Random walk on uniform spanning tree and loop-erased random walk

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Abstract

Random walks on random graphs are associated with diffusion phenomena in disordered media. In this thesis, the graphs of interest are uniform spanning tree (UST) and loop-erased random walk (LERW). Firstly, we will give a quantitative estimate of the number of collisions of two independent simple random walks on the three-dimensional UST. Secondly, we will demonstrate log-logarithmic fluctuations of the quenched heat kernel of the simple random walk on the three-dimensional UST, which is caused by the same type of fluctuation of the volume of intrinsic balls. Finally, we will discuss annealed heat kernel estimates for the simple random walk on high-dimensional LERWs.

Contents

1 Introduction

The exploration of random walks in random environments stands as a critical pursuit in the study of diffusion phenomena. The mathematical models in this area serve as a pivotal avenue for advancing our comprehension of diffusion phenomena, including heat and wave propagation, on disordered media such as polymers, crystals, and porous structures. Historical overview and exploration of pioneering models in this field can be found in [2, 23]. Notably, the examination of random walks in random media has gone beyond the study of diffusion and led to diverse applications, ranging from network analysis and algorithmic development to the formulation of various models in the social and natural sciences.

In this thesis, we investigate the behavior of simple random walks on two types of random graphs, the three-dimensional uniform spanning tree and the high-dimensional loop-erased random walks, defined on the Euclidean lattice $\mathbb{Z}^d$. By delving into the intricacies of these models, we aim to contribute to the broader understanding of random walks in complex environments and their implications across various scientific and analytical domains.

1.1 Uniform spanning trees and loop-erased random walks

Let us first begin with the introduction of uniform spanning forests on $\mathbb{Z}^d$. Pemantle [36] proved that if G_n is a sequence of finite subgraphs which exhausts $\mathbb{Z}^d$, then the sequence of the uniform spanning measures on G_n weakly converges to some probability measure supported on the set of spanning forests of $\mathbb{Z}^d$. The corresponding random graph is called the uniform spanning forest (USF) on $\mathbb{Z}^d$. Pemantle [36] also showed that the uniform spanning forest is a single tree almost surely if $d \leq 4$, in which case the random graph is called the uniform spanning tree (UST) on $\mathbb{Z}^d$, while it consists of infinitely many trees if $d \geq 5$. Since their introduction, the study of uniform spanning forests has played an important role in probability theory, due to its connection to various areas such as electrical networks [10, 12, 20], loop-erased random walk [25, 36, 42], the random cluster model [14, 15], and for $d = 2$, conformally invariant scaling limits [4, 11, 17, 30, 38].

Let us mention that the uniform spanning tree is often considered to be included in the same class as various critical models arising in statistical physics since it has similar properties such as fractal-like scaling limits with non-trivial scaling exponents. Moreover, the uniform spanning tree is one of the few models for which rigorous results have been proved even for the three-dimensional case [1, 21, 31], which is typically the most difficult case to study.

Next, we introduce the loop-erased random walk (LERW) on $\mathbb{Z}^d$. Given a finite path γ, we denote by $\mathrm{LE}(\gamma)$ the chronological loop-erasure of γ (see Section 2.1 for the precise definition). If γ is a random walk path up to a finite time, then $\mathrm{LE}(\gamma)$ is called a finite loop-erased random walk. Let S be the entire path of a simple random walk (SRW) on $\mathbb{Z}^d$. Since S is transient for $d \geq 3$, we can apply the same procedure of erasing loops to S almost surely, and the resulting infinite simple random path is called the infinite loop-erased random walk, while two-dimensional infinite LERW is obtained as the limit of finite LERW on $\mathbb{Z}^2$ (see [28], for example).

Uniform spanning forests and loop-erased random walks are closely related to each other via an algorithm called Wilson's algorithm [10, 42]. While the USF is defined as (the weak limit of) the uniformly distributed random graphs on the set of spanning forests, Wilson's algorithm describes a method to construct USF with independent LERW paths, which enables one to investigate the property of d-dimensional UST or USF by utilizing the known facts concerning LERWs on $\mathbb{Z}^d$.

The behavior of random walks on random graphs strongly depends on the geometric and spectral properties of the random graphs. Motivating the study of the SRW on the uniform spanning trees and the uniform spanning forests is that such a process captures these properties that depend on the dimension d. It has been proved that the random walk displays mean-field behavior for $d \geq 4$, with a logarithmic correction in four dimensions [16, 18]. On the other hand, different (nontrivial) exponents describe the asymptotic behavior of several quantities such as transition density (heat kernel), exit time and mean-square displacement of the random walk below four dimensions [1, 8]. At least, this is confirmed for $d = 2$ and it is strongly believed that this is the case for $d = 3$ (see Remark 1.1). Similar kinds of differences in such properties for different dimensions are also observed for the loop-erased random walks on $\mathbb{Z}^d$. It has been proved that the LERW converges to Brownian motion if $d \geq 4$, with a logarithmic correction in four dimensions, while the mean-square displacement is described with nontrivial exponents for $d = 2$ and 3 [28].

Remark 1.1. Rigorously speaking, it is not clarified that the uniform spanning tree on $\mathbb{Z}^3$ exhibits *different* exponents than the high-dimensional case since the only information about the growth exponent β is that it satisfies $1 < \beta \leq 5/3$. As shown in (1.4), the leading order of the on-diagonal heat kernel is $n^{-\frac{3}{3+\beta}}$ a.s. in three dimensions, while it equals $n^{-\frac{2}{3}}$ for every component of the uniform spanning forest in higher dimensions [16, 18].

1.2 Main theorems

The main part of this thesis consists of three chapters. In this section, we state the main result of each chapter.

1.2.1 Collisions of random walks on the 3D UST

In Chapter 3, we will estimate the number of collisions of two independent random walks on the three-dimensional uniform spanning tree. To be more precise, let us introduce some terminology here. For infinite connected recurrent graph G, let X and Y be independent (discrete time) simple random walks on G. We say that G has the infinite collision property when $|\{n : X_n = Y_n\}| = \infty$ holds almost surely, where $|A|$ denotes the cardinality of A. For classical examples such as $\mathbb{Z}$ and $\mathbb{Z}^2$, it is easy to see that two independent simple random walks collide infinitely often. On the other hand, Krishnapur and Peres [22] gave an example of a recurrent graph for which the number of collisions is almost surely finite. For collisions on random graphs, Barlow, Peres and Sousi [9] proved that a critical Galton-Watson tree, the incipient infinite cluster in high dimensions and the uniform spanning tree on $\mathbb{Z}^2$ all have the infinite collision property almost surely. The infinite collision property of reversible random rooted graphs including uniform spanning trees on $\mathbb{Z}^d$ ($d \leq 4$) and every component of uniform spanning forests on $\mathbb{Z}^d$ ($d \geq 5$) was proved in [19].

The purpose of Chapter 3 is to give a quantitative estimate of the number of collisions until two random walks exit a ball of the three-dimensional UST. Let $\mathcal{U}$ be the uniform spanning tree on $\mathbb{Z}^3$ and $\mathbf{P}$ be its law. Let $\widetilde{X}$ and $\widetilde{Y}$ be two independent simple random walks on $\mathcal{U}$ killed when they exit the intrinsic ball of $\mathcal{U}$ of radius r. We denote by P the law of $(\widetilde{X}, \widetilde{Y})$

started at $(0,0)$ and by E the corresponding expectation. Let Z_{B_r} be the total number of collisions of $\widetilde{X}$ and $\widetilde{Y}$ (see Section 3.1 for the precise definition).

Theorem 1.2. *There exist some universal constants $C > 0$, $c > 0$ and $\delta > 0$ such that for any $r \geq 1$ and for all $0 < \varepsilon < \delta$, there exists some event $K(r,\varepsilon)$ with $\mathbf{P}(K(r,\varepsilon)) \geq 1 - C\varepsilon^c$ such that on $K(r,\varepsilon)$,*

$$\varepsilon r \leq E(Z_{B_r}) \leq 6r, \tag{1.1}$$

$$E(Z_{B_r}^2) \leq 144r^2 + 6r, \tag{1.2}$$

hold. In particular, on $K(r,\varepsilon)$ we have

$$P(\varepsilon r \leq Z_{B_r} \leq 72\varepsilon^{-2} r) \geq \varepsilon^2/12. \tag{1.3}$$

The infinite collision property of the three-dimensional UST directly follows from Theorem 1.2.

Corollary 1.3. *The uniform spanning tree on $\mathbb{Z}^3$ has the infinite collision property $\mathbf{P}$-a.s.*

Remark 1.4. Note that the above statement includes two different probability measures, the law of the three-dimensional UST and that of random walks on it. Corollary 1.3 claims that if we choose a tree according to the law of the three-dimensional UST and check whether two independent simple random walks on the tree collide infinitely often almost surely, then it has the infinite collision property almost surely with respect to the UST measure.

Remark 1.5. In [19], it is proved that the uniform spanning tree on $\mathbb{Z}^d$ $(d = 3,4)$ and each connected component of the uniform spanning forest on $\mathbb{Z}^d$ $(d \geq 5)$ have the infinite collision property. In Section 3.2 of this article, we will derive Corollary 1.3 from Theorem 1.2, which gives another proof for the three-dimensional case. We expect that quantitative moment estimates of the number of collisions for the case $d \geq 4$ can also be derived from various estimates obtained in [16] and [18]. We will not pursue this further in the present article.

1.2.2 Heat kernel fluctuations for the simple random walk on the 3D UST

The aim of Chapter 4 is to demonstrate an oscillatory phenomenon for the volume and heat kernel of the simple random on the three-dimensional uniform spanning tree. Let $\mathcal{U}$ be the uniform spanning tree on $\mathbb{Z}^3$. We write $p_t(x,y)$ for the transition density (heat kernel) of simple random walks on graphs and, in particular, $p_n^{\mathcal{U}}(x,y)$ for the heat kernel of the (discrete-time) simple random walk on $\mathcal{U}$, see Section 2.2 for its precise definition. We also let $\beta \in (1, 5/3]$ be the growth exponent that governs the time-space scaling of the three-dimensional loop-erased random walk, which coincides with the Hausdorff dimension of the scaling limit of the three-dimensional loop-erased random walk [39, 40], see Section 2.1 for details.

Remark 1.6. Numerical estimates suggest that $\beta = 1.624\cdots$ (see [43]).

It was proved in [24] that if a random graph satisfies some assumptions on its volume and effective resistance, the on-diagonal heat kernel $p_t(x,x)$ has upper and lower bounds which are derived from volume and effective resistance estimates. Combining this with estimates for the three-dimensional uniform spanning tree obtained by [1, Theorem 1.6] concludes that there exist deterministic constants b_1, b_2, b_3, $b_4 > 0$ and c_1, $c_2 > 0$ such that with probability one

$$c_1 n^{-\frac{3}{3+\beta}} (\log\log n)^{-b_1} \leq p_{2n}^{\mathcal{U}}(0,0) \leq c_2 n^{-\frac{3}{3+\beta}} (\log\log n)^{b_2}, \tag{1.4}$$

for large n, and also

$$c_1 r^{\frac{3}{\beta}} (\log\log r)^{-b_3} \leq |B_{\mathcal{U}}(0,r)| \leq c_2 r^{\frac{3}{\beta}} (\log\log r)^{b_4}, \tag{1.5}$$

for large r.

The main theorem of Chapter 4 then demonstrates that there exist some exponents of log-logarithmic bounds which cause fluctuations of the on-diagonal heat kernel.

Theorem 1.7. *There exist deterministic constants $a_1, a_2 > 0$ such that one has*

$$\liminf_{n\to\infty} (\log\log n)^{a_1} n^{\frac{3}{3+\beta}} p_{2n}^{\mathcal{U}}(0,0) = 0, \tag{1.6}$$

and also

$$\limsup_{n\to\infty} (\log\log n)^{-a_2} n^{\frac{3}{3+\beta}} p_{2n}^{\mathcal{U}}(0,0) = \infty, \tag{1.7}$$

almost surely.

Similar heat kernel fluctuations have been established for Galton-Watson trees [7, 13] and the uniform spanning tree on $\mathbb{Z}^2$ [5]. We will describe some key differences between these models and the three-dimensional UST at the beginning of Chapter 4, but common ingredients in the proofs of such results are corresponding volume fluctuations. The idea of proof of Theorem 1.7 is similar to that of [5, Corollary 1.2]. Specifically, to prove Theorem 1.7, the crucial step is to demonstrate that the volume of intrinsic balls (with respect to the graph distance) of $\mathcal{U}$ also enjoys log-log fluctuations. To be more precise, let $B_{\mathcal{U}}(0,r)$ be the intrinsic ball in $\mathcal{U}$ of radius r centered at the origin. Then we have the following volume fluctuations.

Theorem 1.8. *There exist deterministic constants $a_3, a_4 > 0$ such that one has*

$$\liminf_{r\to\infty} (\log\log r)^{a_3} r^{-\frac{3}{\beta}} |B_{\mathcal{U}}(0,r)| = 0, \tag{1.8}$$

and also

$$\limsup_{r\to\infty} (\log\log r)^{-a_4} r^{-\frac{3}{\beta}} |B_{\mathcal{U}}(0,r)| = \infty, \tag{1.9}$$

almost surely. Here $|A|$ stands for the cardinality of A.

Remark 1.9. The main contribution of Theorems 1.7 and 1.8 is demonstrating the existence of the exponents a_i that satisfies (1.6), (1.7), (1.8) and (1.9). Determining the optimal exponents for a_i seems to be a difficult problem and we do not pursue this here.

1.2.3 Annealed off-diagonal heat kernel of the simple random walk on high-dimensional LERWs

The main result of Chapter 5 is annealed heat kernel estimates for the random walk on the random graph given by the trace of a LERW in high dimensions. Our main theorem reveals that the annealed (averaged) heat kernel of the random walk satisfies sub-Gaussian estimate, which exhibits an interesting difference from the quenched (typical) heat kernel estimates of Gaussian form with respect to the intrinsic graph metric. Investigating such a difference between quenched and annealed heat kernel estimates rigorously was motivated by a conjecture made in [5, Remark 1.5] for the two-dimensional uniform spanning tree, and naturally leads one to consider to what extent the behavior is typical for random walks on random graphs embedded into an underlying space.

Let us introduce our model of a random walk in a random environment. Throughout the thesis, we let $(L_n)_{n\geq 0}$ be the loop-erasure of the discrete-time simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$, where $d \geq 5$, started from the origin. (See Section 2.1 for a precise definition of this process, which was originally introduced by Lawler in [25].) Given a realisation of $(L_n)_{n\geq 0}$, we define a graph $\mathcal{G}$ to have vertex set

$$V(\mathcal{G}) := \{L_n : \, n \geq 0\},$$

and edge set

$$E(\mathcal{G}) := \{\{L_n, L_{n+1}\} : \, n \geq 0\}.$$

We then let $(X_t^{\mathcal{G}})_{t\geq 0}$ be the continuous-time random walk on $\mathcal{G}$ that has unit mean exponential holding times at each site and jumps from its current location to a neighboring vertex chosen with equal probability. Moreover, we will always suppose that $X_0^{\mathcal{G}} = L_0 = 0$. We define the annealed law of $X^{\mathcal{G}}$ to be the probability measure on the Skorohod space $D(\mathbb{R}_+, \mathbb{R}^d)$ given by

$$\mathbb{P}\left(X^{\mathcal{G}} \in \cdot\right) = \int P^{\mathcal{G}}\left(X^{\mathcal{G}} \in \cdot\right) \mathbf{P}(d\mathcal{G}),$$

where $\mathbf{P}$ is the probability measure on the underlying probability space on which L is built, and $P^{\mathcal{G}}$ is the law of $X^{\mathcal{G}}$ on the particular realization of $\mathcal{G}$ (i.e. the quenched law of $\mathcal{G}$). We use the notation $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

Theorem 1.10. *For any $\varepsilon > 0$, there exist constants $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that, for every $x \in \mathbb{Z}^d$ and $t \geq \varepsilon |x|$,*

$$\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) \leq c_1 \left(1 \wedge |x|^{2-d}\right) \left(1 \wedge t^{-1/2}\right) \exp\left(-c_2 \left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right)$$

and also

$$\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) \geq c_3 \left(1 \wedge |x|^{2-d}\right) \left(1 \wedge t^{-1/2}\right) \exp\left(-c_4 \left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right).$$

The key ingredient of the proof is a time-averaged Gaussian bound on the distribution of the loop-erased random walk. Now, one can check that SRW satisfies pointwise Gaussian bounds of the form

$$cn^{-d/2} e^{-\frac{|x|^2}{cn}} \mathbf{1}_{\{n \geq \|x\|_1\}} \leq \frac{\mathbf{P}(S_n = x) + \mathbf{P}(S_{n+1} = x)}{2} \leq c^{-1} n^{-d/2} e^{-\frac{c|x|^2}{n}}, \qquad \forall x \in \mathbb{Z}^d, \; n \geq 1, \tag{1.10}$$

where c is a constant and we write $\|x\|_1$ for the ℓ_1-norm of x, see [2, Theorem 6.28], for example. (The averaging over two time steps is necessary for parity reasons.) Of course, one can not expect the same bounds for a LERW. Indeed, the 'on-diagonal' part of the distribution $\mathbf{P}(L_n = 0)$ is equal to zero for $n \geq 1$. Instead, we will establish the following theorem, which demonstrates that if one averages $\mathbf{P}(L_n = 0)$ over longer time intervals, then one can see Gaussian estimates.

Theorem 1.11. *The loop-erased random walk $(L_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d \geq 5$, started from the origin satisfies the following bounds: for all $x \in \mathbb{Z}^d \backslash \{0\}$, $n \geq 1$,*

$$\frac{1}{n}\sum_{m=n}^{2n-1} \mathbf{P}(L_m = x) \leq c_1 n^{-d/2} e^{-\frac{c_2|x|^2}{n}},$$

and for all $x \in \mathbb{Z}^d \backslash \{0\}$, $n \geq |x|$,

$$\frac{1}{n}\sum_{m=\lceil c_3 n\rceil}^{\lfloor c_4 n\rfloor} \mathbf{P}(L_m = x) \geq c_5 n^{-d/2} e^{-\frac{c_6|x|^2}{n}},$$

where $c_1, \dots, c_6$ are constants.

To put this result into context, it helps to briefly recall what kind of behavior has been observed for anomalous random walks and diffusions in other settings. In particular, for many random walks or diffusions on fractal-like sets (either deterministic or random), it has been shown that the associated transition density $p_t(x, y)$ satisfies, within appropriate ranges of the variables, upper and lower bounds of the form

$$c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right), \tag{1.11}$$

where $d(x, y)$ is some metric on the space in question. (See [2, 23] for overviews of work in this area.) The exponent d_s is typically called the spectral dimension since it is related to the growth rate of the spectrum of the generator of the stochastic process. The exponent d_w, which is usually called the walk dimension (with respect to the metric d), gives the space-time scaling.

Now, in our setting, we can clearly write

$$\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) = \mathbb{P}\left(X_t^{\mathcal{G}} = x \,\middle|\, x \in \mathcal{G}\right) \mathbf{P}\left(x \in \mathcal{G}\right),$$

and, moreover, using simple facts about the intersection properties of SRW in high dimensions, one can check that $\mathbf{P}(x \in \mathcal{G}) \asymp 1 \wedge |x|^{2-d}$ (where we use the notation $\asymp$ to mean that the left-hand side is bounded above and below by constant multiples of the right-hand side). Hence, Theorem 1.10 gives that the (conditioned) annealed transition probability $\mathbb{P}(X_t^{\mathcal{G}} = x \,|\, x \in \mathcal{G})$ satisfies the sub-Gaussian estimate of the form of (1.11), with $d_s = 1$, $d_w = 4$ and d being the Euclidean metric. We can understand that $d_s = 1$ results from one-dimensional nature of the graph $\mathcal{G}$ with respect to its intrinsic metric $d_{\mathcal{G}}$. Moreover, the exponent $d_w = 4$ gives the space-time scaling of the process $X^{\mathcal{G}}$ with respect to the Euclidean metric. We note that the exponent $d_s = 1$ matches the quenched spectral dimension, while $d_w = 4$ is the multiple of the

‘2’ of the quenched bound, which is the walk dimension of $X^{\mathcal{G}}$ with respect to the intrinsic metric $d_{\mathcal{G}}$, and the ‘2’ that gives the space-time scaling of L. We highlight that the annealed bound is not obtained by simply replacing $d_{\mathcal{G}}(0,x)$ by $|x|^2$ in the quenched bound, though, as doing that does not result in an expression of the form at (1.11).

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2 Definition and Notation

In this chapter, we introduce some notations that will be used in the thesis and discuss some necessary background.

We begin with some notation for subsets of $\mathbb{Z}^d$. We apply the definition below to $d = 3$ in Chapters 3 and 4 and to $d \geq 5$ in Chapter 5. For two points $x, y \in \mathbb{Z}^d$, we let $d_E(x, y) = |x - y|$ be the Euclidean distance between x and y. If A and B are two subsets of $\mathbb{Z}^d$, we let $\mathrm{dist}(A, B) = \inf\{d_E(x, y) : x \in A, y \in B\}$. In particular, for $x \in \mathbb{Z}^d$, we write $d(x, B)$ instead of $d(\{x\}, B)$. For a set $A \subset \mathbb{Z}^d$, let

$$\partial_i A = \{x \in A : \text{there exists } y \in \mathbb{Z}^d \setminus A \text{ such that } d_E(x, y) = 1\},$$
$$\partial A = \{x \in \mathbb{Z}^d \setminus A : \text{there exists } y \in A \text{ such that } d_E(x, y) = 1\}$$

be the inner and outer boundary of A, respectively. We denote balls in the Euclidean metric by

$$B(x, r) = \{y \in \mathbb{Z}^d : d_E(x, y) \leq r\},$$

and balls in l_∞-metric d_∞, *i.e.* cubes, by

$$B_\infty(x, r) = \{y \in \mathbb{Z}^d : d_\infty(x, y) \leq r\}.$$

Throughout the thesis, we let S^z denote a simple random walk on $\mathbb{Z}^d$ started at $z \in \mathbb{Z}^d$ and let P^z denote its law. We take $(S^z)_{z \in \mathbb{Z}^d}$ to be independent.

2.1 Loop-erased random walk

Now we define a loop-erased random walk, which is a model of interest itself in this thesis and plays an important role in the analysis of uniform spanning trees.

Firstly, we introduce some notation for paths on $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if $d_E(x, y) = 1$. A finite or infinite sequence of vertices $\theta = (\theta_0, \theta_1, \cdots)$ is called a **path** if $\theta_{i-1} \sim \theta_i$ for all $i = 1, 2, \cdots$. If θ satisfies $\theta_i \neq \theta_j$ for all $i \neq j$, then θ is called a simple path. We write $\theta[i, j] = (\theta_i, \theta_{i+1}, \cdots, \theta_j)$ for $0 \leq i \leq j$ and $\theta[i, \infty) = (\theta_i, \theta_{i+1}, \cdots)$. For a finite path $\theta = (\theta_0, \cdots, \theta_k)$, we define the length of θ to be $\mathrm{len}(\theta) = k$.

For two paths $\theta = (\theta_0, \theta_1, \cdots, \theta_k)$ and $\theta' = (\theta'_0, \theta'_1, \cdots)$ with $\theta_k = \theta'_0$, we define the concatenation $\theta \oplus \theta'$ of them by

$$\theta \oplus \theta' = (\theta_0, \theta_1, \cdots, \theta_k, \theta'_1, \cdots).$$

Given a path θ on $\mathbb{Z}^d$ and a set $A \subset \mathbb{Z}^d$, we define

$$\tau^\theta_A = \min\{i \geq 0 : \theta_i \in A\}. \tag{2.1}$$

We write $\tau^z(A) := \tau^{S^z}_A$ for the first hitting time of a set $A \subset \mathbb{Z}^d$ by the simple random walk S^z started at z.

Given a path $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_m] \subset \mathbb{Z}^d$ with $\mathrm{len}(\lambda) = m$, we define its (chronological) loop-erasure $\mathrm{LE}(\lambda)$ as follows. Let $\sigma_0 = \max\{k \,:\, \lambda_k = \lambda_0\}$ and also, for $i \geq 1$,

$$\sigma_i = \max\left\{k \,:\, \lambda_k = \lambda_{\sigma_{i-1}+1}\right\}. \tag{2.2}$$

We note that these quantities are well-defined up to the index $j = \min\{i : \lambda_{\sigma_i} = \lambda_m\}$, and we use them to define the loop-erasure of λ by setting

$$\mathrm{LE}(\lambda) = \left[\lambda_{\sigma_0}, \lambda_{\sigma_1}, \ldots, \lambda_{\sigma_j}\right].$$

It follows by construction that $\mathrm{LE}(\lambda)$ is a simple path satisfying $\mathrm{LE}(\lambda) \subseteq \lambda$, $\mathrm{LE}(\lambda)_0 = \lambda_0$ and $\mathrm{LE}(\lambda)_j = \lambda_m$. If $\lambda = [\lambda_0, \lambda_1, \ldots] \subseteq \mathbb{Z}^d$ is an infinite path such that $\{k : \lambda_k = \lambda_i\}$ is finite for each $i \geq 0$, then its loop-erasure $\mathrm{LE}(\lambda)$ can be defined similarly.

2.2 Uniform spanning tree

In this subsection, we introduce the three-dimensional uniform spanning tree, the model of interest in Chapter 3 and Chapter 4.

A subgraph of a connected graph G is called a **spanning tree** on G if it is connected, contains all vertices of G and has no cycle. Let $\mathcal{T}(G)$ be the set of all spanning trees on G. For a finite connected graph G, a random tree chosen according to the uniform measure on $\mathcal{T}(G)$ is called the **uniform spanning tree (UST)** on G. We can define the uniform spanning tree on $\mathbb{Z}^3$, or the three-dimensional uniform spanning tree, as the weak limit of the USTs on the finite boxes $\mathbb{Z}^3 \cap [-n, n]^3$, see [36].

We will assume that the three-dimensional UST $\mathcal{U}$ is built on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we denote the corresponding expectation by $\mathbf{E}$. Note that, $\mathbf{P}$-a.s., $\mathcal{U}$ is a one-ended tree [36]. For any $x, y \in \mathbb{Z}^3$ and any connected subset $A \subset \mathbb{Z}^3$, we write $\gamma(x, y)$ for the unique self-avoiding path between x and y, $\gamma(x, A)$ for the shortest path among $\{\gamma(x, y) : y \in A\}$ if $x \notin A$, and $\gamma(x, A) = \{x\}$ if $x \in A$. We let $\gamma(x, \infty)$ for the unique infinite self-avoiding path started at x. We denote by $d_{\mathcal{U}}$ the intrinsic metric on the graph $\mathcal{U}$, *i.e.* $d_{\mathcal{U}}(x, y) = \mathrm{len}(\gamma(x, y))$. We define balls in the intrinsic metric by

$$B_{\mathcal{U}}(x, r) = \{y \in \mathbb{Z}^3 : d_{\mathcal{U}}(x, y) \leq r\}, \tag{2.3}$$

and let $|B_{\mathcal{U}}(x, r)|$ be the number of points in $B_{\mathcal{U}}(x, r)$.

Now we recall Wilson's algorithm. This method to construct UST with LERW was first introduced to finite graphs [42] and then extended to transient $\mathbb{Z}^d$ including $\mathbb{Z}^3$ [10]. Let $\{v_1, v_2, \cdots\}$ be an ordering of the vertices of $\mathbb{Z}^3$ and let γ_∞ be the infinite LERW started at the origin. We define a sequence of subtrees of $\mathbb{Z}^3$ inductively as follows:

$$\begin{aligned}
\mathcal{U}_0 &= \gamma_\infty, \\
\mathcal{U}_i &= \mathcal{U}_{i-1} \cup \mathrm{LE}(S^{z_i}[0, \tau^{z_i}(\mathcal{U}_{i-1})]),\ i \geq 1, \\
\mathcal{U} &= \cup_i \mathcal{U}_i.
\end{aligned}$$

Then by [10], the random tree $\mathcal{U}$ has the same law as the three-dimensional UST. It follows that the law of $\mathcal{U}$ above does not depend on the ordering of $\mathbb{Z}^3$.

We end this subsection by defining the simple random walk on $\mathcal{U}$. We denote by μ_G the measure on the vertex set V of a (random or deterministic) graph G such that $\mu_G(\{x\})$ is given by the number of edges of G which contain $x \in V$. We write $\mu_G(x) := \mu_G(\{x\})$. For a given realization of $\mathcal{U}$, the simple random walk on $\mathcal{U}$ is the discrete-time Markov process $X^{\mathcal{U}} = ((X_n^{\mathcal{U}})_{n \geq 0}, (P_x^{\mathcal{U}})_{x \in \mathbb{Z}^3})$ which at each step jumps from its current location to a uniformly chosen neighbor in $\mathcal{U}$. For $x \in \mathbb{Z}^3$, the law $P_x^{\mathcal{U}}$ is called the **quenched law** of the simple random walk on $\mathcal{U}$. We write

$$p_n^{\mathcal{U}}(x,y) = \frac{P_x^{\mathcal{U}}(X_n^{\mathcal{U}} = y)}{\mu_{\mathcal{U}}(\{y\})}, \; x, y \in \mathbb{Z}^3, \tag{2.4}$$

for the **quenched heat kernel**.

2.3 Effective Resistance and Green's function

Now we define the effective resistance and Green's function, which is a key tool to derive the key estimates of Chapters 3 and 4.

Definition 2.1. *Let $G = (V, E)$ be a connected graph and let f and g be functions on V. Then we define a quadratic form $\mathcal{E}$ by*

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y \in V \\ x \sim y}} (f(x) - f(y))(g(x) - g(y)).$$

*If we consider G as an electrical network by regarding each edge of G to be a unit resistance, then the **effective resistance** between disjoint subsets A and B of V is defined by*

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : \mathcal{E}(f,f) < \infty, f|_A = 1, f|_B = 0\}. \tag{2.5}$$

If we let $R_{\text{eff}}(x,y) = R_{\text{eff}}(\{x\},\{y\})$, then $R_{\text{eff}}(\cdot,\cdot)$ is a metric on G, see [41].

Definition 2.2. *Let B be a connected subgraph of G. For a simple random walk X with starting point $x \in G$, we define the **Green's function** by*

$$G(x,y) = \sum_{n=0}^{\infty} P^x(X_n = y),$$

and we write $G(x) \coloneqq G(0,x)$. For a simple random walk X^B on G killed when it exits B, the Green's function is defined by

$$G_B(x,y) = \sum_{n=0}^{\infty} P^x(X_n^B = y). \tag{2.6}$$

Let $G = (V,E)$ be a connected recurrent graph with a fixed vertex 0. Recall the definition of μ_G in the previous section. For a finite subset $0 \in B \subset V$, the effective resistance between 0 and B^c and Green's function are related by the following equality:

$$\mu_G(x) R_{\text{eff}}(x, B^c) = G_B(x,x), \tag{2.7}$$

see [34] Section 2.2, for example.

2.4 Simple random walk estimates

Let S be a simple random walk on $\mathbb{Z}^d$, and suppose m and n are real numbers such that $1 \leq m < n$. Moreover, let $A = \{x \in \mathbb{Z}^d : m \leq |x| \leq n\}$, and set $\tau = \tau^S_{A^c}$ to be the first time that S exits A. Then [28, Proposition 1.5.10] gives that, for all $x \in A$,

$$\mathbf{P}^x\left(|S_\tau| \leq m\right) = \frac{|x|^{2-d} - n^{2-d} + O(m^{1-d})}{m^{2-d} - n^{2-d}}. \tag{2.8}$$

Whilst this approximation is good for large m, in this thesis, we also need to consider the situation when $m = 1$ and $|x|$ is large. In this case, $|S_\tau| \leq m$ if and only if $S_\tau = 0$, and the estimate (2.8) is not useful due to the $O(m^{1-d})$ term. However, adapting the argument used to prove [28, Proposition 1.5.10], it is possible to establish that there exists a universal constant $a = a_d > 0$ such that

$$\mathbf{P}^x\left(S_\tau = 0\right) = \frac{a|x|^{2-d} - an^{2-d} + O(|x|^{1-d})}{G(0) - an^{2-d}}, \tag{2.9}$$

where $G(0)$ is as defined in Definition 2.2, which is finite in the dimensions we are considering.

In this thesis, we will also make use of another basic estimate for the simple random walk on $\mathbb{Z}^d$, which is often called the gambler's ruin estimate. We take $\theta \in \mathbb{R}^d$ with $|\theta| = 1$ and set $\widehat{S}_j = S_j \cdot \theta$. Let $\eta_n = \min\{j \geq 0 : \widehat{S}_j \leq 0 \text{ or } \widehat{S}_j \geq n\}$. We denote by $\widehat{\mathbf{P}}^x$ the law of $\widehat{S}$ with starting point $x \in \mathbb{R}$. Then [29, Proposition 5.1.6] guarantees that there exist $0 < \alpha_1 < \alpha_2 < \infty$ such that: for all $1 \leq m \leq n$,

$$\alpha_1 \frac{m+1}{n} \leq \widehat{\mathbf{P}}^m(\widehat{S}_{\eta_n} \geq n) \leq \alpha_2 \frac{m+1}{n}. \tag{2.10}$$

The gambler's ruin estimate gives upper and lower bounds on the probability that a simple random walk on $\mathbb{Z}^d$ projected onto a line escapes from one of the endpoints of a line segment.

3 Quantitative estimates on the collisions of random walks on the three-dimensional uniform spanning tree

In this chapter, we will prove Theorem 1.2. Let us briefly explain the strategy of the proof of the Theorem 1.2. We will obtain some estimates of the moments of Z_{B_r}, the total number of collisions of two independent simple random walks on the three-dimensional UST $\mathcal{U}$ killed when exiting the intrinsic ball $B_{\mathcal{U}}(0,r)$. To this end, we will rewrite Z_{B_r} in terms of the effective resistance of the three-dimensional UST, which can be derived from some geometric properties of graphs. We will construct a "good" event and demonstrate that the three-dimensional UST exhibits such properties with high probability.

This chapter is organized as follows. We will give some definitions and estimates that are needed in the proof of main results in Section 3.1. Then Theorem 1.2 and Corollary 1.3 will be proved in Section 3.2.

3.1 Preliminaries

In this section, we will introduce the growth exponent of the three-dimensional infinite LERW, which represents the time-space scaling of the LERW. We will also present some estimates on Z_{B_r}, which will enable one to bound its moment using the effective resistance.

We run the SRW on $\mathbb{Z}^3$ started at the origin until the first exiting time of $B(0,n)$. Let M_n be the length of the loop erasure of this SRW path. We denote the law of S and the corresponding expectation by P and E, respectively. If the limit

$$\beta := \lim_{n\to\infty} \frac{\log E(M_n)}{\log n}, \tag{3.1}$$

exists, then this constant β is called the growth exponent of the LERW. The existence of the limit is proved in [39] and that $\beta \in (1, 5/3]$ is obtained in [27]. Although the exact value of β has not been discovered yet, it is estimated that $\beta = 1.624\cdots$ by numerical calculations, see [43]. Moreover, the following exponential tail bounds of M_n are obtained in [39].

Theorem 3.1. *([39, Theorem 1.4]) There exists $c > 0$ such that for all $n \geq 1$ and $\kappa \geq 1$,*

$$\mathbf{P}(M_n \geq \kappa E(M_n)) \leq 2\exp\{-c\kappa\},$$

and for any $\varepsilon \in (0,1)$, there exist $0 < c_\varepsilon, C_\varepsilon < \infty$ such that for all $n \geq 1$ and $\kappa \geq 1$,

$$P(M_n \leq \kappa^{-1} E(M_n)) \leq C_\varepsilon \exp\{-c_\varepsilon \kappa^{\frac{1}{\beta}-\varepsilon}\}. \tag{3.2}$$

Next, we define the infinite collision property and introduce some previous results. Let $G = (V,E)$ be a connected graph and let $X = \{X_n\}_{n=0}^\infty$ and $Y = \{Y_n\}_{n=0}^\infty$ be independent discrete time simple random walks on G. For $x, y \in V$, we write $x \sim y$ if x and y are connected with an edge, *i.e.* $\{x,y\} \in E$. We denote by $P_{a,b}$ the law of $\{(X_n, Y_n)\}_{n=0}^\infty$ with starting point $(X_0, Y_0) = (a,b)$.

Definition 3.2. *We define the total number of collisions between X and Y by*

$$Z = \sum_{n=0}^{\infty} \mathbf{1}(X_n = Y_n). \tag{3.3}$$

Let B be a connected subgraph of G and let $X^B = \{X_n^B\}_{n=0}^\infty$ and $Y^B = \{Y_n^B\}_{n=0}^\infty$ be independent discrete-time simple random walks on G killed when they exit B. We define the total number of collisions of X^B and Y^B by

$$Z_B = \sum_{n=0}^{\infty} \mathbf{1}(X_n^B = Y_n^B). \tag{3.4}$$

Definition 3.3. *If*

$$P_{a,a}(Z < \infty) = 1, \tag{3.5}$$

holds for all $a \in G$, then G has the ***finite collision property****. If*

$$P_{a,a}(Z = \infty) = 1, \tag{3.6}$$

holds for all $a \in G$, then G has the ***infinite collision property****.*

Remark 3.4. There is no simple monotonicity property for collisions. Let $\mathrm{Comb}(\mathbb{Z})$ be the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set

$$\{[(x,n),(x,m)] : |m-n| = 1\} \cup \{[(x,0),(y,0)] : |x-y| = 1\}.$$

Then $\mathrm{Comb}(\mathbb{Z})$ has the finite collision property (see [22, Theorem 1.1]) and is a subgraph of $\mathbb{Z}^2$, which has the infinite collision property.

It is proved that for any connected graph, either (3.5) or (3.6) holds.

Proposition 3.5. *([9, Proposition 2.1]) Let G be a (connected) recurrent graph. Then for any starting point $(a,b) \in G \times G$ of the process $\{(X_n, Y_n)\}$,*

$$P_{a,b}(Z = \infty) \in \{0,1\},$$

holds. In particular, for all $a \in G$, either $P_{a,a}(Z = \infty) = 0$ or $P_{a,a}(Z = \infty) = 1$ holds.

