Polynomial association schemes and co-polynomial association schemes

Tatsuro Ito
Kanazawa University

1 Introduction

In [13], distance-polynomial graphs are introduced as a generalization of distance-regular graphs. In this expository article, we revisit both distance-polynomial graphs and distance-regular graphs from the viewpoint of the Weisfeiler-Leman stabilization, following [10]. This approach naturally leads us to define *polynomial graphs* or equivalently *polynomial association schemes*, which lie in between them. As the dual concept, we introduce *co-polynomial association schemes*.

I had not known anything about the Weisfeiler-Leman stabilization until I attended the lecture by Igor Faradjev [4], in which I learnt about the life of Boris Weisfeiler, the founder of the W-L stabilization, and the fact that the concept of coherent configurations [6, 7, 8] appeared earlier in the original paper of [14]. The W-L stabilization is in short an algorithm to get the coherent closure of a combinatorial object. For a long time, this concept was practically not studied outside the graph isomorphism problem. For the original approach and its impact thereafter, readers are referred to the preface [12] by Ilia Ponomarenko to the original paper by Weisfeiler and Leman, which was published in 1968. In this article, we reformulate the W-L stabilization in terms of coherent configurations.

In this article, a graph means a simple graph [3], i.e., a finite undirected graph without loops or multiple edges. We use the following notations throughout:

> For a finite set X, $\mathrm{M}_X(\mathbb{C})$ denotes the full matrix algebra consisting of all the matrices over the complex number field $\mathbb{C}$ whose rows and columns are indexed by the elements of X.

For $A \in \mathrm{M}_X(\mathbb{C})$ and $x, y \in X$, $A(x,y)$ denotes the (x,y)-entry of A. ${}^t\overline{A}$ denotes the conjugate transpose of $A \in \mathrm{M}_X(\mathbb{C})$, i.e., ${}^t\overline{A}(x,y) = \overline{A(y,x)}$ for $x, y \in X$, where $\overline{A(y,x)}$ is the complex conjugate of $A(y,x)$. $A \circ B$ denotes the Hadamard product (entry-wise product) of $A, B \in \mathrm{M}_X(\mathbb{C})$, i.e, $(A \circ B)(x,y) = A(x,y)B(x,y)$ for $x, y, \in X$.

$\mathrm{M}_X(\mathbb{C})^\circ$ denotes the algebra with respect to the Hadamard product over the underlying vector space $\mathrm{M}_X(\mathbb{C})$.

The W-L stabilization proceeds as follows. We are given a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})^\circ$ which is closed under the conjugate transpose, i.e., ${}^t\overline{A} \in \mathcal{F}$ for $A \in \mathcal{F}$. First take the closure in $\mathrm{M}_X(\mathbb{C})^\circ$ of $\mathcal{F}$, and then the closure in $\mathrm{M}_X(\mathbb{C})$ of the resulting subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$. Continue these operations of taking the closures in turn in $\mathrm{M}_X(\mathbb{C})^\circ$ and then in $\mathrm{M}_X(\mathbb{C})$. Then the sequence stops in a finite number of steps and we get the coherent closure of $\mathcal{F}$, i.e., the smallest coherent algebra that contains $\mathcal{F}$. Let $\mathcal{M}(\mathcal{F})$ denote the coherent closure of $\mathcal{F}$. As is well-known, a combinatorial object, which is called a coherent configuration, corresponds with the coherent algebra $\mathcal{M}(\mathcal{F})$.

Basics of coherent algebras, coherent configurations and association schemes will be summarised in Section 2.1. In Section 2.2, the W-L stabilization will be reformulated regorously in terms of coherent algebras. In particular, we introduce the concept of the *coherent length*, denoted by $r(\mathcal{F})$, for a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})^\circ$: the coherent length $r(\mathcal{F})$ of $\mathcal{F}$ is defined as the number of steps we need to reach $\mathcal{M}(\mathcal{F})$ in the process of the W-L stabilization.

In Section 3, connected regular graphs will be treated in the framework of the W-L stabilization. Let Γ be a connected regular graph, X the vertex set of Γ and A the adjacency matrix of Γ. By setting $\mathcal{F} = \{A\}$, we apply the W-L stabilization. Let $r = r(\mathcal{F})$ be the coherent length. If Γ is a distance-regular graph, we have $r = 1$, unless it is a complete graph, in which case $r = 0$. If Γ is a distance-polynomial graph, we have $r \leq 2$. We now define Γ to be a *polynomial graph* by the condition $r = 1$. Then we have the inclusion

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2,$$

where $\mathcal{C}_i$ is the set of distance-regular graphs, polynomial graphs, distance-polynomial graphs for $i = 0, 1, 2$, respectively. It seems that the gap is huge between $\mathcal{C}_0$ and $\mathcal{C}_1$, and also between $\mathcal{C}_1$ and $\mathcal{C}_2$.

A class of symmetric association schemes, which we call *polynomial association schemes*, natually arises from the class $\mathcal{C}_1$ of polynomial graphs. Those that arise from the class $\mathcal{C}_0$ of distance-regular graphs are called P-polynomial association schemes. As is well known, the duals of P-polynomial association schemes can be defined and they are called Q-polynomial association schemes. As the duals of polynomial association schemes, we define *co-polynomial association schemes* in Section 4. So we have the inclusion

$$\mathcal{C}_0^* \subseteq \mathcal{C}_1^*,$$

where $\mathcal{C}_i^*$ is the set of Q-polynomial association schemes, co-polynomial association schemes for $i = 0, 1$, respectively. The class $\mathcal{C}_1^*$ seems much bigger than $\mathcal{C}_0^*$; perhaps the gap is as big as between $\mathcal{C}_0$ and $\mathcal{C}_1$. Similarly, we can naturally define a class $\mathcal{C}_2^*$ as the dual objects of distance-polynomial graphs, using co-distance matrices (see (55) in Section 4), but it will not be covered in this article: it does not have combinatorial meanings, because it is a class of semi-simple algebras (with respect to the ordinary matrix product) that are not necessarily closed under the Hadamard product.

