly coupled QFTs.

One of the strong tools to understand the IR phase of strongly coupled QFT is a quantum anomaly. Quantum anomaly (or anomaly in short) is described as a symmetry breaking by quantum effects. When we define a model of QFT by using a UV action, the symmetry of the action is also defined at the same time. However, this symmetry of the action is a symmetry of classical theory. It is known that some of the classical symmetries are broken in quantum theory.

The information of symmetries (more precisely, global symmetries) in a UV action do not change under the renormalization group flow. When spontaneous symmetry breaking (SSB) occurs, some symmetries are broken in IR, but the information of the symmetries remains. It appears in the IR theory as Nambu-Goldstone bosons (NG bosons) for continuous symmetries, or some topological QFTs (TQFTs) for discrete symmetries. This is why symmetries are important to QFTs. Furthermore, anomalies are also invariant under the renormalization group flow, even though they are no longer symmetries. This is known as the 't Hooft anomaly matching.[2] For this reason, the anomaly can be a strong tool to reveal the IR phases of the strongly coupled QFTs.

The anomaly also has a topological nature. It is known that the anomaly has some connection to mathematics. For example, to classify anomalies, the Chern number plays an important role. Index theorem is also important to consider anomalies, in particular the anomalies of fermions. The anomaly can be useful not only for physics but also for mathematics.

To understand the IR phases of strongly coupled QFTs, we ought to understand anomalies well. One description of the anomaly is, that when we consider the path integral for some fields, this operation breaks some parts of the symmetries in the classical action. This interpretation of the anomalies is useful not only to understand the anomalies but also to calculate them. For example, anomalies of the fermions are known that we can calculate them by considering the change of the fermion path integral measure under the anomalous transformations. This way of calculating the anomalies is known as Fujikawa's method. [3, 4]

In this thesis, we calculate the anomalies of fermions by Fujikawa's method. In particular, we focus on the free Dirac fermions. If we consider N free Dirac fermions without mass terms, we can find flavor symmetries. Let the spacetime dimension D. When D is even number, the chiral symmetry is $U(N)_+ \times U(N)_-$, and $\pm$ denotes the chirality. When D is odd, there is no chirality, however, this system has $U(N)$ symmetry. The anomalous symmetries that we consider in this thesis are these $U(N)_+ \times U(N)_-$ for even D and $U(N)$ for odd D. When we consider Dirac fermions, these flavor symmetries usually appear. Furthermore, these symmetries are important to physics.

For example, it is known that the chiral symmetry in 4 dimensional QCD is important even though QCD in the real world has mass term and does not have chiral symmetry. The IR effective theory of QCD is known as a theory of pion, which is the NG boson that comes from spontaneous breaking of the chiral symmetry (chiral SSB). To understand the pion effective theory, the anomaly of the chiral symmetry is important in some cases. One of them is known as the $U(1)_A$ problem in the 1970's. If we consider 3-flavor QCD, without considering any anomalies, the chiral symmetry $U(3)_+ \times U(3)_-$ breaks to $U(3)_V$. From this symmetry breaking, the broken part is $\left(U(3)_+ \times U(3)_-\right)/U(3)_V$ then 9 pions are expected. However, in the experimental result, the number of pions seems to be only 8, not 9. η' meson can be the 9th pion, but it is much heavier than the other pions. This problem is solved by Witten and Veneziano [5, 6], and anomaly in $U(1)_A$ part plays an important role. The other famous case is the decay of π^0 meson. π^0 meson decays into two photons through the strong interaction, and this is the effect of the anomaly. QCD, which is defined as a UV action, has a anomaly in its chiral symmetry. At the request of the anomaly matching, the pion theory in IR need to reproduce the anomaly in QCD. To reproduce the UV anomaly, we need to add the Wess-Zumino-Witten (WZW) term to the IR pion theory.[7, 8] This term describes the $\pi^0 \to 2\gamma$ decay. These examples are reviewed in section 2.3.

These flavor symmetries have been studied for a long time, however, we study new anomalies for these symmetries. We focus on the free fermions with the mass term, and its mass depends on the spacetime coordinate. This spacetime dependent mass term is equivalent to an external scalar field (or a Higgs field) couples to fermions through the Yukawa coupling. Although the masses of the quarks and leptons in nature are considered to be constant, spacetime dependent mass naturally appears in the standard model and various other models when the value of the Higgs field is not constant. It also appears in hadron physics and condensed matter physics, because the effective mass of fermions can vary depending on some parameters of the environment, such as temperature, chemical potentials, magnetic field, strength of the interaction, etc., which can be spacetime dependent.

Apart from possible applications to realistic systems, the spacetime dependent mass can be used as a theoretical tool to study quantum field theory. For example, it can be regarded as an external source coupled to a fermion bilinear operator. In particular, although the $U(N)_+ \times U(N)_-$ chiral symmetry is explicitly broken to a subgroup when the mass is non-zero, we can make the action invariant under the $U(N)_+ \times U(N)_-$ gauge transformation by promoting the mass to a spacetime dependent external field. Then, we are allowed to discuss the anomaly for this symmetry even though the mass is non-zero. In this sense, the spacetime dependent mass plays a similar role as the external gauge field, with which the action becomes gauge invariant.

In fact, the anomaly for the fermions with spacetime dependent mass (Higgs field) was analyzed in the 80's in [9, 10][1]. The conclusion of these papers was that the mass does not contribute to the anomaly at all. This is true in the case that the mass is bounded and fixed while the cut-off scale is sent to infinity. However, as we will demonstrate, the mass dependence of the anomaly survives when the mass is unbounded. Remarkably, we will also find that the anomaly exists even for odd dimensional cases, when the spacetime dependent mass is introduced. Our discussion is closely related to that of papers by Cordova et al. [12, 13], in which coupling constants including the masses are promoted to external scalar fields, and the anomalies are extended to include them. They analyzed the systems with massive fermions in [12] and found that the space of masses can be considered as a compact space with non-trivial topology by including $|m| \to \infty$, and anomalies in D-dimensional systems are characterized by a $(D+2)$-form, which is a generalization of the usual anomaly polynomial, involving differential forms on the space of masses. This also shows that it is crucial to consider $|m| \to \infty$ to have a non-trivial anomaly that involves the masses.

One of the main parts of this thesis is the derivation of the anomaly of the spacetime

[1] See also section 6.5.1 of [11].

dependent mass. This part corresponds to chapter 3. We show that the anomaly $(D+2)$-form, as well as the anomaly associated with $U(1)_V$ symmetry, are given by the Chern character written in terms of the superconnection introduced by Quillen in [14]. This was also suggested in [12]. We will show this explicitly by using Fujikawa's method. Our formulas (3.3.11) and (3.3.13) can be used for both even and odd dimensional cases, provided that the superconnection of the even and odd types are used accordingly.

In fact, when we consider the string theory, there is a similar object to the structure of these anomalies. To see this similarity, let us focus on the Chern-Simons (CS) terms including the tachyon field in unstable D-brane systems, which are written with the Chern character of the superconnection.[15, 16, 17, 18] As we will discuss in section 3.4, the systems with Dirac fermions in various dimensions can be realized on a D-brane with unstable D9-branes. The mass of the fermion is proportional to the value of the tachyon field and hence the spacetime dependent mass can be naturally obtained by considering a varying tachyon field. The anomaly of the fermions is supposed to be canceled by the contribution from the CS term. Therefore, string theory suggests that the superconnection appears in the formulas of anomaly, which is indeed what we find in the field theory analysis.

The chapter 4 is devoted to the applications of these formulas. We consider the systems with interfaces and boundaries realized by the spacetime dependent mass. Most of the discussion there are consistency checks and demonstrations of our formulas (3.3.11) and (3.3.13). We show in several explicit examples that some known results can be consistently reproduced in a simple and unified way. The results of section 4.2.2 are new. In this section, a system with a spacetime dependent boundary condition is considered and the anomalies due to this boundary condition are obtained. Applications to index theorems are discussed in section 4.3. In section 4.3.1, we study about Atiyah-Patodi-Singer index theorem. Callias-type index theorem is also discussed in section 4.3.2, and this is one way to understand the topological number of the spacetime dependent mass.

Organization of this thesis

This thesis is organized as follows. We start with a review of the chiral symmetry and the ordinary chiral anomaly in chapter 2. We stress the importance of chiral symmetry and its anomaly in QCD. We also review the topological nature of the anomaly, and topological terms in gauge theories and scalar field theories.

Chapter 3 and chapter 4 are the main parts of this thesis. These chapters are based on my work with S. Sugimoto.[1] We start with a brief review of the superconnection in section 3.1. In section 3.2, we derive our main formulas for the anomaly with spacetime dependent mass using Fujikawa's method. The relation between string theory and our results have natural interpretations in string theory as explained in section 3.4. Applications of these formulas are given in chapter 4. The cases with interfaces and boundaries are studied in sections 4.1 and 4.2, respectively, and implications to index theorems are discussed in section 4.3.

Finally, in chapter 5, we summarize our results and see some future directions.

Chapter 2

Review of the chiral anomaly

In this chapter, we review the quantum anomaly. In particular, the quantum anomaly of chiral symmetry in the free fermion system is important to this thesis.

One of the main motivations for studying strongly coupled QFT is to understand QCD. First, we introduce and review QCD. We can learn many basic concepts in QFT through the knowledge of QCD. For example, chiral symmetry is important to understand QCD, as we will see in this chapter. In this thesis, chiral symmetry will also be important in chapter 3. In this chapter, we review QCD and we will introduce chiral symmetry through QCD as an example.

2.1 Review of QCD and its chiral symmetry

The definition of QCD in this thesis is, "4 dimensional $SU(N_c)$ Yang-Mills theory coupled with N_f quarks in fundamental representation (with or without mass term)". The action of QCD is the following.

$$S_{QCD} = \int d^4x \left\{ \bar{\psi} i\gamma^\mu(\partial_\mu + a_\mu)\psi + \bar{\psi}M\psi - \frac{1}{2g^2}\mathrm{tr}\left[f_{\mu\nu}f^{\mu\nu}\right] + \frac{i\theta}{32\pi^2}\mathrm{tr}\left[\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}\right]\right\} . \tag{2.1.1}$$

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \bar{\psi} = \left(\psi_-^\dagger, \psi_+^\dagger\right)$$

$$f_{\mu\nu} = \partial_\mu a_\nu + [a_\mu, a_\nu]$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0 \end{pmatrix}, \quad \sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3 .$$

We only consider Euclidean spacetime with all plus metric ($g_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1)$) in this thesis. We use chiral representation for the γ-matrices (not only for 4dim γ-matrices, but also for general dimension cases).

To understand QCD, we need to understand 4dim Yang-Mills theory and 4dim fermions.

2.1.1 Yang-Mills theory and confinement

4dim Yang-Mills (YM) theory, or 4dim $SU(N_c)$ gauge theory is defined by the following action.

$$\begin{aligned} S_{YM} &= \int d^4x \left\{ -\frac{1}{2g^2} \mathrm{tr}\left[f_{\mu\nu} f^{\mu\nu}\right] + \frac{i\theta}{32\pi^2} \mathrm{tr}\left[\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}\right] \right\} \\ &= \int \left\{ -\frac{1}{2g^2} \mathrm{tr}\left[f \wedge * f\right] + \frac{i\theta}{8\pi^2} \mathrm{tr}\left[f \wedge f\right] \right\} \ . \end{aligned} \tag{2.1.2}$$

The β-function of this theory is negative in one loop. Then, this theory is strongly coupled in IR. It is known that this theory has a mass gap, but the reason why it has a mass gap is unknown.

If we couple this theory with fermions (called quarks), we cannot observe the fermion alone. This is known as quark confinement.

This theory has a θ term, which is the second term in the action (2.1.2). θ term is a topological term because this term can be written as total derivative and does not affect the equation of motion. The existence of the θ term depends on the existence of instantons. YM on S^4 has instantons comes from $\pi_3(SU(N_c)) \simeq \mathbb{Z}$.

2.1.2 Massless QCD

If we couple N_f fermions with YM theory without mass term, we obtain massless QCD. The action is (2.1.1) with $M = 0$ and $\theta = 0$. This theory does not have θ dependence. This is related to the anomaly, and the reason is explained later.

Chiral symmetry

This theory has N_f Dirac fermions in the fundamental representation of $SU(N_c)$. This theory has the following chiral symmetry, up to discrete part.

$$\frac{U(N_f)_+ \times U(N_f)_-}{U(1)_A} = SU(N_f)_+ \times SU(N_f)_- \times U(1)_V \ . \tag{2.1.3}$$

If we consider discrete part, the chiral symmetry becomes,

$$\frac{SU(N_f)_+ \times SU(N_f)_- \times U(1)_V}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_V} \ . \tag{2.1.4}$$

This discrete part is important to consider the generalized symmetry,[19] however, it is not important to this thesis because we just focus on the ordinary symmetry. In (2.1.3), $U(N_f)_+ \times U(N_f)_-$ comes from the phase rotation of the N_f Dirac fermions. We divide this flavor symmetry by $U(1)_A$, which comes from the anomaly between $U(1)_A$ and $SU(N_c)$. This anomaly is described in section 2.2.

The chirality is defined as an eigenvalue of γ_5. In all even dimensions, chirality operators such as γ_5 are defined. We denote the chirality of the flavor symmetry with the index $\pm$, like $U(N_f)_+ \times U(N_f)_-$. For one massless Dirac fermion, we can decompose it into two Weyl fermions with different chirality. This is why we consider two different $U(N_f)$ groups for N_f Dirac fermions.

In this thesis, we will focus on this chiral symmetry for D dimensional fermions. In chapter 3, we just consider N free fermions, however, it is easy to generalize our formulas to some theories with dynamical gauge fields, such as QCD.

In massless QCD, we cannot define the θ term. θ parameter relate to the $U(1)_A$ rotation through the $U(1)_A$ anomaly. Therefore, when we consider $U(1)_A$ transformation for massless QCD action with parameter α,[1] θ parameter is shifted as $\theta \to \theta - 2N_f\alpha$. The effect of the anomaly is described in section 2.2. We can change the action by $U(1)_A$ rotation with any angle α because the classical action is invariant under the $U(1)_A$ rotation. This means we can rotate the θ parameter by any angle, and θ parameter does not have any physical meaning. This massless QCD does not have θ dependence, and cannot have θ term.

Chiral SSB and the pion effective theory

This massless QCD is understood through the spontaneous symmetry breaking of the chiral symmetry (2.1.3). QCD is strongly coupled in IR so we cannot understand it by perturbation techniques. However, we know the information of chiral SSB, we can identify that the IR effective theory of QCD is written by the pion theory.

It is known that the chiral symmetry (2.1.3) breaks to $U(N_f)_V$[2]. Then, IR effective theory can be described by the pion (or NG boson) degree of freedom. This pion takes

[1] The explicit form of this $U(1)_A$ transformation is described in (2.2.1). The shift of θ parameter can be understood through (2.3.5).

[2] The symmetry breaking pattern is known as the Vafa-Witten theorem.[20] For this theorem, the symmetry with mass term is important. The symmetry of QCD with mass term is discussed in the next subsection and (2.1.9).

its value on

$$\frac{SU(N_f)_+ \times SU(N_f)_- \times U(1)_V}{SU(N_f)_V \times U(1)_V} \ . \tag{2.1.5}$$

The effective action of the pion is written as a non-linear sigma model whose target space is (2.1.5). This pion degree of freedom is often written as

$$U(x) = \mathrm{e}^{i\frac{\pi(x)}{f_\pi}} \in SU(N_f) \ . \tag{2.1.6}$$

f_π is called pion decay constant and $\pi(x)$ is a pion field. The first leading term of the pion action is,

$$S_\pi = \int d^4x \, \frac{f_\pi^2}{4} \mathrm{tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] \ . \tag{2.1.7}$$

This term corresponds to the pion kinetic term. We can add many higher derivative terms with respecting the symmetry, but the lowest derivative term is only this term (2.1.7).

2.1.3 Mass term and chiral symmetry

When we turn on the mass term in (2.1.1), the chiral symmetry (2.1.3) is broken by the mass term. This is clear when we consider the mass term with the Weyl fermions. For simplicity, let us consider $N_f = 1$ case and the mass parameter M is just a real scalar m. Then the mass term becomes,

$$\begin{aligned}\bar{\psi} M \psi =& m \left(\psi_-^\dagger, \psi_+^\dagger\right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \\ =& m\psi_-^\dagger \psi_+ + m\psi_+^\dagger \psi_- \ . \end{aligned} \tag{2.1.8}$$

This mass term mixes the two chiralities $+$ and $-$. Therefore, chiral symmetry (2.1.3) breaks to the diagonal part,

$$U(N_f)_V \ . \tag{2.1.9}$$

In QCD with the mass term (2.1.1), the θ parameter is physical. In this case, the $U(1)_A$ symmetry is explicitly broken by the mass term (2.1.9). Therefore, $U(1)_A$ rotation is not only the shift of θ parameter but also the phase rotation of the complex-valued mass parameter. Let us check it by $N_f = 1$ case. If we consider complex mass term with $m \in \mathbb{C}$, the mass term becomes

$$\begin{aligned}\bar{\psi} M \psi =& \left(\psi_-^\dagger, \psi_+^\dagger\right) \begin{pmatrix} m & 0 \\ 0 & m^* \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \\ =& m\psi_-^\dagger \psi_+ + m^*\psi_+^\dagger \psi_- \ . \end{aligned} \tag{2.1.10}$$

m^* is a complex conjugate of m. This mass term is changed under the $U(1)_A$ rotation (2.2.1) as

$$\begin{aligned}\bar{\psi}M\psi \to &\bar{\psi}\mathrm{e}^{i\gamma_5\alpha}M\mathrm{e}^{i\gamma_5\alpha}\psi \\ =&\left(\psi_-^\dagger, \psi_+^\dagger\right)\begin{pmatrix} m\mathrm{e}^{2i\alpha} & 0 \\ 0 & m^*\mathrm{e}^{-2i\alpha}\end{pmatrix}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}.\end{aligned} \tag{2.1.11}$$

This $U(1)_A$ transformation changes the phase of the mass m to $m\mathrm{e}^{2i\alpha}$. This means that the phase of mass m and θ parameter are the same parameter. In other words, we can start from the finite θ action (2.1.1), and rotate in $U(1)_A$ transformation, then we can get the action without θ term but the phase of the mass is changed as $m\mathrm{e}^{-i\theta/2}$. For N_f flavor case, let the phase of mass is α, the set of $\theta - 2N_f\alpha$ is only a physical parameter. This QCD with mass (2.1.1) has θ dependence, but this θ parameter is identified with the phase of mass.

In this thesis, we will consider N massive fermions. However, we consider the spacetime dependent mass (or a background Higgs field). Therefore, the action we consider in chapter 3 is invariant under $U(N)_+ \times U(N)_-$, not (2.1.9) even though it is massive.

2.2 Review of the anomaly for free fermion

One of the most simple cases to calculate the anomaly is 4 dimension free fermion with N_f flavor. For reviews of the perturbative anomalies, please see, e.g., [21, 11, 22, 23]. It is easy to generalize it to other systems with dynamical gauge fields, such as QCD.

2.2.1 Some kinds of anomalies

When we calculate the anomaly, we often couple some background gauge fields to the theory. For example, if we calculate the anomaly of free fermions, we cannot see the continuous part of the anomaly without the background gauge fields. Anomalies are distinguished three types, depending on the dependence of background gauge fields.

Let us consider the following system as an example. One 4 dimensional Dirac fermion has the chiral symmetry $U(1)_V \times U(1)_A$. $U(1)_V$ and $U(1)_A$ transformations are defined as,

$$\begin{aligned}\psi \to &\mathrm{e}^{i\alpha}\psi & \bar{\psi} \to &\mathrm{e}^{-i\alpha}\psi & &(U(1)_V), \\ \psi \to &\mathrm{e}^{i\gamma_5\alpha}\psi & \bar{\psi} \to &\mathrm{e}^{i\gamma_5\alpha}\psi & &(U(1)_A).\end{aligned} \tag{2.2.1}$$

The action of this system couple with $U(1)_V$ gauge field A^V_μ and $U(1)_A$ gauge field A^A_μ is,

$$S = \int d^4x \bar{\psi}(x) i\gamma^\mu \left(\partial_\mu + A^V_\mu + \gamma_5 A^A_\mu\right) \psi(x)$$
$$= \int d^4x \bar{\psi}(x) D\psi(x) \ . \tag{2.2.2}$$

This system has some anomalies, depending on whether these gauge fields are dynamical or background.

’t Hooft anomaly

Anomalies without any dynamical gauge fields are called as ’t Hooft anomalies. For example, if all gauge fields in (2.2.2) are background fields, then this system has ’t Hooft anomalies.

If we consider $U(1)_A$ transformation for this system in equation (2.2.2), we can find anomalies in the following way. First, we consider the partition function of this system. It depends on two gauge fields A^V and A^A as,

$$Z[A^V, A^A] = \int [d\psi d\overline{\psi}] \, \mathrm{e}^{-S} \ . \tag{2.2.3}$$

This partition function is changed under the $U(1)_A$ transformation as,

$$Z[A^V, A^A] \to \mathrm{e}^{\int \frac{i}{4\pi^2}\alpha\left\{F^V \wedge F^V + F^A \wedge F^A\right\}} Z[A^V, A^A] \ . \tag{2.2.4}$$

F^V and F^A are field strengths of background gauge fields A^V and A^A, respectively. The detailed calculation here is described in the next section. In conclusion, the partition function (2.2.3) is not invariant under the $U(1)_A$ transformation, but the difference is just written as a phase. This phase is nothing but the anomaly. If the phase includes only background gauge fields, this anomaly is called an ’t Hooft anomaly.

The ’t Hooft anomalies are useful tools to decide the IR phases of QFTs. The ’t Hooft anomalies are invariant under the renormalization group flow [2].

Gauge anomaly

Let us consider dynamical gauge fields for (2.2.2). If we consider a gauge transformation and its gauge transformation has an anomaly, the anomaly is called a gauge anomaly. Let us write dynamical gauge fields in (2.2.2) as a^V and a^A with small letter a. The partition function is,

$$Z = \int [d\psi d\overline{\psi} da^V da^A] \, \mathrm{e}^{-S} \ . \tag{2.2.5}$$

Indeed, this partition function is ill-defined. When we consider $U(1)_A$ transformation for this action, we know that the path integral measure of the fermion is not invariant and changed as (2.2.4). Let us consider the same transformation for the gauged version (2.2.5). The fermion path integral measure changes as

$$
\begin{aligned}
Z &= \int [d\psi d\overline{\psi} da^V da^A]\, \mathrm{e}^{-S} \\
&\to \int [d\psi d\overline{\psi} da^V da^A]\, \mathrm{e}^{\int \frac{i}{4\pi^2}\alpha\left\{f^V \wedge f^V + f^A \wedge f^A\right\}} \mathrm{e}^{-S} \\
&\neq \int [d\psi d\overline{\psi} da^V da^A]\, \mathrm{e}^{-S} = Z \ .
\end{aligned} \tag{2.2.6}
$$

f^V and f^A are field strengths of dynamical gauge fields a^V and a^A, respectively. This means that this partition function (2.2.5) is not invariant under $U(1)_A$ gauge transformation. We cannot write $U(1)_A$ gauge invariant partition function for this system, so we cannot define any gauge theories for this action (2.2.2) with dynamical gauge fields a^V and a^A.

Gauge anomalies mean that the theories are ill-defined. When a theory has a gauge anomaly, the theory cannot be defined in QFT.

ABJ-type anomaly

ABJ-type anomaly is a mixed anomaly between 't Hooft anomalies and gauge anomalies.