Recall the definition of the effective resistance on a connected graph G given in 2.3. Now let us derive some estimates on Z_B (see (3.4) for definition) for a finite subgraph B from effective resistance and Green's function. For the first moment of Z_B, we have

$$\begin{aligned} E_{0,0}(Z_B) &= \sum_{n=0}^{\infty} \sum_{x \in B} P_{0,0}(X_n^B = Y_n^B = x) \\ &= \sum_{n=0}^{\infty} \sum_{x \in B} P^0(X_n^B = x)^2 \\ &= \sum_{n=0}^{\infty} \sum_{x \in B} P^0(X_n^B = x, X_{2n}^B = 0) \frac{\mu_G(x)}{\mu_G(0)}. \end{aligned}$$

Thus, it holds that

$$\frac{1}{\mu_G(0)} \sum_{n=0}^{\infty} P^0(X_{2n}^B = 0) \le E_{0,0}(Z_B) \le \frac{\max_{x \in B} \mu_G(x)}{\mu_G(0)} \sum_{n=0}^{\infty} P^0(X_{2n}^B = 0). \tag{3.7}$$

Since $P^x(X^B_{2n+1} = x) \le P^x(X^B_{2n} = x)$ for all n, we have that

$$\begin{aligned}\frac{1}{2}G_B(x,x) &= \frac{1}{2}\sum_{n=0}^{\infty}\big(P^x(X^B_{2n} = x) + P^x(X^B_{2n+1} = x)\big)\\ &\le \sum_{n=0}^{\infty} P^x(X^B_{2n} = x) \le G_B(x,x).\end{aligned}$$

Thus, it follows from (3.7) that

$$\frac{1}{2\mu_G(0)}G_B(0,0) \le E_{0,0}(Z_B) \le \frac{\max_{x\in B}\mu_G(x)}{\mu_G(0)}G_B(0,0). \tag{3.8}$$

An upper bound of the second moment is obtained by

$$\begin{aligned}E_{0,0}(Z_B^2) =& \sum_{n=0}^{\infty}\sum_{x\in B} P_{0,0}(X^B_n = Y^B_n = x)\\ &+ 2\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{x\in B}\sum_{y\in B} P_{0,0}(X^B_n = Y^B_n = x, X^B_m = Y^B_m = y)\\ =& E_{0,0}(Z_B)\\ &+ 2\sum_{n=0}^{\infty}\sum_{x\in B} P_{0,0}(X^B_n = Y^B_n = x)\sum_{m=1}^{\infty}\sum_{y\in B} P_{x,x}(X^B_m = Y^B_m = y)\\ \le& \frac{\max_{x\in B}\mu_G(x)}{\mu_G(0)}G_B(0,0) + 2\frac{\max_{x\in B}\mu_G(x)}{\mu_G(0)}G_B(0,0)\max_{x\in B}G_B(x,x),\end{aligned}$$

where we applied the Markov property for the second equality and (3.8) for the last inequality. By plugging (2.7) into the above inequality, we obtain

$$E_{0,0}(Z_B^2) \le \max_{x\in B}\mu_G(x)R_{\text{eff}}(0,B^c) + 2\big(\max_{x\in B}\mu_G(x)\big)^2 R_{\text{eff}}(0,B^c)\max_{x\in B}R_{\text{eff}}(x,B^c). \tag{3.9}$$

3.2 Proof of the main theorem

In this section, we will prove Theorem 1.2. In order to do so, we will first estimate the effective resistance of $\mathcal{U}$ between the origin and $\partial B(0,r)$ in the following theorem.

Let U_r be the connected component of $\mathcal{U} \cap B(0,r)$ which contains the origin. Recall that β is the growth exponent of the three-dimensional LERW defined in (3.1).

Theorem 3.6. *There exists some universal constant $C > 0$ such that for all $r \ge 1$ and $\lambda > 0$,*

$$\mathbf{P}(R_{\text{eff}}(0,\mathcal{U}\setminus U_r) \ge r^{\beta}/\lambda^{1+4\beta}) \ge 1 - C\lambda^{-1}. \tag{3.10}$$

Proof. Note that it suffices to prove the inequality (3.10) for $\lambda \ge \lambda_0$ where λ_0 is a sufficiently large universal constant that does not depend on r.

We first fix $r > 0$ and consider a sequence of subsets of $\mathbb{Z}^3$ including $\partial_i B(0,r)$. For $k = 1, 2, \cdots$, let $\delta_k = \lambda^{-1}2^{-k}$ and $\eta_k = (2k)^{-1}$. We define k_0 to be the smallest positive integer such that $r\delta_{k_0} < 1$. Let

$$A_k = B(0,(1+\eta_k)r) \setminus B(0,(1-\eta_k)r),$$

and let D_k be a finite subset of lattice points of A_k with $|D_k| \le C\delta_k^{-3}$ such that

$$A_k \subset \bigcup_{z \in D_k} B(z, \delta_k r).$$

Next, we perform Wilson's algorithm rooted at infinity (see Section 2.2) to obtain the desired event of the three-dimensional UST. Let $\mathcal{U}_0 = \gamma_\infty$ *i.e.* the infinite LERW started at the origin. Given $\mathcal{U}_k$ ($k \ge 0$), we regard $\mathcal{U}_k$ as the root of Wilson's algorithm and add branches started at vertices in $D_{k+1} \setminus \mathcal{U}_k$ and denote by $\mathcal{U}_{k+1}$ the resulting random subtree at this step. Once we obtain $\mathcal{U}_{k_0}$, we add branches started at vertices in $\mathbb{Z}^3 \setminus \mathcal{U}_{k_0}$ to complete Wilson's algorithm. Note that $\mathcal{U}_k$ ($k = 0, 1, 2, \cdots, k_0$) is a subtree of $\mathcal{U}$ containig all vertices in $\bigcup_{i=1}^k D_i \cup \{0\}$ and the sequence $\{\mathcal{U}_k\}_{k=0}^{k_0}$ is increasing. Since $r\delta_{k_0} < 1$, it holds that $\partial_i B(0, r) \subset D_{k_0} \subset \mathcal{U}_{k_0}$.

Now we are ready to define the events where the effective resistance in (3.10) is bounded below. Firstly, we examine the behavior of the branches started at vertices contained in D_1. For $z \in D_k$ ($k \ge 1$), we denote by y_z be the first point of $\mathcal{U}_{k-1}$ visited by $\gamma(z, 0)$ *i.e.* $d_{\mathcal{U}}(z, y_z) = \min_{y \in \mathcal{U}_{k-1}} d_{\mathcal{U}}(z, y)$. We define the event F_z by

$$F_z = \{\gamma(z, y_z) \cap B(0, \lambda^{-4} r) = \emptyset\}, \tag{3.11}$$

for $z \in D_1$. Since $d_E(0, z) \ge r/2$, by [27, Theorem 1.5.10], there exists some constant $C > 0$ such that for all $\lambda \ge 2$,

$$\mathbf{P}(F_z^c) \le \mathbf{P}(S^z[0, \infty) \cap B(0, \lambda^{-4} r) \ne \emptyset) \le C\lambda^{-4},$$

holds. By taking the union bound, we obtain that

$$\mathbf{P}\left(\bigcup_{z \in D_1} F_z^c\right) \le |D_1| C\lambda^{-4} \le C\lambda^{-1}, \tag{3.12}$$

where the last inequality follows from the fact that $|D_1| \le C\lambda^3$.

Secondly, we bound from below the first time when γ_∞ exits $B(0, \lambda^{-4} r)$, which is denoted by $\tau(B(0, \lambda^{-4} r)^c)$. We define the event $\widetilde{F}$ by

$$\widetilde{F} = \left\{\mathrm{len}\big(\gamma_\infty[0, \tau(B(0, \lambda^{-4} r)^c)]\big) \ge r^\beta / \lambda^{1+4\beta}\right\}. \tag{3.13}$$

By [39, Theorem 1.4], [32, Corollary 1.3] and the fact that $\beta \le 5/3$, there exist some constants $C > 0$ and $c > 0$ such that

$$\mathbf{P}(\widetilde{F}^c) \le C \exp\{-c\lambda^{1/2}\}, \tag{3.14}$$

for all $r \ge 1$ and $\lambda > 0$.

Thirdly, we consider the branches started at vertices in D_k ($k \ge 2$) step by step. Let us begin by defining an event that guarantees the "hittability" of $\gamma(x, \infty)$ for $x \in D_k$. To be precise, for $x \in D_k$ ($k \ge 1$) and $\xi > 0$, we define the event $H_x(\xi)$ by

$$H_x(\xi) = \Big\{\text{There exists some } z \in B(x, \delta_k r) \text{ such that}$$
$$P^z(S^z[0, \tau_{S^z}(B(z, \delta_k^{1/2} r)^c)] \cap \gamma(x, \infty) = \emptyset) \ge \delta_k^\xi\Big\},$$

where S^z is an independent simple random walk started at $z \in \mathbb{Z}^3$ and P^z denotes its law. By [37, Theorem 3.1], there exist some $C > 0$ and $\xi_1 > 0$ such that

$$\mathbf{P}(H_x(\xi_1)) \le C\delta_k^4 \quad \text{for all } r \ge 1,\, k \ge 1 \text{ and } x \in D_k. \tag{3.15}$$

Let

$$\widetilde{H}_k := \bigcap_{x \in D_k} H_x(\xi_1)^c, \tag{3.16}$$

where ξ_1 is as defined in (3.15). Note that $P^z(S^z[0, \tau_{S^z}(B(z, \delta_k^{1/2} r)^c)] \cap \gamma(x, \infty) = \emptyset)$ is a function of $\gamma(x, \infty)$ and thus $H_x(\xi)$ and $\widetilde{H}_k$ are measurable with respect to $\mathcal{U}_k$. Moreover, it follows from (3.15) and the definition of D_k that

$$\mathbf{P}(\widetilde{H}_k) \ge 1 - |D_k| C\delta_k^4 \ge 1 - C'\delta_k, \tag{3.17}$$

where $C' > 0$ is uniform in $r \ge 1$ and $k \ge 1$.

Now we will demonstrate that conditioned on the event $\widetilde{H}_k$, branches $\gamma(z, y_z)$ ($z \in D_{k+1}$) are included in A_k with high conditional probability. Let $M = \lceil 4/\xi_1 \rceil$. For $z \in D_{k+1}$, let

$$I_z = \left\{ S^z[0, \tau_{S^z}(B(z, M\delta_k^{1/2} r)^c)] \cap \mathcal{U}_k = \emptyset \right\}.$$

Since $z \in D_{k+1} \subset A_k$, we can take some $x \in D_k$ with $z \in B(x, \delta_k r)$ and on the event I_z, we have that

$$S^z[0, T^1] \cap \gamma(x, \infty) = \emptyset,$$

holds, where $T^1 = \tau_{S^z}(B(z, \delta_k^{1/2} r)^c)$.

In the rest of this proof, we take $\lambda \ge 6M$ without loss of generality. Since $d_E(z, S^z(T^1 - 1)) \le \delta_k^{1/2} r$, we have that $z_1 := S^z(T^1 - 1) \in A_k$ and we can take $x_1 \in D_k$ with $z_1 \in B(x_1, \delta_k r)$. By the same argument as the above, on the event I_z we have that $S^z[T^1, T^2] \cap \gamma(x_1, \infty) = \emptyset$, where $T^2 = \tau_{S^z}(B(z_1, \delta_k^{1/2} r)^c)$. Iteratively, we obtain the sequences $\{T^i\}$, $\{z_i\} \subset A_k$ and $\{x_i\} \subset D_k$ ($i = 1, 2, \cdots, M$) and we have that

$$I_z \subset \bigcap_{i=1}^{M} \{ S^z[T^{i-1}, T^i] \cap \gamma(x_{i-1}, \infty) = \emptyset \},$$

where we set $T^0 = 0$ and $x_0 = x$. By the strong Markov property, it holds that

$$\begin{aligned} P^z(I_z) &\le P^z\left(\bigcap_{i=1}^{M} \{ R[T^{i-1}, T^i] \cap \gamma(x_{i-1}, \infty) = \emptyset \} \right) \\ &= \prod_{i=1}^{M} P^{z_{i-1}}(S^{z_{i-1}}[0, \tau_{S^{z_{i-1}}}(B(z_{i-1}, \delta_k^{1/2} r))] \cap \gamma(x_{i-1}, \infty) = \emptyset), \end{aligned}$$

from which it follows that

$$\widetilde{H}_k \subset \{ P^z(I_z) \le \delta_k^4 \}.$$

Thus, by Wilson's algorithm, we have that for all $z \in D_{k+1}$,

$$\mathbf{P}\left(\gamma(z, y_z) \not\subset B(z, M\delta_k^{1/2} r) \mid \widetilde{H}_k \right) \le \delta_k^4. \tag{3.18}$$

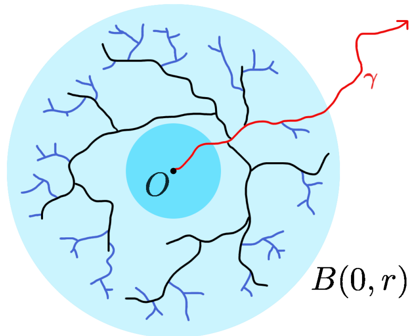


Figure 1: In this figure, two circles represent Euclidean balls centered at the origin: the larger one is of radius r and the smaller one is of radius $\lambda^{-4}r$. On the event K, the branches from D_1 do not enter the smaller ball of radius $\lambda^{-4}r$ and the branches from D_k ($k \geq 2$) hits the already constructed subtree $\mathcal{U}_{k-1}$ before entering $B(0, r/2)$. Moreover, the length of γ_∞ up to the exiting time $\tau(B(0, \lambda^{-4}r)^c)$ is bounded below by $r^\beta/\lambda^{1+4\beta}$.

We define the event $\widetilde{I}_{k+1}$, which is measurable with respect to $\mathcal{U}_{k+1}$, by

$$\widetilde{I}_{k+1} = \bigcap_{z \in D_{k+1}} \left\{ \gamma(z, y_z) \subset B(z, M\delta_k^{1/2} r) \right\}. \tag{3.19}$$

Then by (3.18) and that $|D_{k+1}| \leq C\delta_k^{-3}$, it holds that

$$\mathbf{P}(\widetilde{I}_{k+1} \mid \widetilde{H}_k) \geq 1 - |D_{k+1}|\delta_k^4 \geq 1 - C\delta_k.$$

Combining this with (3.17), we obtain that

$$\mathbf{P}(\widetilde{H}_k \cap \widetilde{I}_{k+1}) \geq 1 - C\delta_k, \tag{3.20}$$

for some universal constant $C > 0$.

Finally, we construct an event where the desired effective resistance bound holds. Let

$$K = \left(\bigcap_{z \in D_1} F_z \right) \cap \widetilde{F} \cap \left(\bigcap_{k=1}^{k_0} (\widetilde{H}_k \cap \widetilde{I}_{k+1}) \right).$$

Recall that F_z, $\widetilde{F}$, $\widetilde{H}_k$ and $\widetilde{I}_{k+1}$ are defined by (3.11), (3.13), (3.16) and (3.19), respectively.

Then combining (3.12), (3.14) and (3.20), we obtain that

$$\mathbf{P}(K^c) \leq C\lambda^{-1} + C\exp\{-c\lambda^{1/2}\} + \sum_{k=1}^{\infty} C\delta_k \leq C\lambda^{-1}. \tag{3.21}$$

We claim that on the event K, the following two statements hold:

(1) $d_{\mathcal{U}}(0, y_z) \geq r^\beta/\lambda^{1+4\beta}$ for all $z \in D_1$.

(2) For $k \geq 2$, $\gamma(z, 0)$ hits $\mathcal{U}_1$ before entering $B(0, r/2)$ for all $z \in D_k$.

Note that (1) is immideate from $K \subset (\bigcap_{z \in D_1} F_z) \cap \widetilde{F}$ and (2) follows from $K \subset (\bigcap_{k=1}^{k_0} (\widetilde{I}_{k+1} \cap \widetilde{H}_k))$.

Suppose that K occurs. Let w be an element of $\{y_z : z \in D_1\}$ which satisfies $d_{\mathcal{U}}(0, w) = \min_{z \in D_1} d_{\mathcal{U}}(0, y_z)$. It follows from the above statements (1) and (2) that every path of $\mathcal{U}$ connecting the origin and $B(0, r)^c$ includes $\gamma(0, w)$ (recall that $\partial_i B(0, r) \subset D_{k_0}$). Thus, by the series law of effective resistance (see [34] Section 2.3, for example), we have that

$$\begin{aligned} R_{\text{eff}}(0, \mathcal{U} \setminus U_r) &= R_{\text{eff}}(0, w) + R_{\text{eff}}(w, \mathcal{U} \setminus U_r) \\ &\geq d_{\mathcal{U}}(0, w) \geq r^{\beta} / \lambda^{1+4\beta}. \end{aligned}$$

Combining this with (3.21) yields the desired result (3.10). □

Now we are ready to prove Theorem 1.2. Recall that Z_B is defined in (3.4). In the rest of the article, we set $B_r = B_{\mathcal{U}}(0, r)$.

Proof of Theorem 1.2. Let us define the event $\widetilde{K}(r, \lambda)$ by

$$\widetilde{K}(r, \lambda) = \{R_{\text{eff}}(0, B_{\mathcal{U}}(0, r)^c) \geq r/\lambda\}. \tag{3.22}$$

By [1, Proposition 4.1], there exist some $C' > 0$ and $c' \in (0, 1)$ such that

$$\mathbf{P}\left(U_r \not\subset B_{\mathcal{U}}(0, \lambda r^{\beta})\right) \leq C' \lambda^{-c'},$$

for all $r > 1$ and $\lambda \geq 1$. On the event $\{U_r \subset B_{\mathcal{U}}(0, \lambda r^{\beta})\}$, by monotonicity

$$R_{\text{eff}}(0, \mathcal{U} \setminus U_r) \leq R_{\text{eff}}(0, B_{\mathcal{U}}(0, \lambda r^{\beta})^c),$$

holds (see [34] Section 2.2, for example). Thus, we have

$$\begin{aligned} &\mathbf{P}\Big(R_{\text{eff}}(0, B_{\mathcal{U}}(0, \lambda r^{\beta})^c) < r^{\beta}/\lambda^{1+4\beta}\Big) \\ &\leq \mathbf{P}\Big(R_{\text{eff}}(0, B_{\mathcal{U}}(0, \lambda r^{\beta})^c) < r^{\beta}/\lambda^{1+4\beta}, U_r \subset B_{\mathcal{U}}(0, \lambda r^{\beta})\Big) + \mathbf{P}\left(U_r \not\subset B_{\mathcal{U}}(0, \lambda r^{\beta})\right) \\ &\leq \mathbf{P}\left(R_{\text{eff}}(0, \mathcal{U} \setminus U_r) < r^{\beta}/\lambda^{1+4\beta}\right) + C' \lambda^{-c'}. \end{aligned}$$

By Theorem 3.6, we obtain that

$$\begin{aligned} \mathbf{P}\left(R_{\text{eff}}(0, B_{\mathcal{U}}(0, \lambda r^{\beta})^c) \geq r^{\beta}/\lambda^{1+4\beta}\right) &\geq \mathbf{P}\left(R_{\text{eff}}(0, \mathcal{U} \setminus U_r) \geq r^{\beta}/\lambda^{1+4\beta}\right) - C' \lambda^{-c'} \\ &\geq 1 - C\lambda^{-1} - C' \lambda^{-c'}. \end{aligned}$$

By reparameterizing $R = \lambda r^{\beta}$, and taking $C' > 0$ properly, we have that

$$\mathbf{P}\left(\widetilde{K}(R, \lambda)\right) \geq 1 - C' \lambda^{-\frac{c'}{2+4\beta}}. \tag{3.23}$$

Next, we make use of the estimates of $E(Z_B)$ and $E(Z_B^2)$ in Section 3.1. Since $1 \leq \mu_{\mathcal{U}}(x) \leq 6$ for all $x \in \mathbb{Z}^3$, it follows from (2.7) and (3.8) that on the event $\widetilde{K}(r, \lambda)$,

$$\frac{r}{2\lambda} \leq E_{0,0}(Z_{B_r}) \leq 6r, \tag{3.24}$$

where we plugged $R_{\text{eff}}(0, B_{\mathcal{U}}(0,r)^c) \le r$ to obtain the second inequality. By reparameterization, (1.1) follows.

On the other hand, since

$$\begin{aligned} R_{\text{eff}}(x, B_{\mathcal{U}}(0,r)^c) &\le R_{\text{eff}}(x,0) + R_{\text{eff}}(0, B_{\mathcal{U}}(0,r)^c) \\ &\le 2r, \end{aligned}$$

for $x \in B_{\mathcal{U}}(0,r)$, plugging this into (3.9) yields that

$$E_{0,0}(Z_{B_r}^2) \le 144r^2 + 6r, \tag{3.25}$$

for any realization $\mathcal{U}$, which gives (1.2).

Now we will apply the second moment method to Z_{B_r} on the event $\widetilde{K}(r,\lambda)$. By (3.24) and (3.25), on the event $\widetilde{K}(r,\lambda)$ we have

$$\begin{aligned} P_{0,0}\left(Z_{B_r} \ge \frac{r}{12\lambda}\right) &\ge P_{0,0}\left(Z_{B_r} \ge \frac{1}{6}E_{0,0}(Z_{B_r})\right) \\ &\ge \frac{25E_{0,0}(Z_{B_r})^2}{36E_{0,0}(Z_{B_r}^2)} \ge \frac{1}{6\cdot(12\lambda)^2}. \end{aligned}$$

By reparameterizing $\varepsilon^{-1} = 12\lambda$, we have that on $\widetilde{K}(r,\varepsilon^{-1}/12)$,

$$P_{0,0}(Z_{B_r} \ge \varepsilon r) \ge \varepsilon^2/6. \tag{3.26}$$

Finally, by Markov's inequality,

$$\begin{aligned} P_{0,0}(Z_{B_r} \ge 72\varepsilon^{-2} r) &\le P_{0,0}(Z_{B_r} \ge 12\varepsilon^{-2}E_{0,0}(Z_{B_r})) \\ &\le \varepsilon^2/12, \end{aligned}$$

holds on the event $\widetilde{K}(r,\varepsilon^{-1}/12)$. Combining this with (3.26) gives (1.3). □

We obtain the infinite collision property of the three-dimensional UST as a corollary.

Proof of Corollary 1.3. Suppose $\omega \in K(r,\varepsilon)$ and let $\mathcal{U}(\omega)$ be the corresponding realization of UST. We take two simple random walks X and Y on $\mathcal{U}(\omega)$. Recall that Z is the total number of collisions between X and Y defined by (3.3). By Theorem 1.2, for any $N \ge 1$ and any fixed $\varepsilon > 0$,

$$P_{0,0}(Z \ge N) \ge P_{0,0}\left(Z_{B_{\varepsilon^{-1}N}} \ge N\right) \ge \varepsilon^2/12,$$

holds. By taking the limit $N \to \infty$ we obtain that $P_{0,0}(Z = \infty) \ge \varepsilon^2/12$, from which the infinite collision property of $\mathcal{U}(\omega)$ follows by Proposition 3.5. Thus,

$$\mathbf{P}(\{\mathcal{U}(\omega) \text{ has the infinite collision property}\}) \ge 1 - C\varepsilon^c.$$

Since ε is arbitrary, we have that

$$\mathbf{P}(\{\mathcal{U}(\omega) \text{ has the infinite collision property}\}) = 1,$$

which completes the proof. □

Remark 3.7. We can also derive the infinite collision property of the three-dimensional UST from (3.23) by applying Corollary 3.3 of [9]. In this article, we gave another proof by using quantitative estimates of the number of collisions in the intrinsic ball Z_{B_r}.

4 Volume and heat kernel fluctuations for the three-dimensional uniform spanning tree

In this chapter, we will prove Theorem 1.7 and Theorem 1.8. We start with explaining the main idea of the proofs of these theorems, which is inspired by the result on the two-dimensional uniform spanning tree [5].

In order to obtain Theorem 1.8, we consider the three-dimensional uniform spanning tree as a collection of small pieces where the probability of events corresponding to those on the whole tree can be calculated. Similarly to [5, Theorem 1.1], we consider unlikely configurations of $\mathcal{U}$, namely "comb" and "spiral" configurations as depicted in Figures 2 and 3 respectively. Here the comb configuration is constructed in such a way that we obtain an intrinsic ball in $\mathcal{U}$ with an unusually large size, which enables us to obtain (1.9). On the other hand, we use the spiral configuration to make an unusually small ball for the sake of the derivation of (1.8). Although the scenario is essentially the same as that for the two-dimensional case in [5], here we need to deal with a central hurdle: since the Beurling projection theorem (a property of the simple random walk on $\mathbb{Z}^2$ that it hits any path of $\mathbb{Z}^2$ with high probability, see [29, Theorem 6.8.1] for example) is not available when $d = 3$, the construction of such unlikely configurations of $\mathcal{U}$ via Wilson's algorithm (see Section 2.2) requires some extra work, which is rather complicated. We overcome this difficulty through careful use of a type of hittability of loop-erased random walks in $\mathbb{Z}^3$, as derived in [37, Theorem 3.1]. Combining Theorem 1.8 with the fact that the behavior of the effective resistance metric on $\mathcal{U}$ is similar to that of the intrinsic metric (see Subsection 4.2.2 below for this), Theorem 1.7 is also proved.

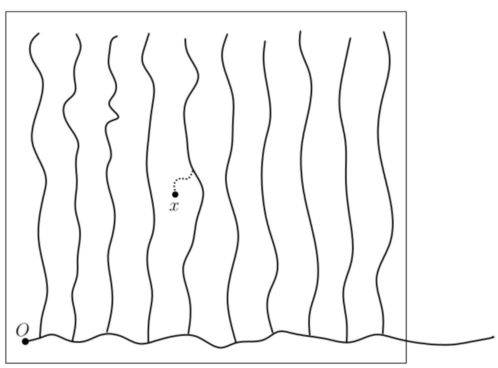


Figure 2: Illustration for the comb configuration. The horizontal solid curve stands for the unique infinite path in $\mathcal{U}$ started at the origin. We force it to keep going to the right with no big backtracking. We also make each vertical solid branch keep going down. For another point x, as the dotted curve illustrates, the branch between x and a solid curve has a small length. As a result, if the Euclidean metric between the origin and x is not small, the intrinsic metric from the origin to x is unusually small.

The rest of this chapter is organized as follows. The claim (1.9) will be proved in Section 4.1 and (1.8) will be proved in Section 4.2. Finally, we will give the proof of Theorem 1.7 in Section 4.3. Throughout this chapter, we refer to the definition and notation for the uniform spanning tree and loop-erased random walk in $\mathbb{Z}^3$ introduced in Chapter 2.

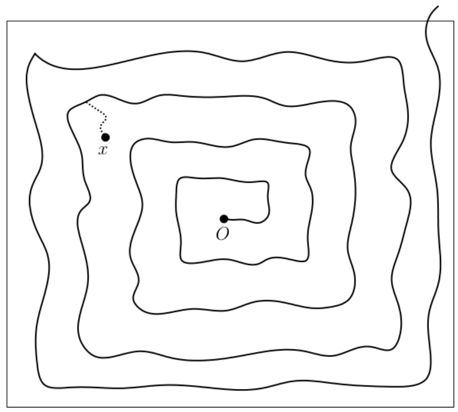


Figure 3: Illustration for the spiral configuration. The solid curve stands for the unique infinite path in $\mathcal{U}$ started at the origin. We make it spiral around the origin many times. This configuration ensures that the intrinsic metric from the origin to x is unusually large if the Euclidean metric between the origin and x is not small.

4.1 Upper volume fluctuations

In this section, we prove (1.9), upper volume fluctuations of log-logarithmic magnitude in Theorem 4.13. The key ingredient of the proof is the following lemma, which provides a lower bound on an upper tail of the volume of intrinsic balls in the three-dimensional UST $\mathcal{U}$.

Recall that β is the growth exponent of the three-dimensional LERW defined by (3.1) and $B_{\mathcal{U}}$ stands for intrinsic balls in $\mathcal{U}$ defined by (2.3).

Proposition 4.1. *Let $\mathcal{U}$ be the three-dimensional UST build on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then there exist $c_1, c_2 > 0$ such that for all $\lambda > 0$ and $r \geq 1$,*

$$\mathbf{P}(|B_{\mathcal{U}}(0,r)| \geq \lambda r^{3/\beta}) \geq c_1 \exp\{-c_2 \lambda^{(\beta-1)/\beta} \log \lambda\}, \tag{4.1}$$

where β is the growth exponent of the three-dimensional LERW.

Remark 4.2. See [1, Proposition 6.1] for an exponential upper bound for the probability in the left-hand side of (4.1).

4.1.1 The comb configuration in the UST

We explain an idea of the proof of (4.1) here, which is inspired by the proof of (4.1) of [5, Lemma 4.1]. We construct a cube of side-length Nm, consisting of N^3 small boxes of side-length m, as follows. For each $j \geq 0$, let

$$x_j = (jm, 0, 0) \in \mathbb{Z}^3, \quad B_{x_j} = B_\infty(x_j, m/2), \tag{4.2}$$

i.e. B_{x_j} is the cube of side-length m centered at x_j (see Chapter 2 for the definition of B_∞). Firstly, we align the boxes B_{x_j} $(0 \leq j \leq N)$, whose center points are located on the x_1 axis. Secondly, we take the boxes $B_\infty((jm, km, 0), m/2)$ for each $0 \leq j \leq N$ and $1 \leq k \leq N$.

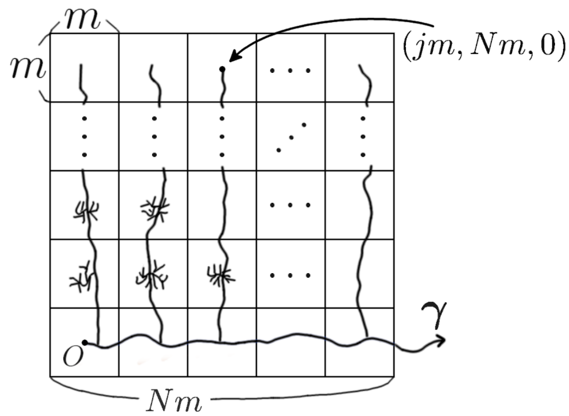


Figure 4: The event $A_{x_1} \cap A_{x_2} \cap A_{x_3}$ to consider for upper volume fluctuations

Described in Figure 4 are the boxes having been constructed at this step. Finally we take the boxes $B_\infty((jm, km, lm), m/2)$ for each $0 \le j \le N$, $1 \le k \le N$ and $1 \le l \le N$.

Let γ_∞ be the infinite LERW started at the origin, which is the first branch in Wilson's algorithm to generate $\mathcal{U}$. We consider the event A_{x_1} which is the intersection of the following events:

- γ_∞ moves toward the right until it exits from a "tube" $\bigcup_{j=1}^N B_{x_j}$ without backtracking.
- The number of points in $\gamma_\infty \cap B_{x_j}$ is bounded above by m^β for all $1 \le j \le N$.
- For some snall $\varepsilon > 0$, with high probability, each point in $B(x_j, \varepsilon m)$ $(j = 1, \cdots N)$ is connected to γ_∞ with a path of length of order m^β.

Next we run SRWs $S^{(j)}$, $j = 1, 2, \cdots, N$ independent of γ and each other started at the points $(jm, Nm, 0)$. We consider the event A_{x_2} where each $S^{(j)}$ moves in a "tube" parallel to the y axis until it hits γ, the number of points in its loop erasure is bounded above by Nm^β and every point in a small Euclidean ball around the center of each box is connected to the loop erasure with a short path.

Finally, we consider the corresponding event A_{x_3} for independent SRWs started at (jm, km, Nm) until they hit the already constructed subtree in the tubes parallel to the z axis.

Note that if the intersection $A_{x_1} \cap A_{x_2} \cap A_{x_3}$ occurs, it leads to a lower bound of the volume of a ball in intrinsic metric. Once we have a lower bound of the probability of the event A, we consider LERWs satisfying the same condition and parallel to the y and z axis.

In the remainder of this subsection, we will establish a lower bound of the probability of the event $A_{x_1} \cap A_{x_2} \cap A_{x_3}$ in Lemma 4.7 and in Lemma 4.8. In order to do so, we follow an argument of [33], Section 4, which makes use of the cut points of the three-dimensional SRW.

We begin with defining several events.

Definition 4.3. *For $a < b$, we define*

$$Q[a, b] = \{(x^1, x^2, x^3) \in \mathbb{Z}^3 : a \le x^1 \le b, -m \le x^2, x^3 \le m\},$$
$$Q(a) = \{(x^1, x^2, x^3) \in \mathbb{Z}^3 : x^1 = a, -m \le x^2, x^3 \le m\}.$$

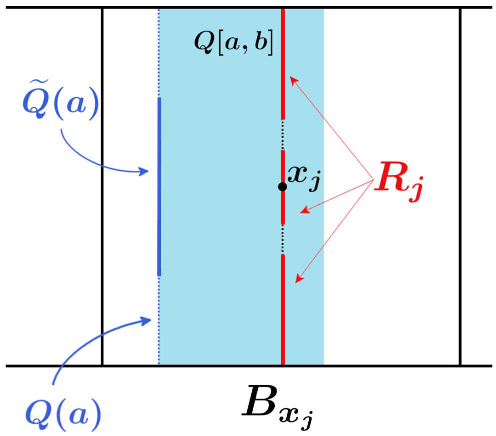


Figure 5: Sets $Q[a,b]$, $Q(a)$, $\widetilde{Q}(a)$ and Rj

We also set

$$\begin{aligned}\widetilde{Q}(a) &= \{(x^1,x^2,x^3) \in \mathbb{Z}^3 : x^1 = a, -m/2 \le x^2, x^3 \le m/2\}, \\ R_j &= \{(x^1,x^2,x^3) \in \mathbb{Z}^3 : x^1 = jm,\ |x^2|^2 + |x^3|^2 < m^2/100\} \\ &\qquad \cup \{(x^1,x^2,x^3) \in \mathbb{Z}^3 : x^1 = jm,\ |x^2|^2 + |x^3|^2 > m^2/64\}. \qquad (4.3)\end{aligned}$$

Note that setting $a_j = (j - 1/2)m$, it follows that $Q[a_j, a_{j+1}] = B_{x_j}$ and that $Q(a_{j+1})$ corresponds to the right face of B_{x_j} (see (4.2) for the definition of B_{x_j})

Now we consider the SRW S on $\mathbb{Z}^3$ started at the origin. By linear interpolation, we may assume that $S(k)$ is defined for every non-negative real k and $S[0,\infty)$ is a continuous curve. For a continuous curve λ in $\mathbb{R}^3$, we define

$$t_\lambda(a) = \inf\{k \ge 0 : \lambda(k) \in Q(a)\}.$$

Let

$$N = (\log\log m)^{1/2}, \qquad (4.4)$$

$$q = m/N^2. \qquad (4.5)$$

Using $t_S(a)$, we define events A_j as the following:

$$A_0 = \{t_S(a_1) < \infty, S(t_S(a_1)) \in \widetilde{Q}(a_1), S[0, t_S(a_1)] \subset B_{x_0}, S[t_S(a_1 - q), t_S(a_1)] \cap Q(a_1 - 2q) = \emptyset\},$$

$$\begin{aligned}A_j = \{&t_S(a_j) < t_S(a_{j+1}) < \infty, S(t_S(a_{j+1})) \in \widetilde{Q}(a_{j+1}), S[t_S(a_j), t_S(a_{j+1})] \subset Q[a_j - q, a_{j+1}] \setminus R_j, \\ &S[t_S(a_{j+1} - q), t_S(a_{j+1})] \subset Q[a_{j+1} - 2q, a_{j+1}]\} \text{ for } j \ge 1. \qquad (4.6)\end{aligned}$$

The event A_0 guarantees that S exits B_{x_0} from $\widetilde{Q}(a_1)$ and has no big backtracking from $t_S(a_1 - q)$ to $t_S(a_1)$. For $j \ge 1$, the event A_j ensures that once S enters B_{x_j}, it keeps moving to the right until hitting $Q(a_{j+1})$. The last condition of A_j requires that S has no big backtracking in $[t_S(a_{j+1} - q), t_S(a_{j+1})]$. We note that the event A_0 (resp. $A_j,\ j \ge 1$) is measurable with respect to $S[0, t_S(a_1)]$ (resp. $S[t_S(a_j), t_S(a_{j+1})]$).

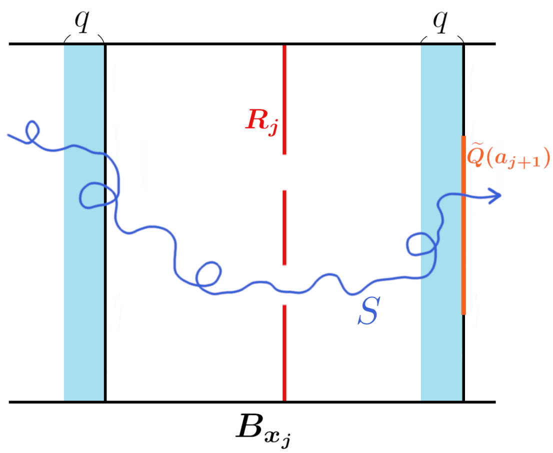


Figure 6: Definition of the event A_j $(j \geq 1)$

We set

$$G_j = \bigcap_{k=0}^{j} A_k. \tag{4.7}$$

We next consider a cut time with special properties for the SRW.

Definition 4.4. *Suppose that the event A_j defined in (4.6) occurs. For each $j \geq 1$, we call k a nice cut time in B_{x_j} if it satisfies the following conditions:*

(i) $t_S(a_j + \frac{q}{2}) \leq k \leq t_S(a_j + q)$,

(ii) $S[t_S(a_j), k] \cap S[k+1, t_S(a_{j+1})] = \emptyset$,

(iii) $S[k, t_S(a_{j+1})] \cap Q(a_j) = \emptyset$,

(iv) $S(k) \in Q[a_j + \frac{q}{2}, a_j + q]$.

If k is a nice cut time in B_{x_j}, then we call $S(k)$ a nice cut point in B_{x_j}.