There is a celebrated conjecture that in the category of primitive symmetric association schemes with sufficiently large diameter, P-polynomial association schemes are Q-polynomial and viceversa. The classification of (P and Q)-polynomial association schemes with sufficiently large diameter is one of the central problems in algebraic combinatorics. Readers are referred to [1, 2].

Recently, Jack Koolen et al [15] showed that P-polynomial association schemes are co-polynomial and that Q-polynomial association schemes are polynomial. Koolen conjectures that symmetric association schemes are co-polynomial if they do not have non-trivial fusion schemes (personal communication). Concerning this conjecture, Eiichi Bannai and Da Zhao made interesting observations on the group association schemes of some finite simple groups (personal communication). At the end of this article, we briefly visit these conjectures and observations.

2 Preliminaries

2.1 Coherent algebras

In this subsection, we summarize basics of coherent algebras for late use. The article [9] can serve as an excellent introduction to the representation theory of coherent algebras.

We keep the notations in the previous section. Recall that $\mathrm{M}_X(\mathbb{C})$ is the full matrix algebra consisting of all the matrices over the complex number field $\mathbb{C}$ whose rows and columns are indexed by the elements of a finite set X, and that $\mathrm{M}_X(\mathbb{C})^\circ$ is the algebra with respect to the Hadamard product over the underlying vector space $\mathrm{M}_X(\mathbb{C})$. Let $I \in \mathrm{M}_X(\mathbb{C})$ (resp. $J \in \mathrm{M}_X(\mathbb{C})^\circ$) denote the identity matrix (resp. the matrix with all entries 1). Note that I (resp. J) is the identity of the algebra $\mathrm{M}_X(\mathbb{C})$ (resp. the identity of the algebra $\mathrm{M}_X(\mathbb{C})^\circ$).

A linear subspace $\mathcal{M}$ of $\mathrm{M}_X(\mathbb{C})$ is said to be a *coherent algebra* if it satisfies the following three conditions:

(i) $\mathcal{M}$ is a subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$, containing J.

(ii) $\mathcal{M}$ is a subalgebra of $\mathrm{M}_X(\mathbb{C})$, containing I.

(iii) $\mathcal{M}$ is closed under the conjugate transpose, i.e., ${}^t\overline{A} \in \mathcal{M}$, $\forall A \in \mathcal{M}$.

The smallest coherent algebra is the linear space $\mathrm{Span}\{I, J\}$ and the largest one is $\mathrm{M}_X(\mathbb{C})$.

For any matrix A, we understand the 0th power of A in $\mathrm{M}_X(\mathbb{C})$ (resp. $\mathrm{M}_X(\mathbb{C})^\circ$) is I (resp. J), and we require a subalgebra to be closed under the operation of taking any power of its elements. This is why a subalgebra of $\mathrm{M}_X(\mathbb{C})$ (resp. $\mathrm{M}_X(\mathbb{C})^\circ$) is required to contain I (resp. J).

For a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})$, the *coherent closure* of $\mathcal{F}$ is defined to be the intersection of coherent algebras $\mathcal{M} \subseteq \mathrm{M}_X(\mathbb{C})$ that contain $\mathcal{F}$:

$$\mathcal{M}(\mathcal{F}) = \bigcap_{\mathcal{F} \subseteq \mathcal{M}\colon \text{coherent alg}} \mathcal{M}. \tag{1}$$

Note that this intersection is not empty, since there exists at least one coherent algebra that contains $\mathcal{F}$, for example $\mathrm{M}_X(\mathbb{C})$. The coherent closure $\mathcal{M}(\mathcal{F})$ turns out to be the smallest coherent algebra that contains $\mathcal{F}$, because the intersection of coherent algebras is again a coherent algebra.

For a non-empty subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})$, the *automorphism group* of $\mathcal{F}$ is defined to be the group consisting of permutaion matrices of $\mathrm{M}_X(\mathbb{C})$ that commute with all elements of $\mathcal{F}$:

$$\mathrm{Aut}(\mathcal{F}) = \{P \in \mathrm{Sym}(X) \mid PA = AP,\ \forall A \in \mathcal{F}\}, \tag{2}$$

where $\mathrm{Sym}(X)$ denotes the symmetric group on X, identified with the set of permutation matrices in $\mathrm{M}_X(\mathbb{C})$. If Γ is a graph and A is the adjacency matrix of Γ, then $\mathrm{Aut}(\mathcal{F})$ with $\mathcal{F} = \{A\}$ coincides with the automorphism group of Γ.

Any subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$ is commutative and semi-simple, because it does not contain nilpotent elements. So let $\{A_i \,|\, i \in \Lambda\}$ denote the primitive idempotents for a subalgebra $\mathcal{M}$ of $\mathrm{M}_X(\mathbb{C})^\circ$. The above three conditions (i)-(iii) for the linear subspace $\mathcal{M} = \mathrm{Span}\{A_i \,|\, i \in \Lambda\}$ are rephrased in terms of them as follows:

(i') Each A_i $(i \in \Lambda)$ is a $(0,1)$ matrix and $J = \sum_{i\in\Lambda} A_i$.

(ii') For any $i, j \in \Lambda$, A_iA_j is a linear combination of $A_k's$, $k \in \Lambda$ and I is a linear combination of $A_k's$, $k \in \Lambda$.

(iii') The set $\{A_i \,|\, i \in \Lambda\}$ is closed under conjugate transpose, i.e., $\{{}^t\overline{A_i} \,|\, i \in \Lambda\} = \{A_j \,|\, j \in \Lambda\}$.

Since A_i is a $(0,1)$ matrix, it corresponds with a relation R_i on X: $(x,y) \in R_i$ if and only if $A_i(x,y) = 1$. A combinatorial structure $\mathfrak{X} = (X, \{R_i\}_{i\in\Lambda})$, where $R_i \subset X \times X$, is called a *coherent configuration* if it comes from a coherent algebra, or equivalently if the set $\{A_i \,|\, i \in \Lambda\}$ of $(0,1)$ matrices, which corresponds with the set $\{R_i\}_{i\in\Lambda}$ of relations on X, satisfies the above conditions (i')-(iii').