For example, let us consider the action (2.2.2) with dynamical $U(1)_V$ gauge field a^V, but $U(1)_A$ part is still non-dynamical. The partition function is,

$$
\begin{aligned}
Z[A^A] &= \int [da^V]\, Z[a^V, A^A] \\
&= \int [d\psi d\overline{\psi} da^V]\, \mathrm{e}^{-S} \ .
\end{aligned} \tag{2.2.7}
$$

If we consider $U(1)_A$ transformation for this action with setting $A^A = 0$[3], the partition function is changed as

$$
\begin{aligned}
Z &= \int [da^V]\, Z[a^V] \\
&\to \int [da^V]\, \mathrm{e}^{\int \frac{i}{4\pi^2}\alpha f^V \wedge f^V} Z[a^V] \\
&\neq Z \ .
\end{aligned} \tag{2.2.8}
$$

[3]The reason why we set $A^A = 0$ is, just to hide the 't Hooft anomaly of $U(1)_A$. After gauging $U(1)_V$, $U(1)_A$ global symmetry disappears from this action. Therefore, there is no 't Hooft anomaly of $U(1)_A$ for this system, because there is no $U(1)_A$ symmetry. This means that the existence of the A^A gauge field is meaningless.

We rewrite $Z[a^V, A^A = 0]$ as $Z[a^V]$. This is a milder situation than the gauge anomaly because the inconsistency in (2.2.8) means just this partition function (2.2.7) does not have $U(1)_A$ symmetry. $U(1)_A$ symmetry is not gauged in this theory, so this theory is well-defined even if there is an anomaly (2.2.8).

The meaning of mixed anomaly between $U(1)_V$ and $U(1)_A$ would be clear when we consider anomaly polynomials in section 2.2.3. The anomaly (2.2.8) corresponds to the following anomaly polynomial

$$\left(\frac{i}{2\pi}\right)^3 F^A \wedge f^V \wedge f^V \ . \tag{2.2.9}$$

ABJ anomaly means that some classical symmetries are broken by the quantum effect. We will check the ABJ anomaly in QCD in section 2.3.

2.2.2 Fujikawa's method

To calculate anomalies of fermions, Fujikawa's method [3, 4] is a useful tool. In this section, we calculate a $U(1)_A$ anomaly of N free fermions in 4 dimension. This is one of the most simple cases of anomalies. Free fermion theory does not include any dynamical gauge fields, so this anomaly is an 't Hooft anomaly. Let us consider the action in (2.2.2) with taking A^V as a $U(N)_V$ gauge field and $A^A = 0$ as

$$\begin{aligned} S &= \int d^4x \bar{\psi}(x) i\gamma^\mu \left(\partial_\mu + A^V_\mu\right) \psi(x) \\ &= \int d^4x \bar{\psi}(x) D\psi(x) \ . \end{aligned} \tag{2.2.10}$$

and $U(1)_A$ rotation in (2.2.1). Here, D is a Dirac operator for this action. The action (2.2.10) is invariant under the $U(1)_A$ rotation (2.2.1) as a classical theory. However, if we consider the quantum theory for the action, this theory is not invariant under the $U(1)_A$ rotation. To see this quantum effect, let us consider the partition function of (2.2.10) like (2.2.3). In this partition function, the path integral measure of fermions $\psi, \overline{\psi}$, written as $[d\psi d\overline{\psi}]$ are included.[4] Fujikawa's method [3, 4] claims this path integral measure is not invariant under the $U(1)_A$ rotation, and its Jacobian shows the anomaly. We follow the calculation in [9, 11] here.

Schematically, the calculation of the anomalies by Fujikawa's method can be written

[4] In many papers and textbooks, $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is used as a path integral measure of the fermions. However, instead of it, we use $[d\psi d\overline{\psi}]$ as the path integral measure because we will use $\mathcal{D}$ as a Dirac operator in this thesis.

as,

$$
\begin{aligned}
Z[A^V] =& \int [d\psi d\overline{\psi}]\, \mathrm{e}^{-S} \\
\rightarrow & \int [d\psi d\overline{\psi}] \mathcal{J}\, \mathrm{e}^{-S} \\
=& \int [d\psi d\overline{\psi}]\, \mathrm{e}^{\int \frac{i}{4\pi^2}\alpha \mathrm{Tr}_f (F\wedge F)} \mathrm{e}^{-S} \ .
\end{aligned} \tag{2.2.11}
$$

Here, "$\rightarrow$" means the $U(1)_A$ rotation with angle α. F is a field strength of A^V, and Tr_f is a trace for N flavor space. The non-trivial points in this calculation are the existence of the Jacobian $\mathcal{J}$ and the explicit form of the Jacobian. We will see these points in this section. From (2.2.11), it is clear that the anomaly corresponds to the log of the Jacobian $\log \mathcal{J}$. The goal of the derivation is to calculate the explicit form of $\log \mathcal{J}$.

In order to calculate the anomaly, we evaluate the Jacobian $\mathcal{J}$ for the $U(1)_A$ rotation (2.2.1). To calculate the Jacobian, we need to consider the regularization for the eigenvalues of the Dirac operator D in (2.2.10). In this thesis, we take the heat kernel regularization and use this form of the regulator

$$
\mathrm{e}^{-\frac{\lambda_n^2}{\Lambda^2}} \ . \tag{2.2.12}
$$

Λ is a UV cut off and λ_n is an eigenvalue of the Dirac operator D. We take both Λ and λ_n positive values because the Dirac operator in (2.2.10) is a Hermitian operator. In the following chapter 3.2.1, we will generalize this calculation for non-Hermitian Dirac operators D.[5]

To calculate the Jacobian $\mathcal{J}$, we expand the fermion fields ψ and $\overline{\psi}$, by using the eigenfunction of D. Let n_ϕ be the number of zero modes of D. We choose the eigenfunctions such that they satisfy the eigenequations

$$
D\phi_n(x) = \lambda_n \phi_n(x) \ , \qquad (n \in \{\, k - n_\phi \,|\, k = 1, 2, 3, \cdots \}) \ , \tag{2.2.13}
$$

and the normalization conditions

$$
\int d^D x \, \phi_m^\dagger(x) \phi_n(x) = \delta_{m,n} \ . \tag{2.2.14}
$$

[5]This Dirac operator (2.2.10) is Hermitian, so we can take this λ_n as an eigenvalue of the Dirac operator. For general Dirac operators, such as in the action (2.2.2), Dirac operators D are non-Hermirian. In these cases, eigenvalues of Dirac operators can be complex values. For these cases, we will take λ_n^2 as an eigenvalue of Hermitian Dirac operators $D^\dagger D$ and $DD^\dagger$. The detail of this treatment will be discussed in section 3.2.1.

Fermions $\psi(x)$ and $\overline{\psi}(x)$ can be expanded as

$$\psi(x) = \sum_n a_n \phi_n(x) \ , \qquad\qquad \overline{\psi}(x) = \sum_n \bar{b}_n \phi_n^\dagger(x) \ , \tag{2.2.15}$$

where a_n and $\bar{b}_n$ are Grassmann odd coefficients, and the action (2.2.10) becomes

$$S = \sum_n \lambda_n \bar{b}_n a_n \ . \tag{2.2.16}$$

From this expression, the path integral measure of the fermion is written as

$$[d\psi d\overline{\psi}] = \prod_x d\psi(x) d\overline{\psi}(x) = \det(\phi_n(x))^{-1} \det(\phi_n^\dagger(x))^{-1} \prod_m da_m \prod_l d\bar{b}_l \ , \tag{2.2.17}$$

where $\det(\phi_n(x))^{-1} \det(\phi_n^\dagger(x))^{-1}$ is the Jacobian which comes from introducing the new variables $\{a_n, \bar{b}_n\}$.

Under the $U(1)_A$ transformation (2.2.1), a_n and $\bar{b}_n$ transforms as

$$\begin{aligned} a_n \to a'_n &\equiv \int d^D x \, \phi_n^\dagger(x) \mathrm{e}^{i\alpha\gamma_5} \psi(x) \simeq \sum_m \left(\delta_{m,n} + i \int d^4 x \, \phi_n^\dagger(x) \alpha \gamma_5 \phi_m(x) \right) a_m \ , \\ &= \sum_m M_{nm} a_m \\ \bar{b}_n \to \bar{b}'_n &\equiv \int d^D x \, \overline{\psi}(x) \mathrm{e}^{i\alpha\gamma_5} \phi_n(x) \simeq \sum_m \bar{b}_m \left(\delta_{m,n} + i \int d^4 x \, \phi_m^\dagger(x) \alpha(x) \gamma_5 \phi_n(x) \right) \ , \\ &= \sum_m \bar{b}_m M_{mn} \end{aligned} \tag{2.2.18}$$

where we have assumed $\alpha(x) \ll 1$. Then, the Jacobian in (2.2.11) is

$$\log \mathcal{J} = -i \int d^4 x \, \alpha(x) \mathcal{I}(x) \ , \tag{2.2.19}$$

where [6]

$$\mathcal{I}(x) \equiv \sum_n \left(\phi_n^\dagger(x) \gamma_5 \phi_n(x) + \phi_n^\dagger(x) \gamma_5 \phi_n(x) \right) = 2 \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x) \ . \tag{2.2.20}$$

[6] This Jacobian $\mathcal{J}$ can be written by using the matrix M_{nm} as $\mathcal{J} = \det(M_{nm})^{-1} \det(M_{mn})^{-1}$. To calculate this Jacobian, we use the relation $\log \det M_{nm} = \mathrm{tr} \log M_{nm}$. The matrix M_{nm} includes infinite small parameter α, so we can calculate it as

$$\mathrm{tr} \log(M_{nm})^{-1} \simeq -\mathrm{tr} \left[i \int d^4 x \, \phi_n^\dagger(x) \alpha(x) \gamma_5 \phi_m(x) \right] = -i \int d^4 x \, \alpha(x) \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x).$$

In this expression, $\mathcal{I}(x)$ diverges. To treat a well-defined expression of $\mathcal{I}(x)$, we need to introduce a regulator. Here, we take (2.2.12) as a regulator.

$$
\begin{aligned}
\mathcal{I}(x) &= \lim_{\Lambda\to\infty} 2\sum_n \mathrm{e}^{-\frac{\lambda_n^2}{\Lambda^2}} \phi_n^\dagger(x)\gamma_5\phi_n(x) \\
&= \lim_{\Lambda\to\infty} \sum_n 2\phi_n^\dagger(x)\mathrm{e}^{-\frac{1}{\Lambda^2}D^2}\phi_n(x) \\
&= \lim_{\Lambda\to\infty} 2\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\,\mathrm{Tr}_s\left(\gamma_5 \mathrm{e}^{-\frac{1}{\Lambda^2}D^2}\right) e^{ikx} \ ,
\end{aligned} \tag{2.2.21}
$$

where Tr_s is the trace over both flavor and spinor indices. D^2 can be written as

$$
\begin{aligned}
D^2 =&(i\gamma^\mu D_\mu)^2 \\
=& -\gamma^\mu\gamma^\nu D_\mu D_\nu \\
=& -(D_\mu)^2 - \frac{1}{4}\left[\gamma^\mu,\gamma^\nu\right] F_{\mu\nu} \ ,
\end{aligned} \tag{2.2.22}
$$

where $D_\mu = \partial_\mu + A_\mu^V$ is a covariant derivative and $F_{\mu\nu}$ is a field strength written by A_μ^V. Then, $\mathcal{I}(x)$ becomes

$$
\begin{aligned}
\mathcal{I}(x) &= \lim_{\Lambda\to\infty} 2\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\,\mathrm{Tr}_s\left(\gamma_5 \mathrm{e}^{-\frac{1}{\Lambda^2}\left(-(D_\mu)^2 - \frac{1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)}\right) e^{ikx} \\
&= \lim_{\Lambda\to\infty} 2\Lambda^4\int \frac{d^4\widetilde{k}}{(2\pi)^4} \mathrm{e}^{-\widetilde{k}_\mu^2}\,\mathrm{Tr}_s\left(\gamma_5 \mathrm{e}^{\frac{1}{\Lambda^2}D_\mu^2 + \frac{2i}{\Lambda}\widetilde{k}^\mu D_\mu + \frac{1}{4\Lambda^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}\right) \ ,
\end{aligned} \tag{2.2.23}
$$

where $\widetilde{k}_\mu \equiv k_\mu/\Lambda$. Be aware that after taking the trace for γ matrices, terms which include $F_{\mu\nu}$ are remained because the trace of γ_5 is equal to zero. To obtain such terms, it is needed to create γ_5 from γ matrices $[\gamma^\mu,\gamma^\nu]$. Therefore, all remaining terms include $(F_{\mu\nu})^2$. To evaluate the trace for γ matrices, we use the following relation

$$
\mathrm{Tr}_\gamma\left[\gamma_5[\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right] F_{\mu\nu}F_{\rho\sigma} = -16\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \ , \tag{2.2.24}
$$

where $\epsilon^{\mu_1\cdots\mu_4}$ is the Levi-Civita symbol with $\epsilon^{1,2,3,4}=1$. After we take $\Lambda\to\infty$ limit, $\mathcal{O}(\Lambda^0)$ term is physical. Therefore, the remaining term includes only a term proportional to $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. We take $1/\Lambda$ expansion for $\mathcal{I}(x)$ and take only $\mathcal{O}(\Lambda^0)$ terms,

$$
\begin{aligned}
\mathcal{I}(x) &= \lim_{\Lambda\to\infty} 2\Lambda^4\int \frac{d^4\widetilde{k}}{(2\pi)^4} \mathrm{e}^{-\widetilde{k}_\mu^2}\,\mathrm{Tr}_f\left(-\frac{1}{2\Lambda^4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) + \mathcal{O}(\Lambda^{-2}) \\
&= -\frac{1}{16\pi^2}\mathrm{Tr}_f\left(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \ .
\end{aligned} \tag{2.2.25}
$$

Tr_f means the trace for flavors.

Finally, we obtain the $U(1)_A$ anomaly as

$$\begin{aligned}
\log \mathcal{J} =& - i \int d^4x\, \alpha(x) \mathcal{I}(x) \\
=& i \int 2\alpha(x) \frac{1}{16\pi^2} \mathrm{Tr}_f \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \\
=& i \int 2\alpha(x) \frac{1}{8\pi^2} \mathrm{Tr}_f \left(F \wedge F \right) \\
=& - i \int 2\alpha(x) \left(\frac{i}{2\pi} \right)^{\frac{D}{2}} \mathrm{Tr}_f \left(\mathrm{e}^F \right) \big|_D \qquad (D = 4) \ .
\end{aligned} \tag{2.2.26}$$

$|_D$ means that take D-form part from the expansion of e^F. This is the anomaly in (2.2.11). The last expression of (2.2.26) is non-trivial from this calculation in the 4 dimension case, however, it is easy to generalize this calculation to D dimensional case for even natural number D. This anomaly can be written by the Chern number,

$$\begin{aligned}
\log \mathcal{J} =& - i \int 2\alpha(x) \left(\frac{i}{2\pi} \right)^{\frac{D}{2}} \mathrm{Tr}_f \left(\mathrm{e}^F \right) \big|_D \\
=& - i \int 2\alpha(x) \left[\mathrm{ch} \left(F \right) \right]_D \ ,
\end{aligned} \tag{2.2.27}$$

with the definition of the Chern character for F as

$$\left(\frac{i}{2\pi} \right)^{\frac{D}{2}} \mathrm{Tr}_f \left(\mathrm{e}^F \right) \big|_D = \left[\mathrm{ch} \left(F \right) \right]_D \ . \tag{2.2.28}$$

In chapter 3, we will discuss the generalization of this derivation.

2.2.3 Anomaly polynomial and descent equation

To understand the anomaly, an anomaly polynomial is an important tool. The anomaly polynomial connects anomalies in D dimension QFT and Chern numbers in $D+2$ dimension.

To classify the anomalies in D dimensional QFTs, it is well known that the anomaly polynomials in $D+2$ dimension are important. We can derive the anomalies in D dimensional QFTs from $D+2$ dimensional anomaly polynomials through the descent equation. In this subsection, we review the anomaly polynomials and the descent equations.

Let us consider the anomaly as a phase that comes from the transformation of the effective action Γ. The effective action is defined as the log of the partition function,

$$e^{-\Gamma[A]} \equiv Z[A] = \int [d\psi d\overline{\psi}]\, e^{-S(\psi, \overline{\psi}, A)} \ . \tag{2.2.29}$$

As we already saw in (2.2.11), this partition function can obtain a non-trivial phase under some chiral gauge transformations even though the action is invariant. (2.2.11) is an example of the anomaly, but we can generalize the anomaly such as (2.2.27). In this section, we consider the anomaly for general spacetime dimension, which comes from the continuous group $G = U(N)_+ \times U(N)_-$ for even dimension or $G = U(N)$ for odd dimension.

Under an infinitesimal chiral gauge transformation ($U_+ = e^{-v_+} \in U(N)_+$, $U_- = e^{-v_-} \in U(N)_-$ with $v_+, v_- \ll 1$) with

$$\delta_v A_+ = dv_+ + [A_+, v_+] \ , \quad \delta_v A_- = dv_- + [A_-, v_-] \ , \tag{2.2.30}$$

The effective action for the massless case $\Gamma[A] \equiv \Gamma[A, m=0]$ defined in (2.2.29) transforms as $\Gamma \to \Gamma + \delta_v \Gamma$ with

$$\delta_v \Gamma[A] = \int I^1_{2r}(v, A) \ , \tag{2.2.31}$$

where $I^1_{2r}(v, A)$ is a $2r$-form obtained as a solution of the descent equations

$$dI^1_{2r} = \delta_v I^0_{2r+1} \ , \quad dI^0_{2r+1} = I_{2r+2} \tag{2.2.32}$$

with

$$I_{2r+2}(A) = -2\pi i \left[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)\right]_{2r+2} \ . \tag{2.2.33}$$

Here, $[\cdots]_{2r+2}$ denotes the $(2r+2)$-form part of the differential form in the square brackets and $\mathrm{ch}(F_\pm) = \mathrm{Tr}\left(e^{\frac{i}{2\pi}F_\pm}\right)$ is the Chern character. $I_{2r+2}(A)$ is called the anomaly polynomial and $I^0_{2r+1}(A)$ is the CS $(2r+1)$-form.[7]

As pioneered by Fujikawa in [3, 4], the chiral anomaly (2.2.31) can be understood as a consequence of the fact that the path integral measure for the fermions is not invariant under the chiral transformation (3.2.7). After a careful regularization, it can be shown that the fermion path integral measure transforms as

$$[d\psi d\overline{\psi}] \to [d\psi d\overline{\psi}]\, \mathcal{J} \tag{2.2.34}$$

with the Jacobian $\mathcal{J}$ given by

$$\log \mathcal{J} = \int I^1_{2r}(v, A) \tag{2.2.35}$$

[7]Here, we consider a flat spacetime. We can extend this formula for curved spacetime. To extend, $\mathrm{ch}(F)$ should be replaced with $\mathrm{ch}(F)\widehat{A}(R)$, where $\widehat{A}(R)$ is the $\widehat{A}$-genus.

under the infinitesimal chiral transformation, reproducing the result in (2.2.31).

The form of the Jacobian $\mathcal{J}$ in (2.2.34) depends on the regularization. In [4, 9], a manifestly gauge covariant form of the anomaly with

$$\log \mathcal{J} = \int I_{2r}^{1\,\mathrm{cov}}(v, A) \ , \tag{2.2.36}$$

where

$$I_{2r}^{1\,\mathrm{cov}}(v, A) = \left(\frac{i}{2\pi}\right)^r \frac{1}{r!} \left(\mathrm{Tr}(v_+ F_+^r) - \mathrm{Tr}(v_- F_-^r)\right) \tag{2.2.37}$$

is obtained with a covariant regularization. (See section 3.2.1.) This form of the anomaly is called the covariant anomaly, while (2.2.31) is called the consistent anomaly. Unlike the consistent anomaly, the covariant anomaly does not satisfy the descent equations (2.2.32) and cannot be written as the gauge variation of a well-defined effective action. The consistent and covariant anomalies are related by the addition of a Bardeen-Zumino counterterm in the associated currents.[24] (See Appendix B.)

In the previous section, we calculated the anomaly for the $U(1)_A$ transformation which corresponds to $v_+ = -i\alpha(x)\, 1_N$ and $v_- = i\alpha(x)\, 1_N$ with a function $\alpha(x)$ and the unit matrix 1_N. More precisely, what we calculated was the mixed anomaly between $U(1)_A$ and $U(N)_V$. In this case (2.2.37) is

$$I_4^{1\,\mathrm{cov}}(-i\alpha\gamma_5, A) = [-i\alpha \mathrm{ch}(F_+) - i\alpha \mathrm{ch}(F_-)]_4 = -2i\alpha\, [\mathrm{ch}(F)]_4 = \frac{\alpha}{\pi} I_4(A) \ . \tag{2.2.38}$$

In the next chapter, we focus on the anomaly for the $U(1)_V$ transformation $v_+ = v_- = -i\alpha(x)\, 1_N$. This corresponds to the mixed anomaly between $U(1)_V$ and $SU(N)_+ \times SU(N)_- \times U(1)_A$. (2.2.37) is

$$I_{2r}^{1\,\mathrm{cov}}(-i\alpha, A) = -i\alpha\, [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} = \frac{\alpha}{2\pi} I_{2r}(A) \ . \tag{2.2.39}$$

2.2.4 Topological numbers in QFT

When D dimension QFT has non-trivial topological numbers in D dimension, we can write θ terms for its QFT. These topological numbers in D dimension do not affect to the anomalies of the D dimensional QFT, but here we check the topological terms to prepare upcoming sections. The topological numbers relate to the homotopy group, but the relationships between topological numbers and homotopy groups are different depending on the theory we consider.

Topological terms in gauge theories

We consider a gauge theory in D dimension. The gauge group is G, and we assume G is a continuous group. For simplicity, we consider the spacetime manifold is a D dimensional sphere S^D. For this case, the homotopy group of $\pi_D(BG)$ is important to distinguish the existence of topological terms. BG is a classifying space of G. In other words, if there are some instantons in D dimensional flat spacetime $\mathbb{R}^D$, then there is a winding number that comes from the configuration of the gauge field in infinite distance. This winding number is classified on S^{D-1}, which surrounds $\mathbb{R}^D$ at the point at infinity. Therefore, the topological number is classified by $\pi_{D-1}(G)$, and $\pi_D(BG) \simeq \pi_{D-1}(G)$.

When D dimensional G-gauge theory has a topological number $\pi_{D-1}(G) \simeq \mathbb{Z}$, it has a θ term.