We define events B_j by

$$B_j = \{S \text{ has a nice cut point in } B_{x_j}\},$$

for each $j \geq 1$. Note that the event B_j is measurable with respect to $S[t_S(a_j), t_S(a_{j+1})]$. We define

$$H_j = \bigcap_{k=1}^{j} B_k. \tag{4.8}$$

Now we consider two random curves ξ_j and ξ'_j defined as follows. We set

$$\xi_j = \mathrm{LE}(S[0, t_S(a_{j+1})]),$$

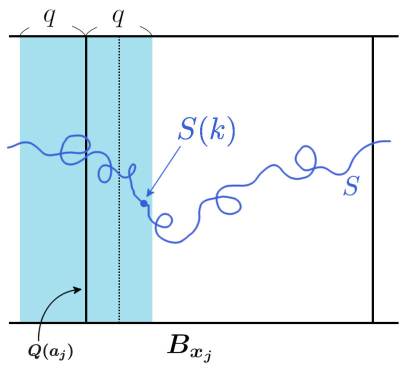
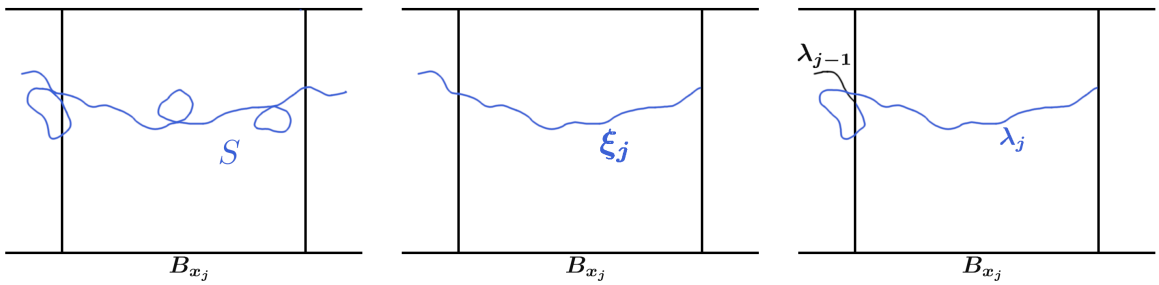


Figure 7: Example of nice cut point in B_{x_j}



(a) Original random walk S

(b) ξ_j

(c) ξ'_j

Figure 8: Examples of ξ_j and ξ'_j

and

$$\lambda_j = \mathrm{LE}(S[t_S(a_j), t_S(a_{j+1})]) \quad \text{for } j \geq 0, \tag{4.9}$$

$$\xi'_0 = \xi_0, \qquad \xi'_j = \xi'_0 \oplus \lambda_1 \oplus \cdots \oplus \lambda_j \quad \text{for } j \geq 1. \tag{4.10}$$

Note that ξ'_j is not necessarily a simple curve and thus $\xi_j \neq \xi'_j$ in general. However, the next lemma from [33] shows that the difference between these two curves is small on the event $G_j \cap H_j$.

Lemma 4.5. *([33, Lemma 4.3]) Let $j \geq 1$. Suppose that $G_j \cap H_j$ defined in (4.7) and (4.8) occurs. Then, for the length of ξ_j and ξ'_j, we have*

$$\mathrm{len}(\xi_j) \leq \mathrm{len}(\xi'_0) + \sum_{k=1}^{j} \left\{ \mathrm{len}(\lambda_k) + |\xi_j \cap Q[a_k - q, a_k + q]| \right\}, \tag{4.11}$$

where for $A \subset \mathbb{R}^3$, we write $|A|$ for the number of points in $A \cap \mathbb{Z}^3$.

Note that $\mathrm{len}(\xi'_j) = \mathrm{len}(\xi_0) + \sum_{k=1}^{j} \mathrm{len}(\lambda_k)$, and thus the above lemma compares the length of ξ_j and ξ'_j.

We will next deal with the length and the hittability of each λ_j. For $C \geq 1$, we define the event $E_j(C)$ by

$$E_0 = E_0(C) = \{\mathrm{len}(\xi_0) \leq C m^{\beta}\}, \quad E_j = E_j(C) = \{\mathrm{len}(\lambda_j) \leq C m^{\beta}\} \text{ for } j \geq 1, \tag{4.12}$$

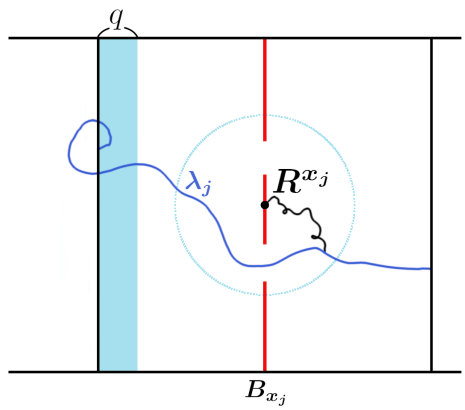


Figure 9: The event $\{\lambda_j \cap R^{x_j}[0, T_{R^{x_j}}(2m/5)] \neq \emptyset\}$

where ξ_0 and $\lambda_j,\ j \geq 1$ are as defined in (4.10). Let R^z be a SRW on $\mathbb{Z}^3$ started at $z \in \mathbb{Z}^3$ and independent of S. We denote by P and P^z the law of S and R^z, respectively. For $N \geq 4$ and $\eta > 0$, we define the event $F_j(\eta)$ by

$$F_j = F_j(\eta) = \{P^{x_j}(\lambda_j \cap R^{x_j}[0, T_{R^{x_j}}(2m/5)] \neq \emptyset) \geq \eta\}, \tag{4.13}$$

where $T_R(r) = \inf\{k \geq 0 : |R(k)| \geq r\}$. Note that $F_j(\eta)$ is measurable with respect to $S[t_S(a_j), t_S(a_{j+1})]$.

The next lemma gives a lower bound on the probability of $A_j \cap B_j \cap E_j(C) \cap F_j(\eta)$ choosing C sufficiently large and η sufficiently small.

Lemma 4.6. *There exist universal constants* $0 < \eta_*, c_*, C_* < \infty$ *such that*

$$P(A_0 \cap E_0(C_*)) \geq c_*,$$

and for all $j \geq 1$,

$$\min_{x \in \widetilde{Q}(a_j)} P^x(A_j \cap B_j \cap E_j(C_*) \cap F_j(\eta_*)) \geq c_* N^{-2}. \tag{4.14}$$

Proof. The first assertion is proved in [33, Lemma 4.4] and we also follow its proof to show that (4.14) holds. By the translation invariance, the minimum in the left-hand side of (4.14) does not depend on j. Hence, we will only consider the case $j = 1$.

It follows from the gambler's ruin estimate ([29, Proposition 5.1.6], for example) that

$$c_1 N^{-2} \leq P^x(A_1) \leq c_2 N^{-2} \quad \text{uniformly in } x \in \widetilde{Q}(a_1), \tag{4.15}$$

and from [26, Corollary 5.2] that

$$P^x(B_1 \mid A_1) \geq c_3 \quad \text{uniformly in } x \in \widetilde{Q}(a_1), \tag{4.16}$$

for some universal constants $0 < c_1, c_2, c_3 < \infty$.

On the event $A_1 \cap B_1$, let k_1 be a nice cut time in B_{x_1} as defined in Definition 4.4. By definition, it follows that $k_1 \leq t_S(a_1 + q)$ and

$$\operatorname{dist}(x_1, \operatorname{LE}(S[k_1, t_S(a_2)])) \leq m/8. \tag{4.17}$$

We check this by contradiction. Suppose that (4.17) does not hold. This implies that $\mathrm{LE}(S[k_1, t_S(a_2)])$ contains some point $z \in R_1$, where R_1 is as defined in (4.3). Thus, it also holds that $z \in S[t_S(a_1), t_S(a_2)]$, which contradicts (4.6).

It follows from (4.17) and the decomposition $\lambda_1 = \mathrm{LE}(S[t_S(a_1), k_1]) \oplus \mathrm{LE}(S[k_1, t_S(a_2)])$ that $\mathrm{dist}(x_1, \lambda_1) \leq m/8$. Hence, by [37, Theorem 3.1], there exist some universal constant $0 < \eta_1, c_4 < 1$ such that

$$P^x\left(F_1(\eta_1) \mid A_1 \cap B_1\right) \geq 1 - c_4 \quad \text{uniformly in } x \in \widetilde{Q}(a_1).$$

Combining this with (4.16), we obtain

$$P^x(B_1 \cap F_1(\eta_1) \mid A_1) = P^x(F_1(\eta_1) \mid A_1 \cap B_1) P^x(B_1 \mid A_1) \geq (1 - c_3)c_4. \tag{4.18}$$

By the similar argument to the proof of [33, Lemma 4.4], we can obtain $E^x(\mathrm{len}(\lambda_1)) \leq Cm^\beta$ uniformly in $x \in \widetilde{Q}(a_1)$ and by the Markov's inequality, there exists a universal constant $0 < C_1 < \infty$ such that

$$P^x(E_1(C_1)^c \mid A_1) \leq c_3(1 - c_4)/2,$$

uniformly in $x \in \widetilde{Q}(a_1)$. Combining this with (4.15), (4.16) and (4.18) yields

$$P^x(A_1 \cap B_1 \cap E_1(C_1) \cap F_1(\eta_1)) \geq \frac{c_1 c_3 (1 - c_4)}{2} N^{-2},$$

which finishes the proof. □

Now we perform Wilson's algorithm around the center of each small cube. Recall that $B(x, r)$ indicates the ball in the Euclidean metric and $\tau_\gamma(A)$ is the first time that γ hits A. Given λ_i (see (4.9) for the definition), we regard it as a deterministic set and consider independent simple random walks started at the points in $B(x_i, \lambda^{-2}m)$ for some $\lambda \geq 1$. We regard these random walks as a step of Wilson's algorithm rooted at ξ_i'.

In the following lemma, we will observe that with high conditional probability, a small Euclidean ball around the center of each box B_{x_i} is included in an intrinsic ball centered at the same point and of radius of order m^β. We define the event $M_j(\lambda)$ by

$$M_j(\lambda) = \{B(x_j, \lambda^{-2}m) \subset B_{\mathcal{U}^N}(x_j, \lambda^{-1}m^\beta\}. \tag{4.19}$$

Lemma 4.7. *There exist $c_4, c_5 > 0$ such that for all $\delta > 0, m \geq 1, \lambda \in [1, m^{(1-\delta)/2})$ and $j \in \{1, \cdots, N\}$,*

$$\mathbf{P}\left(M_j(\lambda)) \mid A_j \cap B_j \cap E_j(C_*) \cap F_j(\eta_*)\right) \geq 1 - c_4\lambda^{-c_5}. \tag{4.20}$$

Proof. It suffices to show (4.20) in the case $j = 1$. Let $\mathbf{P}_{\mathcal{U}^N}(\cdot) := \mathbf{P}(\cdot \mid A_1 \cap B_1 \cap E_1(C_*) \cap F_1(\eta_*))$. We may assume that m and λ are sufficiently large for the same reason as [1, Proposition 4.1]. Thus, we take large m so that

$$\frac{m^{\delta/2}}{\delta \log m + 2} \geq 10, \tag{4.21}$$

for a fixed $\delta > 0$.

Recall that R^z indicates the SRW on $\mathbb{Z}^3$ started at z and independent of S. Given S, we run R^{x_1} until it hits ξ_N. On the event $A_1 \cap B_1 \cap E_1(C_*) \cap F_1(\eta_*)$, we have that $d_E(x_1, \xi_N) \in$

$[m/10, m/8]$ by the definition of A_1 (see (4.6)) and the event $\{\mathrm{LE}(R^{x_1}[0, \tau_{R^{x_1}}(\xi_N)]) \subset B_{x_1}\}$ occurs with positive conditional probability by the definition of F_1 and η_* (see (4.13) and (4.14)). By [35, Corollary 4.5], we have that for $\lambda \geq 40$ the law of $\mathrm{LE}(R^{x_1}[0, \tau_{R^{x_1}}(\xi_N)])$ restricted to $B(x_1, \lambda^{-1}m)$ is comparable to that of the infinite LERW started at x_1 restricted to the same ball. Thus, we can follow the discussion of [1, Proposition 4.1], which gives a tail bound estimate of the volume of intrinsic balls in the three-dimensional UST.

Let σ and $\widetilde{\sigma}$ be the first time that $\gamma_{x_1} := \mathrm{LE}(R^{x_1}[0, \tau_{R^{x_1}}(\xi_N)])$ exits $B(x_i, \lambda^{-2}m)$ and $B(x_i, \lambda^{-1}m)$, respectively. We define the event F by

$$F = \left\{ \gamma_{x_1}[\widetilde{\sigma}, \mathrm{len}(\gamma_{x_1})] \cap B(x_1, 2\lambda^{-2}m) = \emptyset,\ \sigma \leq \frac{1}{2}\lambda^{-1}m^{\beta} \right\}.$$

Then by [28, Proposition 1.5.10], the probability that a SRW started at a point outside $B(x_1, \lambda^{-1}m)$ returns to $B(x_1, \lambda^{-2}m)$ is smaller than $C\lambda^{-1}$ for some universal constant $C < \infty$. This implies $\mathbf{P}_{\mathcal{U}^N}(\gamma_{x_1}[\widetilde{\sigma}, \mathrm{len}(\gamma_{x_1})] \cap B(x_1, 2\lambda^{-2}m) \neq \emptyset) \leq C\lambda^{-1}$. On the other hand, by [39, Theorem 1.4] and [32, Corollary 1.3], the probability that σ is greater than $\frac{1}{2}\lambda^{-1}m^{\beta}$ is bounded above by $C\exp\{-c\lambda^{-1}\}$ for some universal constants $0 < C, c < \infty$. Thus, it follows from the above estimates that

$$\mathbf{P}_{\mathcal{U}^N}(F) \geq 1 - C\lambda^{-1}. \tag{4.22}$$

Next we observe that γ_{x_1} can be hit by another independent SRW started at a point which is close to γ_{x_1} with high probability. For $\zeta > 0$, we define an event $G(\zeta)$ by

$$G(\zeta) = \left\{ \forall y \in B(x_1, 2\lambda^{-2}m),\ P_R^y(R[0, T_{R^y}(x_1, \lambda^{-3/2}m)] \cap \gamma_{x_1} = \emptyset) \leq \lambda^{-\zeta} \right\},$$

where $T_{R^y}(x, l)$ is the first time that R^y exists $B(x, l)$. From [37, Theorem 3.1], there exist universal constants $C < \infty$ and $\zeta_1 \in (0, 1)$ such that for all $m \geq 1$ and $\lambda \geq 2$,

$$\mathbf{P}(G(\zeta_1)) \geq 1 - C\lambda^{-1}. \tag{4.23}$$

Then we take a sequence of subsets of $\mathbb{Z}^3$ including the boundary of $B(x_1, \lambda^{-1}m)$. For each $k \geq 1$, let $\varepsilon_k = \lambda^{-\zeta_1/6}2^{-k-10}$, $\eta_k = (2k)^{-1}$ and

$$A_k = B(x_1, (1 + \eta_k)\lambda^{-2}m) \setminus B(x_1, (1 - \eta_k)\lambda^{-2}m).$$

Write k_0 for the smallest integer satisfying $\lambda^{-2}m\varepsilon_{k_0} < 1$. Note that the condition (4.21) guarantees that both the inner and outer boundary of $B(x_1, \lambda^{-2}m)$ are contained in A_{k_0}. Moreover, let D_k be a set of lattice points in A_k such that $A_k \subset \bigcup_{z \in D_k} B(z, \lambda^{-2}m\varepsilon_k)$. We may suppose that $|D_k| \leq C\varepsilon_k^{-3}$. Since $\lambda^{-2}m\varepsilon_{k_0} < 1$ and $\partial_i B(x_1, \lambda^{-2}m) \subset A_{k_0}$, it follows that $\partial_i B(x_1, \lambda^{-2}m) \subset D_{k_0}$.

Now we perform Wilson's algorithm to prove (4.20). Let $\mathcal{U}_0^N := \xi_N \cup \gamma_{x_1}$.

(i) Consider an independent SRW started at a point in D_1 and run until it hits $\mathcal{U}_0^N$. We add its loop-erasure to $\mathcal{U}_0^N$ and denote the union by $\mathcal{U}_{1,1}^N$. Given $\mathcal{U}_{1,j}^N$, we consider an independent SRW from another point in $D_1 \setminus \mathcal{U}_{1,j}^N$ and let $\mathcal{U}_{1,j+1}^N$ be the union of $\mathcal{U}_{1,j}^N$ and the loop-erasure of the new SRW. We continue this procedure until all points in D_1 are contained in the tree, which we denote by $\mathcal{U}_1^N$.

(ii) We repeat the above procedure for D_2 taking $\mathcal{U}_1^N$ as a root. Let $\mathcal{U}_2^N$ be the output tree. We continue inductively to construct $\mathcal{U}_3^N, \mathcal{U}_4^N, \cdots \mathcal{U}_{k_0}^N$.

(iii) Once we obtain $\mathcal{U}_{k_0}^N$, we perform Wilson's algorithm for all points in $B(x_1, \lambda^{-2}m)$.

(iv) We repeat the same procedure as (i), (ii) and (iii) for all $x_2, x_3, \cdots x_N$.

(v) Finally, we perform Wilson's algorithm for all points in $\bigcup_{j=0}^N B_{x_j}$ to obtain $\mathcal{U}^N$.

By construction, it is clear that $\mathcal{U}_k^N \subset \mathcal{U}_{k+1}^N$, and also $\partial_i B(x_1, \lambda^{-2}r) \subset \mathcal{U}_{k_0}^N$.

By the definition of $G(\zeta_1)$, we have that

$$\mathbf{P}(\gamma(y, \mathcal{U}_0^N) \not\subset B(x_1, \lambda^{-3/2}m) \mid F \cap G(\zeta_1)) \leq \lambda^{-\zeta_1}. \tag{4.24}$$

On the other hand, by stopping conditioning γ_{x_1} on $F \cap G(\zeta_1)$, it follows from [39, Theorem 1.4] and [32, Corollary 1.3] that there exist some universal constant $C,\ c,\ c' > 0$ such that

$$\begin{aligned}
&\mathbf{P}\left(\gamma_{\mathcal{U}^N}(y, \mathcal{U}_0^N) \subset B(x_1, \lambda^{-1}m),\ d_{\mathcal{U}^N}(y, \mathcal{U}_0^N) \geq \frac{1}{2}\lambda^{-1}m^\beta\right) \\
&\quad \leq \frac{\mathbf{P}(\gamma_{\mathcal{U}^N}(y, \mathcal{U}_0^N) \subset B(x_1, \lambda^{-1}m),\ d_{\mathcal{U}^N}(y, \mathcal{U}_0^N) \geq \frac{1}{2}\lambda^{-1}m^\beta)}{\mathbf{P}(F \cap G(\zeta_1))} \\
&\quad \leq C \exp\{-c\lambda^{c'}\}.
\end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25), we have that

$$\mathbf{P}(\gamma(y, \mathcal{U}_0^N) \subset B(x_1, \lambda^{-1}m),\ d_{\mathcal{U}}(y, \mathcal{U}_0^N) \leq \frac{1}{2}\lambda^{-1}m^\beta) \geq 1 - C\lambda^{-\zeta_1}.$$

Let H be the event defined by

$$H = \left\{\gamma(y, \mathcal{U}_0^N) \subset B(x_1, \lambda^{-1}m),\ d_{\mathcal{U}}(y, \mathcal{U}_0^N) \leq \frac{1}{2}\lambda^{-1}m^\beta \text{ for all } y \in D_1\right\}.$$

Then we have

$$\mathbf{P}(H) \geq 1 - C\lambda^{-\zeta_1/2},$$

since $|D_1| \leq C\lambda^{\zeta_1/2}$.

Next, we will consider several events that ensure hittability of branches in the subtree. For $k \geq 1$ and $\zeta > 0$, we define the event $I(k, x, \zeta)$ by

$$\begin{aligned}
&I(k, x, \zeta) \\
&\quad = \left\{P_R^y\left(R\left[0, T_{R^y}(y, \lambda^{-2}m\varepsilon_k^{1/2})\right] \cap (\mathcal{U}_0^N \cup \gamma(x, \mathcal{U}_0^N))\right) \leq \varepsilon_k^\zeta \text{ for all } y \in B(x, \lambda^{-2}m\varepsilon_k)\right\},
\end{aligned} \tag{4.26}$$

Let $I(k, \zeta) = \bigcap_{x \in D_k} I(k, x, \zeta)$. Applying [37, Lemma 3.2], it follows that there exist universal constants $\zeta_2 > 0$ and $C < \infty$ such that for all $k \geq 1, m \geq 1, \lambda \geq 2$ and $x \in D_k$,

$$\mathbf{P}_{\mathcal{U}^N}(I(k, x, \zeta_2)^c) \leq C\varepsilon_k^5.$$

Combining this with $|D_k| \le C\varepsilon_k^{-3}$ yields that

$$\mathbf{P}(I(k,\zeta_2)^c) \le C\varepsilon_k^2 \le C\lambda^{-\zeta_1/3}.$$

We set $A_1' := F \cap G(\zeta_1) \cap H \cap I(1,\zeta_2)$. Note that A_1' is measurable with respect to $\mathcal{U}_1^N$, the subtree obtained after the first step (i) of Wilson's algorithm. We have already seen that $\mathbf{P}_{\mathcal{U}^N}(A_1') \ge 1 - C\lambda^{-\zeta_1/3}$.

Conditioning $\mathcal{U}_1^N$ on the event A_1', we proceed with Wilson's algorithm for the points in D_2. We take $y \in D_2$ and consider the SRW R^y started at y until it hits $\mathcal{U}_1^N$. By the definition of D_1, there exists $x' \in D_1$ such that $d_E(x,y) \le \lambda^{-2}m\varepsilon_1$. Suppose that R^y exits $B(y, \lambda^{-2}m\varepsilon_1^{1/3})$ before it hits $\mathcal{U}_1^N$. Then the event that R^y exits $B(x, \lambda^{-2}m\varepsilon_1)$ before it hits $\mathcal{U}_1^N$ occurs. However by (4.26) and the definition of ζ_2, the probability that the event occurs conditioned on A_1' is lower than $\varepsilon_1^{\zeta_2}$. By iteration, the number of balls of radius $\lambda^{-2}m\varepsilon_1^{1/2}$ that R^y exits before hitting $\mathcal{U}_1^N$ is larger than $\varepsilon_1^{-1/6}$. Hence, we have that

$$P^y(R^y \text{ exits } B(y, \lambda^{-2}m\varepsilon_1^{1/3}) \text{ before it hits } \mathcal{U}_1^N) \le \varepsilon_1^{c\zeta_2\varepsilon_1^{-1/6}},$$

for some universal constant $c > 0$. Moreover, following the same argument as (4.25), we have that

$$P^y\left(\gamma(y,\mathcal{U}_1^N) \not\subset B(y,\lambda^{-2}m\varepsilon_1^{1/3}) \text{ and } d_{\mathcal{U}}(y,\mathcal{U}_1^N) \ge (\lambda^{-2}m)^\beta \varepsilon_1^{1/4}\right) \le C\exp\left\{-c\varepsilon_1^{-1/12}\right\}.$$

With this in mind, we define the event B_2 by

$$B_2 = \left\{\gamma(y,\mathcal{U}_1^N) \subset B(y,\lambda^{-2}m\varepsilon_1^{1/3}) \text{ and } d_{\mathcal{U}}(y,\mathcal{U}_1^N) \le \lambda^{-1}m^\beta\varepsilon_1^{1/4}, \text{ for all } y \in D_2\right\}.$$

Since $|D_2| \le C\varepsilon_2^{-3}$, we have that

$$\mathbf{P}_{\mathcal{U}^N}(B_2 \mid A_1') \ge 1 - C\varepsilon_1^{-3}\exp\left\{-c\varepsilon_1^{-1/12}\right\}.$$

Hence, letting $A_2' := A_1 \cap B_2 \cap I(2,\zeta_2)$, it follows that

$$\mathbf{P}_{\mathcal{U}^N}(A_2' \mid A_1') \ge 1 - C\varepsilon_2^2.$$

Following the above argument, we define the sequences of events $\{A_k'\}, \{B_k\} (k = 2, 3, \cdots, k_0)$ by

$$\begin{aligned}
&B_k = \left\{\gamma(y,\mathcal{U}_{k-1}^N) \subset B(y,\lambda^{-2}m\varepsilon_{k-1}^{1/3}) \text{ and } d_{\mathcal{U}}(y,\mathcal{U}_{k-1}^N) \le \lambda^{-1}m^\beta\varepsilon_{k-1}^{1/4}, \text{ for all } y \in D_2\right\},\\
&A_k' = A_{k-1}' \cap B_k \cap I(k,\zeta_2).
\end{aligned}$$

Then we can conclude that

$$\mathbf{P}_{\mathcal{U}^N}(A_{k_0}') = \mathbf{P}_{\mathcal{U}^N}(A_1')\prod_{k=2}^{k_0}\mathbf{P}_{\mathcal{U}^N}(A_k' \mid A_{k-1}') \ge (1 - C\lambda^{-\zeta_1/3})\prod_{k=1}^{\infty}(1 - C\varepsilon_k^2) \ge 1 - C\lambda^{-\zeta_1/3}. \tag{4.27}$$

On the other hand, on the event A_{k_0}', there exists some universal constant $C > 0$ such that

(1) $d_{\mathcal{U}}(x_1,y) \le \lambda^{-1}m^{\beta}$ for all $y \in \left(\mathcal{U}_0^N \cap B(x_1,\lambda^{-2}m)\right) \cup \left(\bigcup_{y\in D_1}\gamma(y,\mathcal{U}_0^N)\right)$,

(2) $d_{\mathcal{U}}(x_1,y) \le C\lambda^{-1}m^{\beta}$ for all $y \in (\mathcal{U}_0^N \cap B(x_1,\lambda^{-2}m)) \cup \mathcal{U}_{k_0}^N$,

It immediately follows that (1) holds from the definition of F and H. For $y \in \mathcal{U}_{k_0}$, let y_k ($k=1,2,\cdots k_0-1$) be the first point in $\mathcal{U}_k^N$ that appears on $\gamma_{\mathcal{U}^N}(y,\mathcal{U}_0^N)$ (we set $y_k=y$ if $y\in\mathcal{U}_k^N$). On the event A'_{k_0}, we have that

$$\begin{aligned} d_{\mathcal{U}}(x_1,y) &\le d_{\mathcal{U}}(x_1,y_1) + \sum_{k=1}^{k_0-1} d_{\mathcal{U}}(y_k,y_{k+1}) \\ &\le \lambda^{-1}m^{\beta} + \sum_{k=1}^{\infty}\lambda^{-1}m^{\beta}\varepsilon_{k-1}^{1/4} \le C\lambda^{-1}m^{\beta}, \end{aligned}$$

which implies (2).

Once we see that (2) holds on the event A'_{k_0}, we need to estimate the $d_{\mathcal{U}}$ distance between an arbitrary point in $B(x_1,\lambda^{-2}m)$ and $\partial_i B(x_1,\lambda^{-2}m)$. In order to do so, we take another "net": we let $\varepsilon'_k = \lambda^{-\zeta_1/4}2^{-k-10}$ and D'_k be a set of lattice points in $B(x_1,\lambda^{-2}m)$ such that $B(x_1,\lambda^{-2}m) \subset \bigcup_{z\in D'_k} B(z,\lambda^{-2}m\varepsilon'_k)$. We may suppose that $|D'_k| \le C(\varepsilon'_k)^{-3}$. By the similar argument to the estimate of $\mathbf{P}_{\mathcal{U}^N}(A'_{k_0})$, we obtain

$$\mathbf{P}_{\mathcal{U}^N}\left(d_{\mathcal{U}}(y,\partial_i B(x_1,\lambda^{-2}m)) \ge C\lambda^{-1}m^{\beta} \text{ for some } y\in B(x_1,\lambda^{-2}m)\right) \le C\lambda^{-\zeta_1/2}. \tag{4.28}$$

For the lower bound of volume (4.20), we now estimate the distance between x_1 and all points in $B(x_1,\lambda^{-2}m)$. Since $\mathcal{U}_{k_0}^N$ contains $\partial_i B(x_1,\lambda^{-2}m^{\beta})$, it follows from the same argument as (4.25) again that for any $y\in B(x,\lambda^{-2}m)$,

$$\begin{aligned} &\mathbf{P}_{\mathcal{U}^N}\left(d_{\mathcal{U}}(x_1,y) \le C\lambda^{-1}m^{\beta} \text{ for all } y\in B(x_1,\lambda^{-2}m)\right) \\ &\quad\ge \mathbf{P}_{\mathcal{U}^N}(A'_{k_0}) - \mathbf{P}_{\mathcal{U}^N}\left(A'_{k_0} \cap \left\{d_{\mathcal{U}}(x_1,y) > C\lambda^{-1}m^{\beta} \text{ for some } y\in B(x_1,\lambda^{-2}m)\right\}\right) \\ &\quad\ge \mathbf{P}_{\mathcal{U}^N}(A'_{k_0}) - \mathbf{P}_{\mathcal{U}^N}\left(d_{\mathcal{U}}(y,\partial_i B(x_1,\lambda^{-2}m)) > C\lambda^{-1}m^{\beta} \text{ for some } y\in B(x_1,\lambda^{-2}m)\right) \\ &\quad\ge 1-C\lambda^{-c}, \end{aligned}$$

for some universal constant $c>0$, which completes the proof of (4.20). $\square$

It follows from (4.14) and (4.20) that there exists some universal constant $\lambda_*\ge 1$ such that

$$P(A_j\cap B_j\cap E_j(C_*)\cap F_j(\eta_*)\cap M_j(\lambda_*)) = c_*^2N^{-2}, \tag{4.29}$$

for $j=1,2,\cdots N$. We have obtained a lower bound of the probability that the volume of the random tree constructed by Wilson's algorithm in each B_{x_j} is of the order of m^{β}.

Recall that the events G_j and H_j are defined by (4.7) and (4.8) respectively. We define an event I_j by $I_j=\bigcap_{k=1}^{j} M_k(\lambda_*)$. Let $J_j=\bigcap_{k=1}^{j} E_k(C_*)$ and $K_j=\bigcap_{k=1}^{j} F_k(\eta_*)$ for C_* and η_* defined in Lemma 4.6, where $E_k(C)$ and $F_k(\eta)$ are as defined in (4.12) and (4.13), respectively. Recall that $T_S(r)=\inf\{k\ge 0: |S(k)|\ge r\}$.

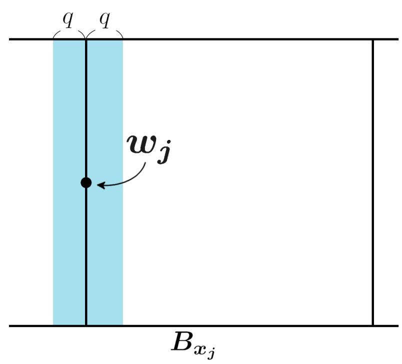


Figure 10: Q_w for $w = w_j = ((j - \frac{1}{2})m, 0, 0)$

By (4.11), in order to estimate $\mathrm{len}(\xi_j)$ on the event $G_j \cap H_j \cap I_j \cap J_j \cap K_j$, we need to give an upper bound on $|\xi_j \cap Q[a_k - q, a_k + q]|$ for $k = 1, \cdots, j$ and for q defined in (4.5). Take

$$w = (w^1, w^2, w^3), \quad R(N) = \exp\{2e^{RN^2} + 1\}, \tag{4.30}$$

and define

$$Q_w = \{y = (y^1, y^2, y^3) : |y^1 - w^1| \leq q,\ |y^i - w^i| \leq m/2 \text{ for } i = 2, 3\},$$
$$N_w = |Q_w \cap \mathrm{LE}(S[0, T_S(R(N)m)])|\,.$$

Then, it follows from [33, Lemma 4.5] that there exist universal constants $0 < c, C < \infty$ such that

$$P(N_w \geq m^\beta) \leq C \exp\{-cN^2\} \ \text{ uniformly in } w \in B(0, R(N)m). \tag{4.31}$$

We define

$$w_j = (a_j, 0, 0), \quad L_j = \Big\{|Q_{w_k} \cap \xi_j| \leq m^\beta \text{ for all } 1 \leq k \leq j/2\Big\} \ \text{ for } j \geq 1,$$
$$U_{2N} = \Big\{S(T_S(R(N))) \in \{(y^1, y^2, y^3) \in \mathbb{R}^3 : y^1 \geq \frac{4}{5}R(N)m\},$$
$$S[t_S(a_{2N+1}), T_S(R(N))] \cap B(0, a_{\frac{7}{4}N}) = \emptyset\Big\}\,.$$

and set

$$A^N = G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N} \cap L_{2N} \cap U_{2N}. \tag{4.32}$$

We estimate the lower bound of the probability of the event A^N.

Lemma 4.8. *There exists a universal constant $c_3 > 0$ such that*

$$P(A^N) \geq c_3^{-1} \exp\{-c_3 N(\log N)\} \tag{4.33}$$

Proof. To prove this, we will make use of the strong Markov property of S as follows. Firstly, by the strong Markov property

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N})$$
$$= P((A_{2N} \cap B_{2N} \cap E_{2N} \cap F_{2N} \cap M_{2N}) \cap (G_{2N-1} \cap H_{2N-1} \cap I_{2N-1} \cap J_{2N-1} \cap K_{2N-1}))$$
$$= P^{S(t_S(a_{2N}))}(A_{2N} \cap B_{2N} \cap E_{2N} \cap F_{2N} \cap M_{2N}) P(G_{2N-1} \cap H_{2N-1} \cap J_{2N-1} \cap K_{2N-1}).$$

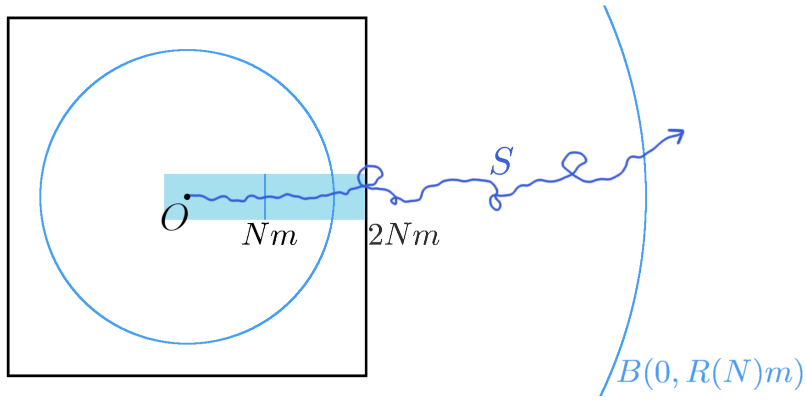


Figure 11: Definition of the event U_{2N}

Then by (4.29), we have that

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N}) \geq c_* N^{-2} P(G_{2N-1} \cap H_{2N-1} \cap I_{2N-1} \cap J_{2N-1} \cap K_{2N-1}),$$

and by iteration, it follows that there exists some universal constant $c > 0$ such that

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N}) \geq (cN^{-2})^{2N}. \tag{4.34}$$

Secondly, again by the strong Markov property

$$\begin{aligned} &P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N} \cap U_{2N}) \\ &\quad = P(U_{2N} \mid G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N}) P(G_{2N} \cap H_{2N} \cap J_{2N} \cap K_{2N}) \\ &\quad = P^{S(t_S(a^{2N+1}))}(U_{2N}) \, P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N}). \end{aligned}$$

Then by [28, Proposition 1.5.10], $P^{S(t_S(a^{2N+1}))}(U_{2N})$ is bounded below by some universal constant $c > 0$. Combining this with (4.34), we obtain

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N} \cap U_{2N}) \geq c \exp\{-cN(\log N)\}. \tag{4.35}$$

Furthermore, following the proof of [33, Proposition 4.6], we obtain that

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N} \cap U_{2N} \cap (L_{2N})^c) \leq CN \exp\{-cN^2\},$$

where we use (4.31) instead. Combining this with (4.35), we obtain

$$P(G_{2N} \cap H_{2N} \cap I_{2N} \cap J_{2N} \cap K_{2N} \cap L_{2N} \cap U_{2N}) \geq c \exp\{-cN(\log N)\},$$

which completes the proof. $\square$

4.1.2 Proof of Proposition 4.1

In the previous subsection, we observed the behavior of $\mathcal{U}$ along the LERW starting at the origin, *i.e.* the first step of Wilson's algorithm. Now we consider several events to complete a lower bound estimate of the upper tail of the volume.

We take $y_j = (jm, Nm, 0) \in \mathbb{Z}^3$, $j = 1, \cdots, N$, and run a simple random walk R^{y_j} started at y_j until it hits $\mathcal{U}^N$. Let $B_y(k)$ be the cube of side-length m centered at $y_{j,k} = (jm, km, 0) \in \mathbb{Z}^3$. We define τ_{y_j} (resp. σ_{y_j}) to be the first time that R^{y_j} hits $B_y(0) = B_{x_j}$ (resp. $\mathcal{U}^N$).