A coherent configuration $\mathfrak{X} = (X, \{R_i\}_{i\in\Lambda})$ is called an *association scheme* if one of R_i, $i \in \Lambda$, is the diagonal relation, in which case we usually choose R_0 to be the diagonal relation, i.e., $A_0 = I$. An association scheme is called *symmetric* if each relation R_i, $i \in \Lambda$, is symmetric, i.e., each A_i, $i \in \Lambda$, is a symmetric matrix. It is well-known and easy to show that if a coherent algebra $\mathcal{M}$ is symmetric, i.e., if all the matrices of $\mathcal{M}$ are symmetric, then $\mathcal{M}$ is commutative (as a subalgebra of $\mathrm{M}_X(\mathbb{C})$). It is also well-known and easy to show that if a coherent algebra $\mathcal{M}$ is commutative, then the corresponding coherent configuration is an association scheme. An association scheme $\mathfrak{X} = (X, \{R_i\}_{i\in\Lambda})$ is said to be *primitive* if for all $i \neq 0$, the graph $\Gamma(X, R_i)$ with X the vertex set and R_i the edge set is connected. An association scheme $\mathfrak{X} = (X, \{R_i\}_{i\in\Lambda})$ is said to have a *fusion*, if there exists a partition $\Lambda = \cup_{j\in\Delta}\Lambda_j$ of Λ such that $\tilde{\mathfrak{X}} = (X, \{R_{\Lambda_j}\}_{j\in\Delta})$ with $R_{\Lambda_j} = \cup_{i\in\Lambda_j} R_i$ is an association scheme. In this case, $\tilde{\mathfrak{X}}$ is called a *fusion scheme* of $\mathfrak{X}$, and $\mathfrak{X}$ is called a *fission scheme* of $\tilde{\mathfrak{X}}$. The fusion is said to be *trivial* if $|\Delta| = 2$ or $|\Delta| = |\Lambda|$.

Note that any subalgebra of $\mathrm{M}_X(\mathbb{C})$ is semi-simple if it is closed under the conjugate transpose. So by (iii), a coherent algebra $\mathcal{M}$ is semi-simple as a subalgebra of $\mathrm{M}_X(\mathbb{C})$, and we can *use representations of $\mathcal{M}$ to analyse the coherent configurations* $\mathfrak{X} = (X, \{R_i\}_{i\in\Lambda})$: the motto of algebraic combinatorics.

2.2 The Weisfeiler-Leman stabilization

In this subsection, we explain the Weisfeiler-Leman stabilization, which is an algorithm to obtain the coherent closure $\mathcal{M}(\mathcal{F})$ for a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})$. A fairly complete picture of the Weisfeiler-Leman algorithm is available in the monograph [5].

Given a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})$, the *closure* in $\mathrm{M}_X(\mathbb{C})$ of $\mathcal{F}$ is defined to be the smallest subalgebra of $\mathrm{M}_X(\mathbb{C})$ that contains $\mathcal{F}$ and I, or equivalently to be the subalgebra of $\mathrm{M}_X(\mathbb{C})$ generated by $\mathcal{F}$ and I:

$$\langle\, \mathcal{F},\, I\,\rangle \subseteq \mathrm{M}_X(\mathbb{C}). \tag{3}$$

Similarly, given a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})^\circ$, the *closure* in $\mathrm{M}_X(\mathbb{C})^\circ$ of $\mathcal{F}$ is defined to be the smallest subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$ that contains $\mathcal{F}$ and J, or equivalently to be the subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$ generated by $\mathcal{F}$ and J:

$$\langle\, \mathcal{F},\, J\,\rangle^\circ \subseteq \mathrm{M}_X(\mathbb{C})^\circ. \tag{4}$$

For a subset $\mathcal{F}$ of $\mathrm{M}_X(\mathbb{C})$, define ${}^t\overline{\mathcal{F}}$ by

$${}^t\overline{\mathcal{F}} = \{{}^t\overline{A} \,|\, A \in \mathcal{F}\}. \tag{5}$$

The coherent closure $\mathcal{M}(\mathcal{F})$ contains ${}^t\overline{\mathcal{F}}$, since $\mathcal{M}(\mathcal{F})$ is closed under the conjugate transpose. In order to find $\mathcal{M}(\mathcal{F})$, we may replace $\mathcal{F}$ by $\mathcal{F} \cup {}^t\overline{\mathcal{F}}$ and assume $\mathcal{F}$ is closed under the conjugate transpose from the beginning: ${}^t\overline{\mathcal{F}} = \mathcal{F}$.

Define a sequence of subalgebras $\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots$ of $\mathrm{M}_X(\mathbb{C})^\circ$ and a sequence of subalgebras $\mathcal{A}_1 \subseteq \mathcal{A}_3 \subseteq \mathcal{A}_5 \subseteq \cdots$ of $\mathrm{M}_X(\mathbb{C})$ by setting $\mathcal{A}_0 = \langle\, \mathcal{F},\, J\,\rangle^\circ \subseteq \mathrm{M}_X(\mathbb{C})^\circ$ and

$$\mathcal{A}_{2i} = \langle\, \mathcal{A}_{2i-1},\, J\,\rangle^\circ \subseteq \mathrm{M}_X(\mathbb{C})^\circ, \quad i = 1, 2, 3 \cdots, \tag{6}$$

$$\mathcal{A}_{2i+1} = \langle\, \mathcal{A}_{2i},\, I\,\rangle \subseteq \mathrm{M}_X(\mathbb{C}), \quad i = 0, 1, 2, \cdots, \tag{7}$$

using the closures (3), (4), inductively. Then we have a sequence of linear subspaces $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \cdots$ of the coherent closure $\mathcal{M}(\mathcal{F})$ of $\mathcal{F}$. If $\mathcal{A}_i = \mathcal{A}_{i+1}$, it holds that $\mathcal{A}_i = \mathcal{A}_{i+1} = \cdots = \mathcal{M}(\mathcal{F})$. Since $\dim\mathcal{M}(\mathcal{F}) \leq \dim\mathrm{M}_X(\mathbb{C}) < \infty$, there exists $r = r(\mathcal{F})$ such that

$$\mathcal{A}_0 \subsetneqq \mathcal{A}_1 \subsetneqq \cdots \subsetneqq \mathcal{A}_{r-1} \subsetneqq \mathcal{A}_r = \mathcal{A}_{r+1} = \cdots = \mathcal{M}(\mathcal{F}). \tag{8}$$

We call $r = r(\mathcal{F})$ the *coherent length* of $\mathcal{F}$. Essentially $r = r(\mathcal{F})$ is what is called the iteration number in [11].