For example, let us consider 4 dimension $SU(N)$ Yang-Mills theory. $\pi_3(SU(N)) \simeq \mathbb{Z}$, so this theory has a integer instanton number and a θ term. The instanton number is defined as

$$n = \int \frac{1}{8\pi^2} \mathrm{tr}\left[f \wedge f\right] \in \mathbb{Z} \ . \tag{2.2.40}$$

We can write the action with the θ term which comes from the integer (2.2.40) as,

$$S_{YM} = -\int \frac{1}{2g^2} \mathrm{tr}\left[f \wedge *f\right] + \int \frac{i\theta}{8\pi^2} \mathrm{tr}\left[f \wedge f\right] \tag{2.2.41}$$

θ parameter has 2π periodicity, because the Boltzmann weight for the θ term can be written by a integer n in (2.2.40) as,

$$\mathrm{e}^{-S_\theta} = \mathrm{e}^{-\int \frac{i\theta}{8\pi^2} \mathrm{tr}[f \wedge f]} = \mathrm{e}^{-i\theta n} \ . \tag{2.2.42}$$

Topological terms in scalar field theories

We can consider the topological number and the θ term not only for gauge theories but also for scalar field theories. Let us consider a scalar field theory in D dimension. Let the target space of the scalar field G. If we consider this G-valued scalar field theory on D dimensional sphere S^D. This scalar field means the map from S^D to G. Then we can define the topological number $\pi_D(G)$. We can define the θ term comes from this topological number. Let us consider S^1 valued scalar field $\phi + 2\pi \sim \phi$ on S^1 for an example. This theory has integer valued topological number comes from $\pi_1(S^1) \simeq \mathbb{Z}$ as,

$$n = \int \frac{1}{2\pi} d\phi \ . \tag{2.2.43}$$

We can consider this scalar field theory with the θ term is,

$$S = \int \frac{1}{2}|d\phi|^2 + \int \frac{i\theta}{2\pi} d\phi \ . \tag{2.2.44}$$

This θ parameter is also 2π periodic. This θ term can be generalized to higher dimensional scalar QFTs or general non-linear sigma models. For example, 2 dimensional $\mathbb{C}P^{N-1}$ model is one of the famous examples of them, and this use $\pi_2(\mathbb{C}P^{N-1}) \simeq \mathbb{Z}$. However, in higher spacetime dimension cases such as $D > 2$, it is needed to D derivatives. In scalar field theories, higher derivative terms are suppressed. When $D > 2$ case, the θ terms have higher derivative than the kinetic term, so they may not matter in their IR phases.

These topological numbers of scalar field theories also appear in WZW terms. Indeed, these θ terms in D dimensional scalar QFTs can be written as total derivative terms, so this term only depends on $D-1$ dimension. In $D-1$ dimensional scalar QFTs, we can consider a topological term corresponding to $\pi_D(G) \simeq \mathbb{Z}$, and this term is called the WZW term. We will consider the WZW term in QCD in section 2.3.

We can consider topological defects made by scalar fields. These defects are similar objects to the instantons in gauge theories. For example, Skyrimon is such an object in a scalar field theory. Let us consider a 2 dimensional scalar field theory and consider the following configuration

$$\phi(x) = u\left(x_1 + ix_2\right) \ , \tag{2.2.45}$$

where ϕ is a scalar field in this theory and u is a complex valued parameter. This ϕ is one component complex scalar field, so it takes the value on $\mathbb{C}$. We take $\mathbb{R}^2$ as the spacetime of this scalar field theory, and x_1 and x_2 are coordinates of $\mathbb{R}^2$. In this case, the topological number or the number of the defect is counted on the point at infinity, whose topology is S^1. The target space of the scalar field is $\mathbb{C}$, but now we are interested in its winding number, so we focus on the value of $\phi/|\phi|$. The space of this value is $\mathbb{C}/|\mathbb{C}| \simeq \mathbb{C}/\mathbb{R}_{\geq 0} \simeq S^1$. Then, the topological number of this system is defined by $\pi_1(S^1) \simeq \mathbb{Z}$. The configuration (2.2.45) correspond to the one soliton case $1 \in \mathbb{Z}$.

The scalar field with this configuration (2.2.45) cannot be treated as a usual QFT, because the value of the scalar field diverges at the point of infinity. However, this topological number will be used later.

2.3 Review of the anomaly in QCD

In this section, we review two examples of anomalies in QCD.

The starting point of this section is, the IR effective theory of QCD is written by the pion theory (2.1.7).

2.3.1 $U(1)_A$ problem

In this subsection, we review the $U(1)_A$ problem in QCD. This was a problem in the 1970's. At that time, chiral SSB was well-known but the effect of the anomaly was not completely known. If we consider QCD without any anomalies, the number of pions is different from the real world. This problem became popular because the experimental data of η' meson mass was much heavier than the mass people expected. This problem was solved by considering the effect of $U(1)_A$ anomaly.[5, 6]

QCD without anomaly

Let us assume that QCD does not have any anomalies. Of course, this assumption is incorrect, we can learn the importance of the anomaly from this example.

If there is no anomaly in QCD, chiral symmetry (2.1.3) is deformed to

$$U(N_f)_+ \times U(N_f)_- \ . \tag{2.3.1}$$

Chiral SSB for this theory is,

$$U(N_f)_+ \times U(N_f)_- \to U(N_f)_V \ , \tag{2.3.2}$$

and pion takes its value on

$$\frac{U(N_f)_+ \times U(N_f)_-}{U(N_f)_V} \ . \tag{2.3.3}$$

This "pion" is different from real pion (2.1.6). Let us consider $N_f = 3$ case, following the history of the $U(1)_A$ problem. When $N_f = 3$, the usual pion (2.1.6) includes 8 degrees of freedom, corresponding to the number of generators in $SU(3)$. This means we have eight pions in three flavor case. On the other hand, if we do not consider anomalies, the SSB pattern (2.3.2) creates pions in $U(3)$. This means that there are nine pions because $U(3) \simeq SU(3) \times U(1)$ has 9 generators. The 9th (or wrong) "pion" corresponds to $U(1)_A$ part, and it is called as η' meson.

In experimental results, this η' meson mass is much heavier than the other pions. This is called $U(1)_A$ problem, and the solution of this problem is $U(1)_A$ anomaly. When we consider ABJ anomaly of $U(1)_A$, QCD does not have global symmetry $U(N_f)_+ \times U(N_f)_-$, but the correct one should be divided by $U(1)_A$ (2.1.3). This means η' meson is not the NG boson of chiral SSB. This effect was pointed out by Witten and Veneziano.[5, 6] After considering this effect, η' meson should be massive and it consistent with the experimental results.

$U(1)_A$ anomaly in QCD

QCD has an ABJ anomaly in $U(1)_A$. More precisely, QCD has a mixed anomaly between $U(1)_A$ and $SU(N_c)$ gauge symmetry, with the anomaly polynomial

$$\left(\frac{i}{2\pi}\right)^3 F^A \wedge \mathrm{tr}\left[f \wedge f\right] \ , \tag{2.3.4}$$

where F^A is a field strength of a background $U(1)_A$ gauge field and f is a field strength of a dynamical $SU(3)$ gauge field or gluons. The calculation of the anomaly is almost the same as the calculation in section 2.2.2. The anomaly can be seen as

$$\begin{aligned} Z =& \int [da]\, Z[a] \\ \to& \int [da] \mathrm{e}^{\int \frac{i}{4\pi^2}\alpha \mathrm{tr}\left[f^V \wedge f^V\right]} Z[a] \\ \neq& Z \ . \end{aligned} \tag{2.3.5}$$

This anomaly means that QCD does not have $U(1)_A$ symmetry. Therefore, we identify the symmetry as (2.1.3).

2.3.2 't Hooft anomaly matching

QCD also has an 't Hooft anomaly in (2.1.3). To respect the 't Hooft anomaly matching, the IR effective theory of QCD, or pion theory (2.1.7), should reproduce the anomaly. However, the naive pion action (2.1.7) cannot match the anomaly, so we need to add some terms to it. This new term to match the anomaly is known as the Wess-Zumino-Witten term (WZW term).[7, 8] WZW term includes coupling between pions and background gauge fields of the chiral symmetry in QCD (2.1.3), and topological term of pions. The anomaly matching requires not only a coupling to gauge fields but also a topological term that includes only pions. The explicit form of the WZW term is written in these papers.[7, 8] This form is too long because the WZW term includes $SU(N_f)_+ \times SU(N_f)_-$ gauge fields, and these couplings with pions are complicated.[8] We check only WZW term without background gauge fields here,[9]

$$S_{WZW} = -\int \frac{N_c}{240\pi^2} \mathrm{tr}\left(U^\dagger dU\right)^5 \ . \tag{2.3.6}$$

[8]The chiral symmetry (2.1.3) also includes $U(1)_V$ part, but this part does not have an anomaly. So we can neglect it in WZW term.

[9]The large number 240 comes from $240 = 2 \cdot 5!$.

This term includes a 5-form, even though we consider 4 dimensional QCD. This is because this WZW term in (2.3.6) can be written as a total derivative, and this term does not depend on the manifold of the 5th direction. This term counts the topological number of $\pi_5(SU(N_f)) \simeq \mathbb{Z}$ when $N_f \geq 3$.[10] The full pion action that respect to the anomaly matching is,

$$S_\pi = \int d^4x \, \frac{f_\pi^2}{4} \mathrm{tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] - \int \frac{N_c}{240\pi^2} \mathrm{tr}\left(U^\dagger dU\right)^5 + (\text{couplings between pions and gauge fields}) + (\text{higher derivative terms}) \ . \tag{2.3.7}$$

WZW term provides not only the coupling between pions, but also the coupling between pion and photon. In SM, QCD couples to $SU(2)_W \times U(1)_Y$ or $U(1)_{EM}$ gauge fields. It is known that the QCD scale is smaller than the electro-weak scale, so it is better to consider that pion couples to photon in $U(1)_{EM}$. These electro-weak gauge couplings are small enough in IR, hence we can treat them as the perturbation theories. In weak coupling limit, we can neglect the dynamics of these gauge fields, then the global symmetries corresponding to these gauge groups are retained. This is why we can treat these symmetries as global symmetries, even though they have dynamical gauge fields.

In our real world, π^0 meson decays into two photons, through the WZW term. This decay process $\pi^0 \to 2\gamma$ is important, because there are no stable pions in our world. This coupling has its origin in the anomaly matching.

[10] In the case of $N_f = 2$, $\pi_5(SU(2)) \simeq \mathbb{Z}_2$. This means that we cannot write the usual WZW term (2.3.6) for $N_f = 2$ case.

Chapter 3

Derivation of the anomaly with spacetime dependent mass

In this chapter, we derive the anomaly of N free fermions with spacetime dependent mass. The main claim of this section is, that we find a new anomaly that comes from the spacetime dependent mass by Fujikawa's method. The anomaly can be described by the anomaly $D+2$ form of (3.3.11).

We introduce the superconnection first and then calculate the anomaly. We show the anomaly can be written by using superconnection. In the last section of this chapter, we check the relation between this anomaly and the string theory. If we consider the string theory with some D-branes, this anomaly has its very natural origin.

3.1 Superconnection

Before the calculation of the anomaly, we review the superconnection, which was introduced by Quillen[14]. From the physical point of view, this superconnection roughly corresponds to the background gauge fields with mass[1]. Its application to physics is considered in the next section. The important thing here is we consider the spacetime dependent mass as a background field like the gauge fields. We just introduce some parts of the superconnection that we use in this thesis. See, e.g., [14, 25] for more general and mathematically rigorous descriptions.

[1] In this paper, the word "superconnection" is used for the background gauge field with mass $\mathcal{A}$ rather than the covariant derivative $d+\mathcal{A}$, which is often used in mathematical literature.

3.1.1 Even dimension case

Let us consider N Dirac fermions in even spacetime dimension. If these fermions are massless, they have $U(N)_+ \times U(N)_-$ chiral symmetry. We can introduce the background gauge fields corresponding to the chiral symmetry, and we denote them as (A_+, A_-). We also introduce a scalar field T, whose representation is bifundamental under $U(N)_+ \times U(N)_-$ chiral symmetry. We can introduce a superconnection as a background field which includes (A_+, A_-) and T for this system.

A superconnection $\mathcal{A}$ of the even type is a matrix-valued field composed of (A_+, A_-) and T as

$$\mathcal{A} = \begin{pmatrix} A_+ & iT^\dagger \\ iT & A_- \end{pmatrix} = A_+ e^+ + A_- e^- + iT^\dagger \sigma^+ + iT\sigma^- \ , \tag{3.1.1}$$

where

$$e^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \quad e^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{3.1.2}$$

In our notation, the gauge fields $A_\pm = A_{\pm\mu}(x)dx^\mu$ are one-forms that take values in anti-Hermitian $N \times N$ matrices. $\sigma^\pm$ in (3.1.1) and dx^μ are treated as fermions, *i.e.*, they anti-commute with each other in the products. The field strength of the superconnection is defined as[2]

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F_+ - T^\dagger T & iDT^\dagger \\ iDT & F_- - TT^\dagger \end{pmatrix} \ , \tag{3.1.3}$$

where $F_\pm$ is a field strength for $A_\pm$, and D is a covariant derivative for the bifundamental scalar field T.

$$\begin{aligned} F_\pm \equiv& dA_\pm + A_\pm^2 \ , \\ DT \equiv& dT + A_- T - TA_+ \ , \qquad DT^\dagger \equiv dT^\dagger + A_+ T^\dagger - T^\dagger A_- \ . \end{aligned} \tag{3.1.4}$$

The Chern character is also defined for $\mathcal{F}$ as

$$\mathrm{ch}(\mathcal{F}) \equiv \sum_{k \geq 0} \left(\frac{i}{2\pi} \right)^{k/2} \left[\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}} \right) \right]_k \ , \tag{3.1.5}$$

where $[\cdots]_k$ denotes the k-form part of the differential form in the square brackets, and 'Str' is the supertrace[3] defined by

$$\mathrm{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{Tr}(a) - \mathrm{Tr}(d) \ . \quad \text{(even case)} \tag{3.1.6}$$

[2]The products of differential forms are the wedge product, though the symbol for the wedge product '$\wedge$' are omitted.

[3]In some literature, the symbol 'Str' is used for the symmetrized trace, which should not be confused with the supertrace in this paper. For the symmetrized trace, we use $\mathrm{Tr}^{\mathrm{sym}}$.

Because of (3.1.6), only the even form part in (3.1.5) can be non-zero.

In the following chapter, we use a one-parameter family of superconnections denoted as $\mathcal{A}_t$ with a parameter $t \in [0,1]$. The following formula for $\mathcal{A}_t$ is useful.

$$\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_1}\right) - \mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_0}\right) = d\left(\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_t}\partial_t\mathcal{A}_t\right)\right) \ , \tag{3.1.7}$$

where $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2$. For $\mathcal{A}_t = \mathcal{A}|_{T\to tT} = \mathcal{A}_0 + t\mathcal{T}$ with $\mathcal{A}_0 = A_+e^+ + A_-e^-$ and $\mathcal{T} = iT^\dagger\sigma^+ + iT\sigma^-$, this formula implies

$$\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right) = \mathrm{Tr}(\mathrm{e}^{F_+}) - \mathrm{Tr}(\mathrm{e}^{F_-}) + d\left(\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_t}\mathcal{T}\right)\right) \ . \tag{3.1.8}$$

Since $\mathrm{Str}(\mathrm{e}^{\mathcal{F}_t}\mathcal{T})$ is gauge invariant, (3.1.8) implies that $\mathrm{ch}(\mathcal{F})$ and $\mathrm{ch}(F_+) - \mathrm{ch}(F_-)$ are equivalent up to an exact form. For a trivial bundle (or, in a local patch) the formula (3.1.7) with $\mathcal{A}_t = t\mathcal{A}$ implies[4]

$$\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right) = d\left(\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{td\mathcal{A}+t^2\mathcal{A}^2}\mathcal{A}\right)\right) \ . \tag{3.1.9}$$

This implies that the Chern character can be expressed locally as

$$\mathrm{ch}(\mathcal{F}) = d\Omega \tag{3.1.10}$$

where Ω is the Chern-Simons (CS) form given by

$$\Omega = \sum_{k\geq 0}\left(\frac{i}{2\pi}\right)^{(k+1)/2}\left[\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{td\mathcal{A}+t^2\mathcal{A}^2}\mathcal{A}\right)\right]_k \ . \tag{3.1.11}$$

This Ω is, in general, not gauge invariant.

3.1.2 Odd dimension case

If we consider N Dirac fermions in odd dimension, the situation is changed. In odd dimension, there is no chirality operator like γ_5 for 4 dimension, so the symmetry is just $U(N)$ for massless fermions. We introduce a $U(N)$ background gauge field A and a scalar field T in the adjoint representation of $U(N)$.

The superconnection of the odd type is given by (3.1.1) with the restrictions $A_+ = A_-$ and $T = T^\dagger$:

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix} = A\,1_2 + iT\sigma_1 \ , \tag{3.1.12}$$

[4] When the gauge group is $U(N_+)\times U(N_-)$ with $N_+ \neq N_-$, the right hand side has an additional constant term $N_+ - N_-$.

where $1_2 = e^+ + e^-$ is the unit matrix of size 2 and $\sigma_1 = \sigma^+ + \sigma^- = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The field strength is

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix} \tag{3.1.13}$$

with $F \equiv dA + A^2$ is a field strength for the gauge field A, and $DT \equiv dT + [A, T]$ is a covariant derivative for T.

The supertrace for the odd case is defined as

$$\mathrm{Str} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \equiv \sqrt{2}\, i^{-3/2}\, \mathrm{Tr}(b) \ . \quad \text{(odd case)} \ . \tag{3.1.14}$$

The reason for putting the normalization factor $\sqrt{2}\, i^{-3/2}$ will become clear later.[5] We also define an analog of the Chern character for the odd case by the same formula as above (3.1.5). In this case, only the odd form part contributes. The formulas (3.1.7)–(3.1.11) also hold for the odd case. In particular, (3.1.8) with $A_+ = A_-$ and $T = T^\dagger$ gives

$$\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right) = d\left(\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_t} iT\sigma_1\right)\right) \ , \tag{3.1.15}$$

where $\mathcal{F}_t = (F - t^2T^2)1_2 + itDT\sigma_1$. Therefore, the Chern character can also be written as

$$\mathrm{ch}(\mathcal{F}) = d\Omega' \ , \tag{3.1.16}$$

where

$$\Omega' = \sum_{k \geq 0} \left(\frac{i}{2\pi}\right)^{(k+1)/2} \left[\int_0^1 dt\, \mathrm{Str}\left(\mathrm{e}^{\mathcal{F}_t} iT\sigma_1\right)\right]_k \ . \tag{3.1.17}$$

Unlike Ω in (3.1.11), this Ω' is gauge invariant.

3.2 Derivation

We calculate the anomaly with the spacetime dependent mass. We focus on the chiral symmetries of the free fermion systems. We consider N Dirac fermions. This theory has $U(N)_+ \times U(N)_-$ symmetry in even dimensions, and $U(N)$ symmetry in odd dimensions. We already know that these systems have some anomalies without mass terms. However, we show that these systems have anomalies even with mass terms.

In this section, we follow the calculation in section 2.2.2.

[5]The sign ambiguity of $i^{-3/2}$ is compensated by that of the $i^{k/2}$ factor in (3.1.5). Namely, the supertrace Str of the odd case always appears in the combination $i^{k/2}\mathrm{Str}$ with odd k in the anomaly, and $i^{k/2}\mathrm{Str}\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) = \sqrt{2}\, i^{(k-3)/2}\mathrm{Tr}(b)$ has no ambiguity.

3.2.1 Even dimension case

Chiral symmetry with a mass term

Let us consider N Dirac fermions ψ in a $D = 2r$-dimensional flat Euclidean spacetime ($r \in \mathbb{Z}_{>0}$). We couple external gauge fields and a spacetime dependent mass to this system. We write the external gauge fields as $A = (A_+, A_-)$, and they associate with $U(N)_+ \times U(N)_-$ chiral symmetry. We also write a spacetime dependent mass m, which belongs to the bifundamental representation of $U(N)_+ \times U(N)_-$. This mass can be regarded as a background scalar field (or a Higgs field), but here we just call it a mass. We treat this mass like a background gauge field. Although we discuss N Dirac fermions, it is easy to get the results for $N_\pm$ positive/negative chirality Weyl fermions by considering a $U(N_+)_+ \times U(N_-)_-$ subgroup of $U(N)_+ \times U(N)_-$ with large enough N. The action is

$$
\begin{aligned}
S &= \int d^D x \left(\overline{\psi}_+ \not{D}_+ \psi_+ + \overline{\psi}_- \not{D}_- \psi_- + \overline{\psi}_- m \psi_+ + \overline{\psi}_+ m^\dagger \psi_- \right) \\
&= \int d^D x \, \overline{\psi} \mathcal{D} \psi \ ,
\end{aligned}
\tag{3.2.1}
$$

where

$$
\psi(x) \equiv \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \ , \qquad \overline{\psi}(x) \equiv \left(\overline{\psi}_+(x), \overline{\psi}_-(x) \right) \ , \tag{3.2.2}
$$

and

$$
\mathcal{D} \equiv \begin{pmatrix} \not{D}_+ & m^\dagger(x) \\ m(x) & \not{D}_- \end{pmatrix} \ , \quad \not{D}_+ \equiv \sigma^{\mu\dagger}(\partial_\mu + A_{+\mu}) \ , \quad \not{D}_- \equiv \sigma^{\mu}(\partial_\mu + A_{-\mu}) \ . \tag{3.2.3}
$$

σ^μ and $\sigma^{\mu\dagger}$ ($\mu = 1, 2, \cdots, D$) are $2^{r-1} \times 2^{r-1}$ matrices satisfying

$$
\sigma^{\mu\dagger}\sigma^\nu + \sigma^{\nu\dagger}\sigma^\mu = \sigma^\nu\sigma^{\mu\dagger} + \sigma^\mu\sigma^{\nu\dagger} = 2\delta^{\mu\nu} \ , \tag{3.2.4}
$$

so that

$$
\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0 \end{pmatrix} \ , \qquad (\mu = 1, 2, \cdots, 2r) \tag{3.2.5}
$$

are D-dimensional gamma matrices in a chiral representation. We choose a representation of γ^μ such that

$$
\gamma^1\gamma^2 \cdots \gamma^{2r} = i^r \begin{pmatrix} 1_{2^{r-1}} & 0 \\ 0 & -1_{2^{r-1}} \end{pmatrix} \equiv i^r \gamma^{2r+1} \tag{3.2.6}
$$

is satisfied, where γ^{2r+1} is the chirality operator. This notation is useful for our purpose, but is not a standard one. A more standard notation is obtained by replacing $\overline{\psi}$ and $\mathcal{D}$ with $\overline{\psi}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{D}$, respectively.

In our action (3.2.1), mass m means not just a parameter but a spacetime dependent background field. Then, the classical action is invariant under $U(N)_+ \times U(N)_-$ chiral gauge transformation that acts on the external (gauge and scalar) fields as well as the dynamical fermions as

$$\begin{aligned}
&\psi_+ \to U_+\psi_+ \ , \quad \overline{\psi}_+ \to \overline{\psi}_+ U_+^{-1} \ , && \psi_- \to U_-\psi_- \ , \quad \overline{\psi}_- \to \overline{\psi}_- U_-^{-1} \ , \\
&A_+ \to U_+ A_+ U_+^{-1} + U_+ dU_+^{-1} \ , && A_- \to U_- A_- U_-^{-1} + U_- dU_-^{-1} \ , \\
&m \to U_- m\, U_+^{-1} \ , && m^\dagger \to U_+ m^\dagger U_-^{-1} \ ,
\end{aligned} \tag{3.2.7}$$

with $(U_+(x), U_-(x)) \in U(N)_+ \times U(N)_-$.

Calculation of the anomaly

In order to calculate the anomaly, we evaluate the Jacobian $\mathcal{J}$ for the $U(N)_+ \times U(N)_-$ transformation (3.2.7). In the following, we demonstrate the derivation of the anomaly in detail focusing on the $U(1)_V$ transformation that acts on the fermions as

$$\psi(x) \to \mathrm{e}^{i\alpha(x)}\psi(x) \ , \qquad \overline{\psi}(x) \to \mathrm{e}^{-i\alpha(x)}\overline{\psi}(x) \ , \tag{3.2.8}$$

which is a special case of the transformation in (3.2.7) with $U_+ = U_- = e^{i\alpha}1_N$. The generalization to general $U(N)_+ \times U(N)_-$ transformations is straightforward.