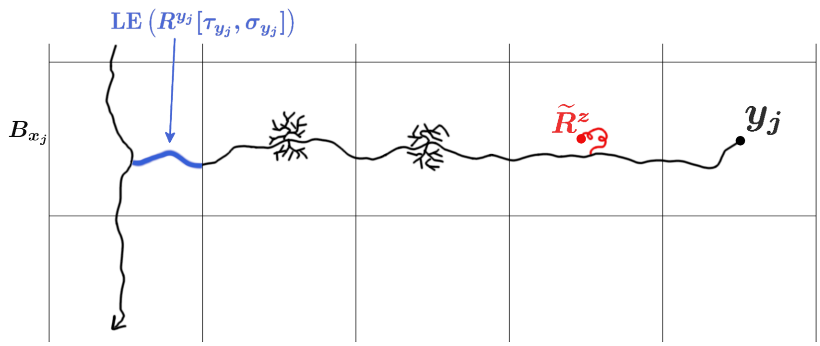


Figure 12: The sets we consider in the event $V_{y_j} \cap W_{y_j}$

Definition 4.9. *Define V_{y_j} to be the intersection of the following events of R^{y_j}:*

- $\{\mathrm{LE}(R^{y_j}[0, \tau_{y_j}]) \subset \bigcup_{k=1}^{N-1} B_y(k)\}$,
- $\{\mathrm{len}(\mathrm{LE}(R^{y_j}[0, \tau_{y_j}])) \leq (N-1)m^{\beta}\}$,
- $\bigcap_{k=1}^{N-1} \left\{ \forall z \in B(y_{j,k}, 2\lambda^{-2}m),\ P^z_{\widetilde{R}}([0, T_{\widetilde{R}^z}(y_{j,k}, \lambda^{-3/2}m)] \cap \mathrm{LE}(R^{y_j}[0, \tau_{y_j}] = \emptyset) \leq \lambda^{-\zeta_1} \right\}$, *where $\widetilde{R}$ is a simple random walk independent of R^{y_j} and ζ_1 is as defined in (4.23),*
- $\bigcap_{k=1}^{N-1} \{B(y_{j,k}, \lambda^{-2}m) \subset B_{\mathcal{U}}(y_{j,k}, \lambda^{-1}m^{\beta})\}$,

and define W_{y_j} by

$$W_{y_j} := \left\{ \mathrm{len}(\mathrm{LE}(R^{y_j}[\tau_{y_j}, \sigma_{y_j}])) \geq cN(\log N)^{100} m^{\beta} \right\}$$

Lemma 4.10. *For each $j = 1, 2, \cdots N$,*

$$P^{y_j}(V_{y_j} \cap W_{y_j} \mid A^N) \geq c \exp\{-CN(\log N)\}. \tag{4.36}$$

Remark 4.11. Since tail bounds for the length of three-dimensional LERW in a general set have not been obtained, we will apply the tail bound for the length of infinite LERW in a Euclidean ball given in [39, Theorem 1.4] and [32, Corollary 1.3], instead of regarding γ_∞ as a deterministic set. Thus, in order to estimate the conditional probability of W_{y_j} on the event A^N from below, we consider the length $cN(\log N)^{100} m^{\beta}$ in the right-hand side of the definition of W_{y_j}, so that $P^{y_j}(W^c_{y_j})$ becomes enough compared to $\mathbf{P}(A^N)$.

Proof. Firstly, applying the same argument as Lemma 4.7 and Lemma 4.8, there exists a universal constant $c > 0$ such that $P^{y_j}(V_{y_j} \mid A^N) \geq c \exp\{-cN \log N\}$. Secondly, we will estimate the upper bound of $P^{y_j}(A^N \cap V_{y_j} \cap W^c_{y_j})$. In order to do so, we stop conditioning on A^N and consider $\mathrm{LE}(R^{y_j}[\tau_{y_j}, \sigma_{y_j}])$ as a part of infinite LERW. By [39, Theorem 1.4] and [32, Corollary 1.3], we have that

$$\begin{aligned} P^{y_j}(A^N \cap V_{y_j} \cap W^c_{y_j}) &\leq P\left(\mathrm{len}(\mathrm{LE}(R^{y_j}[\tau_{y_j}, \sigma_{y_j}])) \geq cN(\log N)^{100} m^{\beta} \right) \\ &\leq \exp\{-CN(\log N)^{100}\}, \end{aligned}$$

from which it follows that

$$P^{y_j}(A^N \cap V_{y_j} \cap W_{y_j}) = P^{y_j}(A^N \cap V_{y_j}) - P^{y_j}(A^N \cap V_{y_j} \cap W_{y_j}^c) \geq \frac{1}{2} P^{y_j}(A^N \cap V_{y_j}),$$

since $P(F_0) \geq c\exp\{-CN \log N\}$ by Lemma 4.8 and Lemma 4.7. Thus, we have

$$\begin{aligned} P^{y_j}(V_{y_j} \cap W_{y_j} \mid A^N) &\geq \frac{1}{2} P^{y_j}(V_{y_j} \mid A^N) \\ &\geq c\exp\{-cN(\log N)\}, \end{aligned}$$

which completes the proof. □

Finally we take $z_{j,k} = (jm, km, Nm) \in \mathbb{Z}^3$, $j, k = 1, \cdots, N$, and run a simple random walk $R^{z_{j,k}}$ started at $z_{j,k}$ until it hits $\mathcal{U}^N$. Let $B_z(l)$ be the cube of side-length m centered at $z_{j,k,l} = (jm, km, lm) \in \mathbb{Z}^3$. We define $\tau_{z_{j,k}}$ (resp. $\sigma_{z_{j,k}}$) to be the first time that $R^{z_{j,k}}$ hits $B_z(0) = B_y(k)$ (resp. already constructed subtree of $\mathcal{U}$). Let $V_{z_{j,k}}$ (resp. $W_{z_{j,k}}$) be an event of $R^{z_{j,k}}$ which corresponds to V_{y_j} (resp. W_{y_j}) with x_2 axis replaced by x_3 axis (see Definition 4.9 for the definition of V_{y_j} and W_{y_j}). By applying the same argument as Lemma 4.10, we have that

$$P^{z_{j,k}}(V_{z_{j,k}} \cap W_{z_{j,k}} \mid A^N \cap V_{y_j} \cap W_{y_j}) \geq c\exp\{-CN(\log N)\}. \tag{4.37}$$

Note that $V_{z_{j,k}} \cap W_{z_{j,k}}$ is independent of $V_{y_{j'}} \cap W_{y_{j'}}$ if $j \neq j'$.

Corollary 4.12. *There exist universal constants c, c', C, C' such that for all $m \geq 1$ and $N \geq 1$,*

$$\mathbf{P}\left(|B_{\mathcal{U}}(0, C'Nm^\beta)| \geq c'(Nm)^3\right) \geq c\exp\{-CN^3(\log N)\}. \tag{4.38}$$

Proof. Recall that $x_j = (jm, 0, 0)$. On the event A^N, we have that for all $y \in B(x_j, \lambda_*^{-2}m)$,

$$\begin{aligned} d_{\mathcal{U}}(0, y) &\leq d_{\mathcal{U}}(0, x_j) + d_{\mathcal{U}}(x_j, y) \\ &\leq CNm^\beta + \lambda_*^{-1} m^\beta \\ &\leq C'Nm^\beta \end{aligned}$$

Since each step of Wilson's algorithm is mutually independent, applying the result of Lemma 4.10 to the "tubes" parallel to x_2 axis, we obtain

$$\begin{aligned} &\mathbf{P}\left(\bigcap_{k=0}^{N} \bigcap_{j=1}^{N} \left\{ d_{\mathcal{U}}(0, y) \leq CNm^\beta \text{ for all } y \in B\left((jm, km, 0), \lambda_*^{-2}m\right) \right\}\right) \\ &\geq \mathbf{P}(A^N) \prod_{j=1}^{N} P^{y_j}(V_{y_j} \cap W_{y_j} \mid A^N) \\ &\geq c\exp\{-CN^2(\log N)\}. \end{aligned}$$

Next, we consider the "tubes" parallel to x_3 and we have

$$\begin{aligned}
&\mathbf{P}\left(\bigcap_{l=0}^{N}\bigcap_{k=0}^{N}\bigcap_{j=1}^{N}\left\{d_{\mathcal{U}}(0,y)\le CNm^{\beta}\text{ for all }y\in B\left((jm,km,lm),\lambda_*^{-2}m\right)\right\}\right)\\
&\ge c\exp\{-CN^2(\log N)\}\prod_{k=1}^{N}\prod_{j=1}^{N}P^{z_{j,k}}\left(V_{z_{j,k}}\cap W_{z_{j,k}}\mid A^N\cap V_{y_j}\cap W_{y_j}\right)\\
&\ge c\exp\{-CN^3(\log N)\},
\end{aligned}$$

where we applied (4.37) in the last inequality. Finally, comparing the left-hand side of the above inequality and (4.38), we obtain (4.38). ☐

Proof of Propositon 4.1. Setting $r=C'Nm^{\beta}$ and $\lambda=c'N^{3(\beta-1)/\beta}/C'^{3/\beta}$ in (4.38) yields the result at (4.1). ☐

Theorem 4.13. $\mathbf{P}$-*a.s.,*

$$\limsup_{r\to\infty}(\log\log r)^{-(\beta-1)/\beta}r^{-3/\beta}|B_{\mathcal{U}}(0,r)|=\infty. \tag{4.39}$$

Proof. We will begin with defining a sequence of scales. Fix $\varepsilon>0$ and let

$$D_i=e^{i^2},\ m_i=D_i/\varepsilon(\log i)^{1/3}.$$

We now run Wilson's algorithm. Let γ_∞ be the infinite LERW started at the origin and let $(S^z)_{z\in\mathbb{Z}^3}$ be the family of independent SRW which is also independent of γ_∞. At stage i ($i\ge1$), we use all the vertices in $B_\infty(0,D_i)$ which have not already been contained and write $\mathcal{U}_i$ for the tree obtained.

By [1, Proposition 4.1], there exists $M>0$ such that the event

$$B_\infty(0,D_i)\subset B_{\mathcal{U}}(0,i^MD_i^{\beta})\subset B_\infty(0,i^{2M}D_i) \tag{4.40}$$

occurs with probabilty greater than $1-ci^{-2}$. Hence, if we run Wilson's algorithm for the vertices contained in $B_\infty(0,D_i)$ taking γ_∞ as the root, then the probability that $\mathcal{U}_i$ leaving $B_\infty(0,i^{2M})$ is less than $c\lambda^{-2}$. By applying the Borel-Cantelli lemma, we obtain that

$$\mathcal{U}_i\subset B_\infty(0,i^{2M}D_i)\subset B_\infty(0,m_{i+1}/2) \tag{4.41}$$

for large i, almost-surely. Moreover, from (4.40), we may also assume that

$$d_{\mathcal{U}}(0,z)\le i^MD_i^{\beta}\le m_{i+1}^{\beta}\quad\text{for all}\quad z\in\mathcal{U}_i \tag{4.42}$$

almost-surely.

Define the event $F(i)$ to be the event that both (4.41) and (4.42) hold. Let $\mathcal{F}_i$ be the σ-field generated by the followings:

- $\gamma_\infty[0,\tau_{B_\infty(0,i^{2M}D_i)}]$,
- All simple random walks added to $\mathcal{U}_{i-1}$ at stage i,

where τ_A represents the first exiting time from A.

Now we bound the probability that the subtree $\mathcal{U}_{i+1}$ obtained at stage $i+1$ also satisfies the diameter estimate and the inclusion corresponding to (4.41) conditioned that $F(i)$ holds. We define an event $G(i)$ by

$$G(i) = \{|\gamma_\infty[0, \tau_{B_\infty(0,m_i)}]| \le m_i^\beta\} \cap A^{\varepsilon(\log i)^{1/3}} \cap \left(\bigcap_{j=1}^{\varepsilon(\log i)^{1/3}} V_{y_j} \cap W_{y_j} \right)$$
$$\cap \left(\bigcap_{k=1}^{\varepsilon(\log i)^{1/3}} \bigcap_{j=1}^{\varepsilon(\log i)^{1/3}} V_{z_{j,k}} \cap W_{z_{j,k}} \right),$$

where replace the scales m and N by m_i and $\varepsilon(\log i)^{1/3}$, respectively. See (4.32) and Definition 4.9 for the definition of the events A^N, $V_{y_j} \cap W_{y_j}$ and $V_{z_{j,k}} \cap W_{z_{j,k}}$.

For $A \subset \mathbb{Z}^3$, let

$$\tau'_A = \sup\{i : \gamma_\infty(i) \in A\},$$

be the last time that γ_∞ exits from A and recall that τ_A indicates the first exiting time. Then, by [35, Proposition 4.6], $\gamma_\infty[0, \tau_{B_\infty(0,i^{2M}D_i)})]$ and $\gamma_\infty[\tau'_{B_\infty(0,i^{4M}D_i)}, \infty)$ is "independent up to constant", *i.e.* there exists a universal constant $C > 0$ such that for any i and any possible paths η_1, η_2,

$$\mathbf{P}(\gamma_\infty[0, \tau_{B_\infty(0,i^{2M}D_i)}] = \eta_1, \gamma_\infty[\tau'_{B_\infty(0,i^{4M}D_i)}, \infty) = \eta_2)$$
$$\ge C\mathbf{P}(\gamma_\infty[0, \tau_{B_\infty(0,i^{2M}D_i)}] = \eta_1)\mathbf{P}(\gamma_\infty[\tau'_{B_\infty(0,i^{4M}D_i)}, \infty) = \eta_2). \quad (4.43)$$

Let $\widehat{\gamma_\infty} = \gamma_\infty[\tau_{B_\infty(0,i^{2M}D_i)}, \tau_{B_\infty(0,i^{4M}D_i)}]$. Then,

$$\mathbf{P}(G(i+1) \mid F_i)$$
$$\ge \mathbf{P}\left(A^{\varepsilon(\log(i+1))^{1/3}} \cap \left(\bigcap_{j=1}^{\varepsilon(\log(i+1))^{1/3}} V_{y_j} \cap W_{y_j} \right) \cap \left(\bigcap_{k=1}^{\varepsilon(\log(i+1))^{1/3}} \bigcap_{j=1}^{\varepsilon(\log(i+1))^{1/3}} V_{z_{j,k}} \right) \right)$$
$$- \mathbf{P}(|\widehat{\gamma_\infty}| \ge m_{i+1}^\beta \mid F_i). \quad (4.44)$$

By Corollary 4.12, we have that the first term of (4.44) is bounded below by $Ci^{-c\varepsilon^3}$.

For the second term of (4.44), we first consider the diameter of $\widehat{\gamma_\infty}$.

Let $\theta_1 = \{\theta_1(0), \cdots, \theta_1(k)\}$ be a path which satisfies

$$\theta_1(0) \in \partial_i B_\infty(0, i^{6M}D_i), \quad \theta_1(k) \in \partial_i B_\infty(0, i^{4M}D_i), \quad \theta_1(1), \cdots, \theta_1(k-1) \in B_\infty(0, i^{4M}D_i)^c,$$

and $\theta_2 = \{\theta_2(0), \cdots, \theta_2(l)\}$ be a path in $B_\infty(0, i^{2M}D_i)$ which satisfies $\theta_2(0) = 0$ and $\theta_2(l) \in \partial_i B_\infty(0, i^{2M}d_i)$. Let X be a random walk on $\mathbb{Z}^3$ started at $z \in \partial_i B(0, i^{6M}D_i)$ and conditioned not to hit θ_2. We define σ to be the first hitting time of $B_\infty(0, i^{4M}D_i)$. Then by calculation of conditional probability, we have that

$$P^z(X[0,\sigma] = \theta_1) = \frac{P^z(S[0,\sigma] = \theta_1, S[\sigma,\infty) \cap \theta_2 = \emptyset)}{P^z(S[0,\infty) \cap \theta_2 = \emptyset)}$$
$$= P^z(S[0,\sigma] = \theta_1)\frac{P^{\theta_1(k)}(R[0,\infty) \cap \theta_2 = \emptyset)}{P^z(S[0,\infty) = \emptyset)}, \quad (4.45)$$

where we applied the strong Markov property for the second equality. Since θ_2 is included in $B_\infty(0, i^{2M} D_i)$, it follows from [28, Proposition 1.5.10] that there exists some universal constant $C > 0$ such that for any θ_1 and θ_2,

$$\frac{1}{C} \leq \frac{P^z(X[0,\sigma] = \theta_1)}{P^z(S[0,\sigma] = \theta_1)} \leq C.$$

Thus, we have

$$\begin{aligned} P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) \neq \emptyset) &\leq \max_{z \in \partial_i B(0, i^{6M} D_i)} P(X[0,\infty) \cap B_\infty(0, i^{4M} D_i) \neq \emptyset) \\ &\leq C \max_{z \in \partial_i B(0, i^{6M} D_i)} P(S[0,\infty) \cap B_\infty(0, i^{4M} D_i) \neq \emptyset). \end{aligned}$$

By applying [28, Proposition 1.5.10] again, we obtain that

$$P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) \neq \emptyset) \leq C i^{-2M}. \tag{4.46}$$

On the other hand, we have that

$$\begin{aligned} &P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) = \emptyset, |\widehat{\gamma_\infty}| \geq m_{i+1}^\beta \mid F_i) \\ &\leq \frac{P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) = \emptyset, |\widehat{\gamma_\infty}| \geq m_{i+1}^\beta)}{P(|\gamma_\infty[0, \tau_{B_\infty(0, i^{2M} D_i)}]| \leq m_i^\beta)} \end{aligned}$$

By applying Lemma 4.8, the denominator is bounded below by

$$P(|\gamma_\infty[0, \tau_{B_\infty(0, i^{2M} D_i)}]| \leq m_i^\beta) \geq P(A^{\varepsilon (\log i)^{1/3}}) \geq i^{-c\varepsilon^3}.$$

For the numerator, now we stop conditioning on $\mathcal{U}_i$ and consider $\widehat{\gamma_\infty}$ as a subset of the infinite LERW started at the origin. Thus, again by [39, Theorem 1.4] and [32, Corollary 1.3], we have that

$$\begin{aligned} P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) = \emptyset, |\widehat{\gamma_\infty}| \geq m_{i+1}^\beta) &\leq P(|\gamma_\infty[0, \tau_{B_\infty(0, i^{6M} D_i)}]| \geq m_{i+1}^\beta) \\ &\leq C e^{-2i}. \end{aligned}$$

It follows from the above inequalities that

$$P(\widehat{\gamma_\infty} \cap B_\infty(0, i^{6M} D_i) = \emptyset, |\widehat{\gamma_\infty}| \geq m_{i+1}^\beta \mid F_i) \leq C i^{c\varepsilon^3} e^{-2i}. \tag{4.47}$$

Substituting (4.46) and (4.47) into (4.44) yields

$$P(G(i+1) \mid F_i) \geq C i^{-c\varepsilon^3}, \tag{4.48}$$

from which it follows that

$$\begin{aligned} \mathbf{P}(G(i+1) \mid \mathcal{F}_i) &\geq \mathbf{P}(G(i+1) \mid \mathcal{F}_i) \mathbf{1}_{F(i)} \\ &\geq \exp\{-c(D_{i+1}/m_{i+1})^3 \log(D_{i+1}/m_{i+1})\} - C i^{2M} \\ &\geq i^{-c\varepsilon^3} \end{aligned}$$

for large i. Since $G(i)$ is $\mathcal{F}_i$-measureable, it follows from the conditional Borel-Cantelli lemma that $G(i)$ occurs infinetely often, almost-surely. Note that on $G(i)$ we have that

$$|B_{\mathcal{U}}(0, C' D_i m_i^{\beta-1})| \geq c' D_i^3.$$

Finally, the reparameterization $r_i = C' D_i m_i^{\beta - 1}$ yields the result. □

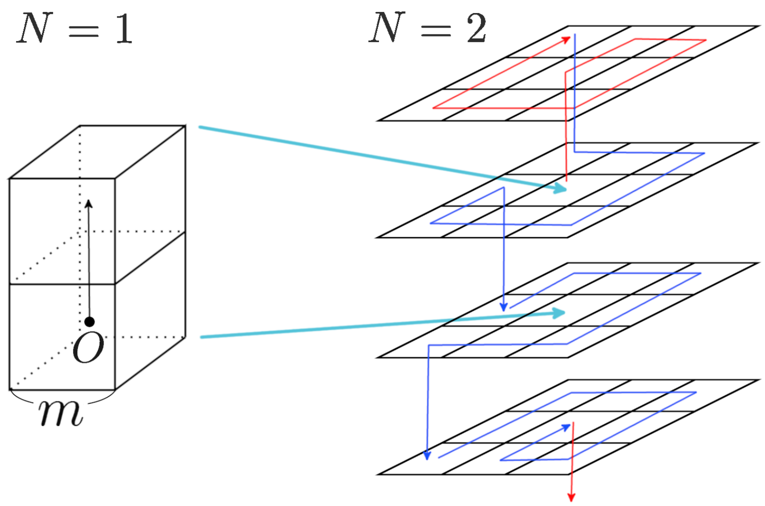


Figure 13: An example of three-dimensional spiral

4.2 Lower volume fluctuations and resistance estimate

In this section, we prove bounds for the volume and effective resistance which are key ingredients of the proof of upper fluctuations for the heat kernel.

4.2.1 Lower volume fluctuations

In this subsection, we prove (1.8), lower volume fluctuations of log-logarithmic magnitude in Theorem 4.22. Recall that β is the growth exponent of the three-dimensional LERW and $B_{\mathcal{U}}$ indicates intrinsic balls in the three-dimensional UST $\mathcal{U}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Proposition 4.14. *There exist $c_6, c_7 > 0$ such that for all $\lambda > 0$ and $r \geq 1$,*

$$\mathbf{P}(|B_{\mathcal{U}}(0,r)| \leq \lambda^{-1} r^{3/\beta}) \geq c_6 \exp\{-c_7 \lambda^{\beta/(3-\beta)} \log \lambda\}. \tag{4.49}$$

Remark 4.15. See [1, Theorem 5.1] for an exponential upper bound for the probability in the left-hand side of (4.49).

Here we follow the idea of [5], Section 3 and 4 again. Recall that γ_∞ is the infinite LERW started at the origin as the first step of Wilson's algorithm. For the result at (4.49), we will take a similar approach to the previous section, in which we construct the UST $\mathcal{U}$ by Wilson's algorithm in a collection of small boxes. We consider a rectangular prism of side-length $(2N-1)m$, $(2N-1)m$, $2Nm$ as a collection of $2N(2N-1)^2$ small boxes of side-length m. We let the origin be located at the center of one of the two boxes closest to the center of the large rectangular prism. Let π be a sequence of boxes that starts at the one containing the origin and spirals outwards.

Now we describe an example of how to construct such a spiral inductively. In the case of $N=1$, we start at the box containing the origin and then move to the other one. Without loss of generality, we can let the latter cube be centered at $(0,0,m)$ and π go upwards. In the case of $N=2$ (also see Figure 13), we first continue to move upwards to the box centered at $(0,0,2m)$ and spiral outwards in the upper face of the cube. Then we spiral down along the side of the cube. Finally, we spiral inwards the lower face of the cube and end up with the box centered at $(0,0,-m)$. Suppose that we have constructed the spiral up to the N-th step. If N is odd, the last step ended up with the box centered at $(0,0,Nm)$, therefore we move to the box centered at $(0,0,(N+1)m)$ and continue in the same procedure as the previous case $(N=2)$. If N is even, the last step ended up with the box centered at $(0,0,-(N-1)m)$, therefore we first move to the box centered at $(0,0,-Nm)$ and spiral outwards the lower face of the cube, spiral up along the side of the cube and then spiral inwards in the upper face of the cube.

Remark 4.16. Note that the following argument can be applied to any spiral π which step by step goes outwards without getting close to the origin.

To prove Proposition 4.14, we consider several events similar to those which we considered in the previous section. Let $\{x_j\}_{j=1}^{2N(2N-1)^2}$ be a sequence of the center of the boxes in the π, *i.e.* $\pi = \{B_{x_j}\}_{j=1}^{2N(2N-1)^2}$, where $B_{x_j} = B_\infty(x_j, m/2)$. Now we define some events for the SRW S started at the origin. By linear interpolation, we may assume that $S[0,\infty)$ is a continuous curve in $\mathbb{R}^3$.

Definition 4.17. *For $j = 1, 2, \cdots, 2N(2N-1)^2$, let Q_j be the face of B_{x_j} which is the closest to x_{j+1} and let $w_j \in \mathbb{R}^3$ be the center of Q_j. Then we define $\widetilde{Q}_j$ by*

$$\widetilde{Q}_j = Q_j \cap B_\infty(w_j, m/4),$$

a subset of the face Q_j which is not too close to its edges. For $a > 0, b \geq 0$, we define

$$\begin{aligned} Q_j[-a,-b] &= \{y \in B_{x_j} : b \leq d_E(y, Q_j) \leq a\}, \\ Q_j(-a) &= \{y \in B_{x_j} : d_E(y, Q_j) = a\}, \\ R_j &= Q_j(-m/2) \cap (B(x_j, m/8)^c \cup B(x_j, m/10)). \end{aligned}$$

For a continuous surve in $\mathbb{R}^3$ and $a \in \mathbb{R}$, we define

$$t_\lambda(Q_j(a)) = \inf\{k \geq 0 : \lambda(k) \in Q_j(a)\}.$$

The sets $\widetilde{Q}_j$, $Q_j[-a,-b]$, $Q_j(-a)$ and R_j we defined above correspond to the idea of $\widetilde{Q}(a_{j+1})$, $Q[a_{j+1}-a, a_{j+1}-b]$, $Q(a_{j+1}-a)$ and R_j, which we defined in Definition 4.3 in Subection 4.1.1, respectively.

Now we define some events for the SRW S. Let $q = m/N$ and

$$\begin{aligned} A'_1 &= \{t_S(Q_1(0)) < \infty, S\left(t_S(Q_1(0))\right) \in \widetilde{Q}_1, S[t_S(Q_1(-q)), t_S(Q_1(0))] \cap Q_1(-2q) = \emptyset\}, \\ A'_j &= \{t_S(Q_{j-1}(0)) < t_S(Q_j(0)) < \infty, S\left(t_S(Q_j(0))\right) \in \widetilde{Q}_j, \\ &\qquad S[t_S(Q_{j-1}(0)), t_S(Q_j(0))] \subset Q_{j-1}[-q, 0] \subset B_{x_j} \setminus R_j, \\ &\qquad S\left[t_S(Q_j(-q)), t_S(Q_j(0))\right] \subset Q_j[-2q, 0]\}, \qquad \text{for } j \geq 2. \end{aligned} \tag{4.50}$$

Note that the event A_1 (resp. A_j, $j \geq 2$) is measurable with respect to $S[0, t_S(Q_j(0))]$ (resp. $S[t_S(Q_{j-1}(0)), t_S(Q_j(0))]$).

We set

$$G'_j = \bigcap_{k=1}^{j} A'_k.$$

Now we define a cut time for S.

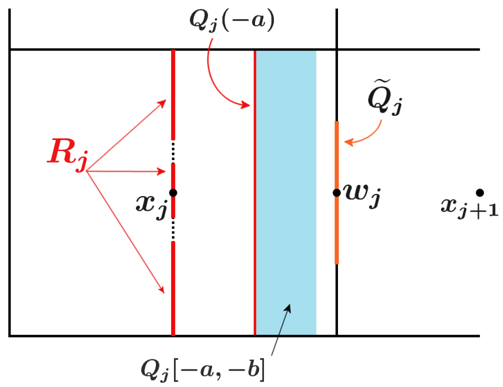


Figure 14: The sets defined in Definition 4.17

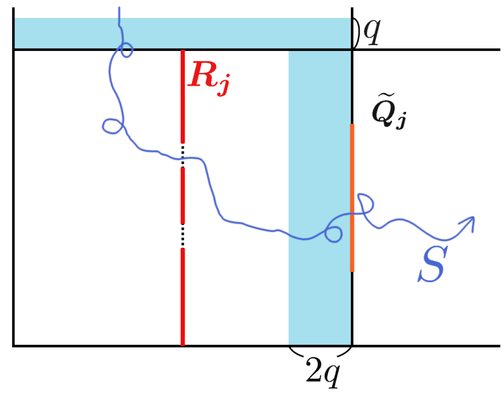


Figure 15: Definition of the event A'_j

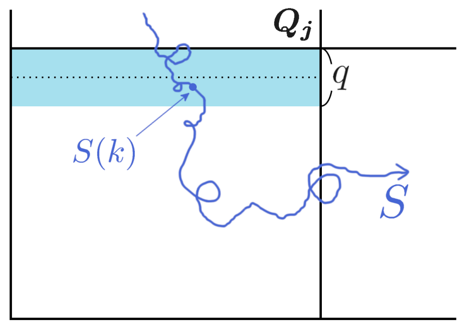


Figure 16: Example of nice cut point in B_{x_j}

Definition 4.18. *Suppose that the event A'_j defined in (4.50) occurs. For each $j \geq 2$, we call k is a nice cut time in B_{x_j} if it satisfies the four conditions in Definition 4.4 with $t_S(a_j + b)$ replaced by $t_S(Q_j(b))$ for $b \in \mathbb{R}$.*

If k is a nice cut time in B_{x_j}, then we call $S(k)$ a nice cut point in B_{x_j}.

We define events B'_j by

$$B'_j = \{S \text{ has a nice cut point in } B_{x_j}\}, \tag{4.51}$$

for each $j \geq 2$. Note that event B'_j is measurable with respect to $S[t_S(Q_{j-1}(0)), t_S(Q_j(0))]$. We define

$$H'_j = \bigcap_{k=2}^{j} B'_k.$$

The events A'_j and B'_j correspond to the idea of the events A_j and B_j, which we defined in Section 4.1.1 to estimate an upper fluctuation of the volume of the three-dimensional UST.

Now we consider the length and the hittability of the loop erasure of S. Suppose that the event $G'_j \cap H'_j$ occurs and let k_j be a nice cut time in B_{x_j}. We set

$$\xi''_j = \mathrm{LE}(S[0, t_S(Q_j(0))]) \ \text{ for } j \geq 1,$$
$$\lambda'_j = \mathrm{LE}(S[k_j, t_S(Q_j(0))]) \ \text{ for } j \geq 2,$$

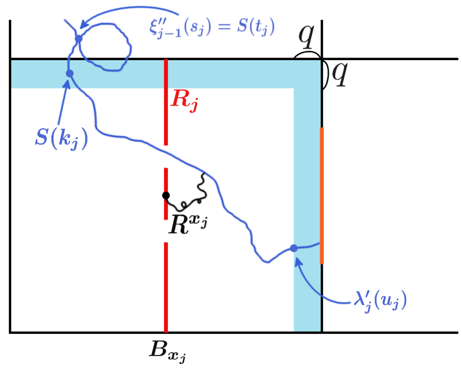


Figure 17: The sets and the points we consider in the events E'_j and F'_j

and let

$$\begin{aligned} s_j &= \inf\{k \geq 0 : \xi''_{j-1} \in S[t_S(Q_{j-1}(0)), t_S(Q_j(0))]\}, \\ t_j &= \sup\{t_S(Q_{j-1}(0)) \leq k \leq t_S(Q_j(0)) : S(k) = \xi''_{j-1}(s_j)\}, \\ u_j &= \inf\{k \geq 0 : \lambda'_j(k) \in Q_j[-q, 0]\}, \end{aligned}$$

for $j \geq 2$. Then we have

$$\xi''_j = \xi''_{j-1}[0, s_j] \oplus \mathrm{LE}(S[t_j, k_j]) \oplus \mathrm{LE}(S[k_j, t_S(Q_j(0))]) \supset \xi''_{j-1}[0, s_j] \cup \lambda'_j,$$

and therefore,

$$\xi_j[0, s_{j+1}] \supset \xi''_{j-1}[0, s_j] \cup \lambda'_j[0, u_j],$$

on the event $G'_j \cap H'_j$. Thus, in order to bound the length of η_j from below, we need to estimate the length of $\lambda'_j[0, u_j]$. For $j \geq 2$ and $C > 0$, we define the event $E'_j(C)$ by

$$E'_j = E'_j(C) = \{\mathrm{len}(\lambda'_j[0, u_j]) \geq C m^{\beta}\}. \tag{4.52}$$

Moreover, we define the event $F'_j(\eta)$ for the hittability of λ'_j for $j \geq 2$ by

$$F'_j = F'_j(\eta) = \{P^{x_j}(\lambda'_j \cap (R^{x_j}[0, T_{R^{x_j}}(2m/5)]) \neq \emptyset) \geq \eta\}, \tag{4.53}$$

where P^z indicates the law of R^z, the simple random walk started at z independent of S.

The next lemma is an analog of Lemma 4.6, which gives a lower bound on the probability that the events as defined above occur simultaneously.

Lemma 4.19. *Let P be the law of S. There exist universal constants $0 < \eta^*, c^* < \infty$ such that*

$$P(A'_1) \geq c^*$$

and for all $j \geq 2$,

$$\min_{x \in \widetilde{Q}_j} P^x(A'_j \cap B'_j \cap E'_j(c^*) \cap F'_j(\eta^*)) \geq c^* N^{-2}. \tag{4.54}$$

Proof. Similarly to the proof of Lemma 4.6, it suffices to show that for a fixed constant c, there exists a universal constant c' such that $P^x(E_2'(c') \mid A_2') \geq c$ holds uniformly in $x \in \widetilde{Q}_1$. We consider a small box $B_\infty(x_2, m/6)$ of side-length $m/3$, which is included in B_{x_2}. Let I be the number of points lying in both λ_2' and $B_\infty(x_2, m/6)$. Then by [39, Lemma 8.9], there exists a universal constant c' such that

$$P^x(I \geq c'm^\beta \mid A_2') \geq c.$$

Since $|\mathrm{len}(\lambda_2'[0, u_2])| \geq I$, we have that

$$P^x(|\lambda_2'[0, u_2]| \geq c'm^\beta \mid A_2') \geq c,$$

uniformly in $x \in \widetilde{Q}_1$, which completes the proof. □

For $j = 1, 2, \cdots, L(l_N)$, let $M_j'(\lambda)$ be an event defined by

$$M_j'(\kappa) = \{B(x_j, \kappa^{-2}m) \subset B_{\mathcal{U}^N}(x_j, \kappa^{-1}m^\beta\}.$$

Let c_* be a constant which satisfies (4.54). By the same argument as Lemma 4.7, there exists some constant $\kappa_* > 0$ depending only on c_* such that

$$\mathbf{P}\left(M_j'(\kappa_*)) \mid A_j' \cap B_j' \cap E_j'(c_*) \cap F_j'(\eta_*)\right) \geq c_*. \tag{4.55}$$

We set $I_j' = \bigcap_{k=2}^j I_k'(\kappa_*)$ for κ_* defined in (4.55).

Let $J_j' = \bigcap_{k=2}^j E_k'(c^*)$ and $K_j' = \bigcap_{k=2}^j F_k'(\eta^*)$ for c^* and η^* defined in Lemma 4.19. We define

$$\begin{aligned} U_N' = \Bigg\{ & S(T_S(R(N))) \in \{(y^1, y^2, y^3) \in \mathbb{R}^3 : y^1 \geq \frac{4}{5}R(N)\}, \\ & S[t_S(Q_{4N(4N-1)^2}), T_S(R(N))] \cap B\left(0, 3\left(N - \frac{1}{2}\right)\right) = \emptyset \Bigg\}, \end{aligned} \tag{4.56}$$

where $R(N) = \exp\{2e^{RN^2} + 1\}$ and set

$$B^N = G'_{2N(2N-1)^2} \cap H'_{2N(2N-1)^2} \cap I'_{2N(2N-1)^2} \cap J'_{2N(2N-1)^2} \cap K'_{2N(2N-1)^2} \cap U_N'. \tag{4.57}$$

Then by the same argument as Lemma 4.8, we obtain the following lemma.

Lemma 4.20. *There exists a universal constant $c_8 > 0$ such that*

$$P(B^N) \geq c_8^{-1} \exp\{-c_8 N^3 (\log N)\}. \tag{4.58}$$

On the event B^N, the first LERW γ_∞ in Wilson's algorithm moves through the spiral π and its length up to the j-th box is bounded below by Cjm^β.

Lemma 4.21. *There exist universal constants c, c', C such that for all $m \geq 1$,*

$$\mathbf{P}\left(B_{\mathcal{U}}\left(0, cN^3 m^\beta\right) \subset B_\infty\left(0, \frac{2}{3}Nm\right)\right) \geq C \exp\{-c'N^3(\log N)\}. \tag{4.59}$$

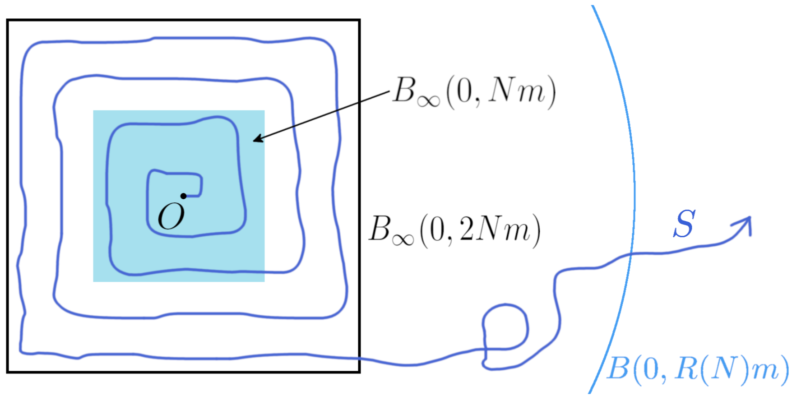


Figure 18: Definition of the event U'_N

Once we prove the above lemma, we will obtain Proposition 4.14 as follows.

Proof of Propositon 4.14. It follows from (4.59) that

$$\mathbf{P}\left(\left|B_{\mathcal{U}}\left(0, cN^3 m^\beta\right)\right| \leq \frac{8}{27} N^3 m^3\right) \geq C \exp\{-c' N^3 (\log N)\}.$$

Reparameterizing $r = cN^3 m^\beta$ and $\lambda = \frac{27}{8} c^{3/\beta} N^{9/\beta - 3}$ yields the result at (4.49). $\square$

Proof of Lemma 4.21. We may assume that m is sufficiently large for the same reason as [1, Proposition 4.1].

We first take a sequence of subsets of $\mathbb{Z}^3$ including the boundary of $B(0, (2N-1)m/3)$. For each $k \geq 1$, let $\varepsilon_k = N^{-4/3} 2^{-k-10}$, $\eta_k = (15k)^{-1}$ and

$$A_k = B_\infty\left(0, \left(\frac{2}{3} + \eta_k\right)(2N-1)\frac{m}{2}\right) \setminus B_\infty\left(0, \left(\frac{2}{3} - \eta_k\right)(2N-1)\frac{m}{2}\right).$$

We let $D_k \subset \mathbb{Z}^3$ be a subset of A_k such that $A_k \subset \bigcup_{z \in D_k} B(z, (2N-1)m\varepsilon_k)$ and we suppose that $|D_k| \leq C\varepsilon_k^{-3}$. Write k_0 for the smallest integer satisfying $(2N-1)m\varepsilon_{k_0} < 1$. Note that for sufficiently large m, both the inner and outer boundary of $B_\infty(0, \frac{1}{3}(2N-1)m)$ are contained in A_{k_0}. Moreover, we have $\partial_i B_\infty(0, \frac{1}{3}(2N-1)m) \subset D_{k_0}$ by the definition of k_0.