We close this subsection with remarks on the sequence of the linear subspaces $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \cdots$ of the coherent closure $\mathcal{M}(\mathcal{F})$ of $\mathcal{F}$ that eventually reach $\mathcal{M}(\mathcal{F})$. Firstly, $\mathrm{Aut}(\mathcal{A}_i) = \mathrm{Aut}(\mathcal{F})$ holds for all i. Secondly, $\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots$ is a sequence of commutative semi-simple algebras of $\mathrm{M}_X(\mathbb{C})^\circ$, and so each of them induces a set of relations on X by means of its primitive idempotents. Thirdly, $\mathcal{A}_1 \subseteq \mathcal{A}_3 \subseteq \mathcal{A}_5 \subseteq \cdots$ is a sequence of semi-simple algebras of $\mathrm{M}_X(\mathbb{C})$, and so their representaions can be used to analyse the interactions between the relations on X that are induced by the sequence $\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots$.

3 Polynomial association schmes

In this section, we analyse distance-polynomial graphs and distance-regular graphs from the viewpoint of the Weisfeiler-Leman stabilization. This approach naturally leads us to define *polynomial graphs* or equivalently *polynomial association schemes*, which lie in between them. Given the 1st eigenmatrix, we can check whether an association scheme is polynomial or not.

Let Γ be a connected graph, X the vertex set of Γ, and A the adjacency matrix of Γ. The distance function of the graph Γ is denoted by ∂, i.e., $\partial(x, y)$ is the length of a shortest path joining x and y $(x, y \in X)$.

First, set $\mathcal{F} = \{A\}$ and apply the W-L stabilization to $\mathcal{F}$. By (6), (7), we have the sequence $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \cdots$, which eventually reaches the coherent closure $\mathcal{M} = \mathcal{M}(\mathcal{F})$ of $\mathcal{F}$. The first three are

$$\mathcal{A}_0 = \mathrm{Span}\{A,\ J\} \subseteq \mathrm{M}_X(\mathbb{C})^\circ, \tag{9}$$

$$\mathcal{A}_1 = \langle\, A, I, J\,\rangle \subseteq \mathrm{M}_X(\mathbb{C}), \tag{10}$$

$$\mathcal{A}_2 = \langle\, \mathcal{A}_1\,\rangle^\circ \subseteq \mathrm{M}_X(\mathbb{C})^\circ. \tag{11}$$

Note that $\mathcal{A}_1 = \langle\, A, I, J\,\rangle$ is the generalized adjacency algebra of Γ.

Define the *ith distance matrix* A_i by

$$A_i(x, y) = 1 \ \text{ if } \partial(x, y) = i,\ 0 \text{ otherwise}. \tag{12}$$

For $i = 0, 1$, we have $A_0 = I$, $A_1 = A$. The following lemma holds.

Lemma 1 *For $i = 2, 3, \cdots$, the distance matrix A_i is contained in $\mathcal{A}_2$.*

We now assume in addition that our graph Γ is regular with valency k. Observe that the all-ones matrix J is a polynomial in A, since $\frac{1}{|X|}J$ is the projection onto the eigenspace belonging to the eigenvalue k of A. So instead of (10), (11), we have

$$\mathcal{A}_1 = \langle\, A \,\rangle = \mathrm{Span}\{A^i \,|\, 0 \leq i \leq D\} \subseteq \mathrm{M}_X(\mathbb{C}), \tag{13}$$

$$\mathcal{A}_2 = \langle\, A^i \,|\, 0 \leq i \leq D \,\rangle^\circ \subseteq \mathrm{M}_X(\mathbb{C})^\circ, \tag{14}$$

where $D+1$ is the degree of the minimal polynomial of A. Note that $\mathcal{A}_1$ is the adjacency algebra of Γ. Let $\mathrm{diam}(\Gamma)$ denote the diameter of Γ. Then we have

$$\mathrm{diam}(\Gamma) \leq D. \tag{15}$$

Let $r = r(\mathcal{F})$ be the coherent length from (8). Then $r = 0$ if and only if Γ is a complete graph or equivalently $\mathrm{diam}(\Gamma) = 1$. This is because by (9),

$$\mathcal{A}_0 = \mathrm{Span}\{A,\ J\} \subseteq \mathrm{M}_X(\mathbb{C})^\circ$$

is required to coincide with the minimun coherent algebra of dimension 2, if $r = 0$.

We are interested in the class of regular connected graphs that have $r = 1$. Suppose $r = 1$, i.e., $\mathcal{A}_1 = \mathcal{A}_2$ and $2 \leq \mathrm{diam}(\Gamma)$. Let $\mathcal{M} = \mathcal{M}(\mathcal{F})$ be the coherent closure of $\mathcal{F} = \{A\}$. Then by (13) we have

$$\mathcal{M} = \mathrm{Span}\{A^i \,|\, 0 \leq i \leq D\}. \tag{16}$$

Let $\{A_\alpha \,|\, \alpha \in \Lambda\}$ denote the set of primitive idempotents of $\mathcal{M}$ with respect to the Hadamard product:

$$\mathcal{M} = \mathrm{Span}\{A_\alpha \,|\, \alpha \in \Lambda\}, \tag{17}$$

$$A_\alpha \circ A_\beta = \delta_{\alpha,\beta} A_\alpha, \quad \sum_{\alpha \in \Lambda} A_\alpha = J, \tag{18}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta. Then there arises a symmetric association scheme