Following [9], we expand the fermion fields ψ and $\overline{\psi}$ with respect to the eigenfunctions of the Hermitian operators $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$, respectively. Let n_ϕ and n_φ be the number of zero modes of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$, respectively, and choose the eigenfunctions such that they satisfy the eigenequations[6]

$$\mathcal{D}^\dagger\mathcal{D}\varphi_n(x) = \lambda_n^2\varphi_n(x) \ , \qquad (n \in \{\, k - n_\varphi \,|\, k = 1, 2, 3, \cdots \}) \ , \tag{3.2.9}$$

$$\mathcal{D}\mathcal{D}^\dagger\phi_n(x) = \lambda_n^2\phi_n(x) \ , \qquad (n \in \{\, k - n_\phi \,|\, k = 1, 2, 3, \cdots \}) \ , \tag{3.2.10}$$

and the normalization conditions

$$\int d^Dx\, \varphi_m^\dagger(x)\varphi_n(x) = \delta_{m,n} \ , \qquad \int d^Dx\, \phi_m^\dagger(x)\phi_n(x) = \delta_{m,n} \ . \tag{3.2.11}$$

[6]Here, we have assumed that the spectra of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are discrete. Later, we will consider the cases with non-compact spacetime. In such cases, the asymptotic behavior of the mass and the gauge fields should be chosen appropriately to have discrete spectra.

Here, the eigenvalues of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are denoted as λ_n^2, because they are non-negative and can be written as the square of real numbers.[7] Without loss of generality, we assume $\lambda_n = 0$ for $n \le 0$ and $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$. Note that the eigenvalues for (3.2.9) and (3.2.10) are the same, because the non-zero modes φ_n and ϕ_n with $n > 0$ are related by

$$\phi_n(x) = \frac{1}{\lambda_n}\mathcal{D}\varphi_n(x) \ , \qquad \varphi_n(x) = \frac{1}{\lambda_n}\mathcal{D}^\dagger\phi_n(x) \ , \qquad (\text{for } n > 0) \ , \tag{3.2.12}$$

up to phase.

Then, fermions $\psi(x)$ and $\overline{\psi}(x)$ can be expanded as

$$\psi(x) = \sum_n a_n\varphi_n(x) \ , \qquad \overline{\psi}(x) = \sum_n \bar{b}_n\phi_n^\dagger(x) \ , \tag{3.2.13}$$

where a_n and $\bar{b}_n$ are Grassmann odd coefficients, and the action (3.2.1) becomes

$$S = \sum_n \lambda_n \bar{b}_n a_n \ . \tag{3.2.14}$$

The fermion path integral measure is formally defined as

$$[d\psi d\overline{\psi}] = \prod_x d\psi(x)d\overline{\psi}(x) = \det(\varphi_n(x))^{-1}\det(\phi_n^\dagger(x))^{-1}\prod_m da_m \prod_l d\bar{b}_l \ , \tag{3.2.15}$$

where $\det(\varphi_n(x))^{-1}\det(\phi_n^\dagger(x))^{-1}$ is the Jacobian for the change of variables from $\{\psi(x), \overline{\psi}(x)\}$ to $\{a_n, \bar{b}_n\}$.

Under the $U(1)_V$ transformation (3.2.8), a_n and $\bar{b}_n$ transforms as

$$\begin{aligned} a_n \to& a_n' \equiv \int d^Dx\, \varphi_n^\dagger(x) e^{i\alpha(x)}\psi(x) \simeq \sum_m \left(\delta_{m,n} + i\int d^Dx\, \varphi_n^\dagger(x)\alpha(x)\varphi_m(x)\right) a_m \ , \\ \bar{b}_n \to& \bar{b}_n' \equiv \int d^Dx\, \overline{\psi}(x) e^{-i\alpha(x)}\phi_n(x) \simeq \sum_m \bar{b}_m \left(\delta_{m,n} - i\int d^Dx\, \phi_m^\dagger(x)\alpha(x)\phi_n(x)\right) \ , \end{aligned} \tag{3.2.16}$$

where we have assumed $\alpha(x) \ll 1$. Then, the Jacobian (2.2.34) is

$$\log\mathcal{J} = -i\int d^Dx\, \alpha(x)\mathcal{I}(x) \ , \tag{3.2.17}$$

where

$$\mathcal{I}(x) \equiv \sum_n \left(\varphi_n^\dagger(x)\varphi_n(x) - \phi_n^\dagger(x)\phi_n(x)\right) \ . \tag{3.2.18}$$

[7] Be aware that λ_n is not the eigenvalue of $\mathcal{D}$. $\mathcal{D}$ is not Hermitian and its eigenvalues are not real in general.

$\mathcal{I}(x)$ can be regularized by introducing a UV cut-off Λ as

$$\begin{aligned}\mathcal{I}(x) &= \lim_{\Lambda\to\infty}\sum_n \mathrm{e}^{-\frac{\lambda_n^2}{\Lambda^2}}\left(\varphi_n^\dagger(x)\varphi_n(x) - \phi_n^\dagger(x)\phi_n(x)\right) \\ &= \lim_{\Lambda\to\infty}\sum_n \left(\varphi_n^\dagger(x)\mathrm{e}^{-\frac{1}{\Lambda^2}\mathcal{D}^\dagger\mathcal{D}}\varphi_n(x) - \phi_n^\dagger(x)\mathrm{e}^{-\frac{1}{\Lambda^2}\mathcal{D}\mathcal{D}^\dagger}\phi_n(x)\right) \\ &= \lim_{\Lambda\to\infty}\int \frac{d^Dk}{(2\pi)^D} e^{-ikx}\,\mathrm{Tr}_s\left(\mathrm{e}^{-\frac{1}{\Lambda^2}\mathcal{D}^\dagger\mathcal{D}} - \mathrm{e}^{-\frac{1}{\Lambda^2}\mathcal{D}\mathcal{D}^\dagger}\right)e^{ikx}\ , \end{aligned} \tag{3.2.19}$$

where Tr_s is the trace over both flavor and spinor indices. The cut-off Λ will be sent to infinity at the end of the calculation.[8]

To evaluate (3.2.19), note that $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ are written as

$$\mathcal{D}^\dagger\mathcal{D} = -\,D_\mu^2 - \Lambda^2\widehat{\mathcal{F}}\ , \qquad\qquad \mathcal{D}\mathcal{D}^\dagger = -\,D_\mu^2 - \Lambda^2\widehat{\mathcal{F}}'\ , \tag{3.2.20}$$

where

$$D_\mu = \begin{pmatrix} \partial_\mu + A_{+\mu} & 0 \\ 0 & \partial_\mu + A_{-\mu} \end{pmatrix}\ , \tag{3.2.21}$$

and

$$\widehat{\mathcal{F}} = \begin{pmatrix} \frac{1}{2\Lambda^2}\sigma^\mu\sigma^{\nu\dagger}F_{+\mu\nu} - \widetilde{m}^\dagger\widetilde{m} & \frac{1}{\Lambda}\sigma^\mu D_\mu\widetilde{m}^\dagger \\ \frac{1}{\Lambda}\sigma^{\mu\dagger}D_\mu\widetilde{m} & \frac{1}{2\Lambda^2}\sigma^{\mu\dagger}\sigma^\nu F_{-\mu\nu} - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix}\ , \tag{3.2.22}$$

$$\widehat{\mathcal{F}}' = \begin{pmatrix} \frac{1}{2\Lambda^2}\sigma^{\mu\dagger}\sigma^{\nu}F_{+\mu\nu} - \widetilde{m}^\dagger\widetilde{m} & -\frac{1}{\Lambda}\sigma^{\mu\dagger} D_\mu\widetilde{m}^\dagger \\ -\frac{1}{\Lambda}\sigma^{\mu}D_\mu\widetilde{m} & \frac{1}{2\Lambda^2}\sigma^{\mu}\sigma^{\nu\dagger} F_{-\mu\nu} - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix} \tag{3.2.23}$$

with $\widetilde{m} \equiv m/\Lambda$. In the following part, we use tilde to denote some variables divided by Λ. Then, (3.2.19) becomes

$$\begin{aligned}\mathcal{I}(x) &= \lim_{\Lambda\to\infty}\int \frac{d^Dk}{(2\pi)^D}\mathrm{Tr}_s\left(\mathrm{e}^{\frac{1}{\Lambda^2}(ik_\mu+D_\mu)^2+\widehat{\mathcal{F}}} - \mathrm{e}^{\frac{1}{\Lambda^2}(ik_\mu+D_\mu)^2+\widehat{\mathcal{F}}'}\right) \\ &= \lim_{\Lambda\to\infty}\Lambda^D\int \frac{d^D\widetilde{k}}{(2\pi)^D}\mathrm{e}^{-\widetilde{k}_\mu^2}\,\mathrm{Tr}_s\left(\mathrm{e}^{\frac{1}{\Lambda^2}D_\mu^2+\frac{2i}{\Lambda}\widetilde{k}^\mu D_\mu+\widehat{\mathcal{F}}} - \mathrm{e}^{\frac{1}{\Lambda^2}D_\mu^2+\frac{2i}{\Lambda}\widetilde{k}^\mu D_\mu+\widehat{\mathcal{F}}'}\right)\ , \end{aligned} \tag{3.2.24}$$

where $\widetilde{k}_\mu \equiv k_\mu/\Lambda$. Using the formula

$$\mathrm{tr}\left(\sigma^{\mu_1}\sigma^{\mu_2\dagger}\cdots\sigma^{\mu_{2k-1}}\sigma^{\mu_{2k}\dagger} - \sigma^{\mu_1\dagger}\sigma^{\mu_2}\cdots\sigma^{\mu_{2k-1}\dagger}\sigma^{\mu_{2k}}\right) = \begin{cases} 0 & (k<r) \\ (2i)^r\epsilon^{\mu_1\cdots\mu_{2r}} & (k=r) \end{cases}\ , \tag{3.2.25}$$

[8]The explicit form of the anomaly actually depends on the choice of the regularization scheme. We adopt this heat kernel regularization in a covariant form.

where $\epsilon^{\mu_1\cdots\mu_{2r}}$ is the Levi-Civita symbol with $\epsilon^{1,2,\cdots,D}=1$, and assuming that the gauge field, $\widetilde{m}$ and $\widetilde{k}_\mu$ as well as their derivatives are all of $\mathcal{O}(1)$ in the $1/\Lambda$ expansion, [9] it is easy to verify

$$\mathcal{I}(x)=\lim_{\Lambda\to\infty}\Lambda^D\int\frac{d^D\widetilde{k}}{(2\pi)^D}\mathrm{e}^{-\widetilde{k}_\mu^2}\,\mathrm{Tr}_s\left(\mathrm{e}^{\widehat{\mathcal{F}}}-\mathrm{e}^{\widehat{\mathcal{F}}'}\right)=\lim_{\Lambda\to\infty}\frac{\Lambda^D}{2^D\pi^{D/2}}\mathrm{Tr}_s\left(\mathrm{e}^{\widehat{\mathcal{F}}}-\mathrm{e}^{\widehat{\mathcal{F}}'}\right)\ ,\tag{3.2.26}$$

and

$$\mathrm{Tr}_s\left(\mathrm{e}^{\widehat{\mathcal{F}}}-\mathrm{e}^{\widehat{\mathcal{F}}'}\right)d^{2r}x=\Lambda^{-2r}(2i)^r\ \left[\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right)\right]_{2r}+\mathcal{O}(\Lambda^{-2r-1})\ ,\tag{3.2.27}$$

where $d^{2r}x=dx^1\cdots dx^{2r}$ and $\mathcal{F}$ is the superconnection defined as

$$\mathcal{F}=\begin{pmatrix}F_+-\widetilde{m}^\dagger\widetilde{m} & iD\widetilde{m}^\dagger\\ iD\widetilde{m} & F_--\widetilde{m}\widetilde{m}^\dagger\end{pmatrix}\ .\tag{3.2.28}$$

Neglecting the $\mathcal{O}(\Lambda^{-1})$ terms, this implies[10]

$$\mathcal{I}(x)d^{2r}x=\left(\frac{i}{2\pi}\right)^r\left[\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right)\right]_{2r}=\left[\mathrm{ch}(\mathcal{F})\right]_{2r}\ ,\tag{3.2.29}$$

and hence we obtain

$$\log\mathcal{J}=-i\int\alpha(x)\left[\mathrm{ch}(\mathcal{F})\right]_D\ ,\tag{3.2.30}$$

which is the desired result. We evaluate this result in the next section.

In section 4, we consider the cases with $A_+=A_-$ and the mass given by a scalar matrix as

$$m=\mu(x)1_N\ ,\tag{3.2.31}$$

where $\mu(x)$ is a complex function and 1_N is the unit matrix of size N. In this case, we have

$$\mathrm{ch}(\mathcal{F})=\frac{i}{2\pi}d\widetilde{\mu}^\dagger d\widetilde{\mu}\,\mathrm{e}^{-|\widetilde{\mu}|^2}\mathrm{ch}(F)\tag{3.2.32}$$

with $F\equiv F_+=F_-$ and $\widetilde{\mu}\equiv\mu/\Lambda$, and the Jacobian (3.2.30) becomes

$$\log\mathcal{J}=\frac{1}{2\pi}\int d\widetilde{\mu}^\dagger d\widetilde{\mu}\,\mathrm{e}^{-|\widetilde{\mu}|^2}\alpha(x)\left[\mathrm{ch}(F)\right]_{D-2}\ .\tag{3.2.33}$$

[9] In section 4, we consider the cases with m being a linear function of x^μ. One may wonder whether $\widetilde{m}$ can be regard as an $\mathcal{O}(1)$ parameter, even though $\widetilde{m}$ diverges at $|x|\to\infty$. In that case, our treatment here can be understood as the evaluation of the $\Lambda\to\infty$ limit of the integration $\int d^Dx\,\alpha(x)\mathcal{I}(x)$ by using rescaled coordinates $\widetilde{x}^\mu=x^\mu/\Lambda$.

[10] This formula (in the $\Lambda\to\infty$ limit with $\widetilde{m}$ kept fixed) corresponds to the local index theorem proved in [26]. See also [27].

3.2.2 Odd dimension case

Flavor symmetry in odd dimensions

In this section, we consider a system with N Dirac fermions ψ in a $D = (2r+1)$-dimensional flat Euclidean spacetime ($r \in \mathbb{Z}_{\geq 0}$). In odd dimension, there is no chirality operator like γ_5 in 4 dimension. Then, the flavor symmetry is just $U(N)$ and the associated external gauge field is denoted as A. We include a spacetime dependent mass m, which is a Hermitian matrix of size N and belongs to the adjoint representation of $U(N)$. The action is

$$S = \int d^D x\, \overline{\psi}\left(\not{D} + m\right)\psi = \int d^D x\, \overline{\psi}\mathcal{D}\psi \;, \tag{3.2.34}$$

where

$$\not{D} \equiv \gamma^\mu(\partial_\mu + A_\mu) \;, \qquad\qquad \mathcal{D} \equiv \not{D} + m \;, \tag{3.2.35}$$

and γ^μ ($\mu = 1, 2, \cdots, 2r+1$) are gamma matrices satisfying $\gamma^{\mu\dagger} = \gamma^\mu$ and $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. For explicit computation, we choose γ^μ to be of the form (3.2.5) for $\mu = 1, \cdots, 2r$ and γ^{2r+1} in (3.2.6) for $\mu = 2r+1$. This action is invariant under the $U(N)$ flavor symmetry:

$$\psi \to U\psi \;, \qquad \overline{\psi} \to \overline{\psi}U^{-1} \;, \qquad A \to UAU^{-1} + UdU^{-1} \;, \qquad m \to UmU^{-1} \tag{3.2.36}$$

with $U(x) \in U(N)$.

Calculation of the anomaly

The Jacobian of the fermion path integral measure for the $U(1)_V$ transformation (3.2.8) in the odd dimensional case can be calculated in a similar way as that for the even dimensional case in section 3.2.1. In particular, (3.2.17), (3.2.19) and (3.2.24) can be used for the $D = (2r+1)$ case with $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}'}$ defined as

$$\widehat{\mathcal{F}} = \frac{1}{2\Lambda^2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \frac{1}{\Lambda}\gamma^\mu D_\mu \widetilde{m} - \widetilde{m}^2 \;, \quad \widehat{\mathcal{F}'} = \frac{1}{2\Lambda^2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \frac{1}{\Lambda}\gamma^\mu D_\mu \widetilde{m} - \widetilde{m}^2 \;. \tag{3.2.37}$$

Note that $\widehat{\mathcal{F}'}$ is obtained by replacing γ^μ with $-\gamma^\mu$ in $\widehat{\mathcal{F}}$. Therefore, when the matrix in the trace in (3.2.24) is expanded with respect to γ^μ , only the terms with odd numbers of γ^μ can contribute. Furthermore, using the relation

$$\mathrm{tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) = \begin{cases} 0 & (k < r) \\ (2i)^r \epsilon^{\mu_1\cdots\mu_{2r+1}} & (k = r) \end{cases} \;, \tag{3.2.38}$$

we find that (3.2.26) also holds for the odd dimensional case, and

$$\mathrm{Tr}_s\left(\mathrm{e}^{\widehat{\mathcal{F}}}-\mathrm{e}^{\widehat{\mathcal{F}}'}\right)d^{2r+1}x=\Lambda^{-(2r+1)}(2i)^{r+1/2}\left[\mathrm{Str}\left(\mathrm{e}^{\mathcal{F}}\right)\right]_{2r+1}+\mathcal{O}(\Lambda^{-(2r+1)-1}) \ , \quad (3.2.39)$$

where $\mathcal{F}$ is the superconnection of the odd type given by (3.1.13) with $T=\widetilde{m}=m/\Lambda$:

$$\mathcal{F}=\begin{pmatrix} F-\widetilde{m}^2 & iD\widetilde{m} \\ iD\widetilde{m} & F-\widetilde{m}^2 \end{pmatrix} \ . \quad (3.2.40)$$

Note that we have taken into account the $\sqrt{2}i^{-3/2}$ factor in the definition of the supertrace 'Str' for the odd case (3.1.14). Then, we obtain

$$\mathcal{I}(x)d^{2r+1}x=\left(\frac{i}{2\pi}\right)^{(2r+1)/2}\left[\mathrm{Str}\left(e^{\mathcal{F}}\right)\right]_{2r+1}=[\mathrm{ch}(\mathcal{F})]_{2r+1} \ . \quad (3.2.41)$$

This implies

$$\log\mathcal{J}=-i\int\alpha(x)[\mathrm{ch}(\mathcal{F})]_{2r+1} \ , \quad (3.2.42)$$

which takes the same form as (3.2.30) for $D=2r+1$.

In particular, when the mass is a scalar matrix given by

$$m=\mu(x)1_N \ , \quad (3.2.43)$$

with a real function $\mu(x)$, we have

$$\mathrm{ch}(\mathcal{F})=\frac{1}{\sqrt{\pi}}\,d\widetilde{\mu}\,\mathrm{e}^{-\widetilde{\mu}^2}\mathrm{ch}(F) \ , \quad (3.2.44)$$

and

$$\log\mathcal{J}=-\frac{i}{\sqrt{\pi}}\int d\widetilde{\mu}\,\mathrm{e}^{-\widetilde{\mu}^2}\alpha(x)\left[\mathrm{ch}(F)\right]_{2r} \ , \quad (3.2.45)$$

where $\widetilde{\mu}\equiv\mu/\Lambda$.

3.3 Meaning of the anomaly

In the previous section 3.2, we derive the anomaly with spacetime dependent mass. This anomaly should be understood as some topological point of view. In this section, we consider anomaly $(D+2)$-form, which is a generalization of anomaly polynomial, for the anomaly with spacetime dependent mass. To check that the anomaly $(D+2)$-form is topological, we introduce the topological number for this system.

In particular, the anomaly $(D+2)$-form (3.3.1) with the spacetime dependent mass can be generalized to any spacetime dimension (3.3.11). The anomaly $(D+2)$-form is also useful to discuss about the non-abelian anomaly. The relation (2.2.39) between the non-Abelian anomaly in $2(r-1)$-dimensions characterized by $I_{2r}(A)$ and the Abelian anomaly given by $I_{2r}^{1\,\mathrm{cov}}(-i\alpha, A)$ in $2r$-dimensions [28, 29] is generalized to the cases with spacetime dependent mass.

3.3.1 Anomaly $(D+2)$-form and descent equation

The results of the previous section 3.2 can be understood by the anomaly polynomial and descent equation, as we already know for the ordinary anomalies in section 2.2.3.

The main claim of this section is that when the spacetime dependent mass m is turned on, the Chern character $\mathrm{ch}(F_+) - \mathrm{ch}(F_-)$ appeared in (2.2.33) and (2.2.39) is replaced with the Chern character written by the superconnection (3.1.5). More explicitly, the anomaly polynomial $I_{2r+2}(A)$ in (2.2.33), the covariant anomaly $I_{2r}^{1\,\mathrm{cov}}(v, A)$ in (2.2.37) and the $U(1)_V$ anomaly $I_{2r}^{1\,\mathrm{cov}}(v, A)$ in (2.2.39) are replaced with

$$I_{2r+2}(A, \widetilde{m}) = -\,2\pi i\,[\mathrm{ch}(\mathcal{F})]_{2r+2}\ , \tag{3.3.1}$$

$$I_{2r}^{1\,\mathrm{cov}}(v, A, \widetilde{m}) = \left(\frac{i}{2\pi}\right)^r \left[\mathrm{Str}\left(v\, e^{\mathcal{F}}\right)\right]_{2r}\ , \tag{3.3.2}$$

$$I_{2r}^{1\,\mathrm{cov}}(-i\alpha, A, \widetilde{m}) = -\,i\alpha\,[\mathrm{ch}(\mathcal{F})]_{2r}\ , \tag{3.3.3}$$

respectively, where $v \equiv \mathrm{diag}(v_+, v_-)$, $\widetilde{m} \equiv m/\Lambda$ is the mass rescaled by the cut-off Λ (see (3.2.19) for the definition) and

$$\mathcal{F} = \begin{pmatrix} F_+ - \widetilde{m}^\dagger \widetilde{m} & iD\widetilde{m}^\dagger \\ iD\widetilde{m} & F_- - \widetilde{m}\widetilde{m}^\dagger \end{pmatrix} \tag{3.3.4}$$

is the field strength of the superconnection (3.1.3) with $T = \widetilde{m}$. (3.3.1) is related to $I_{2r}^1(v, A, \widetilde{m})$ that gives the consistent anomaly

$$\delta_v \Gamma[A, m] = \int I_{2r}^1(v, A, \widetilde{m}) \tag{3.3.5}$$

by the descent equation (2.2.32).[11] Since $I_{2r+2}(A, \widetilde{m})$ is not a polynomial of the field strength $\mathcal{F}$, we refer to it as an anomaly $(2r+2)$-form following [12]. (3.3.2) is the covariant anomaly related to the Jacobian $\mathcal{J}$ defined with a covariant regularization adopted in section 3.2.1 by

$$\log \mathcal{J} = \int I_{2r}^{1\,\mathrm{cov}}(v, A, \widetilde{m})\ . \tag{3.3.6}$$

[11] See section 4.1 for more on the use of the anomaly $(D+2)$-form (3.3.1).

(3.3.3) is obtained from (3.3.2) by setting $v_+ = v_- = -i\alpha 1_N$. In (3.3.5) and (3.3.6), we take $\Lambda \to \infty$ limit after the integration.

Note that when m is bounded, $\widetilde{m}$ vanishes in the limit $\Lambda \to \infty$ and the m dependence drops out.[9, 10] However, there are some physically interesting systems in which the mass is of the order of cut-off scale or unbounded, and the m dependence in the anomaly may survive. For example, a system with a boundary can be realized by setting $m \to \infty$ in a region of the spacetime. Another interesting example is a system with localized massless fermions on an interface (defect) with mass of order cut-off scale in the bulk, such as the domain-wall fermions used in lattice QCD [30]. We will consider such examples in section 4.