We begin with performing Wilson's algorithm in $\bigcup_{j=1}^{2N(2N-1)^2} B_{x_j}$. Let $\mathcal{U}_0^N$ be the subtree of the UST constructed in the event B^N. Then we follow the same construction of the sequence of subtrees $\mathcal{U}_1^N, \mathcal{U}_2^N, \cdots, \mathcal{U}_{k_0}^N$ as (i)-(iv) in the proof of Lemma 4.7 and end up with the subtree $\mathcal{U}^N$, which contains all points in $\bigcup_{j=1}^{2N(2N-1)^2} B(x_j, m/2)$.

Next, we consider the hittability of branches in the constructed subtree. We take $z \in D_1$, then there exists $j \in [1, 2, \cdots, 2N(2N-1)^2]$ such that $z \in B_{x_j}$. Let τ_{B_j} be the first hitting time of $B(x_j, \lambda^{-2}m)$ and σ_{Q_j} be the first exiting time of $B_\infty(x_j, 3m/2)$ by R^z, a simple random walk started at z and independent of S. Then by [28, Proposition 1.5.10], there exists some $p > 0$ such that for all $j \geq 1$ and $z \in B_{x_j}$

$$P^z(\sigma_{Q_j} < \tau_{B_j}) \leq p, \tag{4.60}$$

holds. We define the event $L(1,z)$ by

$$L(1,z) = \{\gamma_{\mathcal{U}^N}(z,\infty) \text{ exits the cube } B_\infty(z, Nm/100) \text{ before hitting the subtree } \mathcal{U}_0^N\},$$

for $z \in D_1$ and suppose that $L(1,z)$ occurs, where $\gamma_{\mathcal{U}^N}(y,\infty)$ stands for the unique infinite path of the subtree $\mathcal{U}^N$ started at y. Then the event $\{\sigma_{Q_j} < \tau_{B_j}\}$ occurs and by (4.60) its probability is smaller than p. By iteration, the number of boxes of side-length $3m/2$ that R^z exist before hitting $\mathcal{U}_0^N$ is larger than $N/200$. Hence, by the strong Markov property, it follows that

$$\mathbf{P}(L(1,z)) \le p^{N/200},$$

for all $z \in D_1$. Since $|D_1| \le C\varepsilon_1^{-3}$, we have that

$$\mathbf{P}\left(\bigcap_{z\in D_1} L(1,z)^c\right) \ge 1 - N^4 p^{N/200}.$$

We next define the event that guarantees the hittability of branches of $\mathcal{U}_k^N$ starting at $z \in D_k$. For $k \ge 1$ and $x \in D_k$, let

$$I'(k,x,\zeta) = \left\{ P_R^y\left(R\left[0, T_{R^y}(y,(2N-1)m\varepsilon_k^{1/2})\right] \cap (\mathcal{U}_0^N \cup \gamma_{\mathcal{U}^N}(x,\mathcal{U}_0^N))\right) \le \varepsilon_k^\zeta \text{ for all } y \in B(x,(2N-1)m\varepsilon_k)\right\}. \tag{4.61}$$

and $I'(k,\zeta) = \bigcap_{x\in D_k} I'(k,x,\zeta)$. Again by [37, Lemma 3.2], there exist universal constants $\zeta_3 > 0$ and $C < \infty$ such that for all $k \ge 1$, $m \ge 1$, and $x \in D_k$,

$$\mathbf{P}_{\mathcal{U}^N}(I'(k,x,\zeta_3)^c) \le C\varepsilon_k^5.$$

Combining this with $|D_k| \le C\varepsilon_k^{-3}$ yields that

$$\mathbf{P}_{\mathcal{U}^N}(I'(k,\zeta_3)^c) \le C\varepsilon_k^2.$$

We finally define an event $L(k,z)$ for $k \ge 2$ and $z \in D_k$ by

$$L(k,z) = \{\gamma_{\mathcal{U}^N}(z,\infty) \text{ exits } B(z,(2N-1)m\varepsilon_{k-1}^{1/3}) \text{ before hitting } \mathcal{U}_{k-1}^N\},$$

and set $M_1 := B^N \cap (\bigcap_{z\in D_1} L(1,z)^c) \cap I'(1,\zeta_3)$ and $M_k := M_{k-1} \cap (\bigcap_{z\in D_1} L(1,z)^c) \cap I'(k,\zeta_3)$ inductively for $k \ge 2$. Suppose that the event M_{k-1} occurs. The number of balls of radius $(2N-1)m\varepsilon_{k-1}^{1/2}$ that R^z exits before hitting $\mathcal{U}_{k-1}^N$ is larger than $\varepsilon_{k-1}^{-1/6}$. By the strong Markov property, it holds that

$$P^z(R^z \text{ exits } B(z,(2N-1)m\varepsilon_1^{1/3}) \text{ before it hits } \mathcal{U}_{k-1}^N) \le \varepsilon_{k-1}^{c\zeta_3\varepsilon_{k-1}^{-1/6}},$$

for some universal constant $c > 0$. Since $|D_k| \le C\varepsilon_k^{-3}$, we have that

$$\mathbf{P}\left(\bigcap_{z\in D_k} L(k,z)^c \,\middle|\, M_{k-1}\right) \ge 1 - C\varepsilon_k^{-3}\varepsilon_{k-1}^{c\zeta_3\varepsilon_{k-1}^{1/6}}, \tag{4.62}$$

It follows from the argument above that

$$\mathbf{P}(M_1 \mid B^N) \geq 1 - N^{-8/3},$$

and

$$\mathbf{P}(M_k \mid M_{k-1}) \geq 1 - C\varepsilon_k^2.$$

Hence we can conclude that

$$\mathbf{P}(M_{k_0} \mid B^N) = \mathbf{P}(M_1 \mid B^N) \prod_{k=2}^{k_0} \mathbf{P}(M_k \mid M_{k-1}) \geq (1 - N^{-8/3}) \prod_{k=1}^{\infty} (1 - C\varepsilon_k^2) \geq 1 - CN^{-8/3}.$$

On the event M_{k_0}, for all $y \in \mathcal{U}^N$,

$$d_E(y, D_1) \leq \frac{Nm}{100} + \sum_{k=2}^{k_0} (2N-1)m\varepsilon_k^{1/6} \leq \frac{Nm}{100} + (2N-1)m \sum_{k=2}^{k_0} \varepsilon_k^{1/6} \leq \frac{Nm}{50},$$

i.e. $\gamma_{\mathcal{U}}(y, 0)$ hits the subtree $\mathcal{U}_0^N$ before it exits the ball centered at y of radius $Nm/50$. Hence we have

$$d_E(0, y) \geq d_E(0, D_1) - \frac{Nm}{50} \geq \frac{11}{20}(2N-1)m.$$

Since $\partial_i B_\infty(0, \frac{1}{3}(2N-1)m) \subset D_{k_0}$, $\gamma_{\mathcal{U}}(z, 0)$ hits $\mathcal{U}_{k_0}^N$ before entering $B_\infty(0, \frac{11}{20}(2N-1)m)$ for all $z \in B_\infty(x, \frac{1}{3}(2N-1)m)$. By the definition of the spiral π, it follows that on the event M_{k_0},

$$z \in B_\infty\left(0, \frac{1}{3}(2N-1)m\right)^c \Longrightarrow d_{\mathcal{U}}(0, z) \geq CN\left(N - \frac{1}{2}\right)^2 m^\beta$$

for some universal constant c, *i.e.* $B_{\mathcal{U}}(0, cN(N-\frac{1}{2})^2 m^\beta) \subset B_\infty(0, \frac{1}{3}(2N-1)m)$. By taking N sufficiently large, we obtain (4.59) for some universal constant c'. □

Theorem 4.22. $\mathbf{P}$-*a.s.*,

$$\liminf_{r\to\infty} (\log\log r)^{(3-\beta)/\beta} r^{-3/\beta} |B_{\mathcal{U}}(0, r)| = 0. \tag{4.63}$$

Proof. Similarly to the proof of upper volume fluctuation in Theorem 4.13, we begin with defining a sequence of scales by

$$D_i = e^{i^2}, \ m_i = D_i/\varepsilon(\log i)^{1/3}.$$

Let γ_∞ be the infinite LERW started at the origin and $(S^z)_{z\in\mathbb{Z}^3}$ be the family of independent SRW which is also independent of γ_∞. Then by the same argument as obtaining (4.41) and (4.42),

$$\mathcal{U}_i \subset B_\infty(0, i^{2M} D_i) \subset B_\infty(0, m_{i+1}/2), \tag{4.64}$$

$$d_{\mathcal{U}}(0, z) \leq i^M D_i^\beta \leq m_{i+1}^\beta \quad \text{for all} \quad z \in \mathcal{U}_i, \tag{4.65}$$

holds for large i, almost-surely.

We define an event $H(i)$ by

$$H(i) = \bigcap_{j=1}^{i} B^{\varepsilon(\log j)^{1/3}},$$

where the event B^N is as defined in (4.57) and at each stage i we rescale by $m = m_i$ and $N = \varepsilon(\log i)^{1/3}$. Then by (4.43), *i.e.* the "independence up to constant" of γ_∞, there exists a universal constant $C > 0$ such that for any i,

$$\begin{aligned}\mathbf{P}(H(i+1) \mid \mathcal{F}_i) &\geq \mathbf{P}(H(i+1) \mid \mathcal{F}_i)\mathbf{1}_{H(i)} \\ &\geq C\mathbf{P}\left(B^{\varepsilon(\log(i+1))^{1/3}}\right) \\ &\geq Ci^{-c\varepsilon^3},\end{aligned}$$

where the last inequality follows from (4.58). Note that on $H(i)$ we have that

$$|B_{\mathcal{U}}(0, cD_i^3 m_i^{\beta-3})| \leq \frac{8}{27} D_i^3.$$

Finally, the reparameterization $r_i = cD_i^3 m_i^{\beta-3}$ yields the result. □

4.2.2 Estimates for effective resistance

In order to demonstrate upper heat kernel fluctuations, we need to estimate the effective resistance of balls in the three-dimensional uniform spanning tree. See (2.5) for the definition of effective resistance.

Suppose that the event B^N defined in (4.58) occurs. We give estimates for bounds of the volume and a lower bound of the effective resistance of UST on the event B^N.

Proposition 4.23. *There exists some universal constant $0 < c_9, c_{10}, c, C < \infty$ and the event $\widetilde{M}^N$ with $\mathbf{P}(\widetilde{M}^N) \geq c_9 \exp\{-c_{10} N^3 \log N\}$ such that on $\widetilde{M}^N$, the followings hold :*

$$\begin{aligned}&|B_{\mathcal{U}}(0, N^3 m^\beta)| \leq \frac{1}{4} N^3 m^3, \\ &|B_{\mathcal{U}}(0, CN^3 m^\beta/(\log N)^{300})| \geq cN^3 m^3/(\log N)^{300}, \\ &R_{\text{eff}}(x, B_{\mathcal{U}}(0, N^3 m^\beta)) \geq cN^3 m^\beta/(\log N)^{300} \text{ for all } x \in B_{\mathcal{U}}(0, cN^3 m^\beta/(\log N)^{300}).\end{aligned} \tag{4.66}$$

We will apply [24, Proposition 3.2] to bound the heat kernel on $\mathcal{U}$ from above. To do so, we need to estimate (i) upper bound of the volume of an intrinsic ball $B_{\mathcal{U}}(0, N^3 m^\beta)$, (ii) lower bound of the volume of a smaller ball $B_{\mathcal{U}}(0, \varepsilon N^3 m^\beta)$ where ε is some small constant and (iii) lower bound of the effective resistance between $x \in B_{\mathcal{U}}(0, \varepsilon N^3 m^\beta)$ and the boundary of the ball $B_{\mathcal{U}}(0, N^3 m^\beta)$, which correspond to the three inequalities above.

Proof. We set $l_N = N/(\log N)^{100}$ and $L(k) = 2k(2k-1)^2$. Let S be the SRW started at the origin and recall that events A'_j, B'_j, E'_j, F'_j and U'_N are as defined in (4.50), (4.51), (4.52),

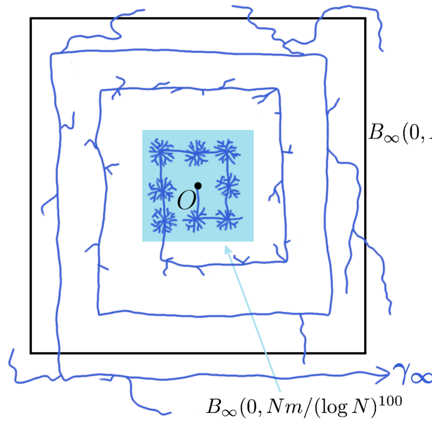


Figure 19: The unlike event which we consider in Proposition 4.23. To bound the effective resistance from below, we need to consider two properties: (i) paths started at a point in $B_{\mathcal{U}}(0, N^3 m^\beta)^c$ do not enter a smaller ball $B_{\mathcal{U}}(0, N^3 m^\beta/(\log N)^{300})$ and (ii) paths branching from γ_∞ at a point close to the origin have limited length.

(4.53) and (4.56), respectively. We define events $B^N_{(1)}$ and $B^N_{(2)}$ by

$$B^N_{(1)} = \bigcap_{j=1}^{L(l_N)} \left(A'_j \cap B'_j \cap E'_j \cap F'_j \cap \{B(x_j, \lambda^{-2} m) \subset B_{\mathcal{U}}(x_j, \lambda^{-1} m^\beta)\} \right),$$

$$B^N_{(2)} = \left\{ \bigcap_{j=L(i_N)+1}^{L(N)} \left(A'_j \cap B'_j \cap E'_j \cap F'_j \cap \{B(x_j, \lambda^{-2} m) \subset B_{\mathcal{U}}(x_j, \lambda^{-1} m^\beta)\} \right) \right\} \cap U'_N.$$

It follows from the same argument as Lemma 4.21 that

$$\mathbf{P}(B^N_{(1)}) \geq C \exp\{-cN^3 (\log N)^{-299}\}, \tag{4.67}$$

$$\mathbf{P}(B^N_{(2)}) \geq C \exp\{-cN^3 (\log N)\}, \tag{4.68}$$

for some constants $C, c > 0$. Note that $(\log N)^{-299}$ in the lower bound of $\mathbf{P}(B^N_{(1)})$ is the result of the number of small boxes l_N. We need this term to avoid the competition between $\mathbf{P}(B^N_{(1)})$ and a bound for the probability that UST paths branching from γ_∞ near the origin do not have a large length.

Suppose that the event $B^N_{(1)} \cap B^N_{(2)}$ occurs. Then the occurence of $\bigcap_{j=1}^{L(N)} A'_j$ guarantees that the part of S after exiting the $L(l_N)$-th box $B_{x_{L(l_N)}}$ does not go back into $\bigcap_{j=1}^{L(l_N)-1} B_{x_j}$. In particular, $\mathrm{LE}(S[0, t_S(Q_{L(l_N)}(0))])$ and $\mathrm{LE}(S[0, t_S(Q_{L(N)}(0))])$ restricted to $\bigcap_{j=1}^{L(l_N/2)} B_{x_j}$ are exactly the same on $B^N_{(1)} \cap B^N_{(2)}$, where $t_\lambda(Q_j(a))$ is as defined in Definition 4.17.

For each $k \geq 1$, let $\varepsilon_k = N^{-4/3} 2^{-10-k}$, $\eta_k = (30k)^{-1}$ and

$$A_k = B\left(0, \left(\frac{1}{3} + \eta_k\right) l_N m\right) \setminus B\left(0, \left(\frac{1}{3} - \eta_k\right) l_N m\right).$$

Write k_0 for the smallest integer satisfying $l_N m \varepsilon_k < 1$. Now we take a "ε_k-net" of A_k, *i.e.* let D_k be a set of lattice points in A_k such that $\bigcup_{z \in D_k} B(z, l_N m \varepsilon_k)$ with $|D_k| \leq C\varepsilon_k^{-3}$.

Now we perform Wilson's algorithm in $\bigcup_{j=1}^{L(N)} B_{x_j}$ to obtain the subtree $\mathcal{U}_{k_0}^{L(N)}$ in the same procedure as we did in Lemma 4.21. Note that $\mathcal{U}_0^{L(N)}$ is the union of the infinite LERW γ_∞ started at the origin and balls $B_{\mathcal{U}}(x_j, \lambda^{-1} m^\beta)$ constructed in the event $I'_{2N(2N-1)^2}$. For $z \in \mathcal{U}_{k_0}^{L(N)}$, let $\gamma_{\mathcal{U}^{L(N)}}(x, \infty)$ be the unique infinite path in $\mathcal{U}_{k_0}^{L(N)}$ starting at z. We set $C_p = 5 \log N / \log(1/p)$ where p be as defined in (4.60). For $z \in D_1$, we define the event $\widetilde{L}(1,z)$ by

$$\widetilde{L}(1,z) = \left\{ \mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \not\subset B\left(z, C_p m\right) \right\} \cup \left\{ |\gamma_{\mathcal{U}^{L(N)}}(z, \mathcal{U}_0^{L(N)})| \geq \frac{N^3}{(\log N)^{10}} m^\beta \right\}.$$

Then

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{L}(1,z) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right) \\
&\quad \leq \mathbf{P}\left(|\mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})])| \geq \frac{N^3}{(\log N)^{10}} m^\beta,\ \mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \subset B\left(z, C_p m\right) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right) \\
&\quad + \mathbf{P}\left(\mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \not\subset B\left(z, C_p m\right) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right). \qquad (4.69)
\end{aligned}$$

Since both of the events we consider in the right-hand side of (4.69) are independent of $B_{(2)}^N$, we can omit the condition on $B_{(2)}^N$ from the conditional probabilities on the right-hand side. For the first term, we stop conditioning on $B_{(1)}^N$ and consider $\gamma_{\mathcal{U}^{L(N)}}(z, \infty)$ as the infinite LERW started at z. Then it follows from [39, Theorem 1.4] and [32, Corollary 1.3] that for some universal constants $C > 0$ and $c > 0$,

$$\begin{aligned}
&P^z\left(|\mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})])| \geq \frac{N^3}{(\log N)^{10}} m^\beta,\ \mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \subset B\left(z, C_p m\right) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right) \\
&\quad \leq \frac{P^z\left(|\mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})])| \geq N^3 m^\beta / (\log N)^{10},\ \mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \subset B\left(z, C_p m\right) \right)}{\mathbf{P}\left(B_{(1)}^N \right)} \\
&\quad \leq C \exp\{-c N^3 (\log N)^{-10-\beta}\},
\end{aligned}$$

where the last inequality follows from (4.67). On the other hand, by the independence, the strong Markov property and the hittability of $\mathcal{U}_0^{L(N)}$,

$$\begin{aligned}
&P^z\left(\mathrm{LE}(S^z[0, \tau(\mathcal{U}_0^{L(N)})]) \not\subset B\left(z, C_p m\right) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right) \\
&\quad \leq p^{-C_p} = N^{-5}.
\end{aligned}$$

Thus, we obtain that

$$\mathbf{P}\left(\widetilde{L}(1,z) \;\middle|\; B_{(1)}^N \cap B_{(2)}^N \right) \leq N^{-5}. \qquad (4.70)$$

Next, we consider events that guarantee the hittability of each branch in $\mathcal{U}_k^{L(N)}$. For $\zeta > 0$, we define an event $\widetilde{I}(k, x, \zeta)$ by

$$\begin{aligned}
&\widetilde{I}(k, x, \zeta) \\
&\quad = \Big\{ P_R^y\left(R\left[0, T_{R^y}(y, l_N m \varepsilon_k^{1/2})\right] \cap (\mathcal{U}_0^{L(N)} \cup \gamma_{\mathcal{U}^{L(N)}}(x, \mathcal{U}_0^{L(N)})) = \emptyset \right) \leq \varepsilon_k^\zeta \\
&\qquad \text{for all } y \in B(x, l_N m \varepsilon_k) \Big\}. \qquad (4.71)
\end{aligned}$$

and let $\widetilde{I}(k,\zeta) = \bigcap_{x\in D_k} \widetilde{I}(k,x,\zeta)$. Then by [37, Lemma 3.2], there exist universal constants $\zeta_4 > 0$ and $C < \infty$ such that for all $k \geq 1$, $m \geq 1$, and $x \in D_k$,

$$\mathbf{P}(\widetilde{I}(k,x,\zeta_4)) \geq 1 - C\varepsilon_k^5,$$

which combined with $|D_k| \leq C\varepsilon_k^{-3}$ yields

$$\mathbf{P}(\widetilde{I}(k,\zeta_4)) \geq 1 - C\varepsilon_k^2. \tag{4.72}$$

Finally, let

$$\begin{aligned}\widetilde{L}(k,z) = &\left\{\gamma_{\mathcal{U}^{L(N)}}(z,\infty) \text{ exits } B\left(z, l_N m \varepsilon_{k-1}^{1/3}\right) \text{ before hitting } \mathcal{U}_{k-1}^{L(N)}\right\} \\ &\cup \left\{\left|\gamma_{\mathcal{U}^{L(N)}}(z,\mathcal{U}_{k-1}^{L(N)})\right| \geq \varepsilon_k^{1/5} \frac{N^3}{(\log N)^{10}} m^\beta\right\},\end{aligned} \tag{4.73}$$

be the event for the length of the attached branch for each $k \geq 2$ and $z \in D_k$. We set $\widetilde{M}_1 = B_{(1)}^N \cap B_{(2)}^N \cap \widetilde{I}(1,\zeta_4) \cap (\bigcap_{z\in D_1} \widetilde{L}(1,z)^c)$ and $\widetilde{M}_k = \widetilde{M}_{k-1} \cap \widetilde{I}(k,\zeta_4) \cap (\bigcap_{z\in D_k} \widetilde{L}(k,z)^c)$ inductively for $k = 2,3,\cdots,k_0$.

Since $|D_1| \leq CN^4$, it follows from (4.70) that

$$\mathbf{P}(\widetilde{M}_1) \geq (1 - N^{-1}) \exp\{-cN^3(\log N)\}.$$

By applying the argument for (4.62) again, we have that,

$$\begin{aligned}&\mathbf{P}\left(\gamma_{\mathcal{U}^{L(N)}}(z,\infty) \text{ exits } B\left(z, \frac{N}{(\log N)^{100}} m\varepsilon_{k-1}^{1/3}\right) \text{ before hitting } \mathcal{U}_{k-1}^{L(N)} \;\middle|\; \widetilde{M}_{k-1}\right) \\ &\leq C\varepsilon_{k-1}^{-3} \varepsilon_{k-1}^{c\zeta_4 \varepsilon_{k-1}^{-1/6}}.\end{aligned}$$

Again we stop conditioning on $\widetilde{M}_1$ and consider $\gamma_{\mathcal{U}^{L(N)}}(z,\infty)$ as the infinite LERW started at z. By [39, Theorem 1.4] and [32, Corollary 1.3],

$$\begin{aligned}&\mathbf{P}(\widetilde{L}(2,z) \mid \widetilde{M}_1) \\ &\leq \mathbf{P}\left(\left\{\gamma_{\mathcal{U}^{L(N)}}(z,\mathcal{U}_1^{L(N)}) \subset B\left(z, l_N m \varepsilon_1^{1/3}\right)\right\} \cap \left\{\left|\gamma_{\mathcal{U}^{L(N)}}(z,\mathcal{U}_1^{L(N)})\right| \geq \varepsilon_1^{1/5} \frac{N^3}{(\log N)^{10}} m^\beta\right\} \mid \widetilde{M}_1\right) \\ &\quad + C\varepsilon_1^{-3}\varepsilon_1^{c\zeta_4\varepsilon_1^{-1/6}} \\ &\leq \frac{\exp\{-\varepsilon_2^{-c\beta/3+1/4} \cdot \varepsilon_2^{-1/20} \cdot N^{3-\beta}(\log N)^{99}\}}{\mathbf{P}(\widetilde{M}_1)} + C\varepsilon_1^{-3}\varepsilon_1^{c\zeta_4\varepsilon_1^{-1/6}} \\ &\leq \exp\{-\varepsilon_2^{-1/12} N^3 (\log N)^{99}\}.\end{aligned}$$

Combining this with (4.72), we obtain

$$\mathbf{P}(\widetilde{M}_2 \mid \widetilde{M}_1) \geq 1 - C\varepsilon_2^2,$$

and by iteration, we have that $\mathbf{P}(\widetilde{M}_k \mid \widetilde{M}_{k-1}) \geq 1 - C\varepsilon_k^2$. Hence we can conclude that

$$\begin{aligned}\mathbf{P}(\widetilde{M}_{k_0}) &= \mathbf{P}(\widetilde{M}_1) \prod_{k=2}^{k_0} \mathbf{P}(\widetilde{M}_k \mid \widetilde{M}_{k-1}) \\ &\geq (1 - N^{-1}) \exp\{-cN^3 \log N\} \prod_{k=1}^{\infty} (1 - C\varepsilon_k^2) \\ &\geq C \exp\{-cN^3 \log N\},\end{aligned}$$

for some universal constant C,$c > 0$. On the event $\widetilde{M}_{k_0}$, we have that there exists some universal constant $C > 0$ such that for all $z \in \partial_i B(0, Nm/3(\log N)^{100})$, $d_{\mathcal{U}}(0,z) \leq CN^3 m^\beta/(\log N)^{10}$ holds.

Once we construct $\mathcal{U}_{k_0}^{L(N)}$, we proceed with Wilson's algorithm. This time, in the same argument as for (4.28), we have that conditioned on $\widetilde{M}_{k_0}$, with conditional probability larger than some universal constant $c > 0$, $d_{\mathcal{U}}(0,z) \leq CN^3 m^\beta/(\log N)^{10}$ holds for all z contained in $B(0, Nm/3(\log N)^{100})$.

Finally, we consider "ε_k-net" of annuli around the boundary of $B(0, Nm)$ and repeat the argument of Lemma 4.21. Then we have that there exist universal constants $0 < C, C', c, c' < \infty$ and an event $\widetilde{M}^N$ with $\mathbf{P}(\widetilde{M}^N) \geq C \exp\{-cN^3 \log N\}$, such that the following statements hold:

(1) $|B_{\mathcal{U}}(0, CN^3 m^\beta/(\log N)^{300})| \geq cN^3 m^3/(\log N)^{300}$,

(2) $\gamma_{\mathcal{U}}(z, \gamma_\infty) \not\subset B(0, c_2 l_N m)$ for all $z \in B(0, l_N m/3$,

(3) $d_{\mathcal{U}}(0,z) \leq C_2 N^3 m^\beta/(\log N)^{10}$ for all $z \in B(0, l_N m/3$,

(4) $|B_{\mathcal{U}}(0, N^3 m^\beta)| \leq \frac{1}{4} N^3 m^3$.

We bound the effective resistance $R_{\text{eff}}(0, B_{\mathcal{U}}(N^3 m^\beta))$ from below. By (2), points outside the Euclidean ball $B(0, l_N m/3)$ is connected to the spiral γ_∞ outside the smaller ball $B(0, c_2 l_N m)$. Combining this with (3), all the paths on $\mathcal{U}$ connecting the origin and $B_{\mathcal{U}}(0, C_2 N^3 m^\beta/(\log N)^{10})$ contains the part of γ_∞ inside the $B(0, c_2 l_N m)$. By the series law of effective resistance (see for example [34], Section 2.3), we have that $R_{\text{eff}}(0, B_{\mathcal{U}}(0, N^3 m^\beta)^c) \geq c_3 N^3 m^\beta/(\log N)^{300}$ for some universal constant $c_3 > 0$. Thus we have $R_{\text{eff}}(x, B_{\mathcal{U}}(0, N^3 m^\beta)^c) \geq c_3 N^3 m^\beta/2(\log N)^{300}$ for all $x \in B(0, c_3 N^3 m^\beta/2(\log N)^{300})$, which completes the proof. □

4.3 Heat kernel fluctuations

In this section, we will show Theorem 1.7, quenched heat kernel fluctuations for the three-dimensional UST. We start with lower fluctuations and then move on to upper fluctuations.

Recall that $p_n^{\mathcal{U}}$ stands for the quenched heat kernel defined in (2.4).

Theorem 4.24. $\mathbf{P}$-*a.s.,*

$$\liminf_{n\to\infty} (\log\log n)^{\frac{\beta-1}{3+\beta}} n^{\frac{3}{3+\beta}} p_{2n}^{\mathcal{U}}(0,0) = 0. \tag{4.74}$$

Proof. By [7, Theorem 4.1], we have that

$$p^{\mathcal{U}}_{2r|B_{\mathcal{U}}(0,r)|}(0,0) \le \frac{2}{|B_{\mathcal{U}}(0,r)|},$$

for any realization of $\mathcal{U}$. Let $t_n = n|B_{\mathcal{U}}(0,n)|$ and $u_n = n^{-3/\beta}|B_{\mathcal{U}}(0,n)|$, then we have

$$\begin{aligned} p^{\mathcal{U}}_{2t_n}(0,0) \le \frac{2}{|B_{\mathcal{U}}(0,0)|} &= \frac{2}{t_n^{3/(3+\beta)} n^{-3/(3+\beta)} |B(0,n)|^{\beta/(3+\beta)}} \\ &= 2t_n^{-3/(3+\beta)} u_n^{-\frac{\beta}{3+\beta}}. \end{aligned}$$

By (4.39), $u_n^{-\frac{\beta}{3+\beta}} > (\log\log n)^{\frac{\beta-1}{3+\beta}}$ infinitely often almost surely, which completes the proof. □

Remark 4.25. We proved the existence of some exponent of the log-logarithmic term which causes lower heat kernel fluctuation. However, The exponent $\dfrac{\beta-1}{3+\beta}$ of the log-logarithmic term of $p^{\mathcal{U}}_{2n}$ which appears in (4.74) is not necesarily a sharp estimate. The critical exponent of the log-logarithmic term has not been obtained even in the critical Galton-Watson tree case.

Proposition 4.26. *There exists some constant $\alpha > 0$ such that* $\mathbf{P}$*-a.s.,*

$$\limsup_{n\to\infty} (\log\log)^{-\alpha} n^{\frac{3}{3+\beta}} p^{\mathcal{U}}_{2n}(0,0) = \infty. \tag{4.75}$$

Proof. Let $\delta > 0$ and

$$N_i = (\delta \log i)^{1/3} \quad m_i = e^{i^2}/N_i,$$

where c_{10} is as in the statement of Proposition 4.23. We follow the construction of a subtree of $\mathcal{U}$ in $B_\infty(0, N_i m_i)$ given in the proof of Proposition 4.23. Let γ_∞ be the infinite LERW started at the origin. Then at stage i ($i \ge 1$), we use all vertices in $B_\infty(0, N_i m_i)$ and write $\mathcal{U}_i$ for the obtained tree. Similarly to (4.41), we have a good separation of scales. By Proposition 4.23, conditioned on $\mathcal{U}_{i-1}$, the event $\widetilde{M}^{N_i}$ occurs with conditional probability greater than i^{-1} if we take δ small enough. Now we apply [24, Proposition 3.2] to this $\mathcal{U}_i$. We set $R = N_i^3 m_i^\beta$, $\lambda = 1$ and $\varepsilon = c/4(\log N_i)^{300}$ where c is as given in (4.66). Let $r : [0,\infty] \to [0,\infty]$ be $r(x) = x$. Then by Proposition 4.23, we can set m (which appears in [24, Proposition 3.2]) by $m = c(\log N_i)^{-300}$. Plugging these into (3.6) of [24], we have

$$p^{\mathcal{U}}_{2n}(0,0) \ge c' N_i^{-3} m_i^{-3} \quad for \quad n \le \frac{c N_i^6 m_i^{3+\beta}}{32(\log N_i)^{300}},$$

for $c > 0$ in (4.66) and some constant $c' > 0$. Thus, taking $T = \frac{c}{32} N_i^6 m_i^{3+\beta} (\log N_i)^{-300}$, it holds that on the event $\widetilde{M}^{N_i}$, we have

$$T^{\frac{3}{3+\beta}} p^{\mathcal{U}}_{2T}(0,0) \ge \left(\frac{c}{64}\right)^{\frac{3}{3+\beta}} c' N_i^{\frac{9-3\beta}{3+\beta}} (\log N_i)^{-\frac{300}{3+\beta}} \ge c''(\log\log T)^{\frac{9-3\beta}{2(3+\beta)}-\varepsilon},$$

for some $c'' > 0$ and $\varepsilon > 0$ which is small. Finally, we apply the Borel-Cantelli argument. Since

$$\sum_i c_9 \exp\{-c_{10} N_i^3 (\log N_i)\} \ge \sum_i i^{-1} = \infty,$$

by the conditional Borel-Cantelli lemma, we obtain the lower bound (4.75). □

5 Annealed transition density of random walk on a loop-erased random walk

In this chapter, we will prove Theorem 1.11 and then prove Theorem 1.10. As remarked in Subsection 1.2.3, it was conjectured in [5] that a similar combination of the various exponents will appear in sub-Gaussian annealed heat kernel bounds for the random walk on the two-dimensional uniform spanning tree. In that case, the spectral dimension of the quenched and annealed bounds is known to be 16/13, the intrinsic walk dimension is 13/5 and the exponent governing the embedding is given by the growth exponent of the two-dimensional LERW, i.e. 5/4, giving an extrinsic walk dimension of 13/4. We can check the corresponding result for our simpler model using the simple observation that

$$\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) = \sum_{m \geq 0} P^{\mathcal{G}}(X^{\mathcal{G}} = L_m)\mathbf{P}(L_m = x), \tag{5.1}$$

and then combining the estimate on the distribution of the LERW from Theorem 1.11 with the deterministic Gaussian bounds on $P^{\mathcal{G}}(X^{\mathcal{G}} = L_m)$ of Lemma 5.17 below.

The remainder of this chapter is organized as follows. In Section 5.1, we study the LERW in more detail, proving Theorem 1.11. In Section 5.2, we derive our heat kernel estimates for $X^{\mathcal{G}}$, proving Theorem 1.10.

5.1 Loop-erased random walk estimates

The aim of this section is to prove Theorem 1.11. Due to the diffusive scaling of the LERW, it is convenient to reparameterize the result. In particular, we will prove the following, which clearly implies Theorem 1.11. Throughout this section, for $x \in \mathbb{Z}^d$, we write $\tau_x = \tau_x^L$ for the first time that the LERW L hits x.

Proposition 5.1. *There exist constants $c_1, c_2 \in (0, \infty)$ such that for every $x \in \mathbb{Z}^d \backslash \{0\}$ and $M > 0$,*

$$\mathbf{P}\left(\tau_x \in \left[M|x|^2, 2M|x|^2 - 1\right]\right) \leq c_1 \left(M|x|^2\right)^{1-d/2} \exp\left(-\frac{c_2}{M}\right).$$

Moreover, there exist constants $c_3, c_4, c_5, c_6 \in (0, \infty)$ such that for every $x \in \mathbb{Z}^d \backslash \{0\}$ and $M \geq |x|^{-1}$,

$$\mathbf{P}\left(\tau_x \in \left[c_3 M|x|^2, c_4 M|x|^2\right]\right) \geq c_5 \left(M|x|^2\right)^{1-d/2} \exp\left(-\frac{c_6}{M}\right).$$

We will break the proof of this result into four pieces, distinguishing the cases $M \in (0, 1)$ and $M \geq 1$, and considering the upper and lower bounds separately. See Propositions 5.2, 5.3, 5.6 and 5.13 for the individual statements.

5.1.1 Upper bound for $M \geq 1$

The aim of this subsection is to establish the following, which is the easiest to prove of the constituent results making up Proposition 5.1.

Proposition 5.2. *There exist constants $c_1, c_2 \in (0, \infty)$ such that for every $x \in \mathbb{Z}^d \backslash \{0\}$ and $M \geq 1$,*

$$\mathbf{P}\left(\tau_x \in \left[M|x|^2, 2M|x|^2 - 1\right]\right) \leq c_1 \left(M|x|^2\right)^{1-d/2} \exp\left(-\frac{c_2}{M}\right).$$

Proof. Recalling the definition of $(\sigma_i)_{i\geq 0}$ from (2.2), we have that

$$\begin{aligned}\mathbf{P}\left(\tau_x \in \left[M|x|^2, 2M|x|^2-1\right]\right) &= \sum_{i=\lceil M|x|^2\rceil}^{\lfloor 2M|x|^2-1\rfloor} \mathbf{P}\left(S_{\sigma_i}=x\right)\\ &\leq \mathbf{E}\left(\#\left\{i\geq \lceil M|x|^2\rceil :\, S_{\sigma_i}=x\right\}\right).\end{aligned}$$

Using that $\sigma_i \geq i$, this implies that

$$\begin{aligned}\mathbf{P}\left(\tau_x \in \left[M|x|^2, 2M|x|^2-1\right]\right) &\leq \mathbf{E}\left(\#\left\{n\geq \lceil M|x|^2\rceil :\, S_n=x\right\}\right)\\ &= \sum_{n=\lceil M|x|^2\rceil}^{\infty} \mathbf{P}(S_n=x)\\ &\leq \sum_{n=\lceil M|x|^2\rceil} Cn^{-d/2}\\ &\leq C\left(M|x|^2\right)^{1-d/2},\end{aligned}$$

where for the second inequality, we have applied the upper bound on the transition probabilities of S from [2, Theorem 6.28]. Since it also holds that $\exp(-c_2/M)\geq C$ uniformly over $M\geq 1$, the result follows. □

5.1.2 Upper bound for $M \in (0,1)$

We will give an upper bound on the probability that τ_x is much smaller than $|x|^2$. More precisely, the goal of this subsection is to prove the following proposition. Replacing M by $2M$, this readily implies the relevant part of Proposition 5.1.