$$\mathfrak{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda}) \tag{19}$$

for which $\mathcal{M}$ is the Bose-Mesner algebra, i.e., R_α is the relation on X corresponding to the $(0, 1)$ matrix $A_\alpha, \alpha \in \Lambda$. Note that there is a special $\alpha_0 \in \Lambda$ for which

$$A_{\alpha_0} = I. \tag{20}$$

Note also

$$\dim \mathcal{M} = D + 1 = |\Lambda|. \tag{21}$$

Definition 2 *A symmetric association scheme* $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ *is called polynomial if its Bose-Mesner algebra* $\mathcal{M}$ *satisfies* (16) *for some* $A \in \mathcal{M}$ *with the property that* $A \circ A = A$ *and* $A \circ I = 0$*. Note that* $A \circ A = A$ *holds if and only if* A *is a* $(0,1)$ *matrix.*

So from a regular connected graph Γ with $r = 1$, a polynomial association scheme naturally arises. In this sense, a regular connected graph Γ with $r = 1$ is called *polynomial*. Conversely start with a polynomial association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$. Then the Bose-Mesner algebra of $\mathfrak{X}$ is generated by a $(0,1)$ matrix A with $A \circ I = 0$. It is easy to see that the graph Γ for which A is the adjacency matrix is regular, connected and has $r = 1$, unless the Bose-Mesner algebra is spanned by I, J, in which case $r = 0$. So from a polynomial association scheme, a polynomial graph natually arises. Note that the $(0,1)$ matrix A with $A \circ I = 0$ that generates the Bose-Mesner algebra of a polynomial association scheme may not be uniquely determined.

For a connected graph Γ, let A_i, $0 \le i \le \mathrm{diam}(\Gamma)$, be the distance matrices from (12). By Lemma 1, it always holds that

$$\mathrm{Span}\{A_i \,|\, 0 \le i \le \mathrm{diam}(\Gamma)\} \subseteq \mathcal{A}_2. \tag{22}$$

If Γ is regular and a stronger condition

$$\mathrm{Span}\{A_i \,|\, 0 \le i \le \mathrm{diam}(\Gamma)\} \subseteq \mathcal{A}_1 \tag{23}$$

holds for Γ, we call Γ *distance-polynomial*. In other words, Γ is distance-polynomial if each distance matrix A_i is a polynomial in the adjacency matrix A of Γ: this is the original definition, in which Γ is not assumed to be regular, but since the super-regularity, particularly the regularity, is derived from the original definition [13], we can add the regularity to the defining condition from the beginning. A distance-polynomial graph Γ is called *distance-regular* if the equality holds in (23), i.e.,

$$\mathcal{A}_1 = \mathrm{Span}\{A_i \,|\, 0 \le i \le \mathrm{diam}(\Gamma)\}. \tag{24}$$

Observe that a distance-polynomial graph Γ is distance-regular if and only if the equality holds in (15), i.e., $\mathrm{diam}(\Gamma) = D$. This is because $\dim \mathcal{A}_1 = D + 1$ by (13). Also observe that the condition (24) implies that $\mathcal{A}_1$ is closed under the Hadamard product and hence $\mathcal{A}_1 = \mathcal{A}_2$.

Therefore if a graph Γ is distance-regular, then Γ is polynomial, and if a graph Γ is polynomial, then Γ is distance-polynomial. We are interested in the gap between the class of distance-regular graphs and that of polynomial graphs, since the gap is formulated in terms of symmetric association schemes.

We start with a symmetric association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$. The Bose-Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$ is

$$\begin{aligned} \mathcal{M} &= \mathrm{Span}\{A_\alpha \,|\, \alpha \in \Lambda\} && (25)\\ &= \mathrm{Span}\{E_\lambda \,|\, \lambda \in \Lambda\}, && (26) \end{aligned}$$

where $\{A_\alpha \,|\, \alpha \in \Lambda\}$ is the set of primitive idempotents of $\mathcal{M}$ with respect to the Hadamard product, which is characterized by (18), and $\{E_\lambda \,|\, \lambda \in \Lambda\}$ is the set of primitive idempotents of $\mathcal{M}$ with respect to the ordinary matrix product:

$$E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda, \quad \sum_{\lambda\in\Lambda} E_\lambda = I. \tag{27}$$

Note that there is a special $\lambda_0 \in \Lambda$ for which

$$E_{\lambda_0} = \frac{1}{|X|} J. \tag{28}$$

Compare (28) with (20). The first eigenmatrix $P = (p_\alpha(\lambda))_{\alpha,\lambda\in\Lambda}$ and the second eigenmatrix $Q = (q_\lambda(\alpha))_{\lambda,\alpha\in\Lambda}$ are the transition matrices between the two bases of $\mathcal{M}$:

$$\begin{aligned} A_\alpha &= \sum_{\lambda\in\Lambda} p_\alpha(\lambda) E_\lambda, && (29)\\ E_\lambda &= \frac{1}{|X|} \sum_{\alpha\in\Lambda} q_\lambda(\alpha) A_\alpha, && (30)\\ PQ &= QP = |X| I. && (31) \end{aligned}$$

For a subset Λ_1 of Λ, set

$$\begin{aligned} A_{\Lambda_1} &= \sum_{\alpha\in\Lambda_1} A_\alpha, && (32)\\ p_{\Lambda_1}(\lambda) &= \sum_{\alpha\in\Lambda_1} p_\alpha(\lambda). && (33) \end{aligned}$$

By (26), (27) and $A_{\Lambda_1} = \sum_{\lambda\in\Lambda} p_{\Lambda_1}(\lambda) E_\lambda$, we immediately have

Lemma 3 *The* $(0, 1)$ *matrix* A_{Λ_1} *generates the subalgebra* $\mathcal{M}$ *of* $\mathrm{M}(\mathbb{C})$ *if and only if* $p_{\Lambda_1}(\lambda)$, $\lambda \in \Lambda$, *are distinct.*

Given the first eigennatrix $P = (p_\alpha(\lambda))_{\alpha,\lambda\in\Lambda}$, we can check in principle whether the symmetric association scheme is polynomial or not, using this lemma.