Another related issue is that, as it was shown in [14], the de Rham cohomology class of (3.3.1) is independent of $\widetilde{m}$ because of the relation (3.1.8), which would mean that the m dependent part of $(2r+2)$-form (3.3.1) does not contribute to the anomaly. This is true in a compact spacetime. However, for an open space, the $\widetilde{m}$ dependent part of the anomaly $(2r+2)$-form can give a non-trivial element of the cohomology with compact support.[12] As we will discuss in section 4.1, this non-trivial element is interpreted as the anomaly of the fermions localized on the interfaces located around the zero locus of the mass profile. The local counterterm that cancels this anomaly is the contribution from the anomaly inflow.

Even dimension case

In this subsection, we give a simple derivation of the anomaly $(D+2)$-form (3.3.1) using the result (3.2.30) for the $U(1)_V$ anomaly for even dimension cases.

We decompose the $U(N)_+ \times U(N)_-$ gauge fields into the $U(1)_V$ gauge field V and the rest, and write the Chern character as

$$\mathrm{ch}(\mathcal{F}) = e^{\frac{i}{2\pi} f^V} \mathrm{ch}(\mathcal{F}_0) \ , \tag{3.3.7}$$

where $f^V \equiv dV$ is the field strength of the $U(1)_V$ gauge field and $\mathcal{F}_0 \equiv \mathcal{F}|_{f^V=0} = \mathcal{F} - f^V 1_{2N}$. First, we try to show (3.3.1) for the case with $f^V = 0$. To this end, let us consider the $U(1)_V$ anomaly (3.2.30) with $f^V = 0$ in a $(D+2)$-dimensional system:

$$I^{1\,\mathrm{cov}}_{D+2}(-i\alpha, A, \widetilde{m})|_{f^V=0} = -i\alpha [\mathrm{ch}(\mathcal{F}_0)]_{D+2} \ . \tag{3.3.8}$$

Note that for this component of the anomaly, there is no difference between the covariant and consistent anomalies.(See Appendix B.1.) The anomaly $(D+4)$-form for the $(D+2)$-

[12] See [12] for more on this point.

dimensional system that reproduces (3.3.8) via the descent equations (2.2.32) is

$$f^V[\mathrm{ch}(\mathcal{F}_0)]_{D+2}\ . \tag{3.3.9}$$

Now, consider a $(D+2)$-dimensional spacetime of the form $S^2 \times M_D$, where M_D is a D-dimensional manifold. We assume that f^V has a flux with $\int_{S^2} f^V = -2\pi i$ and $\mathcal{F}_0$ is independent of the coordinates on S^2. In this case, each fermion in the $(D+2)$-dimensional system has one zero mode on S^2 and hence we get a D-dimensional system with N Dirac fermions in the limit that the radius of the S^2 becomes zero. The anomaly $(D+2)$-form for this D-dimensional system is given by integrating (3.3.9) over S^2, yielding

$$I_{D+2}(A,\widetilde{m})|_{f^V=0} = \int_{S^2} f^V[\mathrm{ch}(\mathcal{F}_0)]_{D+2} = -2\pi i\,[\mathrm{ch}(\mathcal{F}_0)]_{D+2}\ , \tag{3.3.10}$$

which is (3.3.1) for the $f^V=0$ case.

The f^V dependence of the anomaly $(D+2)$-form can be easily recovered by replacing $\mathcal{F}_0$ with $\mathcal{F}$, which completes the derivation of (3.3.1).

Odd dimension case

We can define the anomaly $(D+2)$-form for the anomaly (3.2.42) in odd dimensions. The formulas for odd dimension is the analogous to the even dimension ones (3.3.1), (3.3.2) and (3.3.3):

$$I_{D+2}(A,\widetilde{m}) = -\,2\pi i\,[\mathrm{ch}(\mathcal{F})]_{D+2}\ , \tag{3.3.11}$$

$$I_D^{1\,\mathrm{cov}}(v,A,\widetilde{m}) = \left(\frac{i}{2\pi}\right)^{D/2}\left[\mathrm{Str}\left(v\,e^{\mathcal{F}}\right)\right]_D\ , \tag{3.3.12}$$

$$I_D^{1\,\mathrm{cov}}(-i\alpha,A,\widetilde{m}) = -\,i\alpha\,[\mathrm{ch}(\mathcal{F})]_D\ . \tag{3.3.13}$$

These relations are the same for even dimension ones (3.3.1), (3.3.2) and (3.3.3), therefore these are the generalized form of the anomaly for any D dimensions. In this section, we consider D as an odd number and the odd dimensional analog of the Chern character (3.1.5) defined by the supertrace for the odd case (3.1.14). Unlike the even dimensional cases discussed in section 3.2.1, both (3.3.11) and (3.3.13) vanish when the mass m vanishes. The anomaly appears only when m is turned on.

We will show in section 3.2.2 that the formula (3.2.30) for the $U(1)_V$ transformation (3.2.8) also holds for the odd dimensional cases by examining the Jacobian of the fermion path integral measure using Fujikawa's method. This implies (3.3.13). The derivation can be easily generalized to (3.3.12). (3.3.11) follows from (3.3.13) by an indirect argument given in section 3.3.1.

The meaning of (3.3.11) is somewhat more ambiguous, because, for odd D, we can find a gauge invariant $(D+1)$-form $I^0_{D+1}(A,\widetilde{m})$ satisfying $I_{D+2}(A,\widetilde{m}) = dI^0_{D+1}(A,\widetilde{m})$. (See (3.1.16).) Then, the odd dimensional analogue of the descent equations (2.2.32):

$$dI^1_D = \delta_v I^0_{D+1} \ , \quad dI^0_{D+1} = I_{D+2} \tag{3.3.14}$$

would imply that the anomaly I^1_D simply vanishes. However, as we will see in section 4.1.1, $I^0_{D+1}(A,\widetilde{m})$ is non-vanishing at infinity in our examples with non-trivial interfaces and I_{D+2} can be a non-trivial element of the cohomology with compact support.[13] We will argue that the anomaly of the fermions on the interfaces can be extracted from the formula (3.2.30).

3.3.2 Topological numbers of the mass

As we checked the previous section 2.2.3, ordinary chiral anomalies in D dimension come from the anomaly polynomials and the Chern number in $D+2$ dimension. The anomalies (3.2.30) and (3.2.42) in previous sections include the mass, and we defined the Chern numbers for the anomaly. However, the origin of the topological number has been unclear yet. We check the relation between the anomaly and the topological number that comes from the mass configuration.

The topological number we consider in this subsection is applied in chapter 4, in particular section 4.1 and section 4.3.2.

We already checked topological numbers for gauge fields and scalar fields in the section 2.2.4. In this section, we just consider the mass part. The gauge fields part is the same for the previous section 2.2.3. The spacetime dependent mass is just a background scalar field, so its topological number is defined like topological numbers in non-linear sigma models. When the mass includes some topological defects in its configuration, we can count the topological number of its configuration at the point of infinite. Let us consider the mass configuration in D dimension spacetime. Then the anomaly can be written as

$$\log \mathcal{J} = -i \int \alpha(x) [\mathrm{ch}(\mathcal{F})]_D \ , \tag{3.3.15}$$

where

$$\mathcal{F} = \begin{pmatrix} -\widetilde{m}^\dagger \widetilde{m} & id\widetilde{m}^\dagger \\ id\widetilde{m} & -\widetilde{m}\widetilde{m}^\dagger \end{pmatrix} \tag{3.3.16}$$

[13] A similar statement holds for the mass dependent part of the Chern character $I_{2r}(A,\widetilde{m})$ for the even dimensional case, as mentioned in section 3.2.1 and demonstrated in section 4.1.2.

for D is an even number case. For D is an odd number case, we just take $\widetilde{m}^\dagger = \widetilde{m}$ in (3.3.16). In this section, we check the topological number in the anomaly with the mass (3.3.15).

To consider topological defects in the configuration space of m, we focus on the point of infinity. The divergence of the value of m is important to the anomalies. To identify the configuration space of m at the point of infinity, we consider $g(x)$ which is the value of m divided by its norm,

$$g(x) = \frac{m}{\sqrt{|m^\dagger m|}} \ . \tag{3.3.17}$$

If this $g(x)$[14] has a non-trivial topological number at $r = \sqrt{x_\mu x^\mu} \to \infty$, it should correspond to the number of defects which is made by the configuration of $m(x)$. As we will see in the next section 3.4, this topological number is classified by K-groups as $K(\mathbb{R}^n) \simeq \mathbb{Z}$ or $K^1(\mathbb{R}^n) \simeq \mathbb{Z}$ for even or odd n, respectively. However, we can understand this topological number by homotopy group. This way is written in [31] for even D case and [32] for odd dimension case. In this section, we consider the topological numbers of the mass by homotopy groups of the configuration space of $g(x)$.

Even dimension case

Let us consider even dimension D. To identify the topological numbers of scalar fields m, the configuration space of m is important. In even dimension, we considered $U(N)_+ \times U(N)_-$ symmetry in (3.2.3). The mass field in (3.2.1) and (3.2.3) should be $N \times N$ matrix valued and its components are complex. This mass matrix changes its value under $U(N)_+ \times U(N)_-$ transformation, however, the mass matrix at $r \to \infty$ is changed only under $U(N)$.[15] Here, $U(N)$ is a symmetry of $\overline{\psi}\psi$ term. If we consider m does not depend on the spacetime coordinate, the mass term in (3.2.1) has only $U(N)$ symmetry. At $r \to \infty$ point, the value of the mass should diverge to obtain the anomaly, and the mass term cannot have $U(N)_+ \times U(N)_-$ symmetry. This is almost the same as a chiral SSB in QCD, and in the chiral SSB case, we can consider the WZW term of pions. In this case, we can consider the topological number on

$$\frac{U(N)_+ \times U(N)_-}{U(N)} \simeq U(N) \ . \tag{3.3.18}$$

[14] This g corresponds to g in section 4.3.2.

[15] This can be described by the language of the tachyon condensation in string theory. This case corresponds to Dp-$\overline{\text{D}p}$ system of type IIB string theory. The tachyon field on Dp-$\overline{\text{D}p}$ string has $U(N)_+ \times U(N)_-$ symmetry, but this symmetry decreases to $U(N)$ on the tachyon vacuum. The detail of this is discussed in [31] and the next section 3.4.

This is a configuration space of $g(x)$ in (3.3.17). Topological number of this $g(x)$ field is $\pi_{D-1}(U(N))$, as we saw in section 2.2.4. $\pi_{D-1}(U(N)) \simeq \mathbb{Z}$ when D is even, then this topological number denotes the topological number of the mass.

Odd dimension case

In odd dimension, the mass field is $N \times N$ matrix valued but the mass matrix is Hermitian. This mass matrix has $U(N)$ symmetry, not $U(N)_+ \times U(N)_-$. At the $r \to \infty$ point, this mass matrix is changed $U(N/2) \times U(N/2)$. This is discussed in [32] in the context of tachyon condensation. The configuration space of the mass matrix at $r \to \infty$, or $g(x)$, is $U(N)/(U(N/2) \times U(N/2))$, which is called as Grassmann manifold. This is also the same as the SSB of flavor symmetry in odd dimensions. The homotopy group for this Grassmann manifold is known as

$$\pi_{D-1}\left(\frac{U(N)}{U(N/2) \times U(N/2)}\right) \simeq \mathbb{Z} \qquad \text{(for odd D)} . \tag{3.3.19}$$

This is the topological number of the mass in odd dimensions.

Relation to topological insulator

In fact, this classification appears as the classification of topological insulators. In the context of topological insulators, the structure of the Hamiltonian is important. The classification of the Hamiltonian corresponds to the classification of topological insulators, and Hamiltonians are written as some matrices. This structure of the Hamiltonian is the same as the structure of $g(x)$ in our mass matrix.

Topological insulators are also classified by K-groups.[33, 34] In our classification of the mass matrix for odd and even D corresponds to type A and AIII for topological insulators, respectively.

3.4 Relation to string theory

Many of our results have natural interpretation in string theory. In fact, it is well-known that the CS-terms for unstable D-brane systems (D-brane - anti-D-brane systems and non-BPS D-branes) can be written by using superconnections[16] [15, 16, 17, 18] as

$$S_{\mathrm{CS}}^{\mathrm{D9}} = \int C \, \mathrm{ch}(\mathcal{F}) , \tag{3.4.1}$$

[16] As in the previous sections, we omit the terms with curvature represented by the $\widehat{A}$-genus.

where C is a formal sum of Ramond-Ramond (RR) n-form fields (n is even or odd for type IIA or type IIB string theory, respectively.) and $\mathcal{F}$ is the field strength of the superconnection for the gauge field and tachyon field on them,[17] and it is natural to anticipate the appearance of the superconnection in anomaly analysis of quantum field theory counterparts.

An easy way to realize even dimensional systems having fermions with manifest chiral symmetry is to consider a Dp-brane ($p = -1, 1, 3, 5, 7$) with D9-branes and $\overline{\text{D9}}$-branes in type IIB string theory.[36][18] On the Dp-brane world-volume, $(p+1)$-dimensional fermions are obtained in the spectrum of p-9 strings and p-$\overline{9}$ strings. Here, a p-p' string is an open string stretched between a Dp-brane and a Dp'-brane, and $\overline{p}$ corresponds to a $\overline{\text{D}p}$-brane. It can be shown that p-9 strings and p-$\overline{9}$ strings create positive and negative chirality Weyl fermions, respectively. When we have N D9-$\overline{\text{D9}}$ pairs, there are N flavors of fermions and the $U(N) \times U(N)$ gauge symmetry associated with the D9-$\overline{\text{D9}}$ pairs corresponds to the $U(N)_+ \times U(N)_-$ chiral symmetry for the $(p+1)$-dimensional system realized on the Dp-brane. The CS-term of the D9-$\overline{\text{D9}}$ system is written as (3.4.1) with $\mathcal{F}$ being the field strength of the superconnection of the even type (3.1.3), in which A_+ and A_- are the $U(N) \times U(N)$ gauge fields given by 9-9 strings and $\overline{9}$-$\overline{9}$ strings, respectively, and T is the tachyon field obtained by 9-$\overline{9}$ strings. The tachyon field T is in the bifundamental representation of the $U(N) \times U(N)$ symmetry. It couples with the fermions with Yukawa interaction and the value of the tachyon field plays the role of the mass of the fermions. When $|T| \to \infty$, the fermions decouple, which correspond to the annihilation of the D9-$\overline{\text{D9}}$ pairs.

Similarly, odd dimensional systems with N Dirac fermions can be obtained by placing a Dp-brane ($p = 0, 2, 4, 6, 8$) with N non BPS D9-branes in type IIA string theory. In this case, the CS-term for the non-BPS D9-branes is given by (3.4.1), where $\mathcal{F}$ is the odd type given by (3.1.13). Here, A and T in $\mathcal{F}$ are the $U(N)$ gauge field and the tachyon field, respectively, on the non-BPS D9-branes. The tachyon field T is a Hermitian matrix of size N and transforms as the adjoint representation of the $U(N)$ symmetry. There are N Dirac fermions in the spectrum of p-9 strings, which are in the fundamental representation of $U(N)$, and the value of the tachyon field corresponds to the mass of the fermions.

Although the CS-term (3.4.1) for the unstable D-brane system was originally derived by the computation of the interaction with the RR fields, it can be determined by the requirement of the anomaly cancellation as argued in [38, 39, 40, 41, 42, 43]. For the

[17] See [35] for a generalization.

[18] A T-dual version (N_c D4-branes with N_f D8-$\overline{\text{D8}}$ pairs) is used in [37] to realize QCD in string theory.

brane configuration above, the standard argument shows that the anomaly contribution from the CS-term for the unstable D9-branes (3.4.1) and the Dp-brane

$$S_{\rm CS}^{{\rm D}p} = \int_M C\,{\rm ch}(f)\ , \tag{3.4.2}$$

where M is the $D = p+1$-dimensional Dp-brane world-volume and ${\rm ch}(f) = \exp\left(\frac{i}{2\pi}f\right)$ is the Chern character for the $U(1)$ gauge field on it, is given by the anomaly $(D+2)$-form of the form[19]

$$2\pi i\,[{\rm ch}(\mathcal{F}){\rm ch}(f)]_{D+2}\ . \tag{3.4.3}$$

Note that (3.4.3) can be written as $2\pi i\,[{\rm ch}(\mathcal{F})]_{D+2}$ by absorbing the $U(1)$ gauge field on the Dp-brane into the $U(1)_V$ part of the gauge field of the unstable D9-brane system. This contribution is supposed to cancel the anomaly contribution from the fermions, which is indeed the case with our proposal (3.3.1) and (3.3.11), provided that the tachyon field is identified with the mass as $T = \widetilde{m}$. From the dimensional analysis, the cut-off Λ is of the order of the string scale, though the precise relation between Λ and the string length l_s is not clear.

The argument above suggests that the anomaly is characterized by the anomaly $(D+2)$-form written in terms of the Chern character of the superconnection. However, as we will discuss in section 4.1.1 and 4.1.2, since the T dependent part of the anomaly $(D+2)$-form drops out in the naive use of the anomaly descent relation, it is important to have more evidence for this statement. Let us show that the analysis in section 4.1 is consistent with the D-brane descent relation [44, 31, 32].[20]

It is known that a Dq-brane (q is even/odd for type IIA/IIB) localized at $x^I = 0$ $(I = 1, 2, \cdots, 9-q)$ can be realized as a soliton in the unstable D9-brane system by choosing the tachyon field as in (4.1.30) with $n \equiv 9-q$ and $u \to \infty$. [31, 32] In fact, the tachyon configuration with (4.1.30) is related to the generator of K-groups $K(\mathbb{R}^n) \simeq \mathbb{Z}$ or $K^1(\mathbb{R}^n) \simeq \mathbb{Z}$ for even or odd n, respectively, given by the Atiyah-Bott-Shapiro construction [47], and these K-groups correspond to the Dq-brane charge. When we have the Dp-brane extended along $x^\mu = 0$ $(\mu = 0, 1, \cdots, p)$ with $9-q \le p$, the Dq-brane corresponds to the codimension $(9-q)$ interface considered in section 4.1.3. ($q = 8$ and $q = 7$ correspond to the kink and vortex considered in sections 4.1.1 and 4.1.2, respectively.)

For this intersecting Dp-Dq system, it can be shown that there is a Weyl fermion localized at the $(p+q-8)$-dimensional intersection in the spectrum of p-q strings,

[19] To be more precise, we should consider an anomaly 12-form of the form $2\pi i\,[{\rm ch}(\mathcal{F}){\rm ch}(f)\delta_{9-p}]_{12}$, where δ_{9-p} is a delta function $(9-p)$-form supported on M.

[20] See, e.g., [45, 46] for reviews.

obtained by quantization of the open string. This is consistent with the analysis of the localized fermionic zero modes in section 4.1.

Furthermore, we can obtain k Dq-branes with $U(k)$ gauge field a on them by choosing the tachyon and gauge fields as (4.1.33) and (4.1.34). Then, one can show that the CS-term for the Dq-brane is reproduced from (3.4.1) by inserting (4.1.33) and (4.1.34) into (3.4.1) and integrating over the transverse space [48] (see also [46]), which corresponds to the procedure in (4.1.35). As the anomaly contribution from the CS-terms for the Dp-brane and Dq-branes precisely cancels that of the Weyl fermions created by the p-q strings, the anomaly polynomial for these Weyl fermions is given by (4.1.35), which is completely parallel to the discussion in section 4.1 for the localized fermionic zero modes.

Chapter 4

Applications

The anomalies we calculated in the previous chapter 3 can be applied to some systems. In this chapter, we consider some systems in which the anomaly has non-trivial effects.

4.1 Anomalies on interfaces

In this section, we consider mass profiles with isolated zero loci, which we call interfaces, and show that the anomaly carried by the fermions localized on the interfaces can be easily extracted by the formulas obtained in section 3.2. As pointed out in [12, 49], the anomaly of the localized modes implies the existence of a diabolical point in the space of parameters of the theory, which will be mentioned at the end of section 4.1.3.

4.1.1 Kink (codimension 1 interface)

We consider a $D = (2r+1)$-dimensional system given by (3.2.34) with a kink-like mass profile as

$$m = \mu(y) 1_N = uy 1_N \ , \tag{4.1.1}$$

where $y \equiv x^{2r+1}$ is one of the spatial coordinates and u is a real parameter. Since the mass m diverges at $|y| \to \infty$, the operators $\mathcal{D}^\dagger \mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ have discrete spectra as required in section 3.2.1.

To simplify the discussion, we assume that the gauge field as well as $\alpha(x)$ are independent of y. Then, the integration over y in (3.2.45) can be done and we obtain

$$\log \mathcal{J} = -i \,\mathrm{sgn}(u) \int \alpha(x) \left[\mathrm{ch}(F)\right]_{2r} \ , \tag{4.1.2}$$

where $\mathrm{sgn}(u) = u/|u|$ is a sign function and the integration is taken over the $2r$-dimensional space along $x^{1\sim 2r}$ directions. Note that this result is independent of the cut-off Λ, and hence it survives in the $\Lambda \to \infty$ limit. The dependence on the parameter u is only through its sign. Knowing this fact, for some purposes, it may be convenient to take the $|u| \to \infty$ limit as

$$\lim_{|u|\to\infty} \mathrm{ch}(\mathcal{F}) = \mathrm{sgn}(u)\delta(y)dy\,\mathrm{ch}(F) \ . \tag{4.1.3}$$

In fact, (4.1.2) does not depend on the detail of the profile (4.1.1). As it is clear from (3.2.45), we get the same result (4.1.2) for any function $\mu(y)$ satisfying $\mu(y) \to \pm\infty$ (or $\mu(y) \to \mp\infty$) as $y \to \pm\infty$.

The expression (4.1.2) agrees with the anomaly for Weyl fermions in a $2r$-dimensional spacetime. In fact, (4.1.2) is identical to (2.2.36) with (2.2.39) provided we identify $(F_+, F_-) = (F, 0)$ for $u > 0$ or $(F_+, F_-) = (0, F)$ for $u < 0$. We interpret this as the anomaly contribution from the Weyl fermions localized on the interface at $y = 0$. As a check, it is easy to show that there exist positive or negative chirality Weyl fermions at the interface as the zero modes of the operator $\mathcal{D} = \not{D} + m$ with $u > 0$ or $u < 0$, respectively.[50] To see this, let us consider the Dirac equation $\mathcal{D}\psi = 0$, where $\mathcal{D}$ is defined in (3.2.35). Working in the $A_{2r+1} = 0$ gauge, this equation can be written as

$$\not{D}^{(2r)}\psi + \gamma^{2r+1}\partial_y\psi + \mu(y)\psi = 0 \ , \tag{4.1.4}$$

where

$$\not{D}^{(2r)} = \sum_{\mu=1}^{2r} \gamma^\mu(\partial_\mu + A_\mu) \tag{4.1.5}$$

is the Dirac operator in the $2r$-dimensional space. Then, we find a solution localized around $y = 0$:

$$\psi(\vec{x}, y) = e^{-\frac{1}{2}|u|y^2}\psi^{(2r)}(\vec{x}) \tag{4.1.6}$$

where $\vec{x} = (x^1, \cdots, x^{2r})$ and $\psi^{(2r)}(\vec{x})$ is the $2r$-dimensional Weyl fermion at the interface satisfying

$$\not{D}^{(2r)}\psi^{(2r)} = 0 \ , \qquad \gamma^{2r+1}\psi^{(2r)} = \mathrm{sgn}(u)\psi^{(2r)} \ . \tag{4.1.7}$$

Note, however, that the anomaly contribution of the localized Weyl fermions are known to be canceled by the contribution from the bulk via the anomaly inflow mechanism [51]. Outside the region with $\mu(x) = 0$, the one loop effective action contains a term with the CS $(2r+1)$-form, whose gauge variation precisely cancels the anomaly of the localized fermions. Our result (4.1.2) can be interpreted in two ways. One is that the variation of CS-term simply vanishes when the gauge field and the gauge variation are independent of y, and (4.1.2) is the contribution of the localized fermion. The other is that the anomaly of the localized fermion at $y = 0$ is canceled by the contribution from the CS-term, but the variation of the CS-term also produces the same amount of anomaly at $y = \pm\infty$, which gives (4.1.2). We will make more comments on the relation to the anomaly inflow below.