Proposition 5.3. *There exist constants $c_1, c_2 \in (0,\infty)$ such that for every $x\in\mathbb{Z}^d\backslash\{0\}$ and $M\in(0,2)$,*

$$\mathbf{P}\left(\tau_x \leq M|x|^2\right) \leq c_1\left(M|x|^2\right)^{1-d/2}\exp\left(-\frac{c_2}{M}\right). \tag{5.2}$$

Before diving into the proof, we observe that it is enough to show (5.2) only for the case that both $|x|^{-1}$ and M are sufficiently small. To see this, suppose that there exist some $c_1, c_2\in(0,\infty)$ and $r_0\in(0,1)$ such that the inequality (5.2) holds with constants c_1, c_2 for all x and M satisfying $|x|^{-1}\vee M\leq r_0$. The remaining cases we need to consider are (i) $|x|^{-1}\geq r_0$ and (ii) $M\in[r_0,2)$. We first deal with case (i). If we suppose that $|x|^{-1}\geq r_0$ and $M<r_0^2$, then $M|x|^2<1$, and so the probability on the left-hand side of (5.2) is equal to 0. On the other hand, if $|x|^{-1}\geq r_0$ and $M\in[r_0^2,2)$, by choosing the constant c_1 to satisfy $c_1\geq 2^{d/2-1}r_0^{2-d}\exp\{c_2 r_0^{-2}\}$, we can ensure the right-hand side of (5.2) is greater than 1, and so the inequality (5.2) also holds in this case. Let us move to case (ii). We note that the probability on the left-hand side of (5.2) can be always bounded above by

$$\mathbf{P}\left(\tau_x^S<\infty\right)\leq C|x|^{2-d}$$

for some constant $C\in(0,\infty)$, where we have applied (2.9) to deduce the inequality. Thus, choosing the constant c_1 so that $c_1\geq C2^{d/2-1}\exp\{c_2 r_0^{-1}\}$, the inequality (5.2) holds. Consequently, replacing the constant c_1 by $c_1\vee 2^{d/2-1}r_0^{2-d}\exp\{c_2r_0^{-2}\}\vee C2^{d/2-1}\exp\{c_2r_0^{-1}\}$, the inequality (5.2) holds for all $x\in\mathbb{Z}^d\setminus\{0\}$ and $M\in(0,2)$.

We next give a brief outline of the proof of Proposition 5.3, assuming that both $|x|^{-1}$ and M are sufficiently small. We write $A_x = B_\infty(0, |x|/4\sqrt{d})$ for the box of side length $|x|/2\sqrt{d}$ centered at the origin, and let

$$t_x = \tau^L_{A_x^c}$$

be the first time that L exits A_x. Note that $x \notin A_x$, and so

$$\mathbf{P}\left(\tau_x \le M|x|^2\right) \le \mathbf{P}\left(t_x \le \tau_x \le M|x|^2\right) \le \mathbf{P}\left(\tau_x < \infty \,\middle|\, t_x \le M|x|^2\right) \mathbf{P}\left(t_x \le M|x|^2\right). \quad (5.3)$$

Writing

$$p_{x,M} = \mathbf{P}\left(\tau_x < \infty \,\middle|\, t_x \le M|x|^2\right) \quad \text{and} \quad q_{x,M} = \mathbf{P}\left(t_x \le M|x|^2\right), \quad (5.4)$$

we will show that

$$p_{x,M} \le C|x|^{2-d}, \qquad q_{x,M} \le C\exp\{cM^{-1}\}$$

in Lemmas 5.4 and 5.5 below, respectively. Proposition 5.3 is then a direct consequence of these lemmas.

We start by dealing with $p_{x,M}$, as defined in (5.4).

Lemma 5.4. *There exists a constant $C \in (0,\infty)$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $M \in (0,2)$ with $\mathbf{P}(t_x \le M|x|^2) > 0$,*

$$p_{x,M} \le C|x|^{2-d}.$$

Proof. Let

$$\Lambda = \left\{\lambda \,:\, \mathbf{P}\left(t_x \le M|x|^2,\, L[0,t_x] = \lambda\right) > 0\right\}$$

be the set of all possible paths for $L[0,t_x]$ satisfying $t_x \le M|x|^2$. For $\lambda \in \Lambda$, we write $R = R^\lambda$ for a random walk conditioned on the event that $R[1,\infty) \cap \lambda = \emptyset$. Note that R is a Markov chain (see [29, Section 11.1]). We use $\mathbf{P}^y_R$ to denote the law of R started from $R_0 = y$. Then the domain Markov property for L (see [28, Proposition 7.3.1]) ensures that

$$p_{x,M} = \frac{\sum_{\lambda\in\Lambda} \mathbf{P}_R^{\lambda_{\mathrm{len}(\lambda)}}\left(x \in \mathrm{LE}\left(R[0,\infty)\right)\right) \mathbf{P}(L[0,t_x] = \lambda)}{q_{x,M}} \le \max_{\lambda\in\Lambda} \mathbf{P}_R^{\lambda_{\mathrm{len}(\lambda)}}\left(x \in R[0,\infty)\right).$$

Therefore, it suffices to show that there exists a constant $C \in (0,\infty)$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$, $M \in (0,2)$ with $\mathbf{P}(t_x \le M|x|^2) > 0$ and $\lambda \in \Lambda$,

$$\mathbf{P}_R^{\lambda_{\mathrm{len}(\lambda)}}\left(x \in R[0,\infty)\right) \le C|x|^{2-d}.$$

With the above goal in mind, let us fix $\lambda \in \Lambda$. We set $u := \tau^R_{B(|x|/2)^c}$ for the first time that R exits $B(|x|/2)$. (Note that $A_x \subseteq B(|x|/2)$ by our construction.) Using the strong Markov property for R at time u, we have

$$\mathbf{P}_R^{\lambda_{\mathrm{len}(\lambda)}}\left(x \in R[0,\infty)\right) \le \max_{y\in\partial B(|x|/2)} \mathbf{P}^y_R\left(x \in R[0,\infty)\right).$$

On the other hand, it follows from (2.8) that

$$\min_{y\in\partial B(|x|/2)} \mathbf{P}^y\left(S[0,\infty) \cap \lambda = \emptyset\right) \ge \min_{y\in\partial B(|x|/2)} \mathbf{P}^y\left(S[0,\infty) \cap A_x = \emptyset\right) \ge c_0$$

for some constant $c_0 > 0$. Combining these estimates and using (2.9), we see that, for each $y \in \partial B(|x|/2)$,

$$\mathbf{P}_R^y\left(x \in R[0,\infty)\right) \leq \frac{\mathbf{P}^y\left(x \in S[0,\infty)\right)}{\mathbf{P}^y\left(S[0,\infty) \cap \lambda = \emptyset\right)} \leq \frac{1}{c_0}\mathbf{P}^y\left(x \in S[0,\infty)\right) \leq C|x|^{2-d}$$

for some constant $C \in (0,\infty)$. This finishes the proof. $\square$

Recall that $q_{x,M}$ was defined at (5.4). We will next estimate $q_{x,M}$ as follows.

Lemma 5.5. *There exist constants $c, C \in (0,\infty)$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $M \in (0,2)$,*

$$q_{x,M} \leq C\exp\{-cM^{-1}\}. \tag{5.5}$$

Proof. As per the discussion after (5.2), it suffices to prove (5.5) only in the case that both $|x|^{-1}$ and M are sufficiently small. In particular, throughout the proof, we assume that

$$M \leq (3200d)^{-1}. \tag{5.6}$$

Furthermore, we may assume

$$|x|M \geq (4\sqrt{d})^{-1}, \tag{5.7}$$

since $q_{x,M} = 0$ when $|x|M < (4\sqrt{d})^{-1}$. (Notice that it must hold that $t_x \geq |x|(4\sqrt{d})^{-1}$.)

Now, define the increasing sequence of boxes $\{A^i\}_{i=1}^N$, where $N = \lfloor (1600dM)^{-1} \rfloor$, by setting

$$A^i = B_\infty\left(0, 400\sqrt{d}\,|x|Mi\right)$$

for $1 \leq i \leq N$. Observe that the particular choice of N ensures that $A^N \subseteq A_x = B_\infty(0, |x|/4\sqrt{d})$, and the assumption (5.6) guarantees that

$$N \geq (3200dM)^{-1}. \tag{5.8}$$

Also, we note that $\operatorname{dist}(\partial A^{i-1}, \partial A^i)$ is bigger than $400\sqrt{d}\,|x|M - 1$, which is in turn bounded below by 99 because of (5.7). As a consequence, it is reasonable to compare the number of lattice points in the set $L \cap (A^i \setminus A^{i-1})$ with $|x|^2M^2$. To be more precise, let $t^0 = 0$, and, for $i \geq 1$, set

$$t^i = \tau^L_{(A^i)^c}$$

to be the first time that L exits A^i. Then [6, Corollary 3.10] shows that there exists a deterministic constant $c_1 \in (0,1)$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $M \in (0,2)$ satisfying (5.6) and (5.7),

$$\mathbf{P}\left(t^i - t^{i-1} \geq c_1|x|^2M^2 \,\middle|\, L[0,t^{i-1}]\right) \geq c_1, \qquad 1 \leq i \leq N. \tag{5.9}$$

With the inequality (5.9) and a constant $a > 0$ satisfying

$$2\sqrt{\frac{2a}{1-c_1}} < \frac{1}{6400d}\log\frac{1}{1-c_1}, \tag{5.10}$$

it is possible to apply [3, Lemma 1.1] to deduce the result of interest. In particular, the following table explains how the quantities of this article are substituted into [3, Lemma 1.1].

[3, Lemma 1.1]	X	Y_i	n	p	b	x
This article	t_x	$t^i - t^{i-1}$	N	$1-c_1$	$2\,\lvert x\rvert^{-2}M^{-2}$	$aM\lvert x\rvert^2$

Then, from [3, Lemma 1.1], one has

$$\begin{aligned}\mathbf{P}\left(t_x \le aM|x|^2\right) &\le \exp\left\{2M^{-1}\sqrt{\frac{2a}{1-c_1}} - N\log\frac{1}{1-c_1}\right\}\\ &\le \exp\left\{\left(2\sqrt{\frac{2a}{1-c_1}} - \frac{1}{3200d}\log\frac{1}{1-c_1}\right)M^{-1}\right\}\\ &\le \exp\left\{-\frac{M^{-1}}{6400d}\log\frac{1}{1-c_1}\right\},\end{aligned}$$

where for the second and third inequalities, we apply (5.8) and (5.10), respectively. Rewriting $aM = M'$ completes the proof. □

Proof of Proposition 5.3. Proposition 5.3 follows directly from (5.3) and Lemmas 5.4 and 5.5 (and the basic observation that $M^{1-d/2} \ge 2^{1-d/2}$ for $M \in (0,2)$). □

5.1.3 Lower bound for $M \in (0,1)$

Recall that for $x \in \mathbb{Z}^d \setminus \{0\}$, τ_x indicates the first time that L hits x. The aim of this subsection is to bound below the probability that τ_x is much smaller than $|x|^2$. In particular, the following is the main result of this subsection, which readily implies the part of the lower bound of Proposition 5.1 with $|x|^{-1} \le M < 1$.

Proposition 5.6. *There exist constants $c, c', R \in (0,\infty)$ such that for every $x \in \mathbb{Z}^d\setminus\{0\}$ and $|x|^{-1} \le M < 1$,*

$$\mathbf{P}\left(\tau_x \in [R^{-1}M|x|^2, RM|x|^2]\right) \ge c'(M|x|^2)^{1-d/2}\exp\left(-\frac{c}{M}\right). \tag{5.11}$$

Before moving on to the proof, we will first show that once we prove that there exists a constant $n_0 \ge 1$ such that (5.11) holds for $n_0|x|^{-1} \le M < 1$, we obtain (5.11) for every $x \in \mathbb{Z}^d \setminus \{0\}$ and $|x|^{-1} \le M < 1$ by adjusting c, c' and R as needed. Let us consider the following three events:

- $S[0,\tau_x^S]$ is a simple path of length $\lceil M|x|^2\rceil$,
- $S[\tau_x^S+1, \tau_{B(0,2r)^c}^S]$ is a simple path that does not intersect $S[0,\tau_x^S]$,
- $S[\tau_{B(0,2r)^c}^S, \infty) \cap B(0,\frac{3}{2}|x|) = \emptyset$,

where we set $r = |x| \vee n_0$. It is straightforward to see that $\tau_x \in [M|x|^2, 2M|x|^2]$ holds on the intersection of these events. By constructing a simple random walk path that satisfies the first two conditions up to the first exit time from the Euclidean ball $B(0,2r)$ and then applying (2.8) and the strong Markov property, we have

$$\mathbf{P}\left(\tau_x \in [M|x|^2, 2M|x|^2]\right) \ge a(2d)^{-2M|x|^2}(2d)^{-2r},$$

for some $a > 0$ that does not depend on M or x. Suppose $1 \leq M|x| \leq n_0$. If $|x| < n_0$, then the right-hand side is bounded below as follows:

$$a(2d)^{-2M|x|^2}(2d)^{-2r} \geq a(2d)^{-(2n_0^2+2n_0)} \geq C \geq c'(M|x|^2)^{1-d/2}e^{-\frac{c}{M}}.$$

On the other hand, if $|x| \geq n_0$, then the right-hand side satisfies

$$a(2d)^{-2M|x|^2}(2d)^{-2r} \geq C\left(M|x|^2\right)^{1-d/2}e^{-cM|x|^2-c'|x|} \geq C\left(M|x|^2\right)^{1-d/2}e^{-c''/M}.$$

In particular, by replacing R by $R \vee 2$ and adjusting c, c' appropriately, the result at (5.11) can be extended to $1 \leq M|x| < |x|$.

The structure of this subsection is as follows. Firstly, we define several subsets of $\mathbb{R}^d$. These will be used to describe a number of events involving the simple random walk S whose loop-erasure is L. Secondly, we provide some key estimates on the probabilities of these events in Lemmas 5.9 and Lemma 5.12. Finally, applying these results, we prove Proposition 5.6 at the end of this subsection.

We begin by defining "a tube connecting the origin and x", which will consist of a number of boxes of side-length $M|x|$. To this end, for $M \in (0,1)$, let

$$N_M = \left\lceil \frac{1}{M} + \frac{1}{2} \right\rceil.$$

Moreover, for $x = (x^1, \dots, x^d) \in \mathbb{Z}^d \setminus \{0\}$ and $M \in (0,1)$, define a sequence $\{b_i\}$ of vertices of $\mathbb{R}^d$ by setting

$$b_i = \left(iMx^1, \dots, iMx^d\right) \tag{5.12}$$

for $i \in \{0, 1, \dots, 2N_M\}$. Let us consider a rotation around the origin that aligns the x^1-axis with the line through the origin and x. We denote by B and Q the images of $[-M|x|/2, M|x|/2]^d$ and $\{0\} \times [-M|x|/2, M|x|/2]^{d-1}$ under this rotation, respectively. For $y = (y^1, \dots, y^d) \in \mathbb{R}^d$, we let

$$\widetilde{B}(y,r) = \left\{\left(y^1 + rz^1, \dots, y^d + rz^d\right) : (z^1, \dots, z^d) \in B\right\}, \tag{5.13}$$

be the tilted cube of side-length $rM|x|$ centered at y, and we write B_i for $\widetilde{B}(b_i, 1)$. For $i = 0, 1, \dots, 2N_M$ and $a, b \in \mathbb{R}$ with $a < b$, also let

$$Q(y,r) = \left\{(y^1 + rz^1, \dots, y^d + rz^d) : (z^1, \dots, z^d) \in Q\right\},$$

$$B_i[a,b] = \left\{\left(z^1 + sMx^1, \dots, z^d + sMx^d\right) : s \in [a,b],\ (z^1, \dots, z^d) \in Q_i(0)\right\},$$

where

$$Q_i(b) = Q\left(\left((i - \frac{1}{2} + b)M|x|, \dots, (i - \frac{1}{2} + b)M|x|\right), 1\right).$$

We also set

$$\widetilde{Q}_i(b) = Q\left(\left((i - \frac{1}{2} + b)M|x|, \dots, (i - \frac{1}{2} + b)M|x|\right), \frac{1}{2}\right),$$

and note that, by definition, $\widetilde{Q}_i(b) \subseteq Q_i(b)$ for all $i \geq 0$ and $b \in \mathbb{R}$. Observe that every $\widetilde{Q}_i(b)$ is perpendicular to the line through the origin and x, and that $Q_i := Q_i(0)$ is the "left face" of the cube $B_i = B_i[0,1]$. Finally, we write

$$\widetilde{Q}_i := \widetilde{Q}_i(0), \qquad i = 1, 2, \dots, 2N_M + 1, \tag{5.14}$$

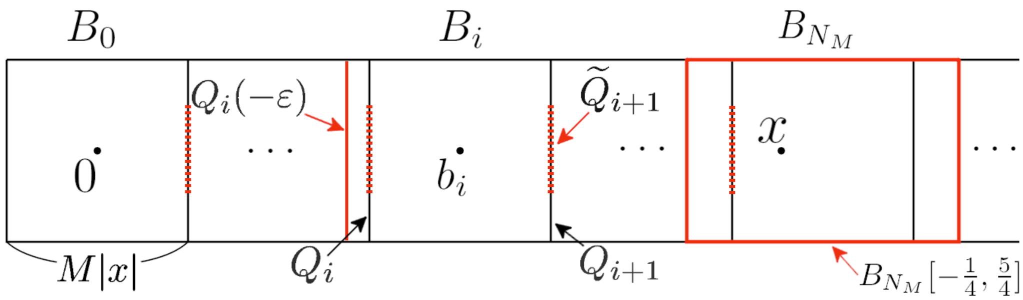


Figure 20: Illustration of B_i, Q_i and $\widetilde{Q}_i$ for a given x.

and set $Q_0 = \widetilde{Q}_0 = \{0\}$ for convenience. See Figure 20 for a graphical representation of the situation.

In this subsection, it will be convenient to consider S (recall that $L = \mathrm{LE}(S[0,\infty))$) as a continuous curve in $\mathbb{R}^d$ by linear interpolating between discrete time points, and thus we may assume that $S(k)$ is defined for all non-negative real k. If λ is a continuous path in $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$, we write

$$\tau^\lambda(A) = \inf\{t \geq 0 \,:\, \lambda(t) \in A\},$$

and also, for $x \in \mathbb{R}^d$, we set $\tau_x^\lambda = \tau^\lambda(\{x\})$, analogous to the notation of first hitting times for discrete paths (2.1).

In order to obtain the lower bound (5.11), we consider events under which the LERW L, started at the origin, travels through the "tube" $\bigcup_{i=0}^{N_M} B_i$ until it hits x. See Figure 21 for a graphical representation.

Definition 5.7. *We define the events F_i, $i = 0, 1, \ldots, 2N_M$, as follows. Firstly,*

$$F_0 = \left\{ \begin{array}{c} \tau^S(Q_1) < \infty,\ S(\tau^S(Q_1)) \in \widetilde{Q}_1,\ S[0, \tau^S(Q_1)] \subset B_0, \\ S[\tau^S(Q_1(-\varepsilon)), \tau^S(Q_1)] \cap Q_1(-2\varepsilon) = \emptyset \end{array} \right\}.$$

For $i = 1, 2, \ldots, N_M - 1$,

$$F_i = \left\{ \begin{array}{c} \tau^S(Q_i) < \tau^S(Q_{i+1}) < \infty,\ S(\tau^S(Q_{i+1})) \in \widetilde{Q}_{i+1},\ S[\tau^S(Q_i), \tau^S(Q_{i+1})] \subset B_i[-\varepsilon, 1], \\ S[\tau^S(Q_{i+1}(-\varepsilon)), \tau^S(Q_{i+1})] \cap Q_{i+1}(-2\varepsilon) = \emptyset \end{array} \right\}.$$

Moreover,

$$F_{N_M} = \left\{ \begin{array}{c} \tau^S(Q_{N_M}) < \tau_x^S < \tau^S\left(Q_{N_M+1}\left(\frac{1}{4}\right)\right) < \infty, \\ S\left(\tau^S\left(Q_{N_M+1}\left(\frac{1}{4}\right)\right)\right) \in \widetilde{Q}_{N_M+1}\left(\frac{1}{4}\right),\ S[\tau^S(Q_{N_M}), \tau^S(Q_{N_M+1})] \subset B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right], \\ S\left[\tau_x^S, \tau^S\left(Q_{N_M+1}\left(\frac{1}{4}\right)\right)\right] \cap \mathrm{LE}(S[\tau^S(Q_{N_M}), \tau_x^S]) = \emptyset \end{array} \right\},$$

$$F_{N_M+1} = \left\{ \begin{array}{c} \tau^S\left(Q_{N_M+1}\left(\frac{1}{4}\right)\right) < \tau^S(Q_{N_M+2}),\ \tau^S(Q_{N_M+2}) \in \widetilde{Q}_{N_M+2}, \\ S\left[\tau^S\left(Q_{N_M+1}\left(\frac{1}{4}\right)\right), \tau^S(Q_{N_M+2})\right] \subset B_{N_M+1}\left[\frac{1}{4} - \varepsilon, 1\right] \end{array} \right\},$$

and, for $i = N_M + 1, \ldots, 2N_M$,

$$F_i = \left\{ \begin{array}{c} \tau^S(Q_i) < \tau^S(Q_{i+1}) < \infty,\ S(\tau^S(Q_{i+1})) \in \widetilde{Q}_{i+1}, \\ S[\tau^S(Q_i), \tau^S(Q_{i+1})] \subset B_i[-\varepsilon, 1] \end{array} \right\}.$$

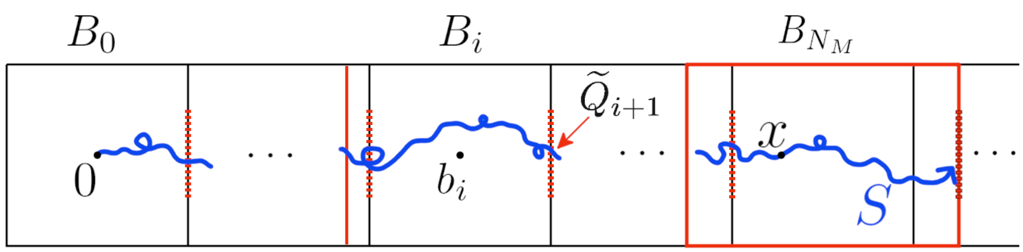


Figure 21: Illustration of the events F_i, $i = 0, 1, \dots, N_M$.

The first three conditions of the definition of each F_i, $i = 0, 1, \dots, 2N_M$, require that S travels inside the "tube" and it exits each B_i at a point which is not close to ∂Q_{i+1}. Furthermore, the last condition in the definition of F_0, the last two conditions in that of each F_i, $i = 1, 2, \dots, N_M - 1$, and the third condition in that of F_{N_M} control the range of backtracking of S. Finally, the last condition in the definition of F_{N_M} and events F_i, $i = N_M + 1, \dots, 2N_M$ guarantee that x remains in $\mathrm{LE}(S[0, \tau^S(Q_{2N_M+1})])$.

Next, we define events that provide upper and lower bounds for the length of the loop-erasure of S in each B_i. For $i \in \{0, 1, \dots, N_M - 1\}$, let

$$\xi_i = \mathrm{LE}\left(S[0, \tau^S(Q_{i+1})]\right), \tag{5.15}$$

and also set $\xi_{N_M} = \mathrm{LE}(S[0, \tau^S_x])$. We further define

$$\begin{aligned} \lambda_i &= \mathrm{LE}\left(S[\tau^S(Q_i), \tau^S(Q_{i+1})]\right), \qquad 1 \le i \le N_M - 1, \\ \lambda_{N_M} &= \mathrm{LE}\left(S[\tau^S(Q_{N_M}), \tau^S_x]\right), \\ \xi'_0 &= \xi_0, \\ \xi'_i &= \xi_0 \oplus \lambda_1 \oplus \cdots \oplus \lambda_i, \qquad i \ge 1. \end{aligned} \tag{5.16}$$

Since ξ_i is not necessarily a simple curve, ξ_i and ξ'_i do not coincide in general. However, the restriction on the backtracking of S on the events F_i and a cut time argument (see Definition 5.10 and Definition 5.11 below) enable us to handle the difference between ξ_i and ξ'_i. This will be discussed later, in the proof of Proposition 5.6.

We now define events upon which the length of λ_i is bounded above.

Definition 5.8. *For $C > 0$, the event $G_0(C)$ is given by*

$$G_0(C) = \left\{\mathrm{len}(\xi_0) \le CM^2|x|^2\right\},$$

and for $i = 1, 2, \dots, N_M$, the event $G_i(C)$ is given by

$$G_i(C) = \left\{\mathrm{len}(\lambda_i) \le CM^2|x|^2\right\}.$$

In the following lemma, we demonstrate that G_i occurs with high conditional probability. Recall that $\widetilde{Q}_i$ was defined at (5.14).

Lemma 5.9. *For any $\delta > 0$, there exists a constant $C_+ > 0$ such that*

$$\mathbf{P}^z\left(G_i(C_+) \,|\, F_i\right) \ge 1 - \delta, \tag{5.17}$$

uniformly in $x \in \mathbb{Z}^d \setminus \{0\}$, $|x|^{-1} \le M < 1$, $i \in \{0, 1, \dots, N_M\}$ and $z \in \widetilde{Q}_i$.

Proof. For $i = 0, 1, \dots, N_M - 1$, $y \in B_i[-\varepsilon, 1]$ and $z \in \widetilde{Q}_i$, we have that

$$\mathbf{P}^z\left(y \in \lambda_i \mid F_i\right) \le \mathbf{P}^z(\tau_y^S < \tau^S(Q_{i+1}) \mid F_i).$$

Moreover, by (2.10) and translation invariance, we have that there exists some constant $c > 0$ such that

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(F_i) \ge c\varepsilon, \tag{5.18}$$

for all $i = 1, 2, \dots, N_M - 1$. For $i = 0$, the same argument yields that $\mathbf{P}(F_0) \ge c\varepsilon$. Thus, it follows from (2.9) and (5.18) that

$$\mathbf{P}^z(\tau_y^S < \tau^S(Q_{i+1}) \mid F_i) \le \frac{\mathbf{P}^z(\tau_y^S < \tau^S(Q_{i+1}))}{\mathbf{P}^z(F_i)} \le C\mathbf{P}^z(\tau_y^S < \infty) \le C\left(|y - z|^{2-d} \wedge 1\right),$$

for some constant $C > 0$. By taking the sum over $y \in B_i[-\varepsilon, 1]$, we have that

$$\mathbf{E}^z(\mathrm{len}(\lambda_i) \mid F_i) = \sum_{y \in B_i[-\varepsilon,1]} \mathbf{P}^z(y \in \lambda_i \mid F_i) \le C \sum_{y \in B_i[-\varepsilon,1]} \left(|y - z|^{2-d} \wedge 1\right) \le CM^2|x|^2. \tag{5.19}$$

The same argument also applies to the case $i = 0$, and thus we have

$$\mathbf{E}^0(\mathrm{len}(\xi_0) \mid F_0) \le CM^2|x|^2. \tag{5.20}$$

Similarly, for the case $i = N_M$, recalling that λ_{N_M} was defined at (5.16), we have that

$$\begin{aligned}
\mathbf{P}^z\left(y \in \lambda_{N_M} \mid F_{N_M}\right) &\le \mathbf{P}^z\left(\tau_y^S < \tau_x^S \mid F_{N_M}\right) \\
&\le \frac{\mathbf{P}^z\left(\tau_y^S < \tau_x^S < \tau^S(B_{N_M}[-\frac{1}{4}, \frac{5}{4}]^c)\right)}{\mathbf{P}^z\left(F_{N_M}\right)} \\
&= \frac{\mathbf{P}^z\left(\tau_y^S < \tau_x^S \wedge \tau^S(B_{N_M}[-\frac{1}{4}, \frac{5}{4}]^c)\right)\mathbf{P}^y\left(\tau_x^S < \tau^S(B_{N_M}[-\frac{1}{4}, \frac{5}{4}]^c)\right)}{\mathbf{P}^z\left(F_{N_M}\right)},
\end{aligned} \tag{5.21}$$

where we used the strong Markov inequality to obtain the last inequality. We will prove that $\mathbf{P}^z(F_{N_M}) \ge C'(M|x|)^{2-d}$ later in this subsection, see (5.38).

Now we bound above the sum of the numerator of (5.21) over $y \in B_{N_M}[-\frac{1}{4}, \frac{5}{4}]$, separating into three cases depending on the location of y.

(i) For $y \in B(z, \frac{1}{18}M|x|)$, it follows from (2.9) that

$$\mathbf{P}^z\left(\tau_y^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right) \wedge \tau_x^S\right) = C(|y - z|^{2-d} - (M|x|/2)^{2-d}) + O(|y - z|^{1-d}),$$

$$\mathbf{P}^y\left(\tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \le C(M|x|)^{2-d}.$$

Thus we have that

$$\begin{aligned}
&\sum_{y \in B(z, \frac{1}{18}M|x|)} \mathbf{P}^z\left(\tau_y^S < \tau_x^S \wedge \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right)\mathbf{P}^y\left(\tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \\
&\quad \le C(M|x|)^{2-d} \sum_{k=1}^{\frac{1}{18}M|x|} \sum_{|y-z|=k} (k^{2-d} - (M|x|/2)^{2-d} + O(k^{1-d})) \\
&\quad \le C(M|x|)^{4-d}.
\end{aligned}$$

(ii) For $y \in B(x, \frac{1}{18}M|x|)$, a similar argument to case (i) yields that

$$\sum_{y \in B(x, \frac{1}{18}M|x|)} \mathbf{P}^z\left(\tau_y^S < \tau_x^S \wedge \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \mathbf{P}^y\left(\tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right)$$
$$\leq C(M|x|)^{4-d}.$$

(iii) For $y \in B_{N_M}[-\frac{1}{4}, \frac{5}{4}] \setminus (B(z, \frac{1}{18}M|x|) \cup B(x, \frac{1}{18}M|x|))$, we have that

$$\mathbf{P}^z\left(\tau_y^S < \tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \leq \mathbf{P}^z(\tau_y^S < \infty) \leq C(M|x|)^{2-d},$$
$$\mathbf{P}^y\left(\tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \leq \mathbf{P}^y(\tau_x^S < \infty) \leq C(M|x|)^{2-d},$$

which gives

$$\sum_{\substack{y \in B_{N_M}[-\frac{1}{4}, \frac{5}{4}] \\ |y-z|, |y-x| \geq \frac{1}{18}M|x|}} \mathbf{P}^z\left(\tau_y^S < \tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right) \mathbf{P}^y\left(\tau_x^S < \tau^S\left(B_{N_M}\left[-\frac{1}{4}, \frac{5}{4}\right]^c\right)\right)$$
$$\leq C \sum_{\substack{y \in B_{N_M}[-\frac{1}{4}, \frac{5}{4}] \\ |y-z|, |y-x| \geq \frac{1}{18}M|x|}} (M|x|)^{4-2d} \leq C(M|x|)^{4-d}.$$

Thus, by (5.21), it holds that

$$\mathbf{E}^z(\mathrm{len}(\lambda_{N_M}) \mid F_{N_M}) = \sum_{y \in B_{N_M}[-\frac{1}{4}, \frac{5}{4}]} \mathbf{P}^z(y \in \lambda_{N_M} \mid F_{N_M}) \leq \frac{C(M|x|)^{4-d}}{c' \cdot C'(M|x|)^{2-d}} \leq CM^2|x|^2. \tag{5.22}$$

Combining (5.19), (5.20) and (5.22) with Markov's inequality, it holds that

$$\mathbf{P}^z(\mathrm{len}(\lambda_i) \geq C_+ M^2|x|^2 \mid F_i) \leq C/C_+,$$

for $i = 0, 1, \ldots, N_M$. By taking $C_+ = \delta^{-1}C$, we obtain (5.17). □

Now we will deal with events involving S upon which the length of λ_i is bounded below and, at the same time, the gap between the lengths of ξ_i and ξ_i' is bounded above. Firstly, we define a special type of cut time of S in each B_i.

Definition 5.10. *A nice cut time in B_i is a time k satisfying the following conditions:*

- $k \in \left[\tau^S(Q_i(\varepsilon/3)), \tau^S(Q_i(\varepsilon))\right]$,
- $S(k) \in B(S(\tau^S(Q_i)), \varepsilon M|x|/2)$,
- $S[\tau^S(Q_i), k] \cap S\left[k+1, \tau^S(Q_i(\varepsilon))\right] = \emptyset$,
- $S[k+1, \tau^S(Q_i(\varepsilon))] \cap Q_i = \emptyset$.

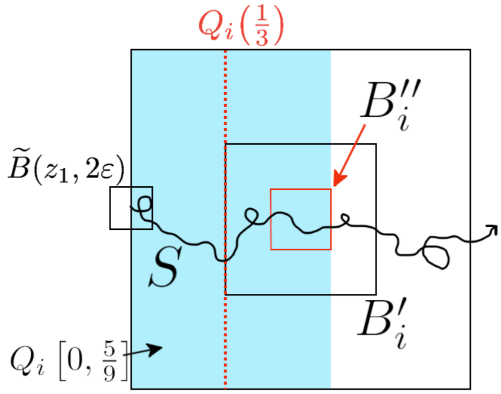


Figure 22: Illustration of B_i' and B_i''.

Secondly, let B_i' (respectively B_i'') be the cube of side-length $M|x|/3$ (respectively $M|x|/9$) centered at b_i whose faces are parallel to those of B_i, i.e.

$$B_i' = \widetilde{B}\left(b_i, \frac{1}{3}\right), \qquad B_i'' = \widetilde{B}\left(b_i, \frac{1}{9}\right),$$

where b_i and $\widetilde{B}(y,r)$ are as defined in (5.12) and (5.13), respectively. We denote by Q_i^L (respectively Q_i^R) the "left (respectively right) face" of B_i'. See Figure 22.

Let $\rho_i = \inf\{n \geq \tau^S(B_i'') : S(k) \in (B_i')^c\}$ be the first time that S exits B_i' after it first entered B_i''. We define a set of local cut times of S by

$$K_i = \left\{\tau^S(B_i'') \leq k \leq \rho_i : S(k) \in B_i'',\ S[\tau^S(B_i'), k] \cap S[k+1, \rho_i] = \emptyset\right\}.$$

Finally, we define events $H_i^{(j)}$ ($j = 1,2,3,4$) as follows.

Definition 5.11. *For $1 \leq i \leq N_M - 1$ and $l > 0$,*

$$\begin{aligned} H_i^{(1)} &= \{S \text{ has a nice cut time in } B_i\} \cap \left\{0 < \tau^S(Q_i(\varepsilon)) - \tau^S(Q_i) \leq C\varepsilon^2 M^2 |x|^2\right\}, \\ H_i^{(2)} &= \left\{ \begin{array}{c} \tau^S(B_i') < \tau^S(Q_{i+1}),\ S(\tau^S(Q_i(1/3))) \in Q_i^L, \\ S[\tau^S(Q_i(\varepsilon)),\ \tau^S(Q_i(1/3))] \cap Q_i(\varepsilon/2) = \emptyset \end{array} \right\}, \\ H_i^{(3)}(l) &= \left\{\#K_i \geq lM^2|x|^2,\ S(\rho_i) \in Q_i^R,\ S[\tau^S(B_i'), k] \in Q_i[0, 5/9] \text{ for all } k \in K_i\right\}, \\ H_i^{(4)} &= \left\{S[\rho_i, \tau^S(Q_{i+1})] \cap Q_i[0, 11/18] = \emptyset\right\}, \end{aligned}$$

where $\#A$ denotes the cardinality of set A. Moreover, $H_i(l) = H_i^{(1)} \cap H_i^{(2)} \cap H_i^{(3)}(l) \cap H_i^{(4)}$.

Note that, on the event $H_i(l)$, a local cut time $k \in K_i$ satisfies

$$S[\tau^S(Q_i), k] \cap S[k+1, \tau^S(Q_{i+1})] = \emptyset,$$

and thus $\mathrm{len}(\lambda_i) \geq \#K_i$ holds.

Lemma 5.12. *Let $H_i(l)$ be as defined above. Then there exists constants $c > 0$, $\varepsilon > 0$, $l > 0$ and $R' > 0$ such that*

$$\mathbf{P}^z\left(H_i(l) \mid F_i\right) \geq c, \tag{5.23}$$

uniformly in x and M with $M|x| > R'$, $i \in \{1, 2, \ldots, N_M - 1\}$ and $z \in \widetilde{Q}_i$.

Proof. By the strong Markov property, we have that

$$
\begin{aligned}
&\mathbf{P}^z\left(H_i(l) \mid F_i\right) \\
&\geq \inf_{z_1,z_2,z_3,z_4} \frac{1}{\mathbf{P}^z(F_i)} \prod_{j=1}^{3} \mathbf{P}^{z_j}\left(H_i^{(j)} \cap \{S[\tau^S(z_j),\tau^S(z_{j+1})] \subseteq B_i[-\varepsilon,1]\}\right) \\
&\qquad \times \mathbf{P}^{z_4}\left(H_i^{(4)} \cap \left\{ \begin{array}{c} S[\tau^S(z_3),\tau^S(z_4)] \subseteq B_i[-\varepsilon,1],\ S(\tau^S(Q_{i+1})) \in \widetilde{Q}_{i+1}, \\ S[\tau^S(Q_{i+1}(-\varepsilon)),\ \tau^S(Q_{i+1})] \cap Q_{i+1}(-2\varepsilon) \end{array} \right\}\right) \\
&\geq \inf_{z_1} \mathbf{P}^{z_1}(H_i^{(1)}) \inf_{z_2} \mathbf{P}^{z_2}(H_i^{(2)} \mid \{S[\tau^S(z_2),\tau^S(z_3)] \subseteq B_i[-\varepsilon,1]\}) \inf_{z_3} \mathbf{P}^{z_3}(H_i^{(3)}(l)) \\
&\qquad \times \inf_{z_4} \mathbf{P}^{z_4}\left(H_i^{(4)} \middle| \begin{array}{c} S[\tau^S(z_3),\tau^S(z_4)] \subseteq B_i[-\varepsilon,1],\ S(\tau^S(Q_{i+1})) \in \widetilde{Q}_{i+1}, \\ S[\tau^S(Q_{i+1}(-\varepsilon)),\ \tau^S(Q_{i+1})] \cap Q_{i+1}(-2\varepsilon) \end{array}\right),
\end{aligned} \tag{5.24}
$$

where the infima are taken over $z_1 \in \widetilde{Q}_i$, $z_2 \in \partial\widetilde{B}(z_1,2\varepsilon)$, $z_3 \in Q_i^L$ and $z_4 \in Q_i^R$ (see (5.13) for the definition of $\widetilde{B}(y,r)$).