We now return to a polynomial association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$. The Bose-Mesner algebra of $\mathfrak{X}$ is $\mathcal{M} = \mathrm{Span}\{A_\alpha \,|\, \alpha \in \Lambda\}$, where A_α, $\alpha \in \Lambda$, is the relation matrix of R_α. By the definition of a polynomial association scheme, $\mathfrak{X}$ is a symmetric association scheme and its Bose-Mesner algebra $\mathcal{M}$ satisfies the following property: there exists a (0, 1) matrix $A \in \mathcal{M}$ such that $A \circ I = 0$ and

$$A_\alpha = v_\alpha(A),\ \alpha \in \Lambda \tag{34}$$

for some polynomial $v_\alpha(x) \in \mathbb{C}[x]$. Let $\varphi(x)$ denote the minimal polynomial of A. Set $\deg \varphi(x) = D + 1$: the degree of $\varphi(x)$ is $D + 1$. We may assume

$$\deg v_\alpha(x) \le D,\ \alpha \in \Lambda. \tag{35}$$

Regarding $\mathcal{M}$ as a subalgebra of $\mathrm{M}_X(\mathbb{C})$, we have

$$\begin{aligned}\mathcal{M} &= \langle\, A \,\rangle = \mathrm{Span}\{A^i \,|\, 0 \le i \le D\}\\ &\simeq \mathbb{C}[x]/(\varphi(x)) = \langle\, x \,\rangle = \mathrm{Span}\{x^i \,|\, 0 \le i \le D\},\end{aligned}$$

where $(\varphi(x))$ is the ideal of the polynomial ring $\mathbb{C}[x]$ generated by $\varphi(x)$, and

$$\mathcal{M} = \mathrm{Span}\{v_\alpha(A) \,|\, \alpha \in \Lambda\} \tag{36}$$

$$\simeq \mathbb{C}[x]/(\varphi(x)) = \mathrm{Span}\{v_\alpha(x) \,|\, \alpha \in \Lambda\}. \tag{37}$$

Since $(A - kI)J = 0$ and $J = \sum_{\alpha\in\Lambda} A_\alpha = \sum_{\alpha\in\Lambda} v_\alpha(A)$, we have

$$\varphi(x) = \frac{1}{c}(x - k)\sum_{\alpha\in\Lambda} v_\alpha(x), \tag{38}$$

where k is the constant row sum of A and c is the leading coefficient of $\sum_{\alpha\in\Lambda} v_\alpha(x)$.

Let Γ be the polynomial graph for which the adjacency matrix is A. Define a partition

$$\Lambda = \bigcup_{0\le i\le \mathrm{diam}(\Gamma)} \Lambda_i \tag{39}$$

of Λ by setting inductively as

$$\Lambda_0 = \{\alpha_0\},\ A_{\alpha_0} = I, \tag{40}$$

$$\Lambda_i = \{\alpha \in \Lambda \,|\, A^i \circ A_\alpha \ne 0\} - (\Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{i-1}). \tag{41}$$

Then the ith distance matrix of Γ from (12) is given by

$$A_{\Lambda_i} := \sum_{\alpha \in \Lambda_i} A_\alpha, \quad 0 \le i \le \mathrm{diam}(\Gamma). \tag{42}$$

Note that the notation of the ith distance matrix is changed in (42) to be A_{Λ_i}.

The polynomial graph Γ becomes distance-regular if and only if $\mathrm{diam}(\Gamma) = D$, which is equivalent to the condition

$$|\Lambda_i| = 1, \quad 0 \le i \le \mathrm{diam}(\Gamma), \tag{43}$$

because $|\Lambda| = D + 1$ by (21). In this case, it is well-known that

$$\Lambda_i = \{\alpha_i\}, \ \deg v_{\alpha_i}(x) = i, \quad 0 \le i \le D. \tag{44}$$

The polynomial association scheme $\mathfrak{X}$ is called *P-polynomial* if the polynomial graph Γ, from which $\mathfrak{X}$ arises, is distance-regular. In terms of the first eigenmatrix $P = (p_\alpha(\lambda))_{\alpha,\lambda \in \Lambda}$, a symmetric association scheme $\mathfrak{X}$ is P-polynomial if and only if there exists an ordering $\alpha_0, \alpha_1, \cdots, \alpha_D$ of the indexes in Λ ($D = |\Lambda| - 1$) such that $p_{\alpha_i}(\lambda) = v_{\alpha_i}(p_{\alpha_1}(\lambda))$ for some polynomial $v_{\alpha_i}(x)$ of degree i, $0 \le \forall i \le D$. It seems that Lemma 3 suggests a huge gap between the class of polynomial association schemes and that of P-polynomial association schemes. However, when we check distribution diagrams of generously transitive groups, we are inclined to think the following conjecture plausibly holds for $N = 8$.

Conjecture 4 *There exists an absolute constant N such that any polynomial graph Γ that satisfies $|\Lambda_i| = 1, \ 0 \le i \le N$, for Λ_i from* (41) *is distance-regular, i.e.,*

$$|\Lambda_i| = 1, \ 0 \le i \le N \Longrightarrow |\Lambda_i| = 1, \ \forall i.$$

4 Co-polynomial association schemes

In this section, we discuss co-polynomial association schemes as the dual objects of polynomial association schemes, tracing the previous arguments with $\{A_\alpha \,|\, \alpha \in \Lambda\}$ and $\{E_\lambda \,|\, \lambda \in \Lambda\}$ interchanged. Given the 2nd eigenmatrix, we can check whether an association scheme is co-polynomial or not.