Let us next discuss the anomaly $(D+2)$-form (3.3.11). Inserting (4.1.1) into (3.3.11), we obtain

$$I_{2r+3}(A, \widetilde{m}) = -2\sqrt{\pi} i\, \mathrm{e}^{-\widetilde{m}^2} d\widetilde{m}\, [\mathrm{ch}(F)]_{2r+2} = df(\widetilde{m})\, I_{2r+2}(A) \ , \tag{4.1.8}$$

where $I_{2r+2}(A) \equiv -2\pi i[\mathrm{ch}(F)]_{2r+2}$ and

$$f(x) \equiv \frac{1}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-y^2} dy = \frac{1}{2}\mathrm{erf}(x) \ . \tag{4.1.9}$$

A possible choice of I^0_{2r+2} satisfying the relation $I_{D+2} = dI^0_{D+1}$ in (3.3.14) with $D = 2r+1$ is

$$I^0_{2r+2}(A, \widetilde{m}) = f(\widetilde{m}) I_{2r+2}(A) \ . \tag{4.1.10}$$

Since this is invariant under the $U(N)$ transformation, we have $\delta_v I^0_{2r+2}(A, \widetilde{m}) = 0$ and the anomaly I^1_{2r+1} related to I^0_{2r+2} by the decent relation (3.3.14) vanishes. However, this does not mean the m dependent anomaly $(D+2)$-form $I_{2r+3}(A, \widetilde{m})$ is useless. In fact, we can extract the information of the anomaly from the fermions localized at the interface from (4.1.8) as follows.

The point is that the factor $f(\widetilde{m})$ in (4.1.10) does not vanish but approaches $\pm\frac{1}{2}\mathrm{sgn}(u)$ at $y = \pm\infty$. Therefore, the relation $I_{2r+3} = dI^0_{2r+2}$ with a gauge invariant $(2r+2)$-form I^0_{2r+2} does not imply that I_{2r+3} is trivial as an element of cohomology with compact support. To find the anomaly for the localized modes, we decompose I^0_{2r+2} in (4.1.10) into a local part that vanishes at $y \to \pm\infty$ and a closed form that does not contribute in the relation $I_{2r+3} = dI^0_{2r+2}$ as

$$I^0_{2r+2}(A, \widetilde{m}) = I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m}) + d\omega_{2r+1}(A, \widetilde{m}) \tag{4.1.11}$$

with

$$I^{0\,\mathrm{local}}_{2r+2}(A, \widetilde{m}) \equiv -\, df(\widetilde{m}) I^0_{2r+1}(A) \ , \qquad \omega_{2r+1}(A, \widetilde{m}) \equiv f(\widetilde{m}) I^0_{2r+1}(A) \ , \tag{4.1.12}$$

where $I^0_{2r+1}(A)$ is the CS $(2r+1)$-form satisfying $I_{2r+2}(A) = dI^0_{2r+1}(A)$.

We interpret $I^{0\,\mathrm{local}}_{2r+2}(A,\widetilde{m})$ as the part that gives the anomaly localized at the interface. Integrating $I^{0\,\mathrm{local}}_{2r+2}(A,\widetilde{m})$ over the y direction, one obtains a CS $(2r+1)$-form

$$I^{0\,\mathrm{local}}_{2r+1}(A) \equiv -\int_{\{y\}} I^{0\,\mathrm{local}}_{2r+2}(A,\widetilde{m}) = \mathrm{sgn}(u) I^0_{2r+1}(A) \ , \tag{4.1.13}$$

which is related to the anomaly $I^{1\,\mathrm{local}}_{2r}(v,A)$ for the Weyl fermions localized at the interface by the descent relation $\delta_v I^{0\,\mathrm{local}}_{2r+1}(A) = dI^{1\,\mathrm{local}}_{2r}(v,A)$ in (3.3.14). Here, $\int_{\{y\}}$ denotes the integral over y. The anomaly $(2r+2)$-form for the localized fermions is given by

$$I^{\mathrm{local}}_{2r+2}(A) \equiv \int_{\{y\}} I_{2r+3}(A,\widetilde{m}) = \mathrm{sgn}(u) I_{2r+2}(A) \ . \tag{4.1.14}$$

The second term in (4.1.11) corresponds to the anomaly contribution from the bulk that cancels the anomaly localized at the interface around $y = 0$ through the anomaly inflow [51]. To see this explicitly, it is convenient to take the $|u| \to \infty$ limit, in which $f(\widetilde{m})$ and $df(\widetilde{m})$ approach a step function and a delta function 1-form with support at $y = 0$, respectively:

$$f(\widetilde{m}) \to \frac{1}{2}\mathrm{sgn}(u)\,\mathrm{sgn}(y) \ , \qquad\qquad df(\widetilde{m}) \to \mathrm{sgn}(u)\delta(y)dy \ . \tag{4.1.15}$$

Then, $I^{0\,\mathrm{local}}_{2r+2}(A,\widetilde{m})$ is completely localized at $y = 0$ and ω_{2r+1} becomes

$$\omega_{2r+1}(A,\widetilde{m}) \to -\frac{1}{2}\mathrm{sgn}(\widetilde{m}) I^0_{2r+1}(A) \ , \tag{4.1.16}$$

which can be interpreted as the CS-term in the bulk induced from the path integral of the massive fermions, which precisely cancels the anomaly localized at the interface.

4.1.2 Vortex (codimension 2 interface)

Next, consider a $D = (2r+2)$-dimensional system (3.2.1) with a vortex-type mass profile given by

$$m = \mu(z) 1_N = uz 1_N \ , \tag{4.1.17}$$

where $z = x^{2r+1} - ix^{2r+2}$ and u is a complex parameter. Here, we assume that the gauge fields as well as the parameter α are independent of z, and satisfy $A_+ = A_- \equiv A$ and $A_{2r+1} = A_{2r+2} = 0$, for simplicity.

Then, (3.2.33) implies

$$\log \mathcal{J} = -i \int \alpha(x) \, [\mathrm{ch}(F)]_{2r} \ . \tag{4.1.18}$$

This agrees with the anomaly of a $2r$-dimensional system with Weyl fermions and it is interpreted as the anomaly contribution from the Weyl fermion localized on the interface at $z = \overline{z} = 0$.

Again, we can explicitly find localized Weyl fermions as follows.[52, 53, 51] For this purpose, it is convenient to choose $\sigma^\mu = \gamma^\mu_{(2r)}$ ($\mu = 1, \cdots, 2r+1$) and $\sigma^{2r+2} = -i1_{2^r}$, where $\gamma^\mu_{(2r)}$ ($\mu = 1, \cdots, 2r$) are gamma matrices in $2r$-dimensions and $\gamma^{2r+1}_{(2r)}$ is the chirality operator for them. In this case, the Dirac equation $\mathcal{D}\psi = 0$ can be written as

$$\not{D}^{(2r)}\psi_+ + 2(P_+\partial_z - P_-\partial_{\overline{z}})\psi_+ + \overline{uz}\psi_- = 0 \ , \tag{4.1.19}$$

$$\not{D}^{(2r)}\psi_- + 2(P_+\partial_{\overline{z}} - P_-\partial_z)\psi_- + uz\psi_+ = 0 \ , \tag{4.1.20}$$

where $\not{D}^{(2r)}$ is defined in (4.1.5) and $P_\pm \equiv \frac{1}{2}(1_{2^r} \pm \gamma^{2r+1}_{(2r)})$ is a projection operator that projects to positive/negative chirality spinors in $2r$-dimensions. Then, we find a solution localized around $z = 0$:

$$\psi_+(\vec{x}, z, \overline{z}) = \psi_-(\vec{x}, z, \overline{z}) = \mathrm{e}^{-\frac{1}{2}u|z|^2} \psi^{(2r)}(\vec{x}) \ , \tag{4.1.21}$$

where we have assumed u to be real and positive without loss of generality, and $\psi^{(2r)}$ is a positive chirality massless Weyl fermion in $2r$-dimensions. A negative chirality mode can also be obtained when the mass is $m = u\overline{z}\, 1_N$, which represents an anti-vortex.

The role of the anomaly $(D+2)$-form (3.3.1) can be discussed in a similar way as the codimension 1 interface considered in section 4.1.1. For the mass profile (4.1.17), the anomaly $(D+2)$-form (with $D = 2r+2$) becomes

$$I_{2r+4}(A, \widetilde{m}) = df_1(\widetilde{m}) \, I_{2r+2}(A) \ , \tag{4.1.22}$$

where $I_{2r+2}(A) \equiv -2\pi i [\mathrm{ch}(F)]_{2r+2}$ is the anomaly polynomial for a Weyl fermion in $2r$-dimensions and f_1 is a 1-form given by

$$f_1(\widetilde{m}) \equiv \frac{i}{4\pi} \left(1 - \mathrm{e}^{-|\widetilde{m}|^2}\right) \left(d \log \widetilde{m} - d \log \widetilde{m}^\dagger\right) \ . \tag{4.1.23}$$

Note that f_1 is non-vanishing at $|z| \to \infty$, while its derivative

$$df_1(\widetilde{m}) = \frac{i}{2\pi} d\widetilde{m}^\dagger d\widetilde{m} \, \mathrm{e}^{-|\widetilde{m}|^2} \tag{4.1.24}$$

decays exponentially as $|z| \to \infty$, and approaches a delta function 2-form with support at $z = \overline{z} = 0$ in the $u \to \infty$ limit. The integral of df_1 over the z-plane is normalized as

$$\int df_1 = 1 \ , \tag{4.1.25}$$

The CS-form $I^0_{2r+3}(A, \widetilde{m})$ satisfying $I_{2r+4}(A, \widetilde{m}) = dI^0_{2r+3}(A, \widetilde{m})$ can be chosen as

$$I^0_{2r+3}(A, \widetilde{m}) = f_1(\widetilde{m}) I_{2r+2}(A) = I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) + d\omega_{2r+2}(A, \widetilde{m}) \tag{4.1.26}$$

where

$$I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) \equiv df_1(\widetilde{m}) I^0_{2r+1}(A) \ , \qquad \omega_{2r+2}(A, \widetilde{m}) \equiv -\, f_1(\widetilde{m}) I^0_{2r+1}(A) \ . \tag{4.1.27}$$

Here, $I^0_{2r+1}(A)$ is the CS-form satisfying $I_{2r+2}(A) = dI^0_{2r+1}(A)$.

The anomaly contribution of the fermions localized at the interface, denoted as $I^{1\,\mathrm{local}}_{2r}(A)$, is related to

$$I^{0\,\mathrm{local}}_{2r+1}(A) \equiv \int_{\{z,\overline{z}\}} I^{0\,\mathrm{local}}_{2r+3}(A, \widetilde{m}) = I^0_{2r+1}(A) \ . \tag{4.1.28}$$

where $\int_{\{z,\overline{z}\}}$ denotes the integral over the z-plane, by the descent relation $dI^{1\,\mathrm{local}}_{2r} = \delta_v I^{0\,\mathrm{local}}_{2r+1}$. In other words, it is characterized by the anomaly polynomial

$$I^{\mathrm{local}}_{2r+2}(A) \equiv \int_{\{z,\overline{z}\}} I_{2r+4}(A, \widetilde{m}) = I_{2r+2}(A) \ . \tag{4.1.29}$$

On the other hand, $\omega_{2r+2}(A, \widetilde{m})$ gives the bulk contribution of the anomaly that cancels the anomaly on the interface.

4.1.3 Interfaces of higher codimension

The discussion in sections 4.1.1 and 4.1.2 can be generalized to the cases with interfaces of higher codimensions. We are interested in the interfaces with Weyl fermions on them.

A codimension n interface in $D = (2r+n)$-dimensional spacetime can be constructed by giving a mass of the form

$$m(x) = u \sum_{I=1}^{n} \Gamma^I x^I \ , \tag{4.1.30}$$

where Γ^I $(I = 1, 2, \cdots, n)$ are matrices of size $N = 2^{[(n-1)/2]}$ related to n-dimensional gamma matrices $\widehat{\gamma}^I$ by

$$\widehat{\gamma}^I = \Gamma^I \qquad \text{(for odd } n \text{ and } D\text{)} \ , \tag{4.1.31}$$

$$\widehat{\gamma}^I = \begin{pmatrix} & \Gamma^I \\ \Gamma^{I\dagger} & \end{pmatrix} \qquad \text{(for even } n \text{ and } D\text{)} \ . \tag{4.1.32}$$

In this case, it can be shown that there is a Weyl fermion on the interface at $x^1 = \cdots = x^n = 0$ obtained as a localized fermion zero mode, as we have seen this explicitly in sections 4.1.1 and 4.1.2 for $n = 1, 2$. This generalization comes from the string theory, which we considered in section 3.4. We will give an indirect argument for this fact for general n in connection to index theorems in section 4.3.2.

It is also possible to get k Weyl fermions by replacing Γ^I in (4.1.30) by $1_k \otimes \Gamma^I$ as

$$m(x) = u \sum_{I=1}^{n} 1_k \otimes \Gamma^I x^I \ . \tag{4.1.33}$$

In this case, the gauge group is $U(kN)$ or $U(kN)_+ \times U(kN)_-$ for odd or even D, respectively, and the vector-like $U(k)$ subgroup of the form $g \otimes 1_N$ with $g \in U(k)$ is unbroken. Then, k Weyl fermions coupled with $U(k)$ gauge field a can be obtained by setting $U(kN)$ gauge field A as

$$A = a \otimes 1_N \ . \tag{4.1.34}$$

It is straightforward to check that the anomaly for these Weyl fermions on the interface can be obtained by inserting the mass profile (4.1.33) and the gauge field (4.1.34) into our formulas (3.3.1)–(3.3.3) and (3.3.11)–(3.3.13). In particular, the expressions (4.1.14) and (4.1.29) of the anomaly $(2r+2)$-form for the localized fermions are generalized as

$$I^{\text{local}}_{2r+2}(a) \equiv \int_n I_{2r+n+2}(A, \widetilde{m}) \ , \tag{4.1.35}$$

where $\int_n$ denotes the integral over x^I $(I = 1, 2, \cdots, n)$. This agrees with the anomaly polynomial for $2r$-dimensional Weyl fermions coupled to the $U(k)$ gauge field a.

As discussed in [12, 49], the anomaly contributions from fermion zero modes localized on the interfaces implies that there is at least one point in the space of parameters of the theory, called a diabolical point, at which the theory is not trivially gapped. In our examples, it is of course clear that the massless point $m = 0$ is the diabolical point. However, since the anomaly takes a discrete value, the existence of the diabolical point is robust against continuous deformations of the theory. In fact, as we have seen, the anomaly depends only on the asymptotic behavior of the mass profile. The existence of the diabolical point can be shown without examining the theory at the massless point. This point is more explicit in the Callias-type index theorem (4.3.16) discussed in section 4.3.2.

4.2 Anomaly in spacetime with boundaries

Since the fermions cannot propagate in a region with infinite mass, it is possible to realize a spacetime with boundaries by considering a spacetime dependent mass that blows up in some regions. In this subsection, we discuss the anomaly driven by the boundary condition imposed on the fermions, using our formulas obtained in section 3.2.

4.2.1 Odd dimensional cases

Let us first consider a $D = (2r+1)$-dimensional system of N Dirac fermions with $y \equiv x^{2r+1}$ dependent mass given by

$$m(y) = \mu(y)1_N = \begin{cases} (m_0 + u'(y-L))1_N & (L < y) \\ m_0 1_N & (0 \le y \le L) \\ (m_0 + uy)1_N & (y < 0) \end{cases} , \tag{4.2.1}$$

where u, u' and m_0 are real parameters.[1] We assume that the gauge field is independent of y in the $y < 0$ and $L < y$ regions.

When $|u|$ and $|u'|$ are large enough, this system can be regarded as that of N Dirac fermions with mass m_0 living in an interval $0 \le y \le L$ with boundaries at $y = 0$ and $y = L$. The boundary conditions for the fermion fields follow from the requirement that they do not blow up at $y \to \pm\infty$. The discussion around (4.1.4)–(4.1.7) implies that the corresponding boundary conditions are

$$\left(\gamma^{2r+1}\psi - \mathrm{sgn}(u)\psi\right)|_{y=0} = 0 , \qquad \left(\gamma^{2r+1}\psi - \mathrm{sgn}(u')\psi\right)|_{y=L} = 0 , \tag{4.2.2}$$

which are equivalent to one of the boundary conditions considered in [54].

In this setup, the formula (3.2.45) implies that the Jacobian is

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha \left[\mathrm{ch}(F)\right]_{2r} + i\kappa_+ \int_{y=L} \alpha \left[\mathrm{ch}(F)\right]_{2r} , \tag{4.2.3}$$

with

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u) + f(\widetilde{m}_0) , \qquad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u') - f(\widetilde{m}_0) \tag{4.2.4}$$

[1] Strictly speaking, since $\partial_y^2 m$ has delta function singularities at $y = 0, L$, the assumption that we made above (3.2.26) is not satisfied. However, it can be shown that these singularities do not contribute and the result is unchanged. Alternatively, one could replace $\mu(y)$ with a smooth function with the same asymptotic behavior as (4.2.1), which also gives the same result.

where $\widetilde{m}_0 \equiv m_0/\Lambda$ and $f(z)$ is the function defined in (4.1.9), and α is assumed to be independent of y in the $y < 0$ and $L < y$ regions. When the cut-off Λ is sent to infinity, while keeping m_0 finite, $f(\widetilde{m}_0)$ simply vanishes and we get

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u) \ , \qquad\qquad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u') \ . \tag{4.2.5}$$

Note that each term in (4.2.3) with (4.2.5) is proportional to the anomaly contribution from a Weyl fermion in $2r$-dimensions. However, since the coefficients $\kappa_\pm$ are not integers, it is not possible to interpret this result as the contribution from the Weyl fermions localized at the boundaries. This is because the wave function of the fermions are not completely localized at the boundary in our setup, unless we take the $|\widetilde{m}_0| \to \infty$ limit. One way to understand (4.2.5) is to use the anomaly inflow argument given in section 4.1.1. Namely, the anomaly contributions from the modes localized at $y = 0$ and/or $y = L$ are canceled by the bulk CS-terms, but the gauge variation of the (half-level) CS-terms implies non-vanishing surface terms at $y = \pm\infty$, which gives (4.2.3) with (4.2.5) as α and F are independent of y for $y < 0$ and $L < y$. On the other hand, one can argue that $\kappa_\pm$ can be shifted as $\kappa_\pm \to \kappa_\pm \pm \beta$ by adding a local counterterm of the form

$$S_{\mathrm{c.t.}} = \beta \int V[\mathrm{ch}(F)]_{2r} \tag{4.2.6}$$

where V is the $U(1)$ gauge field, and including its gauge variation in the Jacobian (4.2.3). Therefore, only the combination $\kappa_+ + \kappa_- = \frac{1}{2}(\mathrm{sgn}(u) + \mathrm{sgn}(u'))$ is free from this ambiguity.

It is nonetheless useful to find the anomaly contribution of the localized fermionic zero modes. Assuming that m_0 is very large and the y-dependence of the gauge field is negligible, the solutions of the Dirac equation (4.1.4) in the region $0 \le y < L$ are approximately a linear combination of exponentially increasing and decreasing modes as

$$\psi(\vec{x}, y) \simeq e^{-m_0 y}\psi_+^{(2r)}(\vec{x}) + e^{m_0 y}\psi_-^{(2r)}(\vec{x}) \ , \tag{4.2.7}$$

where $\psi_\pm^{(2r)}$ satisfies

$$\not{D}^{(2r)}\psi_\pm^{(2r)} = 0 \ , \qquad\qquad \gamma^{2r+1}\psi_\pm^{(2r)} = \pm\,\psi_\pm^{(2r)} \ . \tag{4.2.8}$$

Then, the boundary conditions (4.2.2) imply that there are Weyl fermions localized near the boundary with chirality $\mathrm{sgn}(u)$ and $\mathrm{sgn}(u')$ localized around $y = 0$ and $y = L$, if $\mathrm{sgn}(m_0) = \mathrm{sgn}(u)$ and $\mathrm{sgn}(m_0) = -\mathrm{sgn}(u')$, respectively. The anomaly contributions of these localized modes are obtained by formally taking the limit $|\widetilde{m}_0| \to \infty$ in (4.2.4),[2] in which we have

$$\kappa_- = \frac{1}{2}(\mathrm{sgn}(u) + \mathrm{sgn}(m_0)) \ , \qquad \kappa_+ = \frac{1}{2}(\mathrm{sgn}(u') - \mathrm{sgn}(m_0)) \ . \tag{4.2.9}$$

4.2.2 Even dimensional cases

In this subsection, we consider a $D = 2r$-dimensional spacetime with boundaries realized by the mass profile

$$m(x) = \mu(y)g(x) = \begin{cases} u'(y-L)g(x) & (L < y) \\ 0 & (0 \le y \le L) \\ uyg(x) & (y < 0) \end{cases} \ , \tag{4.2.10}$$

where $y \equiv x^{2r}$, $g(x) \in U(N)$ and $u, u' \in \mathbb{C}$. Since the phases of u and u' can be absorbed in $g(x)$, we assume $u, u' > 0$ without loss of generality. We take a gauge with $A_{+y} = A_{-y} = 0$ and assume that the gauge fields (A_+, A_-) and $g(x)$ are independent of y in the $y \le 0$ and $L \le y$ regions. Since $\mu(y)$ vanishes in the region $0 < y < L$, the $g(x)$ dependence in this region drops out. Therefore, we can choose $g(x)$ to be discontinuous in the region $\epsilon < y < L - \epsilon$ with $0 < \epsilon \ll L$, and the configuration of $g(x)$ at $y = 0$ and $y = L$ can be topologically different.

As discussed around (4.2.2) for the odd dimensional case, by the requirement that the fermion fields do not blow up at $y \to \pm\infty$, the boundary conditions corresponding to the mass profile (4.2.10) are obtained as

$$\left(\gamma^{2r}\psi^g - \psi^g\right)|_{y=0,L} = 0 \ , \tag{4.2.11}$$

where $\psi^g \equiv \begin{pmatrix} g & \\ & 1 \end{pmatrix} \psi = \begin{pmatrix} g\psi_+ \\ \psi_- \end{pmatrix}$.[3] Therefore, this system can be regarded as that of massless N Dirac fermions on the interval $0 \le y \le L$ with a boundary condition (4.2.11). Note that this boundary condition (4.2.11) depends on the spacetime coordinates through $g(x)$. With fixed $g(x)$, the boundary condition (4.2.11) breaks the $U(N)_+ \times U(N)_-$ gauge symmetry down to the $U(N)$ subgroup that consists of elements $(U_+, U_-) \in U(N)_+ \times U(N)_-$ with $U_- = gU_+g^{-1}$. However, as it is evident from our construction, the boundary condition (4.2.11) is invariant under the gauge transformation

$$A_+ \to A_+^{U_+} \ , \qquad A_- \to A_-^{U_-} \ , \qquad g \to U_-\, g\, U_+^{-1} \ , \tag{4.2.12}$$

[2]In this limit, only the localized zero modes are expected to contribute, since the modes with energy greater than Λ are suppressed by the heat kernel regularization (3.2.19).