Firstly, we estimate the conditional probability of $H_i^{(1)}$. Recall that $B(x,r)$ denotes the Euclidean ball of radius r with center point x. We consider the event of S up to the first exit time of the small box $\widetilde{B}(z_1,2\varepsilon)$. Let $k_1 = \tau^S(B(z,\frac{\varepsilon}{2}M|x|)^c)$ and $k_2 = \tau^S(\widetilde{B}(z_1,2\varepsilon)^c)$. Then

$$
\begin{aligned}
\mathbf{P}^{z_1}(H_i^{(1)}) &\geq \mathbf{P}^{z_1}\left(S \text{ has a nice cut time } k \in [k_1,k_2],\ 0 < k_2 - \tau^S(Q_i) \leq C'\varepsilon^2 M^2|x|^2\right) \\
&\geq \mathbf{P}^{z_1}\left(\begin{array}{c} S \text{ has a nice cut time } k \in [k_1,k_2], \\ \#S[\tau^S(Q_i),k_1] \geq C'^{-1}\varepsilon^2M^2|x|^2,\ 0 < k_2 - \tau^S(Q_i) \leq C'\varepsilon^2M^2|x|^2 \end{array}\right) \\
&\geq \mathbf{P}^{z_1}\left(S \text{ has a nice cut time } k \in [C'^{-1}\varepsilon^2M^2|x|^2, C'\varepsilon^2M^2|x|^2]\right) \\
&\qquad -\mathbf{P}^{z_1}\left(\#S[\tau^S(Q_i),k_1] \geq C'^{-1}\varepsilon^2M^2|x|^2\right) \\
&\qquad -\mathbf{P}^{z_1}\left(0 < k_2 - \tau^S(Q_i) \leq C'\varepsilon^2M^2|x|^2\right).
\end{aligned} \tag{5.25}
$$

If we take $C' > 1$ sufficiently large, then the second and third terms on the right-hand side of (5.25) are bounded below by some small constant, while it follows from [26, equation (1)] that the first term is bounded below by some universal constant. Thus, we have

$$\mathbf{P}^{z_1}(H_i^{(1)}) \geq c_1 \tag{5.26}$$

for some constant $c_1 > 0$.

Secondly, we consider the conditional probability of $H_i^{(2)}$. By (2.8), there exists some universal constant $C > 0$ such that

$$\mathbf{P}^{z_2}\left(S\left[\tau^S(B(z,\varepsilon M|x|/2)^c),\tau^S(B_i')\right] \cap (B(z,\varepsilon^2M|x|/2)) \neq \emptyset\right) \leq C\varepsilon^{d-2},$$

for $M|x| \geq \varepsilon^{-d}$. It follows from (2.10) that

$$
\begin{aligned}
c_2\varepsilon &\leq \mathbf{P}^{z_2}\left(\tau^S(B_i') < \tau^S(Q_{i+1}),\ S[\tau^S(z_2),\tau^S(z_3)] \subseteq B_i[-\varepsilon,1]\right) \\
&\leq \mathbf{P}^{z_2}\left(S[\tau^S(z_2),\tau^S(z_3)] \subseteq B_i[-\varepsilon,1]\right) \\
&\leq c_3\varepsilon,
\end{aligned}
$$

uniformly in $z_2 \in B(z_1,\varepsilon M|x|/2)$. Thus we have that

$$\mathbf{P}^{z_2}\left(H_i^{(2)} \middle| S[\tau^S(z_2),\tau^S(z_3)] \subseteq B_i[-\varepsilon,1]\right) \geq \frac{c_2\varepsilon - C\varepsilon^{d-2}}{c_3\varepsilon} \geq c, \tag{5.27}$$

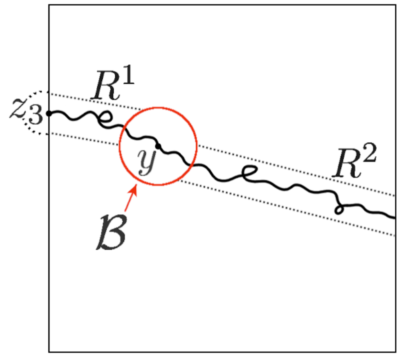


Figure 23: Illustration of the event $I_1 \cap I_2 \cap J_1 \cap J_2$.

for some constant $c > 0$ and sufficiently small ε.

Again by (2.10), it holds that

$$\mathbf{P}^{z_4}\left(H_i^{(4)} \,\middle|\, \begin{array}{c} S[\tau^S(z_4), \tau^S(Q_{i+1})] \subseteq B_i[-\varepsilon, 1],\ S(\tau^S(Q_{i+1})) \in \widetilde{Q}_{i+1}, \\ S[\tau^S(Q_{i+1}(-\varepsilon)), \tau^S(Q_{i+1})] \cap Q_{i+1}(-2\varepsilon) \end{array} \right) \geq c \tag{5.28}$$

for some constant $c > 0$.

We will next derive a lower bound for $\mathbf{P}^{z_3}(H_i^{(3)})$ by applying the second moment method. We consider the ball $\mathcal{B} := B\left(y, M|x|/18\right)$ and two independent simple random walks R^1 and R^2 with starting point y. Let $w_j = R^j(\tau^{R^j}(\mathcal{B}^c))$ for $j = 1, 2$. We define two events of R^1 and R^2 as follows:

$$I_1 = \left\{ R^1[1, \tau^{R^1}(\mathcal{B}^c)] \cap R^2[1, \tau^{R^2}(\mathcal{B}^c)] = \emptyset \right\},$$
$$I_2 = \left\{ \mathrm{dist}(\{w_1\}, R^2[1, \tau^{R^2}(\mathcal{B}^c)]) \vee \mathrm{dist}(\{w_2\}, R^1[1, \tau^{R^1}(\mathcal{B}^c)]) \geq M|x|/36 \right\}.$$

Let us denote by P the joint distribution of R^1 and R^2. By [6, Lemma 3.2], we have

$$\mathbf{P}(I_2 \mid I_1) \geq c_4, \tag{5.29}$$

while it follows from [28, equation (3.2)] that $\mathbf{P}(I_1) \geq c_4$ for some constant $c_4 > 0$. On I_2, without loss of generality, we suppose that $|w_1 - z_3| \leq |w_2 - z_3|$. Let

$$J_1 = \left\{ \tau_{z_3}^{R^1} < \tau^{R^1}(B_i'^c),\ \mathrm{dist}(R^1(k), l^1) \leq M|x|/200 \text{ for all } k \in [\tau^{R^1}(\mathcal{B}^c), \tau_{z_3}^{R^1}] \right\}$$
$$J_2 = \left\{ R^2(\tau^{R_2}({B_i}^c)) \in Q_i^R,\ \mathrm{dist}(R^2(k), l^2) \leq M|x|/200 \text{ for all } k \in [\tau^{R^2}(\mathcal{B}^c), \tau^{R^2}(B_i'^c)] \right\},$$

where l^1 (respectively l^2) is the line segment between the points w_1 and z_3 (or between w_2 and $R^2(\tau^{R^2}(B_i'^c))$, respectively). (See Figure 23 for a depiction of $I_1 \cap I_2 \cap J_1 \cap J_2$.) Since $\mathrm{dist}(w_2, B_i'^c)$ is comparable to $M|x|$, we have that

$$\mathbf{P}(J_2) \geq c',$$

for some constant $c' > 0$. Moreover, by the strong Markov property and (2.9),

$$\begin{aligned}
\mathbf{P}(J_1) &\geq \mathbf{P}\left(\operatorname{dist}(R^1(k), l^1) \leq M|x|/200 \text{ for all } k \in [\tau^{R^1}(\mathcal{B}^c), \tau^{R^1}(B(z_3, M|x|/400))]\right) \\
&\qquad \times \mathbf{P}^{R^1(\tau^{R^1}(B(z_3,M|x|/400)))}\left(\tau^{R^1}_{z_3} < \tau^{R^1}(B_i'^c)\right) \times \mathbf{P}^{z_3}\left(R^1(\tau^{R^1}(B(z_3, M|x|/200))) \notin B_i'\right) \\
&\geq c|w_1 - z_3|^{2-d}, \qquad (5.30)
\end{aligned}$$

for some constant $c > 0$. By the strong Markov property, we bound from below the expectation of $\#K_i$ on the event $A \coloneqq \{S(\rho_i) \in Q_i^R, S[\tau^S(B_i'), k] \in Q_i[0, 5/9] \text{ for all } k \in K_i\}$ by

$$\begin{aligned}
\mathbf{E}^{z_3}\left(\#K_i \mathbf{1}_A\right) &\geq \sum_{y \in B_i''} \mathbf{P}\left(I_1 \cap I_2 \cap J_1 \cap J_2\right) \\
&= \sum_{y \in B_i''} \mathbf{P}(I_1)\mathbf{P}(I_2 \mid I_1)\mathbf{P}(J_1)\mathbf{P}(J_2) \\
&\geq \sum_{y \in B_i''} c_4^2 \cdot c|y - z_3| \cdot c' \\
&\geq cM^2|x|^2. \qquad (5.31)
\end{aligned}$$

On the other hand, the first and second moment of K_i is bounded above as follows. Since $|y - z_3| \geq \frac{1}{9}M|x|$ for $y \in B_i''$ and $z \in Q_i^L$,

$$\mathbf{E}^{z_3}(\#K_i) \leq \mathbf{E}^{z_3}\left(\sum_{y \in B_i''} \mathbf{1}(\tau_y^S < \infty)\right) \leq \sum_{y \in B_i''} \mathbf{P}^{z_3}\left(\{\tau_y^S < \infty\}\right) \leq \sum_{y \in B_i''} C|y - z_3|^{2-d} \leq CM^2|x|^2, \qquad (5.32)$$

$$\begin{aligned}
\mathbf{E}^{z_3}((\#K_i)^2) &\leq \mathbf{E}^{z_3}\left(\left(\sum_{y \in B_i''} \mathbf{1}(\tau_y^S < \infty)\right)^2\right) \\
&\leq CM^2|x|^2 + \sum_{y \in B_i''}\sum_{y' \in B_i''}\left(\mathbf{P}^{z_3}(\tau_y^S < \tau_{y'}^S < \infty) + \mathbf{P}^{z_3}(\tau_{y'}^S < \tau_y^S < \infty)\right) \\
&\leq CM^2|x|^2 + \sum_{y \in B_i''}\sum_{y \in B_i''}\left(\mathbf{P}^{z_3}(\tau_y^S < \infty) + \mathbf{P}^{z_3}(\tau_{y'}^S < \infty)\right)\mathbf{P}^y(\tau_{y'}^S < \infty) \\
&\leq CM^2|x|^2 + \sum_{y \in B_i''}\sum_{k=1}^{\frac{1}{18}M|x|}\sum_{\substack{y \in B_i'' \\ |y-y'|=k}}\left(|y - z_3|^{2-d} + |y' - z_3|^{2-d}\right)k^{2-d} \\
&\qquad + \sum_{y \in B_i''}\sum_{k \geq \frac{1}{18}M|x|+1}\sum_{\substack{y \in B_i'' \\ |y-y'|=k}}\left(|y - z_3|^{2-d} + |y' - z_3|^{2-d}\right)(M|x|/9)^{2-d} \\
&\leq C'M^4|x|^4, \qquad (5.33)
\end{aligned}$$

where C and C' depend only on d. Now, for $0 \leq \theta \leq 1$, we have that

$$\mathbf{E}^{z_3}\left(\#K_i\mathbf{1}_A\right) \leq \theta\mathbf{E}^{z_3}\left(\#K_i\mathbf{1}_A\right) + \mathbf{E}^{z_3}\left(\#K_i\mathbf{1}_A\mathbf{1}\left(\#K_i\mathbf{1}_A > \theta\mathbf{E}^{z_3}(\#K_i\mathbf{1}_A)\right)\right).$$

From this, since $\#K_i \geq 0$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\mathbf{P}^{z_3}(\{\#K_i > \theta\mathbf{E}^{z_3}(\#K_i\mathbf{1}_A)\} \cap A) &\geq \mathbf{P}^{z_3}(\{\#K_i\mathbf{1}_A > \theta\mathbf{E}^{z_3}(\#K_i\mathbf{1}_A)\} \cap A) \\ &\geq (1-\theta)^2\frac{\mathbf{E}^{z_3}(\#K_i\mathbf{1}_A)^2}{\mathbf{E}^{z_3}((\#K_i)^2)}.\end{aligned}$$

By substituting (5.31), (5.32) and (5.33) into the above estimate, we obtain that

$$\mathbf{P}^{z_3}(H_i^{(3)}(l)) \geq \mathbf{P}^{z_3}\left(\#K_i \geq \frac{l}{C}\mathbf{E}^{z_3}(\#K_i)\right) \geq \left(1-\frac{l}{C}\right)^2\frac{\mathbf{E}^{z_3}(\#K_i\mathbf{1}_A)^2}{\mathbf{E}^{z_3}((\#K_i)^2)} \geq c. \tag{5.34}$$

Finally, substituting (5.26), (5.27), (5.28) and (5.34) into (5.24) gives (5.23). □

We are now ready to prove Proposition 5.6. Recall that F_i, G_i and H_i were defined in Definitions 5.7, 5.8 and 5.11, respectively. Let

$$U_{N_M} = \left\{S[\tau^S(Q_{2N_M+1}), \infty] \cap B(0, N_M \cdot M|x|) = \emptyset\right\},$$

and

$$\Theta = \Theta(C,l) = \left(\bigcap_{i=0}^{2N_M} F_i\right) \cap \left(\bigcap_{i=0}^{N_M} G_i(C)\right) \cap \left(\bigcap_{i=1}^{N_M-1} H_i(l)\right) \cap U_{N_M}.$$

Proof of Proposition 5.6. We will first demonstrate that the bound $\tau_x \in [R^{-1}M|x|^2, RM|x|]$, as appears in the probability on the left-hand side of (5.11), holds on $\Theta(C,l)$. Suppose that $\Theta(C,l)$ occurs. By the definition of F_i, $i = 0, 1, \ldots, 2N_M$,

$$\mathrm{LE}(S[0, \tau_x^S]) \cap S[\tau_x^S + 1, \infty] = \emptyset$$

holds. Thus $x \in L$ and $\tau_x = \mathrm{len}(S[0, \tau_x^S])$. Let k_i be a nice cut time of S in B_i (see Definition 5.10), and recall that ξ_i and ξ_i' are as defined in (5.15) and (5.16), respectively. Let

$$\begin{aligned} s_i &= \inf\left\{n \geq 0 : \xi_{i-1}(n) \in S[\tau^S(Q_i), \tau^S(Q_{i+1})]\right\}, \\ t_i &= \sup\left\{n \in [\tau^S(Q_i), \tau^S(Q_{i+1})] : S(n) = \xi_{i-1}(s_i)\right\}. \end{aligned}$$

Then we have that

$$\lambda_i = \mathrm{LE}\left(S[\tau^S(Q_i), k_i]\right) \oplus \mathrm{LE}\left(S[k_i, \tau^S(Q_{i+1})]\right), \tag{5.35}$$

and also

$$\xi_i = \xi_{i-1}[0, s_i] \oplus \mathrm{LE}\left(S[t_i, k_i]\right) \oplus \mathrm{LE}\left(S[k_i, \tau^S(Q_{i+1})]\right) \subseteq \xi_{i-1} \cup S[t_i, k_i] \cup \lambda_i,$$

for $i = 1, 2, \ldots, N_M - 1$, where we have applied (5.35) for the inclusion. Furthermore,

$$\xi_{N_M} = \xi_{N_M-1}[0, s_{N_M}] \oplus \mathrm{LE}(S[t_{N_M}, \tau_x^S]) \subseteq \xi_{N_M-1} \cup \lambda_{N_M}.$$

Thus, by induction, it follows that

$$\bigcup_{i=1}^{N_M-1} \mathrm{LE}\left(S[k_i, \tau^S(Q_{i+1})]\right) \subseteq \xi_{N_M} \subseteq \xi_0 \cup \bigcup_{i=1}^{N_M-1}\left(S[\tau^S(Q_i), \tau^S(Q_i(\varepsilon))] \cup \lambda_i\right) \cup \lambda_{N_M}.$$

Note that, on $H_i(l)$, $k' \in K_i$ is a cut time of the path $S[k_i, \tau^S(Q_{i+1})]$, and thus $S(k') \in LE(S[k_i, \tau^S(Q_{i+1})])$. By the definition of $G_i(C)$ and $H_i(l)$ (see Definitions 5.8 and 5.11, respectively), we have that

$$\begin{aligned}
\mathrm{len}(\xi_{N_M}) &\geq \sum_{i=1}^{N_M-1} \#K_i \geq \frac{1}{2} lM|x|^2, \\
\mathrm{len}(\xi_{N_M}) &\leq \mathrm{len}(\xi_0) + \sum_{i=1}^{N_M-1} \left(\#S[\tau^S(Q_i), \tau^S(Q_i(\varepsilon))] + \mathrm{len}(\lambda_i) \right) + \mathrm{len}(\lambda_{N_M}) \\
&\leq 3(C+\varepsilon^2)M|x|^2,
\end{aligned}$$

on $\Theta(C,l)$. Hence choosing R suitably large gives the desired bound on $\Theta(C,l)$.

Consequently, to complete the proof, it will be enough to show that $\mathbf{P}(\Theta)$ is bounded below by the right-hand side of (5.11). By Lemma 5.12, there exist constants $c_5, \varepsilon, l > 0$ such that $\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(H_i(l) \mid F_i) \geq c_5$ for $i = 1, 2, \ldots, N_M$. Moreover, by Lemma 5.9, there exists a constant $C > 0$ such that $\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(G_i(C) \mid F_i) \geq 1 - c_5/2$ for $i = 0, 1, \ldots, N_M$. Thus we have

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(G_i(C) \cap H_i(l) \mid F_i) \geq \frac{c_5}{2}, \qquad i \in \{1, 2, \ldots, N_M - 1\}, \tag{5.36}$$

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(G_i(C) \mid F_i) \geq 1 - \frac{c_5}{2}, \qquad i = 0, N_M. \tag{5.37}$$

As already noted in the proof of Lemma 5.9, we also have that

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(F_i) \geq c_6 \varepsilon, \tag{5.38}$$

for all $i = 1, 2, \ldots, N_M - 1$, and a similar bound holds for $\mathbf{P}(F_0)$. And, repeating a similar argument to the lower bound for $\mathbf{P}^{z_3}(H_i^{(3)})$ from the proof of Lemma 5.12, from (5.29) to (5.30) we have that

$$\inf_{z \in \widetilde{Q}_{N_M}} \mathbf{P}^z(F_{N_M}) \geq c_6 M^{2-d} |x|^{2-d},$$

where $c_6 > 0$ is adjusted if necessary. By combining these estimates on $\mathbf{P}(F_i)$ with (5.36) and (5.37), we obtain that

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(F_i \cap G_i(C) \cap H_i(l)) \geq \frac{c_5 c_6 \varepsilon}{2}, \qquad i \in \{1, 2, \ldots, N_M - 1\},$$

$$\mathbf{P}(F_0 \cap G_0(C)) \geq \left(1 - \frac{c_5}{2}\right) c_6,$$

$$\inf_{z \in \widetilde{Q}_{N_M}} \mathbf{P}^z(F_{N_M} \cap G_{N_M}(C)) \geq \left(1 - \frac{c_5}{2}\right) c_6 M^{2-d} |x|^{2-d}.$$

Furthermore, similarly to the case with $i = 1, 2, \ldots, N_M - 1$, it holds that

$$\inf_{z \in \widetilde{Q}_i} \mathbf{P}^z(F_i) \geq c_6 \varepsilon, \qquad i \in \{N_M + 1, N_M + 2, \ldots, 2N_M\},$$

and it follows from (2.8) that

$$\inf_{z \in \widetilde{Q}_{2N_M+1}} \mathbf{P}^z(U_{N_M}) \geq c$$

for some constant $c>0$. Finally, by the strong Markov property, we have that

$$\begin{aligned}
&\mathbf{P}(\Theta(C,l))\\
&\geq \mathbf{P}(F_0\cap G_0(C))\prod_{i=1}^{N_M-1}\inf_{z\in\widetilde{Q}_i}\mathbf{P}^z\left(F_i\cap G_i(C)\cap H_i(l)\right)\\
&\qquad\qquad\times\inf_{z\in\widetilde{Q}_{N_M}}\mathbf{P}^z(F_{N_M}\cap G_{N_M}(C))\times\prod_{i=N_M+1}^{2N_M}\inf_{z\in\widetilde{Q}_i}\mathbf{P}^z(F_i)\times\inf_{z\in\widetilde{Q}_{2N_M+1}}\mathbf{P}^z(U_{N_M})\\
&\geq CM^{2-d}|x|^{2-d}e^{-\frac{c}{M}}\\
&\geq CM^{1-d/2}|x|^{2-d}e^{-\frac{c}{M}},
\end{aligned}$$

where the third inequality holds simply because $M\leq 1$. □

5.1.4 Lower bound for $M\geq 1$

We now turn to the proof of the lower bound of Proposition 5.1 with $M\geq 1$. In particular, we will establish the following.

Proposition 5.13. *There exist constants $c,c',R\in(0,\infty)$ such that for every $x\in\mathbb{Z}^d\backslash\{0\}$ and $M\geq 1$,*

$$\mathbf{P}\left(\tau_x\in[R^{-1}M|x|^2,RM|x|^2]\right)\geq c'(M|x|^2)^{1-d/2}\exp\left(-\frac{c}{M}\right).$$

As in the previous subsection, the basic strategy involves the construction of a set of particular realizations of L that we can show occur with suitably high probability. To do this, we will use a certain reversibility property of the simple random walk, as is set out in the next lemma. In the statement of this, for a finite path $\lambda=[\lambda(0),\lambda(1),\cdots,\lambda(m)]$, we write $\lambda^R=[\lambda(m),\lambda(m-1),\cdots,\lambda(0)]$ for its time reversal.

Lemma 5.14. *Let $x,z\in\mathbb{Z}^d$, $x\neq z$, and write S^x, S^z for independent simple random walks in $\mathbb{Z}^d$ started at x, z, respectively. Moreover, write $\tau^z_x:=\inf\{j:S^z_j=x\}$, $\tau^x_z:=\inf\{j:S^x_j=z\}$,*

$$\sigma_1=\sup\{j\leq\tau^x_z\ :\ S^x_j=x\},\quad u=\inf\{j\geq\tau^x_z\ :\ S^x_j=x\},\quad \sigma_2=\sup\{j<u\ :\ S^x_j=z\}.$$

It then holds that

$$\left\{\lambda\ :\ \mathbf{P}\left((S^z[0,\tau^z_x])^R=\lambda\middle|\tau^z_x<\infty\right)>0\right\}=\left\{\lambda\ :\ \mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda\,|\,\tau^z_x<\infty\right)>0\right\},\quad(5.39)$$

and, denoting the set above Λ,

$$\mathbf{P}\left((S^z[0,\tau^z_x])^R=\lambda\middle|\tau^z_x<\infty\right)=\mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda\,|\,\tau^x_z<\infty\right),\qquad\forall\lambda\in\Lambda.\qquad(5.40)$$

Proof. Since (5.39) is easy to see, we only check (5.40). Take $\lambda=[\lambda(0),\lambda(1),\cdots,\lambda(m)]\in\Lambda$. Note that $\mathbf{P}(\tau^z_x<\infty)=\mathbf{P}(\tau^x_z<\infty)$ by symmetry. It follows that

$$\mathbf{P}\left((S^z[0,\tau^z_x])^R=\lambda,\ \tau^z_x<\infty\right)=\mathbf{P}\left(S^z[0,\tau^z_x]=\lambda^R,\ \tau^z_x<\infty\right)=\mathbf{P}\left(S^z[0,\tau^z_x]=\lambda^R\right)=(2d)^{-m}.$$

On the other hand, we have

$$
\begin{aligned}
\mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda,\ \tau_z^x<\infty\right) &= \mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda\right)\\
&= \sum_{k\geq 0}\mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda,\ \sigma_1=k\right)\\
&= \sum_{k\geq 0}\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x,\ S^x[k,k+m]=\lambda,\ \sigma_2=k+m\right)\\
&= \sum_{k\geq 0}\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x,\ S^x[k,k+m]=\lambda,\ S^x_{k+m}=z,\ F\right),
\end{aligned}
$$

where $F:=\{z\notin S^x[k+m+1,u')\}$ with $u'=\inf\{j\geq k+m:\ S^x_j=x\}$. Therefore, the Markov property ensures that

$$
\begin{aligned}
&\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x,\ S^x[k,k+m]=\lambda,\ S^x_{k+m}=z,\ F\right)\\
&=\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x\right)\mathbf{P}\left(S^x[0,m]=\lambda\right)\mathbf{P}(F')\\
&=(2d)^{-m}\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x\right)\mathbf{P}(F'),
\end{aligned}
$$

where $F':=\{z\notin S^z[1,\tau^z_x]\}$. Writing

$$
\xi_x=\inf\{j\geq 1:\ S^x_j=x\}\qquad\text{and}\qquad p=\mathbf{P}\left(\xi_x<\infty,\ z\notin S^x[0,\xi_x]\right),
$$

we note that

$$
\sum_{k\geq 0}\mathbf{P}\left(z\notin S^x[0,k],\ S^x_k=x\right)=\frac{1}{1-p}.
$$

Moreover, by symmetry again, it holds that $\mathbf{P}(F')=1-p$. Hence we conclude that

$$
\mathbf{P}\left(S^x[\sigma_1,\sigma_2]=\lambda,\ \tau^x_z<\infty\right)=(2d)^{-m},
$$

which gives (5.40). □

In order to explain our application of the previous result, we need to introduce some notation. Let $x\in\mathbb{Z}^d\backslash\{0\}$ and $M\geq 1$. Moreover, set $J=C\sqrt{M|x|^2}$ for some $C\geq 1$ that will be determined later, and, for $i\in\mathbb{Z}$, write $\widehat{b}_i=(2iJ,0,\dots,0)\in\mathbb{R}^d$ and

$$
\widehat{B}_i=B_\infty\left(\widehat{b}_i,J\right),
$$

which represent adjacent cubes of side length $2J$. We also introduce the following smaller cubes centred at $b'=(\frac{9}{4}J,\frac{J}{2},0,\dots,0)\in\mathbb{R}^d$,

$$
\widehat{B}'=B_\infty(b',J/6),\qquad \widehat{B}''=B_\infty(b',J/18)
$$

Note that $\widehat{B}''\subset\widehat{B}'\subset\widehat{B}_1$. See Figure 24 for a sketch showing the cubes $\widehat{B}_{-1}$, $\widehat{B}_0$, $\widehat{B}_1$ and $\widehat{B}'$, as well as some of the other objects that we now define. In particular, we introduce a collection of surfaces:

$$
Q^*=\left\{\frac{5}{2}J\right\}\times[-J,J]^{d-1},
$$

$$
Q_*=\{3J\}\times[-J,J]^{d-1},
$$

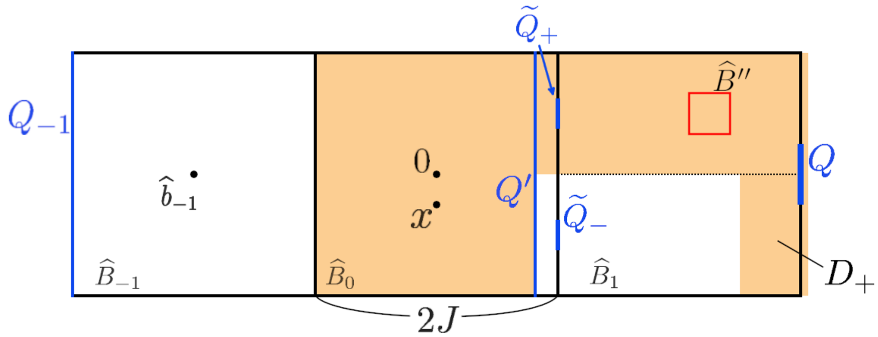


Figure 24: Cubes and other regions appearing in the proof of Proposition 5.13.

$$Q = \{3J\} \times \left[-\frac{J}{4}, \frac{J}{4}\right] \times [-J, J]^{d-2} \subset Q_*,$$

$$\widetilde{Q} = \{J\} \times [-J, J]^{d-1}, \qquad Q' = \left\{(y^1 - J/16, y^2, \cdots, y^d) : (y^1, y^2, \cdots, y^d) \in \widetilde{Q}\right\},$$

$$\widetilde{Q}_\pm = \{J\} \times \left[\pm\frac{J}{2} - \frac{J}{8}, \pm\frac{J}{2} + \frac{J}{8}\right] \times [-J, J]^{d-2} \subset \widetilde{Q},$$

$$Q_{-1} = \widehat{B}_{-1} \cap \{-3J\} \times \mathbb{R}^{d-1};$$

the hyperplane $\mathbb{H}^{(1)}_{85J/36}$, where for $a \in \mathbb{R}$ and $i \in \{1, \ldots, d\}$, we denote

$$\mathbb{H}^{(i)}_a = \left\{(x_1, \cdots, x_d) \in \mathbb{R}^d : x_i = a\right\}$$

(see Figure 26 below for the location of $\mathbb{H}^{(1)}_{85J/36}$ in particular); and also the following regions:

$$D_\pm = \left[-J, \frac{49}{16}J\right] \times [-J, J]^{d-1} \setminus \left[\frac{15}{16}J, \frac{5}{2}J\right] \times \left[-\frac{J}{2} \mp \frac{J}{2}, \frac{J}{2} \mp \frac{J}{2}\right] \times [-J, J]^{d-2},$$

$$\widetilde{D}_\pm = D_\pm \cap \left[\frac{15}{16}J, \infty\right) \times \mathbb{R}^{d-1}.$$

We highlight that D_+ is shown as the shaded region in Figure 24.

Roughly speaking, to establish the main result of this subsection, we will show that, with high enough probability, the loop-erased random walk L passes from 0 to (somewhere close to) Q through D_+, spending a suitable time in $\widehat{B}''$ on the way, before returning to x through D_-, and then escapes to ∞ via Q_{-1}. To make this precise, we will consider an event based on the simple random walk started from 0; see Figure 25. Controlling the probability of this will involve an appeal to Lemma 5.14, through which we obtain a bound that depends on three independent random walks, one started from 0 and two started from x (see Lemma 5.15 below).

Concerning notation, as in the statement of Lemma 5.14, for each $z \in \mathbb{Z}^d$, we will write S^z for a simple random walk started from z. We assume that the elements of the collection $(S^z)_{z\in\mathbb{Z}^d}$ are independent. We moreover write $(\widetilde{S}^z)_{z\in\mathbb{Z}^d}$ for an independent copy of $(S^z)_{z\in\mathbb{Z}^d}$. We also set

$$\tau^z_A := \inf\{k \geq 0 : S^z_k \in A\}, \qquad \sigma^z_A = \sup\{k \geq 0 : S^z_k \in A\},$$

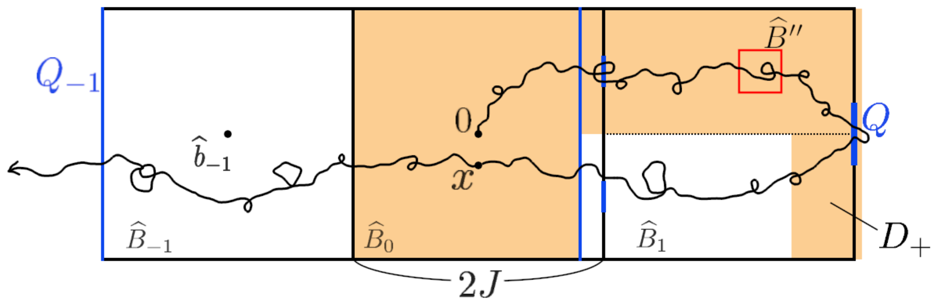


Figure 25: A sketch of a realisation of S^0 yielding $\tau_x \geq M|x|^2$.

$\tau_x^z = \tau_{\{x\}}^z$ and $\sigma_x^z = \sigma_{\{x\}}^z$. A particularly important point in the argument that follows is given by

$$\rho = S^0_{\tau_Q^0},$$

i.e. the location where S^0 hits Q, which is defined when $\tau_Q^0 < \infty$. Additionally, we introduce

$$\widetilde{\tau} = \inf\left\{k \geq 0 : \widetilde{S}_k^x \in \mathbb{R}^d \setminus (\widehat{B}_0 \cup \widehat{B}_{-1})\right\},$$

and, to describe a collection of local cut points for a path λ,

$$\Gamma(\lambda[i,j]) = \{\lambda(k) : \lambda[i,k] \cap \lambda[k+1,j] = \emptyset\}.$$

The following result gives the key decomposition of the simple random walk underlying L that we will consider later in the subsection. It already takes into account the time-reversal property of Lemma 5.14. We will break the complicated event that appears in the statement into several more convenient pieces below.

Lemma 5.15. *In the setting described above,* $\mathbf{P}(\tau_x \geq M|x|^2)$ *is bounded below by the probability of the following event:*

$$\left\{\begin{array}{c} \tau_Q^0 < \infty,\ S^0[0,\tau_Q^0] \subset D_+,\ S^0[0,\sigma^0_{\widehat{B}''}] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset,\ S^0[\tau^0_{Q^*},\tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset, \\ \#(\Gamma(S^0[0,\tau_Q^0]) \cap \widehat{B}'') \geq M|x|^2,\ \tau_\rho^x < \infty,\ S^x[0,\sigma_\rho^x] \subset D_-, \\ (S^0[0,\tau_Q^0] \cap S^x[0,\sigma_\rho^x]) \cap \widehat{B}_0 = \emptyset,\ \widetilde{S}^x \cap (S^0[0,\tau_Q^0] \cup S^x[0,\sigma_\rho^x]) = \emptyset \end{array}\right\}.$$

Proof. Clearly,

$$\left\{\begin{array}{c} \tau_Q^0 < \tau_x^0 < \infty,\ S^0[0,\tau_Q^0] \subset D_+,\ S^0[0,\sigma^0_{\widehat{B}''}] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset,\ S^0[\tau^0_{Q^*},\tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset, \\ \#(\Gamma(S^0[0,\tau_Q^0]) \cap \widehat{B}'') \geq M|x|^2,\ S^0[\tau_Q^0,\tau_x^0] \subset D_-, \\ (S^0[0,\tau_Q^0] \cap S^0[\tau_Q^0,\tau_x^0]) \cap \widehat{B}_0 = \emptyset,\ S^0[\tau_x^0,\infty) \cap (S^0[0,\tau_Q^0] \cup S^0[\tau_Q^0,\tau_x^0]) = \emptyset \end{array}\right\}$$

is a subset of the event $\{\tau_x \geq M|x|^2\}$. Now, conditioning on the value of ρ and applying the strong Markov property at times τ_Q^0 and τ_x^0, we have that the probability of the above event is equal to

$$\sum_{z \in Q} \mathbf{P}\left(\begin{array}{c} \tau_Q^0 < \infty,\ \rho = z,\ \tau_x^z < \infty, \\ S^0[0,\tau_Q^0] \subset D_+,\ S^0[0,\sigma^0_{\widehat{B}''}] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset,\ S^0[\tau^0_{Q^*},\tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset, \\ \#(\Gamma(S^0[0,\tau_Q^0]) \cap \widehat{B}'') \geq M|x|^2,\ S^z[0,\tau_x^z] \subset D_-, \\ (S^0[0,\tau_Q^0] \cap S^z[0,\tau_x^z]) \cap \widehat{B}_0 = \emptyset,\ \widetilde{S}^x \cap (S^0[0,\tau_Q^0] \cup S^z[0,\tau_x^z]) = \emptyset \end{array}\right).$$

Applying Lemma 5.14, we can replace τ^z_x and $S^z[0,\tau^z_x]$ in the above expression by τ^x_z and $S^x[\sigma_1,\sigma_2]$, respectively, where σ_1, σ_2 are defined as in the statement of that result. Since $0 \le \sigma_1 \le \sigma_2 \le \sigma^x_z$, it holds that $S^x[\sigma_1,\sigma_2] \subseteq S^x[0,\sigma^x_z]$. Consequently, we obtain that the above sum is bounded below by

$$\sum_{z\in Q} \mathbf{P}\left(\begin{array}{c} \tau^0_Q < \infty,\ \rho = z,\ \tau^x_z < \infty, \\ S^0[0,\tau^0_Q] \subset D_+,\ S^0[0,\sigma^0_{\widehat{B}''}] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset,\ S^0[\tau^0_{Q^*},\tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset, \\ \#(\Gamma(S^0[0,\tau^0_Q]) \cap \widehat{B}'') \ge M|x|^2,\ S^x[0,\sigma^x_z] \subset D_-, \\ (S^0[0,\tau^0_Q] \cap S^x[0,\sigma^x_z]) \cap \widehat{B}_0 = \emptyset,\ \widetilde{S}^x \cap (S^0[0,\tau^0_Q] \cup S^x[0,\sigma^x_z]) = \emptyset \end{array}\right),$$

and replacing the sum with a union inside the probability completes the proof. □

Now, we will rewrite the event we defined in the statement of Lemma 5.15 as the intersection of various smaller events concerning S^0, S^x and $\widetilde{S}^x$. For convenience, we will write

$$\eta_0 := S^0_{\tau^0_{\widetilde{Q}}}, \qquad \eta_x := S^x_{\tau^x_{\widetilde{Q}}}, \qquad \widetilde{\eta} := \widetilde{S}^x_{\widetilde{\tau}}$$

in the remainder of this subsection. We moreover define the event E_1 by setting

$$E_1 = \left\{\begin{array}{c} \tau^0_{\widetilde{Q}} < \infty,\ \eta_0 \in \widetilde{Q}_+,\ \tau^x_\rho < \infty,\ \eta_x \in \widetilde{Q}_-,\ \widetilde{\tau} < \infty,\ \widetilde{\eta} \in Q_{-1}, \\ (S^0[0,\tau^0_{\widetilde{Q}}] \cap S^x[0,\tau^x_{\widetilde{Q}}]) \cap \widehat{B}_0 = \emptyset, \\ S^0[0,\tau^0_{\widetilde{Q}}] \subset D_+,\ S^x[0,\tau^x_{\widetilde{Q}}] \subset D_-,\ \widetilde{S}^x[0,\widetilde{\tau}] \cap (S^0[0,\tau^0_{\widetilde{Q}}] \cup S^x[0,\tau^x_{\widetilde{Q}}]) = \emptyset \end{array}\right\}.$$

On E_1, the paths S^0, S^x and $\widetilde{S}^x$ do not have an intersection and move along the different courses until they first exit the union of $\widehat{B}_{-1}$ and $\widehat{B}_0$.