Definition 5 *A symmetric association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha \in \Lambda})$ is called co-polynomial if its Bose-Mesner algebra $\mathcal{M}$ is generated as a subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$ by some $E \in \mathcal{M}$ with the property that $EE = E$ and $EJ = 0$. An element $E \in \mathcal{M}$ is called an idempotent if $EE = E$ holds with respect to the ordinary matrix product.*

Let $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ be a symmetric association scheme. Let $\mathcal{M}$ be the Bose-Mesner algebra of $\mathfrak{X}$. Then $\mathcal{M}$ is given by (25), (26) with the properties (18), (20), (27), (28):

$$\begin{aligned}
&\mathcal{M} = \mathrm{Span}\{A_\alpha \,|\, \alpha \in \Lambda\} = \mathrm{Span}\{E_\lambda \,|\, \lambda \in \Lambda\},\\
&A_\alpha \circ A_\beta = \delta_{\alpha,\beta} A_\alpha, \quad \sum_{\alpha\in\Lambda} A_\alpha = J, \quad A_{\alpha_0} = I,\\
&E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda, \quad \sum_{\lambda\in\Lambda} E_\lambda = I, \quad E_{\lambda_0} = \frac{1}{|X|} J.
\end{aligned}$$

Let $P = (p_\alpha(\lambda))_{\alpha,\lambda\in\Lambda}$, and $Q = (q_\lambda(\alpha))_{\lambda,\alpha\in\Lambda}$ be the first eigenmatrix and the second eigenmatrix of $\mathfrak{X}$, respectively. So by (29), (30), (31)

$$A_\alpha = \sum_{\lambda\in\Lambda} p_\alpha(\lambda) E_\lambda, \quad E_\lambda = \frac{1}{|X|} \sum_{\alpha\in\Lambda} q_\lambda(\alpha) A_\alpha, \quad PQ = QP = |X| I.$$

For a subset Λ_1 of Λ, set

$$E_{\Lambda_1} = \sum_{\lambda\in\Lambda_1} E_\lambda, \quad q_{\Lambda_1}(\alpha) = \sum_{\lambda\in\Lambda_1} q_\lambda(\alpha).$$

By (25), (18) and $E_{\Lambda_1} = \frac{1}{|X|} \sum_{\alpha\in\Lambda} q_{\Lambda_1}(\alpha) A_\alpha$, we immediately have

Lemma 6 *The idempotent matrix E_{Λ_1} generates the subalgebra $\mathcal{M}$ of* $\mathrm{M}(\mathbb{C})^\circ$ *if and only if $q_{\Lambda_1}(\alpha)$, $\alpha \in \Lambda$, are distinct.*

Given the second eigenmatrix $Q = (q_\lambda(\alpha))_{\lambda,\alpha\in\Lambda}$, we can check in principle whether the symmetric association scheme is co-polynomial or not, using this lemma.

Now we assume that the symmetric association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ is co-polynomial. By the definition of a co-polynomial association scheme, there exists an idempotent $E \in \mathcal{M}$ such that $EJ = 0$ and in the subalgebra $\mathcal{M}$ of $\mathrm{M}_X(\mathbb{C})^\circ$

$$E_\lambda = \frac{1}{|X|} v_\lambda^*(|X|E), \ \lambda \in \Lambda \tag{45}$$

for some polynomial $v_\lambda^*(x) \in \mathbb{C}[x]$. Let $\varphi^*(x)$ denote the minimal polynomial of $|X|E$ in the subalgebra $\mathcal{M}$ of $\mathrm{M}_X(\mathbb{C})^\circ$. Set $\deg \varphi^*(x) = D^* + 1$. So

$$\dim \mathcal{M} = D^* + 1 = |\Lambda|. \tag{46}$$

We may assume

$$\deg v_\lambda^*(x) \le D^*, \ \lambda \in \Lambda. \tag{47}$$

Regarding $\mathcal{M}$ as a subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$, which is denoted by $\mathcal{M}^\circ$, we have

$$\begin{aligned}\mathcal{M}^\circ &= \langle\, E\,\rangle^\circ = \mathrm{Span}\{(|X|E)^{\circ i} \,|\, 0 \leq i \leq D^*\}\\ &\simeq \mathbb{C}[x]/(\varphi^*(x)) = \langle\, x\,\rangle = \mathrm{Span}\{x^i \,|\, 0 \leq i \leq D^*\},\end{aligned}$$

where $(|X|E)^{\circ i}$ is the ith power of $|X|E$ with respexct to the Hadamard product, and

$$\mathcal{M}^\circ = \mathrm{Span}\{v^*_\lambda(|X|E) \,|\, \lambda \in \Lambda\} \tag{48}$$

$$\simeq \mathbb{C}[x]/(\varphi^*(x)) = \mathrm{Span}\{v^*_\lambda(x) \,|\, \lambda \in \Lambda\}. \tag{49}$$

Since $(|X|E - mJ) \circ I = 0$ and $I = \sum_{\lambda\in\Lambda} E_\lambda = \frac{1}{|X|}\sum_{\lambda\in\Lambda} v^*_\lambda(|X|E)$, we have

$$\varphi^*(x) = \frac{1}{c^*}(x-m)\sum_{\lambda\in\Lambda} v^*_\lambda(x), \tag{50}$$

where m is the rank of E and c^* is the leading coefficient of $\sum_{\lambda\in\Lambda} v^*_\lambda(x)$.

Define the subset Λ^*_i of Λ inductively by

$$\Lambda^*_0 = \{\lambda_0\},\ \ E_{\lambda_0} = \frac{1}{|X|}J, \tag{51}$$

$$\Lambda^*_i = \{\lambda \in \Lambda \,|\, E^{\circ i}E_\lambda \neq 0\} - (\Lambda^*_0 \cup \Lambda^*_1 \cup \cdots \cup \Lambda^*_{i-1}). \tag{52}$$

The *co-diameter* of $\mathfrak{X}$ is defined to be

$$\mathrm{diam}^*(\mathfrak{X}) = \max\{i \in \mathbb{Z}_{>0} \,|\, \Lambda^*_i \neq \emptyset\}. \tag{53}$$

Then we get a partition

$$\Lambda = \bigcup_{0\leq i\leq \mathrm{diam}^*(\mathfrak{X})} \Lambda^*_i \tag{54}$$

and the *ith co-distance matrix*

$$E_{\Lambda^*_i} := \sum_{\lambda\in\Lambda^*_i} E_\lambda, \quad 0 \leq i \leq \mathrm{diam}^*(\mathfrak{X}). \tag{55}$$

Note that

$$\mathrm{diam}^*(\mathfrak{X}) \leq D^* \tag{56}$$

and compare it with (15).