[3]This type of boundary condition with constant g was introduced in the bag model of hadrons. [55] The cases with $g = 1$ or $g = i$ were considered recently in [56, 54].

and it makes sense to consider the anomaly with respect to $U(N)_+ \times U(N)_-$ even at the boundaries.

For this configuration, the field strength of the superconnection (3.3.4) becomes

$$
\begin{aligned}
\mathcal{F} &= \begin{pmatrix} g^{-1} & \\ & 1_N \end{pmatrix} \begin{pmatrix} F_+^g - \widetilde{\mu}^2 1_N & i(d\widetilde{\mu} 1_N - (A_- - A_+^g)\widetilde{\mu}) \\ i(d\widetilde{\mu} 1_N + (A_- - A_+^g)\widetilde{\mu}) & F_- - \widetilde{\mu}^2 1_N \end{pmatrix} \begin{pmatrix} g & \\ & 1_N \end{pmatrix} \\
&= \begin{pmatrix} g^{-1} & \\ & 1_N \end{pmatrix} \left(-\widetilde{\mu}^2 1_{2N} + F_+^g e^+ + F_- e^- + i d\widetilde{\mu}\sigma_1 + \widetilde{\mu}(A_- - A_+^g)\sigma_2\right) \begin{pmatrix} g & \\ & 1_N \end{pmatrix} ,
\end{aligned}
\tag{4.2.13}
$$

where $\widetilde{\mu} \equiv \mu/\Lambda$, $A_+^g \equiv g A_+ g^{-1} + g d g^{-1}$ and $F_+^g \equiv g F_+ g^{-1}$. The second line of (4.2.13) is written in the notation introduced in (3.1.1) with $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Then, we obtain

$$
\mathrm{Str}(\mathrm{e}^{\mathcal{F}}) = \mathrm{e}^{-\widetilde{\mu}^2} \mathrm{Str}\left(\mathrm{e}^{F_+^g e^+ + F_- e^- + \widetilde{\mu}(A_- - A_+^g)\sigma_2}\left(1 + i d\widetilde{\mu}\sigma_1\right)\right) ,
\tag{4.2.14}
$$

and, hence, the Jacobian (3.2.30) becomes

$$
\log \mathcal{J} = -i \int_{0<y<L} \alpha \left[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)\right]_{2r} - i \int_{y=L} \alpha[\omega]_{2r-1} + i \int_{y=0} \alpha[\omega]_{2r-1} ,
\tag{4.2.15}
$$

where we have assumed that α is independent of y in the $y < 0$ and $L < y$ regions, and defined

$$
\omega \equiv i \sum_{r \geq 1} \left(\frac{i}{2\pi}\right)^r \int_0^\infty dt\, \mathrm{e}^{-t^2} \left[\mathrm{Str}\left(\mathrm{e}^{F_+^g e^+ + F_- e^- + t(A_- - A_+^g)\sigma_2}\sigma_1\right)\right]_{2r-1} .
\tag{4.2.16}
$$

This ω is a formal sum of differential forms on the boundaries (*i.e.* $y = 0$ and $y = L$ planes). The 1-form and 3-form components of ω are

$$
[\omega]_1 = \frac{i}{2\pi} \mathrm{Tr}\,(A_- - A_+^g) ,
\tag{4.2.17}
$$

$$
[\omega]_3 = -\frac{1}{8\pi^2} \mathrm{Tr}\left((A_- - A_+^g)(F_- + F_+^g) - \frac{1}{3}(A_- - A_+^g)^3\right) .
\tag{4.2.18}
$$

One can show that this a generalization of CS-forms satisfying

$$
d\omega|_{y=0,L} = \left(\mathrm{ch}(F_-) - \mathrm{ch}(F_+)\right)|_{y=0,L} ,
\tag{4.2.19}
$$

and it is manifestly invariant under the gauge transformation (4.2.12). To show (4.2.19), consider the $L \leq y$ region and note that ω at $y = L$ can be written as

$$
\omega|_{y=L} = \int_{\{L \leq y\}} \mathrm{ch}(\mathrm{e}^{\mathcal{F}}) ,
\tag{4.2.20}
$$

where $\int_{\{L\leq y\}}$ denotes the integration over y with $L \leq y$. Then, applying the exterior derivative $d = d_x + d_y$, where $d_x \equiv \sum_{\mu=1}^{2r-1} dx^\mu \partial_\mu$ and $d_y \equiv dy\,\partial_y$, and using the fact that $\mathrm{ch}(e^{\mathcal{F}})$ is a closed form, we obtain

$$d\omega|_{y=L} = \int_{\{L\leq y\}} d_x \mathrm{ch}(\mathrm{e}^{\mathcal{F}}) = -\int_{\{L\leq y\}} d_y \mathrm{ch}(\mathrm{e}^{\mathcal{F}}) = -\mathrm{ch}(\mathrm{e}^{\mathcal{F}})|_{y=L} \ , \tag{4.2.21}$$

which implies (4.2.19).

An important observation is that even if the gauge fields are set to zero, (4.2.16) can be non-vanishing. In fact, for $A_+ = A_- = 0$, we obtain

$$[\omega]_{2r-1} = \left(\frac{-i}{2\pi}\right)^r \frac{(r-1)!}{(2r-1)!} \mathrm{Tr}((gdg^{-1})^{2r-1}) \ . \tag{4.2.22}$$

When the spacetime is of the form $S^{2r-1} \times \{y\}$, the integral of this form over S^{2r-1} gives a winding number in $\pi_{2r-1}(U(N))$ represented by the map $g : S^{2r-1} \to U(N)$. If the winding number at $y = 0$ and $y = L$ are the same, a function $g : S^{2r-1} \times \{y\} \to U(N)$ that interpolates the configuration of g at $y = 0$ and $y = L$ can be found and the Jacobian (4.2.15) can be canceled by the gauge variation of a local counterterm

$$S_{\mathrm{c.t.}} = -\int_{0<y<L} V[\omega]_{2r-1} \ , \tag{4.2.23}$$

where V is the $U(1)_V$ gauge field and $[\omega]_{2r-1}$ is given by (4.2.22). However, when the winding numbers at $y = 0$ and $y = L$ are different, this is not allowed and there is an anomaly.

Another interesting situation is the case with $g(x) = \mathrm{e}^{i\phi(x)} 1_N$ and $A \equiv A_+ = A_-$. In this case, the formula (4.2.16) implies

$$\omega = -\frac{d\phi}{2\pi} \mathrm{ch}(F) \ . \tag{4.2.24}$$

Therefore, when the spacetime is of the form $S^1 \times S^{2r-2} \times \{y\}$ and the winding number of $e^{i\phi}$ on S^1 for $y = 0$ and $y = L$ are different, there is an anomaly for the $U(1)_V$ symmetry in the presence of a non-vanishing background vector-like gauge field on S^{2r-2}.

4.3 Index theorems

From (3.2.11) and the first expression in (3.2.19), we find that the integral of $\mathcal{I}(x)$ gives the index of operator $\mathcal{D}$:

$$\int d^D x\, \mathcal{I}(x) = n_\varphi - n_\phi = \dim\ker \mathcal{D} - \dim\ker \mathcal{D}^\dagger \equiv \mathrm{Ind}(\mathcal{D}) \ , \tag{4.3.1}$$

and the result (3.2.29) implies an index theorem written in terms of the superconnection:[4]

$$\mathrm{Ind}(\mathcal{D}) = \int [\mathrm{ch}(\mathcal{F})]_D \ . \tag{4.3.2}$$

When we set $m = 0$ and $A_- = 0$ in an even dimensional case, this formula reduces to a more familiar form of the Atiyah-Singer (AS) index theorem: $\mathrm{Ind}(\not{D}) = \int \mathrm{ch}(F_+)$. Thus, (4.3.2) is a generalization of the AS index theorem, which includes spacetime dependent mass and is supposed to hold even when the spacetime manifold is odd dimensional and/or non-compact, provided that the spectra of $\mathcal{D}\mathcal{D}^\dagger$ and $\mathcal{D}^\dagger\mathcal{D}$ are discrete.

Here, we discuss some of the implications of this formula. We will not try to make the statements mathematically rigorous.[5] Nevertheless, we hope they are useful and worth mentioning.

4.3.1 Atiyah-Patodi-Singer index theorem

The Atiyah-Patodi-Singer (APS) index theorem [57] is an index theorem for a Dirac operator on an even dimensional manifold N with boundary, stated as

$$\mathrm{Ind}(\not{D}) = \int \mathrm{ch}(F)\widehat{A}(R) - \frac{1}{2}\eta(i\not{D}_b) \ , \tag{4.3.3}$$

where $\not{D}$ is a Dirac operator on N, $\eta(i\not{D}_b)$ is the eta invariant of a Dirac operator on the boundary denoted as $\not{D}_b$ (see (4.3.7)).[6]

In this subsection, we first generalize (4.3.3) to include the spacetime dependent mass m and then apply it to the system considered in section 4.2.2. Let us consider a system in section 3.2.1 with $D = 2r$-dimensional spacetime of the form $N = M \times I$, where M is a $(2r-1)$-dimensional manifold with coordinates x^μ $(\mu = 1, 2, \cdots, 2r-1)$ and $I = [y_-, y_+] \subset \mathbb{R}$ is an interval parameterized by $y \equiv x^{2r} \in I$. For simplicity, as in the previous sections, we assume M to be flat and the $\widehat{A}$-genus is omitted.

It is convenient to choose σ^μ in (3.2.3) such that $\sigma^{2r} = 1_{2^{r-1}}$ and $\sigma^\mu = i\gamma^\mu$ $(\mu = 1, 2, \cdots, 2r-1)$ with γ^μ being the $(2r-1)$-dimensional gamma matrices. Then the operator $\mathcal{D}$ defined in (3.2.3) and its conjugate $\mathcal{D}^\dagger$ can be written as

$$\mathcal{D} = \partial_y + H_y \ , \qquad \mathcal{D}^\dagger = -\partial_y + H_y \ , \tag{4.3.4}$$

[4] A quick way to get the expression of the index from the results of the Jacobian in the previous sections is to set $\alpha = i$ in $\log \mathcal{J}$ as $\mathrm{Ind}(\mathcal{D}) = \log \mathcal{J}|_{\alpha=i}$.

[5] See, e.g., [27] for mathematically rigorous description of index theorems using the superconnection.

[6] See [58, 59, 60] for recent physicists-friendly formulations and derivations. See also [61] and Appendix A.

in the $A_{+y} = A_{-y} = 0$ gauge, where

$$H_y \equiv \begin{pmatrix} -i\not{D}_+^{(2r-1)} & m^\dagger \\ m & i\not{D}_-^{(2r-1)} \end{pmatrix} , \tag{4.3.5}$$

$$\not{D}_+^{(2r-1)} = \sum_{\mu=1}^{2r-1} \gamma^\mu(\partial_\mu + A_{+\mu}) , \qquad \not{D}_-^{(2r-1)} = \sum_{\mu=1}^{2r-1} \gamma^\mu(\partial_\mu + A_{-\mu}) . \tag{4.3.6}$$

Note that although H_y is y-dependent, it does not contain the derivative with respect to y and it can be regarded as a Hermitian operator acting on spinors on M. Here, the mass m can depend on both x^μ and y. When M is non-compact, the mass should diverge at infinity, as the examples considered in sections 4.1 and 4.2, so that H_y has a discrete spectrum.

The eta invariant of a Hermitian operator H is defined as

$$\eta(H) \equiv \lim_{s\to 0} \eta(s, H) , \qquad \eta(s, H) \equiv \frac{2}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^s \, \mathrm{Tr}_{\mathcal{H}} \left(H \mathrm{e}^{-t^2 H^2} \right) , \tag{4.3.7}$$

where the trace $\mathrm{Tr}_{\mathcal{H}}$ is over the Hilbert space $\mathcal{H}$ on which the operator H is acting and $s \to 0$ limit is taken after analytic continuation of $\eta(s, H)$ on the complex s-plane. [57] $\eta(s, H)$ can be written as a sum over eigenvalues λ of H as

$$\eta(s, H) = \sum_\lambda \mathrm{sgn}(\lambda)|\lambda|^{-s} . \tag{4.3.8}$$

Here and in the following, we assume that H does not have a zero eigenvalue, whenever it is used in $\eta(H)$ or $\eta(s, H)$. For the massless case, the eta invariant of H_y reduces to the difference of the eta invariant of the Dirac operators $i\not{D}_+^{(2r-1)}$ and $i\not{D}_-^{(2r-1)}$ as

$$\eta(H_y)|_{m=0} = -\eta(i\not{D}_+^{(2r-1)}) + \eta(i\not{D}_-^{(2r-1)}) \tag{4.3.9}$$

Then, as it is explained in Appendix A, the index of $\mathcal{D}$ is given by

$$\mathrm{Ind}(\mathcal{D}|_I) = \lim_{\Lambda\to\infty} \int_{y_- < y < y_+} [\mathrm{ch}(\mathcal{F})]_{2r} + \frac{1}{2} \left[\eta(H_y)\right]_{y=y_-}^{y=y_+} , \tag{4.3.10}$$

where $\mathcal{F}$ is the field strength of the superconnection (3.3.4) with $\Lambda \to \infty$ taken after the integration, $[f(y)]_{y=y_-}^{y=y_+} \equiv f(y_+) - f(y_-)$ and $\mathrm{Ind}(\mathcal{D}|_I)$ denotes the index of the operator $\mathcal{D}$ acting on spinors on $M \times I$ with the following APS boundary conditions. For the operator $\mathcal{D}$, when the wave function at $y = y_\pm$ is expanded with respect to eigenfunctions of $H_{y_\pm}$, the components with the negative (for $y = y_+$) or positive (for $y = y_-$) eigenvalues of $H_{y_\pm}$ have to vanish. The conditions for the operator $\mathcal{D}^\dagger$ are the same as $\mathcal{D}$ with the replacement $H_{y_\pm} \to -H_{y_\pm}$. These boundary conditions follow from the requirement that wave function of the fermion does not blow up at $y \to \pm\infty$, when the system is extended to the $y < y_-$ and $y_+ < y$ regions with a y-independent configuration for $y \le y_-$ and $y_+ \le y$. (See Appendix A.)

Let us apply (4.3.10) to the system considered in section 4.2.2. Using (4.3.9), the formula (4.3.10) with $[y_-, y_+] = [0, L]$ becomes

$$\mathrm{Ind}(\mathcal{D}|_{[0,L]}) = \int_{0<y<L} [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} - \frac{1}{2}\left[\eta(i\not{D}_+^{(2r-1)}) - \eta(i\not{D}_-^{(2r-1)})\right]_{y=0}^{y=L} , \tag{4.3.11}$$

which is the APS index theorem for the massless Dirac operator defined by $\mathcal{D}|_{m=0}$ with the APS boundary conditions. On the other hand, for $[y_-, y_+] = [-\infty, +\infty]$, (4.3.2) can be used, and from (4.2.15), we obtain

$$\mathrm{Ind}(\mathcal{D}) = \int_{0<y<L} [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} + \int_{y=L} [\omega]_{2r-1} - \int_{y=0} [\omega]_{2r-1} . \tag{4.3.12}$$

This is interpreted as the index theorem for the massless fermions in the interval $[0, L]$ with the boundary condition given by (4.2.11).

The difference between (4.3.11) and (4.3.12) can be evaluated by applying (4.3.10) to the cases with $[y_-, y_+] = [L, +\infty]$ and $[-\infty, 0]$ (More precisely, (A.0.12) and (A.0.13) with $\eta_0 = 0$.) :

$$\begin{aligned}\mathrm{Ind}(\mathcal{D}|_{[L,+\infty]}) &= \int_{y=L} [\omega]_{2r-1} + \frac{1}{2}\left(\eta(i\not{D}_+^{(2r-1)}) - \eta(i\not{D}_-^{(2r-1)})\right)\Big|_{y=L} , \\ \mathrm{Ind}(\mathcal{D}|_{[-\infty,0]}) &= -\int_{y=0} [\omega]_{2r-1} - \frac{1}{2}\left(\eta(i\not{D}_+^{(2r-1)}) - \eta(i\not{D}_-^{(2r-1)})\right)\Big|_{y=0} .\end{aligned} \tag{4.3.13}$$

In particular, it implies a well-known relation between eta invariant of a Dirac operator and the CS-form ω defined by (4.2.16):

$$\int [\omega]_{2r-1} = \frac{1}{2}\left(\eta(i\not{D}_-^{(2r-1)}) - \eta(i\not{D}_+^{(2r-1)})\right) \qquad (\mathrm{mod}\ \mathbb{Z}) . \tag{4.3.14}$$

4.3.2 Callias-type index theorem

To illustrate the importance of the mass parameter (or the Higgs field) in the formula (4.3.2), let us consider the case where the gauge fields are turned off. The spacetime manifold is chosen to be a D-dimensional plane $\mathbb{R}^D$, where D can be either even or odd. In order to have discrete spectrum, we assume that the mass diverges at infinity. To be specific, the asymptotic behavior of the mass is assumed to be as

$$\widetilde{m} \to r g(x) \qquad (\text{as } r \to \infty) , \tag{4.3.15}$$

where $r = \sqrt{x_\mu x^\mu}$ is the radial coordinate of $\mathbb{R}^D$ and $g(x) \in U(N)$ is a unitary matrix that only depends on the angular coordinates of $\mathbb{R}^D$. For odd D, $g(x)$ is also required to be Hermitian.[7]

Then, the right hand side of (4.3.2) can be easily evaluated by using (3.1.8). The result is

$$\mathrm{Ind}(\mathcal{D}) = \int \mathrm{ch}(\mathcal{F}) = \begin{cases} \left(\frac{-i}{2\pi}\right)^{\frac{D}{2}} \frac{\left(\frac{D}{2}-1\right)!}{(D-1)!} \int_{S^{D-1}} \mathrm{Tr}\left((gdg^{-1})^{D-1}\right) \ , & \text{(for even } D\text{)} \\ \left(\frac{i}{8\pi}\right)^{\frac{D-1}{2}} \frac{1}{2\left(\frac{D-1}{2}\right)!} \int_{S^{D-1}} \mathrm{Tr}\left((dg)^{D-1} g\right) \ , & \text{(for odd } D\text{)} \end{cases} , \tag{4.3.16}$$

where S^{D-1} is the sphere at $r \to \infty$. The former (even D case) is the same as the integral of (4.2.22) over S^{D-1} and the latter (odd D case) agrees with expression of the index for Callias's index theorem.[62]

We can apply these formulas for the configuration given by (4.1.30), in which $g(x)$ is given by

$$g(x) = \frac{1}{r} \sum_{I=1}^{n} \Gamma^I x^I \ . \tag{4.3.17}$$

Inserting this into (4.3.16), we obtain $\mathrm{Ind}(\mathcal{D}) = (-1)^{\left[\frac{D-1}{2}\right]}$, which is consistent with the fact that there is a fermionic zero mode as suggested in section 4.1.3 from the existence of the anomaly.

[7] Here, we assume $g(x)$ to be unitary for computational simplicity. However, this condition can be relaxed to $g(x) \in GL(N, \mathbb{C})$, as an invertible matrix (or invertible Hermitian matrix) can be continuously deformed to a unitary matrix (or unitary Hermitian matrix, respectively), keeping the invertibility.

Chapter 5

Conclusion

5.1 Conslusion

In this thesis, we have investigated about the anomalies of free fermions with spacetime dependent mass. The main result of chapter 3 is, that the anomalies of spacetime dependent mass can be written by the anomaly $D+2$-form (a generalization of the anomaly polynomial, which is described in section 3.3.1) of (3.3.11). This anomaly can be calculated by Fujikawa's method. This anomaly has some applications, which we discussed in chapter 4. In section 4.1, we considered the interfaces made by the spacetime dependent mass on which Weyl fermions are localized and confirmed that our formulas can be used to extract the anomaly of these localized Weyl fermions. The boundaries of spacetime realized by making the mass large in some regions were studied in section 4.2. A notable example was a system with the spacetime dependent boundary conditions (4.2.11) considered in section 4.2.2. It was found that there are contributions to the anomaly from the boundaries, even when the gauge fields are turned off. Implications to the index theorems were discussed in section 4.3, in which the AS and APS index theorems for the operator $\mathcal{D}$ defined in (3.2.3) and (3.2.35) were given, and the application to the Callias-type index theorems was briefly described.

5.2 Future direction

In this paper, we have considered complex Dirac fermions. An obvious interesting problem would be to generalize our discussion to systems with real or pseudo-real fermions, for which there are 8 families of theories. For this purpose, the concept of real superconnections and their realization on unstable D-brane systems considered in [48] would be useful.

We can understand this anomaly as a mixed anomaly between the chiral (or flavor) symmetry and the parameter space of the mass.[12, 13] The topological nature of the parameter space is important to condensed matter physics, and there is an application to this direction.[63]

This anomaly can also be applied to the anomaly matching. As we saw in the rewiew part (2.3.7), the anomaly matching has been discussed only about gauge fields. Spacetime dependent mass can be regarded as a background field, so we can treat it as a kind of background gauge field. To discuss the anomaly matching for this spacetime dependent mass, creating the WZW term with the background mass term might be useful. If we consider the WZW term for this anomaly, we can apply the anomaly matching for more various kinds of QFTs.

Although we have seen that the formulas for the anomaly with the superconnection are quite useful in some applications, we have not explored much on the significance of the superalgebra acting on it. It would be interesting if a deeper meaning behind this structure could be uncovered.

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Appendix A

The APS index theorem

In this appendix, we give a heuristic derivation of (4.3.10) following the argument given in the appendix of [61]. The setup is the same as that of section 4.3.1. As mentioned below (4.3.10), we extend the system to $-\infty < y < +\infty$ by choosing a y-independent configuration in the regions $y \le y_-$ and $y_+ \le y$.

First, we derive one of the key relations:

$$\int_M d^{2r-1}x\,\mathcal{I}(x) = \frac{1}{\sqrt{\pi}} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \mathrm{Tr}_{\mathcal{H}} \left((\partial_y H_y)\, \mathrm{e}^{-\frac{1}{\Lambda^2} H_y^2} \right) \ . \tag{A.0.1}$$

Inserting

$$\mathcal{D}^\dagger \mathcal{D} = H_y^2 - \partial_y^2 - \partial_y H_y \ , \qquad \mathcal{D}\mathcal{D}^\dagger = H_y^2 - \partial_y^2 + \partial_y H_y \ , \tag{A.0.2}$$

into (3.2.19), we obtain

$$\begin{aligned}&\int_M d^{2r-1}x\,\mathcal{I}(x)\\ &= \lim_{\Lambda\to\infty} \Lambda \int \frac{d\widetilde{k}}{2\pi}\, \mathrm{e}^{-\widetilde{k}^2} \mathrm{Tr}_{\mathcal{H}} \left(\mathrm{e}^{\frac{1}{\Lambda^2}\partial_y^2 + \frac{2i}{\Lambda}\widetilde{k}\partial_y - \frac{1}{\Lambda^2}(H_y^2 - \partial_y H_y)} - \mathrm{e}^{\frac{1}{\Lambda^2}\partial_y^2 + \frac{2i}{\Lambda}\widetilde{k}\partial_y - \frac{1}{\Lambda^2}(H_y^2 + \partial_y H_y)} \right) \ ,\end{aligned} \tag{A.0.3}$$

where $\widetilde{k} = k_{2r}/\Lambda$. As we did around (3.2.24), we expand the right hand side with respect to $1/\Lambda$ regarding $\widetilde{k}$ and H_y/Λ to be of $\mathcal{O}(1)$. The leading term in the $1/\Lambda$ expansion gives (A.0.1).