Next, we will define some events that restrict the behavior of S^0 after $\tau^0_{\widetilde{Q}}$. Recall that $b' = (\frac{9}{4}J, \frac{J}{2}, 0, \cdots, 0) \in \mathbb{R}^d$ and $\widehat{B}' = B_\infty(b', \frac{J}{6})$. We define the "left" and "right" faces of $\widetilde{B}'$ by

$$Q_L = \left\{\frac{25}{12}J\right\} \times \left[\frac{J}{3}, \frac{2}{3}J\right] \times \left[-\frac{J}{6}, \frac{J}{6}\right]^{d-2}, \qquad Q_R = \left\{\frac{29}{12}J\right\} \times \left[\frac{J}{3}, \frac{2}{3}J\right] \times \left[-\frac{J}{6}, \frac{J}{6}\right]^{d-2}.$$

Moreover, we define a subset of $\widetilde{D}_+$ by setting

$$\widehat{B}'_L = \left[\frac{17}{18}J, \frac{79}{36}J\right] \times \left[\frac{J}{3}, \frac{2}{3}J\right] \times \left[-\frac{J}{2}, \frac{J}{2}\right]^{d-2}.$$

See Figure 26. Then, writing $u^y = \inf\{n \ge \tau^y_{\widetilde{Q}} : S^y_n \in Q'\}$ and $\sigma' = \inf\{n \ge \tau^y_{\widehat{B}''} : S^y_n \in (\widehat{B}')^c\}$, let

$$F_1(y) = \left\{\tau^y_{Q_L} < u^y, S^y[\tau^y_{\widetilde{Q}}, \tau^y_{Q_L}] \subset \widehat{B}'_L\right\}, \tag{5.41}$$

$$F_2(y) = \left\{\begin{array}{c} \tau^y_{\widehat{B}''} < \inf\{n \ge \tau^y_{Q_L} : S^y_n \in (\widehat{B}'_L)^c\} < \infty,\ \sigma' \in Q_R, \\ \#\left\{k \in [\tau^y_{\widehat{B}''}, \sigma'] : S^y[\tau^y_{Q_L}, k] \cap S^y[k+1, \sigma'] = \emptyset,\ S^y_k \in \widehat{B}''\right\} \ge M|x|^2 \end{array}\right\}, \tag{5.42}$$

$$F_3(y) = \left\{\tau^y_{Q^*} < \tau^y_Q < \infty,\ S^y[\sigma', \tau^y_Q] \subset D_+ \cap \left[\frac{85}{36}J, \infty\right) \times \mathbb{R}^{d-1}\right\}, \tag{5.43}$$

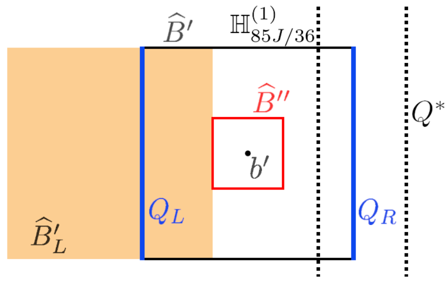


Figure 26: Illustration of the sets used in controlling the number of cut points of S^0 in $\widehat{B}''$.

and set $E_2 = F_1(0) \cap F_2(0) \cap F_3(0)$. In particular, on the event E_2, we have the existence of cut points of $S[\tau^y_{Q_L}, \sigma']$ contained in $\widehat{B}''$. Finally, let

$$E_3 = \left\{ \tau^x_\rho < \infty,\ S^x[\tau^x_{\widetilde{Q}}, \sigma^x_\rho] \subset \widetilde{D}_- \right\},$$
$$E_4 = \left\{ \widetilde{S}^x[\widetilde{\tau}, \infty] \cap (S^0[0, \tau^0_Q] \cup S^x[0, \sigma^x_\rho]) = \emptyset \right\},$$

be events that restrict the regions where S^x and $\widetilde{S}^x$ can explore, respectively.

We continue by checking the local cut points that we construct on the event $F_2(0)$ are cut points of the loop-erasure of $S^0[0, \tau^0_Q]$. Note that on the event E_2, it follows from the definition of $F_2(0)$ and $F_3(0)$ that

$$S[\tau^0_{Q^*}, \tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset, \qquad S[\sigma', \tau^0_{Q_*}] \cap \widehat{B}'' = \emptyset.$$

Moreover, on the event $E_1 \cap E_2$, we have that

- $S^0[0, \tau^0_{\widetilde{Q}}] \cap Q^* = \emptyset$,
- $S^0[\tau^0_{\widetilde{Q}}, \tau^0_{Q_L}] \cap Q^* = \emptyset$,
- $S^0[\tau^0_{Q_L}, \sigma'] \cap Q^* = \emptyset$.

The first and second statements follow from the definitions of E_1 and $F_1(0)$, respectively, while the third statement is derived from the definitions of $F_2(0)$ and σ' (recall the definitions of the sets defined above, which are also shown in Figure 26). From these statements, we immediately conclude that

$$S^0[0, \sigma'] \cap Q^* = \emptyset.$$

For the rest of the path $S^0[0, \tau^0_Q]$, the definition of $F_3(0)$ implies that

$$S^0[\sigma', \tau^0_{Q^*}] \cap \widehat{B}'' = \emptyset, \qquad S^0[\tau^0_{Q^*}, \tau^0_Q] \cap \mathbb{H}^{(1)}_{85/36J} = \emptyset.$$

Combining the preceding three statements, we see that, on $E_1 \cap E_2$,

$$S^0[0, \sigma^0_{\widehat{B}''}] \cap \mathbb{H}^{(1)}_{85/36J} = \emptyset, \qquad S^0[\tau^0_{Q^*}, \tau^0_Q] \cap \mathbb{H}^{(1)}_{85J/36} = \emptyset,$$

where we recall that is $\sigma^0_{\widehat{B}''}$ be the last exit time of $\widehat{B}''$ by S^0 (we assume here that S^0 is stopped at τ^0_Q). Thus, the local cut points of the event $F_2(0)$ are indeed cut points of the loop-erasure of $S^0[0,\tau_Q^0]$ and the probability of the event we defined in the statement of Lemma 5.15 is bounded below by

$$\mathbf{P}(E_1 \cap E_2 \cap E_3 \cap E_4).$$

In what follows, we will bound below this probability below. To start with, we will prove that S^0, S^x and $\widetilde{S}^x$ do not have an intersection and are separated in a cube with positive probability. Let $T_r^z = \tau^z_{B_\infty(0,r)}$. We define the event G_n by setting

$$G_n = \left\{ S^0[0,T_n^0] \cap S^x[0,T_n^x] = S^0[0,T_n^0] \cap \widetilde{S}^x[0,\widetilde{T}_n^x] = S^x[0,T_n^x] \cap \widetilde{S}^x[1,\widetilde{T}_n^x] = \emptyset \right\},$$

and let Z_n be given by

$$\min\left\{ d(S^0_{T_n^0}, S^x[0,T_n^x] \cup \widetilde{S}^x[0,\widetilde{T}_n^x]), d(S^x_{T_n^x}, S^0[0,T_n^0] \cup \widetilde{S}^x[0,\widetilde{T}_n^x]), d(\widetilde{S}^x_{\widetilde{T}_n^x}, S^0[0,T_n^0] \cup S^x[0,T_n^x]) \right\},$$

where d here is the Euclidean distance, i.e. Z_n is the minimum of the distance between the point from which either S^0, S^x or $\widetilde{S}^x$ exits $B_\infty(0,n)$ and the union of the other two paths up to their exit times.

Lemma 5.16. *There exists $c > 0$ and $n_0 \geq 1$ such that: for all $n \geq n_0$,*

$$\mathbf{P}\left(G_n \cap \left\{ Z_n \geq \frac{n}{2} \right\} \right) \geq c.$$

Proof. For readability, we assume that $x = (0, |x|, 0, \cdots, 0)$. (Other cases will follow by a small modification of the argument.) We follow the idea of [6, Lemma 3.2]. Let $e_1 = (1, 0, 0, \cdots, 0) \in \mathbb{Z}^d$ and $e_2 = (0, 1, 0, \cdots, 0) \in \mathbb{Z}^d$. We define the event I_1 by setting

$$I_1 = \left\{ S_i^0 = ie_2,\ S_i^x = ie_1,\ \widetilde{S}_i^x = -ie_1 \text{ for } 1 \leq i \leq k \right\},$$

where $k \geq 1$ will be fixed later. Then we have $\mathbf{P}(I_1) = (2d)^{-3k}$.

We will show that the probability that S^0, S^x and $\widetilde{S}^x$ do not intersect before they first exit from $B_\infty(0,n)$ conditioned on I_1 is bounded above by arbitrarily small ε by taking k sufficiently large. Let $K(j)$ be the number of intersections of $S^0[j,T_n^0]$, $S^x[j,T_n^x]$ and $\widetilde{S}^x[j,\widetilde{T}_n^x]$, i.e.

$$K(j) = \#\left((S^0[j,T_n^0] \cap S^x[j,T_n^x]) \cup (S^0[j,T_n^0] \cap \widetilde{S}^x[j,\widetilde{T}_n^x]) \cup (S^x[j,T_n^x] \cap \widetilde{S}^x[\max\{1,j\},\widetilde{T}_n^x]) \right).$$

Then by the Markov inequality,

$$\begin{aligned}
\mathbf{P}(K(0) > 0 \mid I_1) &\leq \mathbf{P}(K(k) \geq 1 \mid I_1) \\
&\leq \mathbf{E}(K(k) \mid I_1) \\
&\leq 3 \max_{x,y\colon d(x,y) \geq k} \mathbf{E}\left(\#(S^x[0,T_n^x] \cap S^y[0,T_n^y])\right) \\
&\leq 3 \max_{x,y\colon d(x,y) \geq k} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \sum_{z \in \mathbb{Z}^d} \mathbf{P}^x(S_m^x = z) \mathbf{P}^y(S_{m'}^y = z) \\
&\leq 3 \max_{x,y\colon d(x,y) \geq k} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \mathbf{P}^x(S_{m+m'}^x = y).
\end{aligned}$$

By substituting the Gaussian estimate of the transition probability of the simple random walk, the right-hand side is bounded above as follows:

$$
\begin{aligned}
3 \max_{x,y:\, d(x,y)\geq k} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \mathbf{P}^x(S^x_{m+m'} = y) &\leq 3 \sum_{l=1}^{\infty} l \cdot C l^{-d/2} e^{-ck^2/l} \\
&\leq C \sum_{l=1}^{\infty} l^{1-d/2} e^{-ck^2/l}, \qquad (5.44)
\end{aligned}
$$

for some C, $c > 0$, where we applied the Gaussian estimate for the off-diagonal heat kernel of the simple random walk on $\mathbb{Z}^d$ for the last inequality (see (1.10)). Since $d \geq 5$, the right-hand side of (5.44) converges to 0 as $k \to \infty$.

Our next step is to construct subsets where each simple random walk path is constrained to move until it first exits from $B_\infty(0, n)$. We define by

$$
H_L = \left\{-\frac{n}{2}\right\} \times \left[-\frac{n}{4}, \frac{n}{4}\right]^{d-1}, \qquad H_R = \left\{\frac{n}{2}\right\} \times \left[-\frac{n}{4}, \frac{n}{4}\right]^{d-1},
$$

the subsets of the left and right face of $B_\infty(0, \frac{n}{2})$ in the direction of x_1-axis, respectively, and by

$$
H_+ = \left[-\frac{n}{4}, \frac{n}{4}\right] \times \left\{\frac{n}{2}\right\} \times \left[-\frac{n}{4}, \frac{n}{4}\right]^{d-2},
$$

the subset of the upper face of $B_\infty(0, \frac{n}{2})$ in the direction of x_2-axis. Let

$$
\begin{aligned}
I_2^0 &= \left\{ S^0_{T^0_{n/2}} \in H_+,\ S^0_{T^0_n} \in \mathbb{H}^{(2)}_n,\ S^0[T^0_{n/2}, T^0_n] \cap (\mathbb{H}^{(1)}_{n/3} \cup \mathbb{H}^{(1)}_{-n/3} \cup \mathbb{H}^{(2)}_{n/3}) = \emptyset \right\}, \\
I_2^x &= \left\{ S^x_{T^x_{n/2}} \in H_R,\ S^x_{T^x_n} \in \mathbb{H}^{(1)}_n,\ S^x[T^x_{n/2}, T^x_n] \cap (\mathbb{H}^{(1)}_{n/3} \cup \mathbb{H}^{(2)}_{n/3}) = \emptyset \right\}, \\
\widetilde{I}_2^x &= \left\{ \widetilde{S}^x_{\widetilde{T}^x_{n/2}} \in H_L,\ \widetilde{S}^x_{\widetilde{T}^x_n} \in \mathbb{H}^{(1)}_{-n},\ \widetilde{S}^x[\widetilde{T}^x_{n/2},\ \widetilde{T}^x_n] \cap (\mathbb{H}^{(1)}_{-n/3} \cup \mathbb{H}^{(2)}_{n/3}) = \emptyset \right\},
\end{aligned}
$$

and $I_2 = I_2^0 \cap I_2^x \cap \widetilde{I}_2^x$. It is an elementary exercise to check that there exists some $\varepsilon > 0$ such that $\mathbf{P}(I_2 \mid I_1) \geq \varepsilon$ holds uniformly in $n \geq 1$ and $x \in \mathbb{Z}^d$. Note that on the event $I_1 \cap I_2$, it holds that

$$
S^x_{T^x_n} \in \mathbb{H}^{(1)}_n, \qquad S^0[0, T^0_n] \cup \widetilde{S}^x[0, \widetilde{T}^x_n] \subset (-\infty, \frac{n}{2}] \times \mathbb{R}^{d-1},
$$

so that $d(S^x_{T^x_n}, S^0[0, T^0_n] \cup \widetilde{S}^x[0, \widetilde{T}^x_n]) \geq \frac{n}{2}$. Similarly, the same bound holds for $d(S^0_{T^0_n}, S^x[0, T^x_n] \cup \widetilde{S}^x[0, \widetilde{T}^x_n])$ and $d(\widetilde{S}^x_{\widetilde{T}^x_n}, S^0[0, T^0_n] \cup S^x[0, T^x_n])$. Thus we have $Z_n \geq \frac{n}{2}$ on the event in question. Finally, by taking k large so that $\mathbf{P}(K(0) > 0 \mid I_1) \leq \frac{\varepsilon}{2}$, we obtain that

$$
\begin{aligned}
\mathbf{P}\left(G_n \cap \left\{Z_n \geq \frac{n}{2}\right\}\right) &\geq \mathbf{P}(I_1 \cap I_2 \cap \{K(0) = 0\}) \\
&= \mathbf{P}(I_1)\left(\mathbf{P}(I_2 \mid I_1) - \mathbf{P}(K(0) > 0 \mid I_1)\right) \\
&\geq (2d)^{-3k} \varepsilon/2,
\end{aligned}
$$

which completes the proof. □

We are now ready to complete the proof of the main result of this subsection.

Proof of Proposition 5.13. For any $R > 1$, we have that

$$\mathbf{P}\left(\tau_x \in [R^{-1}M|x|^2, RM|x|^2]\right) \geq \mathbf{P}\left(\tau_x \geq M|x|^2\right) - \mathbf{P}\left(\tau_x > RM|x|^2\right)$$

Now, by the argument used to prove Proposition 5.2, we have that

$$\mathbf{P}\left(\tau_x > RM|x|^2\right) \leq C\left(RM|x|^2\right)^{1-d/2}.$$

Thus, by taking R suitably large, and applying Lemma 5.15 and the argument above Lemma 5.16, to complete the proof it suffices to prove that

$$\mathbf{P}\left(E_1 \cap E_2 \cap E_3 \cap E_4\right) \geq cJ^{2-d}$$

for some constant c.

Firstly, we derive an estimate for $\mathbf{P}(E_1)$ from the result of Lemma 5.16. We take $C \geq 1$ large so that $J = C\sqrt{M|x|^2} \geq 3n_0$. We then define the event F' by setting

$$F' = \left\{ \begin{array}{c} \tau^0_{\widetilde{Q}} < \infty,\ \eta_0 \in \widetilde{Q}_+,\ \tau^x_\rho < \infty,\ \eta_x \in \widetilde{Q}_-,\ \widetilde{\tau} < \infty,\ \widetilde{\eta} \in Q_{-1}, \\ S^0[T^0_{J/3}, \tau^0_{\widetilde{Q}}] \cap \left(\mathbb{H}^{(1)}_{J/9} \cup \mathbb{H}^{(2)}_{5J/36}\right) = \emptyset,\ S^x[T^x_{J/3}, \tau^x_{\widetilde{Q}}] \cap \left(\mathbb{H}^{(1)}_{J/9} \cup \mathbb{H}^{(2)}_{5J/36}\right) = \emptyset, \\ \widetilde{S}^x[\widetilde{T}^x_{J/3}, \widetilde{\tau}] \cap \mathbb{H}^{(1)}_{-J/9} = \emptyset \end{array} \right\}.$$

Recall the definition of the events I_1 and I_2 and the random variable $K(j)$ from the proof of Lemma 5.16. It is straightforward to check that if we take $n = J/3$, then

$$I_1 \cap I_2 \cap \{K(0) = 0\} \cap F' \subset E_1.$$

Moreover, by the strong Markov property and the approximation to Brownian motion, there exists some constant $c > 0$ such that $\mathbf{P}(F' \mid I_1 \cap I_2 \cap \{K(0) = 0\}) \geq c$, uniformly in x and M. By Lemma 5.16, we thus obtain that

$$\mathbf{P}(E_1) \geq \mathbf{P}\left(F' \mid I_1 \cap I_2 \cap \{K(0) = 0\}\right) \mathbf{P}(I_1 \cap I_2 \cap \{K(0) = 0\}) \geq c^2. \tag{5.45}$$

Secondly, we estimate $\mathbf{P}(E_2 \mid E_1)$. It follows from the strong Markov property that

$$\mathbf{P}\left(E_2 \mid E_1\right) \geq \inf_{a_1 \in \widetilde{Q}_+} \mathbf{P}^{a_1}\left(F_1(a_1) \cap F_2(a_2) \cap F_3(a_3)\right),$$

where $F_1(y)$, $F_2(y)$ and $F_3(y)$ are as defined in (5.41), (5.42) and (5.43), respectively. Thus it suffices to bound from below the right-hand side of the above inequality. We begin with a lower bound for $\mathbf{P}^{a_1}(F_1)$. By the gambler's ruin estimate (2.10), we have that $\mathbf{P}^{a_1}(F_1) \geq c$ for some universal constant $c > 0$. Next, applying a similar argument to that used to obtain (5.34) in the proof of Lemma 5.12 and the strong Markov property, we obtain that $\mathbf{P}^{a_1}(F_2 \mid F_1) \geq c$. Finally, again by (2.10) and the strong Markov property, we have that $\mathbf{P}^{a_1}(F_3 \mid F_1 \cap F_2) \geq c$. Since $c > 0$ does not depend on $a_1 \in \widetilde{Q}_+$, we can conclude that

$$\mathbf{P}^{a_1}(E_2 \mid E_1) \geq c^3. \tag{5.46}$$

By the independence of S^0 and S^x and the strong Markov property, we also have that

$$\mathbf{P}(E_3 \mid E_1 \cap E_2) \geq \inf_{a_2 \in \widetilde{Q}_-,\, z \in Q} \mathbf{P}^{a_2}\left(\tau^{a_2}_z < \infty,\ S^{a_2}[0, \sigma^{a_2}_z] \subset \widetilde{D}_-\right).$$

Let $a_2 \in \widetilde{Q}_-$ and $z \in Q$. Again by the strong Markov property,

$$\mathbf{P}^{a_2}\left(\tau_z^{a_2} < \infty,\ S^{a_2}[0, \sigma_z^{a_2}] \subset \widetilde{D}_-\right) = \mathbf{P}^{a_2}\left(\tau_z^{a_2} < \infty,\ S^{a_2}[0, \tau_z^{a_2}] \subset \widetilde{D}_-\right) \mathbf{P}^z\left(S^z[0, \sigma_z^z] \subset \widetilde{D}_-\right). \tag{5.47}$$

We will give lower bounds for the two probabilities on the right-hand side. Let $l_{a_2,z}$ be the piecewise linear curve that runs from a_2 in the direction of e_2 until its second coordinate reaches $5J/2$, and then runs along the line from that point to z. Similarly to (5.30) in Lemma 5.12, we obtain that

$$\begin{aligned}\mathbf{P}^{a_2}\left(\tau_z^{a_2} < \infty, S^{a_2}[0, \tau_z^{a_2}] \subset \widetilde{D}_-\right) &\geq \mathbf{P}^{a_2}\left(\tau_z^{a_2} < \infty, \mathrm{dist}(S^{a_2}(k), l_{a_2,z}) \leq J/16 \text{ for all } k \in [0, \tau_z^{a_2}]\right) \\ &\geq cJ^{2-d}, \end{aligned} \tag{5.48}$$

uniformly in a_2 and z for some $c > 0$. Furthermore, we have that

$$\begin{aligned}\mathbf{P}^z\left(S^z[0, \sigma_z^z] \subset \widetilde{D}_-\right) &\geq 1 - \mathbf{P}^z\left(S^z[0, \sigma_z^z] \cap B(z, J/16)^c \neq \emptyset\right) \\ &\geq 1 - \sup_{w \in \partial B(z, J/16)} \mathbf{P}^w(\tau_z^w < \infty) \\ &\geq 1 - \frac{a}{G(0)} J^{2-d},\end{aligned}$$

where we applied (2.9) with $n \to \infty$ to the last inequality. Thus, by increasing the value of the constant $C > 0$ in $J = C\sqrt{M|x|^2}$ if necessary, we have that

$$\mathbf{P}^z\left(S^z[0, \sigma_z^z] \subset \widetilde{D}_-\right) \geq c, \tag{5.49}$$

for some uniform constant $c > 0$. Plugging (5.48) and (5.49) into (5.47) yields that

$$\mathbf{P}(E_3 \mid E_1 \cap E_2) \geq cJ^{2-d}. \tag{5.50}$$

Now we will give a lower bound for $\mathbf{P}(E_4 \mid E_1 \cap E_2 \cap E_3)$. Recall that $Q_{-1} = \widehat{B}_{-1} \cap \{-3J\} \times \mathbb{R}^{d-1}$. By the strong Markov property and the definition of the events E_1, E_2 and E_3, we have that

$$\mathbf{P}(E_4 \mid E_1 \cap E_2 \cap E_3) \geq \inf_{a_3 \in Q_{-1}} \widetilde{\mathbf{P}}^{a_3}(\widetilde{S}^{a_3}[0, \infty] \cap (D_+ \cup D_-) = \emptyset).$$

From this, it is an easy application of (2.8) to deduce that

$$\mathbf{P}\left(E_4 \mid E_1 \cap E_2 \cap E_3\right) \geq c, \tag{5.51}$$

for some universal constant $c > 0$.

Finally, by multiplying each side of (5.45), (5.46), (5.50) and (5.51), we obtain the desired lower bound. $\square$

5.2 Heat kernel estimates for the associated random walk

The aim of this section is to prove Theorem 1.10. As explained in the introduction, the main input concerning the loop-erased random walk will be Theorem 1.11. To estimate $\mathbb{P}(X_t^{\mathcal{G}} = x)$ using the decomposition at (5.1), we also require control over $P^{\mathcal{G}}(X_t^{\mathcal{G}} = L_m)$, where in this section $(L_m)_{m\geq 0}$ is always the infinite LERW started from 0. Since the structure of the graph $\mathcal{G}$ is simply that of $\mathbb{Z}_+$ equipped with nearest-neighbour bonds, we have the obvious identity

$$P^{\mathcal{G}}(X_t^{\mathcal{G}} = L_m) = q_t(0, m),$$

where $(q_t(x, y))_{x,y\in\mathbb{Z}_+,\, t>0}$ gives the transition probabilities of the continuous-time simple random walk on $\mathbb{Z}_+$ with unit mean holding times. For this, we have the following estimates from [2]. (We note that although the result we will cite in [2] is stated for the simple random walk on $\mathbb{Z}$, it is easy to adapt to apply to the half-space $\mathbb{Z}_+$.)

Lemma 5.17. *For any $\varepsilon > 0$, there exist constants $c_1, c_2, c_3, c_4, c_5, c_6 \in (0, \infty)$ such that for every $m \in \mathbb{Z}_+$ and $t \geq \varepsilon m$,*

$$q_t(0, m) \leq c_1 \left(1 \wedge t^{-1/2}\right) \exp\left(-\frac{c_2 m^2}{1 \vee t}\right)$$

and also

$$q_t(0, m) \geq c_3 \left(1 \wedge t^{-1/2}\right) \exp\left(-\frac{c_4 m^2}{1 \vee t}\right).$$

Moreover, for $m \geq 1$ and $t < \varepsilon m$, we have that

$$q_t(0, m) \leq c_5 \exp\left(-c_6 m \left(1 + \log(m/t)\right)\right).$$

Proof. From [2, Theorem 6.28(b)], we obtain the relevant bounds for $t \geq 1 \vee m$. Moreover, the bounds for $m = 0$, $t \in (0, 1)$, follow from [2, Theorem 6.28(d)]. As for $m \geq 1$, $t \in (\varepsilon m, m)$, we can apply [2, Theorem 6.28(c)] to deduce that $q_t(0, m)$ is bounded above and below by an expression of the form:

$$c \exp\left(-c^{-1} m \left(1 + \log(m/t)\right)\right).$$

This can be bounded above and below by an expression of the form $c\exp(-c^{-1}m)$, and that in turn by $c(1 \wedge t^{-1/2}) \exp(-\frac{c^{-1}m^2}{1\vee t})$, uniformly over the range of m and t considered. This completes the proof of the first two inequalities in the statement of the lemma. The third inequality is given by again applying [2, Theorem 6.28(c)]. □

We are now ready to proceed with the proof of Theorem 1.10.

Proof of Theorem 1.10. Clearly, if $x = 0$, then Lemma 5.17 immediately yields

$$\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) = q_t(0, 0) \asymp 1 \wedge t^{-1/2},$$

which gives the result in this case.

We next suppose $x \neq 0$. In this case, applying Lemma 5.17 with $\varepsilon = 1$, we find that

$$\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) &= \sum_{m=1}^{\infty} \mathbb{P}\left(X_t^{\mathcal{G}} = L_m\right) \mathbf{P}\left(L_m = x\right) \\
&= \sum_{m=1}^{\infty} q_t(0,m) \mathbf{P}\left(L_m = x\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \sum_{m=1}^{t} \exp\left(-\frac{c_2 m^2}{1 \vee t}\right) \mathbf{P}\left(L_m = x\right) \\
&\qquad + c_3 \sum_{m=t+1}^{\infty} \exp\left(-c_4 m \left(1 + \log(m/t)\right)\right) \mathbf{P}\left(L_m = x\right). \qquad (5.52)
\end{aligned}$$

Now, the second sum here is readily bounded as follows:

$$c_3 \sum_{m=t+1}^{\infty} \exp\left(-c_4 m \left(1 + \log(m/t)\right)\right) \mathbf{P}\left(L_m = x\right) \leq c_3 \sum_{m=t+1}^{\infty} \exp\left(-c_4 m\right) \leq c_3 \exp\left(-c_5 t\right).$$

Moreover, since we are assuming $t \geq \varepsilon |x| \geq \varepsilon$, the final expression is readily bounded above by one of the form

$$c_6 \left(1 \wedge |x|^{2-d}\right) \left(1 \wedge t^{-1/2}\right) \exp\left(-c_7 \left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right).$$

Thus, to complete the proof of the upper bound in the statement of Theorem 1.10, it remains to derive a similar bound for the first sum on the right-hand side at (5.52). For this, we have that

$$\begin{aligned}
&c_1 \left(1 \wedge t^{-1/2}\right) \sum_{m=1}^{t} \exp\left(-\frac{c_2 m^2}{1 \vee t}\right) \mathbf{P}\left(L_m = x\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \sum_{k=0}^{\infty} \exp\left(-\frac{c_2 (2^k)^2}{1 \vee t}\right) \sum_{m=2^k}^{2^{k+1}-1} \mathbf{P}\left(L_m = x\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \sum_{k=0}^{\infty} \exp\left(-\frac{c_2 (2^k)^2}{1 \vee t}\right) (2^k)^{1-d/2} \exp\left(-\frac{c_3 |x|^2}{2^k}\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \sum_{m=1}^{\infty} m^{-d/2} \exp\left(-\frac{c_2 m^2}{1 \vee t} - \frac{c_3 |x|^2}{m}\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \int_1^{\infty} u^{-d/2} \exp\left(-\frac{c_2 u^2}{1 \vee t} - \frac{c_3 |x|^2}{u}\right) du, \qquad (5.53)
\end{aligned}$$

where we have applied Theorem 1.11 for the second inequality. To bound the integral, we first note that, for any $\delta > 0$, it is possible to find a constant $C < \infty$ such that $a^{d/2} \leq C e^{\delta a}$ for all $a \geq 0$. In particular, choosing $\delta = c_3/2$, this implies that

$$u^{-d/2} = |x|^{-d} \left(\frac{u}{|x|^2}\right)^{-d/2} \leq C |x|^{-d} \exp\left(\frac{c_3 |x|^2}{2u}\right).$$

Hence, applying this estimate and the change of variable $v = u/((1 \vee t)|x|^2)^{1/3}$, we obtain

$$
\begin{aligned}
&\int_1^\infty u^{-d/2} \exp\left(-\frac{c_2 u^2}{1 \vee t} - \frac{c_3|x|^2}{u}\right) du \\
&\leq \; C|x|^{-d} \int_1^\infty \exp\left(-\frac{c_2 u^2}{1 \vee t} - \frac{c_3|x|^2}{2u}\right) du \\
&\leq \; C|x|^{2-d} \left(\frac{|x|^4}{1 \vee t}\right)^{-1/3} \int_0^\infty \exp\left(-\left(c_2 v^2 + \frac{c_3}{2v}\right) \times \left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right) dv. \qquad (5.54)
\end{aligned}
$$

Now, let $f(v) := c_2 v^2 + \frac{c_3}{2v}$, and note that this is a function that has a unique minimum v_0 on $(0, \infty)$ such that $f(v_0) > 0$. Thus, for $|x|^4 \geq 1 \vee t$, the remaining integral above is estimated as follows:

$$
\begin{aligned}
&\int_0^\infty \exp\left(-\left(c_2 v^2 + \frac{c_3}{2v}\right) \times \left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right) dv \\
&\leq \; \int_0^\infty \exp\left(-\left(f(v) - f(v_0)\right)\right) dv \exp\left(-f(v_0)\left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right) \\
&= \; C \exp\left(-c\left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right).
\end{aligned}
$$

Putting this together with (5.53) and (5.54), we deduce the desired result in the range $|x|^4 \geq 1 \vee t$. If $|x|^4 < 1 \vee t$, then we follow a simpler argument to deduce:

$$
\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) &= \sum_{m=1}^\infty q_t(0, m) \mathbf{P}\left(L_m = x\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \sum_{m=1}^\infty \mathbf{P}\left(L_m = x\right) \\
&= c_1 \left(1 \wedge t^{-1/2}\right) \mathbf{P}\left(L_m = x \text{ for some } m \geq 0\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) \mathbf{P}\left(S_m = x \text{ for some } m \geq 0\right) \\
&\leq c_1 \left(1 \wedge t^{-1/2}\right) |x|^{2-d},
\end{aligned}
$$

where we have applied Lemma 5.17 for the first inequality, and (2.9) for the third. This is enough to establish that the upper bound of Theorem 1.10 holds in this case as well.

For the lower bound when $x \neq 0$, we follow a similar argument to the upper bound, but with additional care about the range of summation/integration. In what follows, we set $\alpha = c_4/c_3$, where, here and for the rest of the proof, c_3, c_4 are the constants of Theorem 1.11. Clearly, we can assume that $c_3 \leq 1 < c_4$, so that $\alpha > 1$. Recall that we are also assuming $t \geq \varepsilon|x|$, and without loss of generality, we may suppose $\varepsilon \in (0, 1)$. Applying the bounds of Lemma 5.17 with ε given by

$$
\varepsilon' := \min\left\{\frac{\varepsilon}{1 + \alpha^2}, \frac{\alpha \varepsilon^{4/3}}{4c_4(1 + \alpha^2)}\right\},
$$

we deduce that

$$\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}}=x\right) &= \sum_{m=1}^{\infty} q_t(0,m)\mathbf{P}\left(L_m=x\right)\\
&\geq c\left(1\wedge t^{-1/2}\right)\sum_{m=1}^{\lfloor t/\varepsilon'\rfloor}\exp\left(-\frac{Cm^2}{1\vee t}\right)\mathbf{P}\left(L_m=x\right)\\
&\geq c\left(1\wedge t^{-1/2}\right)\sum_{k=0}^{\lfloor\log_\alpha(\lfloor t/\varepsilon'\rfloor)\rfloor-1}\exp\left(-\frac{C(\alpha^k)^2}{1\vee t}\right)\sum_{m=\lceil\alpha^k\rceil}^{\lfloor\alpha^{k+1}\rfloor}\mathbf{P}\left(L_m=x\right),
\end{aligned}$$

where for the second inequality, we have applied that

$$\left[1,\lfloor t/\varepsilon'\rfloor\right]\supseteq\bigcup_{k=0}^{\lfloor\log_\alpha(\lfloor t/\varepsilon'\rfloor)\rfloor-1}\left[\alpha^k,\alpha^{k+1}\right]$$

and the observation that each m can appear in at most two of the intervals $[\lceil\alpha^k\rceil,\lfloor\alpha^{k+1}\rfloor]$. (We also note that our choice of c ensures $\lfloor\log_\alpha(\lfloor t/\varepsilon'\rfloor)\rfloor-1\geq 1$, and so the sum is non-empty.) Consequently, applying Theorem 1.11 with $n=\alpha^k/c_3$, we find that

$$\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}}=x\right) &\geq c\left(1\wedge t^{-1/2}\right)\sum_{k=0\vee\lceil\log_\alpha(c_3|x|)\rceil}^{\lfloor\log_\alpha(\lfloor t/\varepsilon'\rfloor)\rfloor-1}\exp\left(-\frac{C(\alpha^k)^2}{1\vee t}\right)(\alpha^k)^{1-d/2}\exp\left(-\frac{C|x|^2}{\alpha^k}\right)\\
&\geq c\left(1\wedge t^{-1/2}\right)\sum_{m=1\vee\lceil\alpha c_3|x|\rceil}^{\lfloor\alpha^{-1}\lfloor t/\varepsilon'\rfloor\rfloor} m^{-d/2}\exp\left(-\frac{Cm^2}{1\vee t}-\frac{C|x|^2}{m}\right)\\
&\geq c\left(1\wedge t^{-1/2}\right)\int_{2c_4|x|}^{\alpha t/(1+\alpha^2)\varepsilon'} u^{-d/2}\exp\left(-\frac{Cu^2}{1\vee t}-\frac{C|x|^2}{u}\right)du,
\end{aligned}$$

where we have used that $1\vee\lceil\alpha c_3|x|\rceil=1\vee\lceil c_4|x|\rceil=\lceil c_4|x|\rceil$ to obtain the bottom limit of the integral, and the choice of ε' to obtain the top one. Making the change of variable $v=u\varepsilon^{4/3}/((1\vee t)|x|^2)^{1/3}$ yields a lower bound for the integral of

$$c\left(((1\vee t)|x|^2)^{1/3}\right)^{1-d/2}\int_{2c_4\varepsilon\times(\varepsilon|x|/t)^{1/3}}^{\frac{\alpha\varepsilon^{7/3}}{(1+\alpha^2)\varepsilon'}\times(t/\varepsilon|x|)^{2/3}} v^{-d/2}\exp\left(-C\left(v^2+\frac{1}{v}\right)\times\left(\frac{|x|^4}{1\vee t}\right)^{1/3}\right)dv,$$

and, since $t\geq\varepsilon|x|$, our choice of ε' implies that this is bounded below by

$$\begin{aligned}
&c\left(((1\vee t)|x|^2)^{1/3}\right)^{1-d/2}\int_{2c_4\varepsilon}^{4c_4\varepsilon} v^{-d/2}\exp\left(-C\left(v^2+\frac{1}{v}\right)\times\left(\frac{|x|^4}{1\vee t}\right)^{1/3}\right)dv\\
&\geq c((1\vee t)|x|^2)^{1/3-d/6}\exp\left(-C\left(\frac{|x|^4}{1\vee t}\right)^{1/3}\right).
\end{aligned}$$

Hence, if $|x|^4 \geq 1 \vee t$, we can put the pieces together to find that

$$\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) &\geq c\left(1 \wedge t^{-1/2}\right)\left((1 \vee t)|x|^2\right)^{1/3-d/6}\left(\frac{|x|^4}{1 \vee t}\right)^{1/3-d/6} \exp\left(-C\left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right) \\
&= c\left(1 \wedge t^{-1/2}\right)|x|^{2-d} \exp\left(-C\left(\frac{|x|^4}{1 \vee t}\right)^{1/3}\right),
\end{aligned}$$

as required. Finally, for $|x|^4 < 1 \vee t$, continuing to suppose that c_4 is the constant of Theorem 1.11, we have that

$$\begin{aligned}
\mathbb{P}\left(X_t^{\mathcal{G}} = x\right) &\geq \sum_{m=1}^{\lfloor \sqrt{4c_4^2 t} \rfloor} q_t(0,m)\mathbf{P}\left(L_m = x\right) \\
&\geq c\left(1 \wedge t^{-1/2}\right) \sum_{m=1}^{\lfloor \sqrt{4c_4^2 t} \rfloor} \mathbf{P}\left(L_m = x\right) \\
&\geq c\left(1 \wedge t^{-1/2}\right) \sum_{m=\lceil c_3 \lceil |x| \rceil \rceil}^{\lfloor c_4 \lceil |x| \rceil \rfloor} \mathbf{P}\left(L_m = x\right) \\
&\geq c\left(1 \wedge t^{-1/2}\right)|x|^{2-d},
\end{aligned}$$

where we have applied Lemma 5.17 with $\varepsilon = 1/4c_4^2$ for the second inequality, that $c_4\lceil |x| \rceil \leq 2c_4|x| \leq \sqrt{4c_4^2 t}$ for the third, and Theorem 1.11 for the final one. □

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