On the other hand, (4.3.7) implies

$$\begin{aligned}\partial_y \eta(H_y) =& \frac{2}{\sqrt{\pi}} \int_0^\infty dt\, \mathrm{Tr}_{\mathcal{H}} \left((\partial_y H_y)(1 - 2t^2 H_y^2)\, \mathrm{e}^{-t^2 H_y^2} \right) \\ =& \frac{2}{\sqrt{\pi}} \int_0^\infty dt\, \partial_t \mathrm{Tr}_{\mathcal{H}} \left(t(\partial_y H_y)\, \mathrm{e}^{-t^2 H_y^2} \right) \\ =& -\frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \mathrm{Tr}_{\mathcal{H}} \left(\epsilon(\partial_y H_y)\, \mathrm{e}^{-\epsilon^2 H_y^2} \right) \ .\end{aligned} \tag{A.0.4}$$

Combining this with (A.0.1), we obtain

$$-\frac{1}{2}\partial_y \eta(H_y) = \int_M d^{2r-1}x\, \mathcal{I}(x) \ , \tag{A.0.5}$$

which can be used when H_y does not have a zero eigenvalue.

Let us assume that H_y has zero eigenvalues at finite values of y denoted as y_i $(i = 1, 2, \cdots, k)$ with $y_- < y_1 < y_2 < \cdots < y_k < y_+$. From the expression in (4.3.8), we see that the value of $\eta(H_y)$ jumps by $+2$ or -2 at $y = y_i$ when one of the eigenvalues of H_y crosses zero from below or above, respectively, while increasing y from $y = y_i - \epsilon$ to $y = y_i + \epsilon$ with a positive small parameter $0 < \epsilon \ll 1$. It is known that the index of the operator $\mathcal{D}$ is equal to a half of the sum over these jumps [57]:[1]

$$\begin{aligned}
\mathrm{Ind}(\mathcal{D}|_I) =& \frac{1}{2}\sum_{i=1}^{k} (\eta(H_{y_i+\epsilon}) - \eta(H_{y_i-\epsilon})) \\
=& \frac{1}{2}\left(\eta(H_{y_+}) - \eta(H_{y_-})\right) - \frac{1}{2}\sum_{i=0}^{k}\left(\eta(H_{y_{i+1}-\epsilon}) - \eta(H_{y_i+\epsilon})\right) \\
=& \frac{1}{2}\left(\eta(H_{y_+}) - \eta(H_{y_-})\right) - \frac{1}{2}\sum_{i=0}^{k}\int_{y_i}^{y_{i+1}} dy\, \partial_y \eta(H_y) \ ,
\end{aligned} \tag{A.0.6}$$

where $y_0 \equiv y_-$ and $y_{k+1} \equiv y_+$. Using (A.0.5) and (3.2.29), we obtain the desired result (4.3.10):

$$\mathrm{Ind}(\mathcal{D}|_I) = \frac{1}{2}\left(\eta(H_{y_+}) - \eta(H_{y_-})\right) + \lim_{\Lambda\to\infty}\int_{y_-<y<y_+} [\mathrm{ch}(\mathcal{F})]_{2r} \ . \tag{A.0.7}$$

Here, the boundary conditions for the fermions are such that the wave function does not blow up at $y \to \pm\infty$. In these regions, the Dirac equation $\mathcal{D}\psi = 0$ with (4.3.4) can be solved by

$$\psi = \mathrm{e}^{-\lambda_\pm y}\psi_{\lambda_\pm} \ , \tag{A.0.8}$$

where $\psi_{\lambda_\pm}$ is an eigenfunction of $H_{y_\pm}$ with the eigenvalue $\lambda_\pm$. Therefore, the modes with $\lambda_+ < 0$ and $\lambda_- > 0$ are discarded, which gives the APS boundary conditions.

[1] This fact can be easily understood in the adiabatic limit,[64, 65, 66]: in which H_y is slowly varying with respect to y. In such cases, the Dirac equation $\mathcal{D}\psi = 0$ has an approximate solution of the form $\psi = \mathrm{e}^{-\int^y dy\, \lambda}\psi_\lambda$, where ψ_λ is an eigenfunction of H_y with eigenvalue $\lambda(y)$. This solution is normalizable when $\lambda > 0$ and $\lambda < 0$ as $y \to +\infty$ and $y \to -\infty$, respectively. Similarly, a normalizable approximate solution of $\mathcal{D}^\dagger\psi = 0$ is given by $\psi = \mathrm{e}^{+\int^y dy\, \lambda}\psi_\lambda$ with $\lambda < 0$ and $\lambda > 0$ as $y \to +\infty$ and $y \to -\infty$, respectively. Therefore, the index is given by the difference of the number of eigenvalues that cross zero from below and above when y is increased from y_- to y_+.

Note that the formula (4.3.10) is valid only for the finite interval $I = [y_-, y_+]$. When, one wish to apply it for the cases with $y_- \to -\infty$ and/or $y_+ \to +\infty$, one should be careful about the order of the limit $y_\pm \to \pm\infty$ and $\Lambda \to \infty$, because they do not commute when the mass diverges at $y \to \pm\infty$, as we have seen in many examples in section 4. Let us consider a system defined on $M \times \mathbb{R}$ with mass diverging at $y \to \pm\infty$. Suppose $|y_\pm|$ are large enough so that H_y does not have a zero eigenvalue for any y satisfying $y < y_-$ or $y_+ < y$. Then, (A.0.6) implies that the index $\mathrm{Ind}(\mathcal{D}|_I)$ is the same as that for $I = \mathbb{R}$. Therefore, in this case, comparing (4.3.2) and (4.3.10), we obtain

$$\frac{1}{2}\eta(H_{y_+}) - \lim_{\Lambda\to\infty}\int_{y_+<y}[\mathrm{ch}(\mathcal{F})]_{2r} = \frac{1}{2}\eta(H_{y_-}) + \lim_{\Lambda\to\infty}\int_{y<y_-}[\mathrm{ch}(\mathcal{F})]_{2r} \ . \tag{A.0.9}$$

Since the field configuration of the left hand side and the right hand side are independent, we find

$$\eta(H_{y_+}) = 2\lim_{\Lambda\to\infty}\int_{y_+<y}[\mathrm{ch}(\mathcal{F})]_{2r} + \eta_0 \ , \tag{A.0.10}$$

$$\eta(H_{y_-}) = -2\lim_{\Lambda\to\infty}\int_{y<y_-}[\mathrm{ch}(\mathcal{F})]_{2r} + \eta_0 \ , \tag{A.0.11}$$

with a field-independent constant η_0. Using these relations, we obtain

$$\mathrm{Ind}(\mathcal{D}|_{[y_-,+\infty]}) = \frac{1}{2}\left(\eta_0 - \eta(H_{y_-})\right) + \lim_{\Lambda\to\infty}\int_{y_-<y}[\mathrm{ch}(\mathcal{F})]_{2r} \ , \tag{A.0.12}$$

$$\mathrm{Ind}(\mathcal{D}|_{[-\infty,y_+]}) = \frac{1}{2}\left(\eta(H_{y_+}) - \eta_0\right) + \lim_{\Lambda\to\infty}\int_{y<y_+}[\mathrm{ch}(\mathcal{F})]_{2r} \ . \tag{A.0.13}$$

These formulas are formally the same as (4.3.10) with $[y_-, y_+]$ replaced with $[y_-, +\infty]$ or $[-\infty, y_+]$, and $\eta(H_{\pm\infty})$ replaced with η_0. Note that the second term in the right hand side of (A.0.12) and (A.0.13) is the generalized (gauge invariant) CS-form given in (4.2.20) integrated over M.

For example, let us consider the case with compact M. As a simple field configuration, we choose $A_- = A_+ = 0$ and $m = uy1_N$ with a real non-zero constant u. In this case, we have

$$H_y = \begin{pmatrix} -i\gamma^\mu\partial_\mu & uy \\ uy & i\gamma^\mu\partial_\mu \end{pmatrix} \ , \qquad H_y^2 = \begin{pmatrix} -\partial^2 + (uy)^2 & 0 \\ 0 & -\partial^2 + (uy)^2 \end{pmatrix} \ , \tag{A.0.14}$$

and $\eta(H_y)$ is trivially zero for any $y \neq 0$. This implies $\eta_0 = 0$.

Appendix B

Consistent vs. covariant anomalies

For the massless cases, it is well-known that the consistent and covariant anomalies are related by the Bardeen-Zumino counterterm.[24] In this appendix, we review the relation between consistent and covariant anomalies, and sketch the derivation of the Bardeen-Zumino counterterms for the cases with spacetime dependent mass in the covariant anomaly for completeness. Our strategy is to find a counterterm to be added to the covariant anomaly so that it satisfies the Wess-Zumino consistency condition. Note, however, that this approach is not powerful enough to fix the mass dependence of the anomaly $(D+2)$-form for the consistent anomaly. We also point out that anomalous violation of current conservation laws can be written in terms of supermatrix-valued currents.

B.1 Wess-Zumino consistency condition

Let us first introduce the notations for the consistent and covariant anomalies as

$$G(v) \equiv \delta_v \Gamma[A, m] \ , \tag{B.1.1}$$

$$G^{\text{cov}}(v) \equiv \int_M I_D^{1\,\text{cov}}(v, A, \widetilde{m}) \ , \tag{B.1.2}$$

respectively, where $\Gamma[A, m]$ is the effective action defined in (2.2.29), M is the D-dimensional spacetime and $I_D^{1\,\text{cov}}$ is given in (3.3.2) and (3.3.12). By definition, the consistent anomaly $G(v)$ satisfies the Wess-Zumino consistency condition [67]

$$\delta_{v_1} G(v_2) - \delta_{v_2} G(v_1) = G([v_1, v_2]) \ . \tag{B.1.3}$$

On the other hand, it is easy to check from the explicit expression that the covariant anomaly satisfies

$$\delta_{v_1} G^{\mathrm{cov}}(v_2) = G^{\mathrm{cov}}([v_1, v_2]) \ , \tag{B.1.4}$$

which implies

$$\delta_{v_1} G^{\mathrm{cov}}(v_2) - \delta_{v_2} G^{\mathrm{cov}}(v_1) = 2 G^{\mathrm{cov}}([v_1, v_2]) \ , \tag{B.1.5}$$

and hence the Wess-Zumino consistency condition is not satisfied.

The claim is that $G(v)$ and $G^{\mathrm{cov}}(v)$ are related (up to surface terms and the gauge variation of local counterterms) by

$$G(v) = G^{\mathrm{cov}}(v) + \alpha(v) \tag{B.1.6}$$

with

$$\alpha(v) \equiv \left(\frac{i}{2\pi}\right)^{D/2} \int_M \int_0^1 dt\, t \, \left[\mathrm{Str}^{\mathrm{sym}} \left(\mathscr{D} v \, \mathrm{e}^{t d\mathcal{A} + t^2 \mathcal{A}^2} \mathcal{A} \right)\right]_D \ , \tag{B.1.7}$$

where $\mathcal{A}$ is the superconnection (3.1.1) or (3.1.12) for even or odd dimensions, respectively, with $T = \widetilde{m} = m/\Lambda$ and

$$\mathscr{D} v \equiv dv + [\mathcal{A}, v] = \delta_v \mathcal{A} \ . \tag{B.1.8}$$

Here, $\mathrm{Str}^{\mathrm{sym}}$ denotes the symmetrized supertrace, in which $\mathscr{D}v$, $t d\mathcal{A} + t^2 \mathcal{A}^2$ and $\mathcal{A}$ are symmetrized (taking into account the sign flip when the odd elements, such as $\mathscr{D}v$ and $\mathcal{A}$, are exchanged) before taking the supertrace.

Let us show that the right hand side of (B.1.6) satisfies the Wess-Zumino consistency condition (B.1.3). For this purpose, it is convenient to rewrite $\alpha(v)$ as

$$\alpha(v) = -\left(\frac{i}{2\pi}\right)^{D/2} \int_{M \times I} \left[\mathrm{Str}\left(\delta_v \widetilde{\mathcal{A}} \, \mathrm{e}^{\widetilde{\mathcal{F}}}\right)\right]_{D+1} \ , \tag{B.1.9}$$

where $I \equiv [0, 1] \ni t$ and

$$\widetilde{\mathcal{A}} \equiv t\mathcal{A} \ , \qquad \widetilde{\mathcal{F}} \equiv \widetilde{d}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}^2 = t d\mathcal{A} + t^2 \mathcal{A}^2 + dt \mathcal{A} \ , \qquad \widetilde{d} \equiv d + dt \frac{\partial}{\partial t} \ . \tag{B.1.10}$$

We also define covariant derivatives $\mathscr{D}$ and $\widetilde{\mathscr{D}}$ as

$$\mathscr{D}\eta \equiv d\eta + \mathcal{A}\,\eta - (-1)^{|\eta|} \eta\, \mathcal{A} \ , \qquad \widetilde{\mathscr{D}}\widetilde{\eta} \equiv \widetilde{d}\widetilde{\eta} + \widetilde{\mathcal{A}}\,\widetilde{\eta} - (-1)^{|\widetilde{\eta}|} \widetilde{\eta}\, \widetilde{\mathcal{A}} \ , \tag{B.1.11}$$

where η and $\widetilde{\eta}$ are supermatrix-valued fields in M and $M \times I$, respectively, and $|\eta|$ and $|\widetilde{\eta}|$ denote their fermion numbers (mod 2).[1]

Using the relations

$$\delta_{v_1}\delta_{v_2}\mathcal{A} - \delta_{v_1}\delta_{v_2}\mathcal{A} = \delta_{[v_1,v_2]}\mathcal{A} \ , \tag{B.1.12}$$

$$\delta_v\widetilde{\mathcal{F}} = \widetilde{d}\delta_v\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}\delta_v\widetilde{\mathcal{A}} + \delta_v\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathscr{D}}\delta_v\widetilde{\mathcal{A}} \ , \tag{B.1.13}$$

and the Bianchi identity

$$\widetilde{\mathscr{D}}\widetilde{\mathcal{F}} = \widetilde{d}\widetilde{\mathcal{F}} + \widetilde{\mathcal{A}}\widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}\widetilde{\mathcal{A}} = 0 \ , \tag{B.1.14}$$

One can show

$$\begin{aligned} \delta_{v_1}\alpha(v_2) - \delta_{v_2}\alpha(v_1) - \alpha([v_1,v_2]) &= -\left(\frac{i}{2\pi}\right)^{D/2}\int_{M\times I}\mathrm{Str}^{\mathrm{sym}}\left(\widetilde{\mathscr{D}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,\mathrm{e}^{\widetilde{\mathcal{F}}}\right)\right) \\ &= -\left(\frac{i}{2\pi}\right)^{D/2}\int_{M\times I}\widetilde{d}\,\mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,\mathrm{e}^{\widetilde{\mathcal{F}}}\right) \ . \end{aligned} \tag{B.1.15}$$

Using Stokes' theorem and dropping the surface terms on the boundary of M,[2] the right hand side of (B.1.15) is evaluated as

$$\begin{aligned} \int_{M\times I}\widetilde{d}\,\mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\widetilde{\mathcal{A}}\,\delta_{v_2}\widetilde{\mathcal{A}}\,\mathrm{e}^{\widetilde{\mathcal{F}}}\right) &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(\delta_{v_1}\mathcal{A}\,\delta_{v_2}\mathcal{A}\,\mathrm{e}^{\mathcal{F}}\right) \\ &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(\mathscr{D}v_1\mathscr{D}v_2\,\mathrm{e}^{\mathcal{F}}\right) \\ &= \int_M \left(d\,\mathrm{Str}^{\mathrm{sym}}\left(v_1\mathscr{D}v_2\,\mathrm{e}^{\mathcal{F}}\right) - \mathrm{Str}^{\mathrm{sym}}\left(v_1\mathscr{D}^2v_2\,\mathrm{e}^{\mathcal{F}}\right)\right) \\ &= \int_M \mathrm{Str}^{\mathrm{sym}}\left(v_1[v_2,\mathcal{F}]\,\mathrm{e}^{\mathcal{F}}\right) \\ &= \int_M \mathrm{Str}\left([v_1,v_2]\,\mathrm{e}^{\mathcal{F}}\right) \ , \end{aligned} \tag{B.1.16}$$

where we have used

$$D\mathcal{F} = d\mathcal{F} + \mathcal{A}\mathcal{F} - \mathcal{F}\mathcal{A} = 0 \ , \qquad \mathscr{D}^2 v = d\mathscr{D}v + \mathcal{A}\mathscr{D}v + \mathscr{D}v\mathcal{A} = [\mathcal{F}, v] \ . \tag{B.1.17}$$

Therefore, we get

$$\delta_{v_1}\alpha(v_2) - \delta_{v_2}\alpha(v_1) - \alpha([v_1,v_2]) = -G^{\mathrm{cov}}([v_1,v_2]) \ , \tag{B.1.18}$$

[1] Recall that the differential form dx^μ and $\sigma^\pm$ are treated as fermions. See section 3.1.

[2] We only keep the parts that contribute to the anomaly $(D+2)$-form for the consistent anomaly.

which implies that the right hand side of (B.1.6) satisfies the Wess-Zumino consistency condition (B.1.3).

In section 3.3.1, we used the fact that there is no difference between the consistent and covariant anomalies for the $U(1)_V$ transformation when the background $U(1)_V$ gauge field V is turned off. This fact can be easily seen from the expression of $\alpha(v)$ in (B.1.7). When v is proportional to the unit matrix and the $U(1)_V$ gauge field V is set to zero, $\alpha(v)$ in (B.1.7) can be written as

$$\alpha(v) = \int_M \delta_v V \beta(\mathcal{A}_0) = \int_M \delta_v (V\beta(\mathcal{A}_0)) \ , \tag{B.1.19}$$

where $\mathcal{A}_0 \equiv \mathcal{A}|_{V=0}$ and

$$\beta(\mathcal{A}_0) \equiv \left(\frac{i}{2\pi}\right)^{D/2} \int_0^1 dt\, t \left[\mathrm{Str}^{\mathrm{sym}}\left(\mathrm{e}^{td\mathcal{A}_0 + t^2\mathcal{A}_0^2}\mathcal{A}_0\right)\right]_{D-1} \ . \tag{B.1.20}$$

Therefore, this part can be canceled by the gauge variation of a local counterterm.

B.2 Currents and the Bardeen-Zumino counterterm

The gauge variation of the effective action $\Gamma[A,m]$ can be written as

$$\delta_v \Gamma[A,m] = \int d^D x \ ((\mathscr{D}_\mu v)^a J^\mu_a + (\mathscr{D}v)^\alpha J_\alpha) \ , \tag{B.2.1}$$

where

$$J^\mu_a(x) \equiv \frac{\delta\Gamma[A,m]}{\delta A^a_\mu(x)} \ , \qquad J_\alpha(x) \equiv \frac{\delta\Gamma[A,m]}{\delta\widetilde{m}^\alpha(x)} \ . \tag{B.2.2}$$

Here, A^a_μ and $\widetilde{m}^\alpha = m^\alpha/\Lambda$ are the components of the gauge field and the mass rescaled by a constant Λ, and $(\mathscr{D}_\mu v)^a = (\delta_v A_\mu)^a$ and $(\mathscr{D}v)^\alpha = (\delta_v \widetilde{m})^\alpha$ are their infinitesimal gauge variations. (See (B.1.8).) J^μ_a and J_α in (B.2.2) are the vacuum expectation values of the currents $\delta S/\delta A^\mu_a$ and the fermion bilinear operators $\delta S/\delta\widetilde{m}^\alpha$, respectively. Note that Λ here is just an arbitrary parameter. In fact, (B.2.1) does not depend on Λ.

J^μ_a and J_α can be considered as components of a supermatrix-valued current analogous to the superconnection (3.1.1). To see this explicitly, we choose a basis of the supermatrices $\{T_a, T_\alpha\}$ such that the superconnection can be written as $\mathcal{A} = A^a_\mu dx^\mu T_a + \widetilde{m}^\alpha T_\alpha$ and introduce a dual basis $\{T^a, T^\alpha\}$ satisfying

$$\mathrm{Str}(T_a T^b) = \delta^b_a \ , \quad \mathrm{Str}(T_\alpha T^\beta) = \delta^\beta_\alpha \ , \quad \mathrm{Str}(T_a T^\beta) = 0 \ , \quad \mathrm{Str}(T_\alpha T^b) = 0 \ . \tag{B.2.3}$$

A supermatrix-valued current is defined as

$$\mathscr{J}(x) \equiv *J_a^{(1)}(x)\, T^a + *J_\alpha^{(0)}(x)\, T^\alpha \ , \tag{B.2.4}$$

where $*$ is the Hodge star operator:

$$*J_a^{(1)}(x) \equiv \frac{1}{(D-1)!}\epsilon_{\mu_1\cdots\mu_D} J_a^{\mu_1}(x)\, dx^{\mu_2}\cdots dx^{\mu_D} \ , \tag{B.2.5}$$

$$*J_\alpha^{(0)}(x) \equiv J_\alpha(x)\, dx^1 \cdots dx^D \ . \tag{B.2.6}$$

Using this, (B.2.1) can be written as

$$\delta_v \Gamma[A,m] = \int \mathrm{Str}(\mathscr{D}v\, \mathscr{J}) \ , \tag{B.2.7}$$

and the anomaly equation, obtained as the functional derivative of (B.1.1) with respect to $v(x)$, becomes

$$*(\mathscr{D}\mathscr{J})_a = -\frac{\delta G(v)}{\delta v^a} \ , \tag{B.2.8}$$

which shows that the consistent anomaly $G(v)$ represents the anomalous violation of the current conservation law. For example, for the axial $U(1)$ symmetry (with $v_+ = -v_- = -i\alpha 1_N$) in 4-dimensions (2.2.1), the left hand side of (B.2.8) becomes

$$*(\mathscr{D}\mathscr{J})_{U(1)_A} = \partial_\mu \left\langle \overline{\psi}\gamma^\mu\gamma^5\psi \right\rangle + 2im \left\langle \overline{\psi}\gamma^5\psi \right\rangle \ , \tag{B.2.9}$$

and, together with the right hand side obtained from (3.3.5)[3], (B.2.8) reduces to the well-known formula for the axial $U(1)$ anomaly.

From the expression (B.1.7), we find that $\alpha(v)$ can be written in the form

$$\alpha(v) = \int_M d^D x \left((\mathscr{D}_\mu v)^a P_a^\mu + (\mathscr{D}v)^\alpha P_\alpha \right) = \int \mathrm{Str}(\mathscr{D}v\, \mathscr{P}) \tag{B.2.10}$$

where P_a^μ and P_α are local functions of the gauge field and the mass, and $\mathscr{P} \equiv *P_a T^a + *P_\alpha T^\alpha$. Then, the relation (B.1.6) implies that the covariant anomaly is understood as the anomalous violation of conservation laws

$$*(\mathscr{D}\mathscr{J}^{\,\mathrm{cov}})_a = -\frac{\delta G^{\mathrm{cov}}(v)}{\delta v^a} \ , \tag{B.2.11}$$

for the covariant currents defined by

$$J_a^{\mathrm{cov}\,\mu}(x) \equiv J_a^\mu(x) - P_a^\mu(x) \ , \quad J_\alpha^{\mathrm{cov}}(x) \equiv J_\alpha(x) - P_\alpha(x) \ , \quad \mathscr{J}^{\,\mathrm{cov}}(x) \equiv \mathscr{J}(x) - \mathscr{P}(x) \ . \tag{B.2.12}$$

These P_a^μ, P_α and $\mathscr{P}$ are the Bardeen-Zumino counterterms generalized to include the space-time dependent mass.

[3] In this local expression without integration over spacetime, the $\widetilde{m}$ dependence in (3.3.5) drops out in the $\Lambda \to \infty$ limit with fixed m.

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