The co-polynomial association schme $\mathfrak{X}$ is called *Q-polynomial* if $\mathrm{diam}^*(\mathfrak{X}) = D^*$, which is equivalent to the condition

$$|\Lambda_i^*| = 1, \quad 0 \le i \le \mathrm{diam}^*(\mathfrak{X}), \tag{57}$$

because $|\Lambda| = D^* + 1$ by (46). In this case, it is well-known that

$$\Lambda_i^* = \{\lambda_i\}, \ \deg v_{\lambda_i}^*(x) = i, \quad 0 \le i \le D^*. \tag{58}$$

In terms of the second eigenmatrix $Q = (q_\lambda(\alpha))_{\lambda,\alpha\in\Lambda}$, a symmetric association scheme $\mathfrak{X}$ is Q-polynomial if and only if there exists an ordering $\lambda_0, \lambda_1, \cdots, \lambda_{D^*}$ of the indexes in Λ ($D^* = |\Lambda|-1$) such that $q_{\lambda_i}(\alpha) = v_{\lambda_i}^*(q_{\lambda_1}(\alpha))$ for some polynomial $v_{\lambda_i}^*(x)$ of degree i, $0 \le \forall i \le D^*$. It seems that Lemma 6 suggests a huge gap between the class of co-polynomial association schemes and that of Q-polynomial association schemes. We are not sure if the dual version of Conjecture 4 holds, since we have not yet started to collect examples of co-polynomial association schemes.

Question 7 *Does there exist an absolute constant N^* such that any co-polynomial association scheme $\mathfrak{X}$ that satisfies $|\Lambda_i^*| = 1$, $0 \le i \le N^*$, for Λ_i^* from* (52) *is Q-polynomial? In other words,*

$$|\Lambda_i^*| = 1, \ 0 \le i \le N^* \Longrightarrow |\Lambda_i^*| = 1, \ \forall i \ ?$$

Recently, Jack Koolen et al proved [15]:

Theorem 8 *A P-polynomial asociation scheme is co-polynomial and a Q-polynomial association scheme is polynomial.*

He conjectures (personal communication):

Conjecture 9 *Symmetric association schemes are co-polynomial if they do not have non-trivial fusion schemes.*

In fact, a stronger version of Theorem 8 is shown in [15]. Modifying Definition 2 (resp. Definition 5), let us call a symmetric association scheme $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ *strongly polynomial* (resp. *strongly co-polynomial*) if its Bose-Mesner algebra $\mathcal{M}$ is generated by some primitive idempotent A_α of $\mathcal{M}^\circ$ as a subalgebra of $\mathrm{M}_X(\mathbb{C})$ (resp. generated by some primitive idempotent E_λ of $\mathcal{M}$ as a subalgebra of $\mathrm{M}_X(\mathbb{C})^\circ$). Obviously, a strongly polynomial (resp. strongly co-polynomial) association scheme is polynomial (resp. co-polynomial). In [15], it is shown that a P-polynomial association scheme is strongly co-polynomial and a Q-polynomial association scheme is strongly polynomial. However, Conjecture 9 does not hold, if the 'co-polynomial' property is replaced by the 'strongly co-polynomial' property. Eiichi Bannai and Da Zhao observed that the group association scheme of the Janko group J_1 (resp. the alternating group A_{10}) has a fusion scheme of class 4 (resp. class 21) that is not strongly polynomial and has no non-trivial fusions (personal communication). They made similar observations for the group association schemes of simple groups $A_5, A_6, PSL(2,8), PSL(2,13)$.

We close this expository article with the well-known conjecture [1]:

Conjecture 10 *In the class of primitive symmetric association schemes* $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\Lambda})$ *with sufficiently large* $|\Lambda|$, $\mathfrak{X}$ *is P-polynomial if and only if* $\mathfrak{X}$ *is Q-polynomial.*

Acknowledgements This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings Lecture Note Series, Menlo Park, 1984.

[2] Eiichi Bannai, Etsuko Bannai, Tatsuro Ito and Rie Tanaka, Algebraic Combinatorics, De Gruyter Series in Discrete Mathematics and Applications, Volume 5, 2021.

[3] N. L. Biggs, Algebraic Graph Theory, Cambridge Univ. Press, Cambridge, 1974.

[4] I. A. Faradjev, "Symmetry vs Regularity", How it started and what it led to, https://www.iti.zcu.cz/wl2018/slides.html

[5] M. Grohe, Descriptive complexity, canonisation, and definable graph structure theory, Cambridge University Press, 2017

[6] D. G. Higman, Coherent configurations, I, Rend. Semin. Mat. Univ. Padova 44 (1970) 1-25.

[7] D. G. Higman, Coherent configurations, I, Ordinary representation theory, Geom. Dedic. 4 (1975) 1-32.

[8] D. G. Higman, Coherent configurations, II, Weights, Geom. Dedic. 5 (1976) 413-424.

[9] D. G. Higman, Coherent algebras, Linear Algebra Appl. 93 (1987) 209–239.

[10] T. Ito, Terwilliger algebras and the Weisfeiler-Leman stabilization, in: Combinatorial Algebra and the Monster Group, edited by A. A. Ivanov, Cambridge University Press, 2022.

[11] M. Lichter, I. Ponomarenko and P. Schweitzer, Walk refinement, walk logic, and the iteration number of the Weisfeiler-Leman algorithm, Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science, June 2019, Article No. 33, Pages 1–13.

[12] I. Ponomarenko, The original paper by Weisfeiler and Leman: The preface, https://www.iti.zcu.cz/wl2018/wlpaper.html

[13] P. M. Weichsel, On Distance-Regularity in Graphs, J. Comb. Theory (B) 32 (1982) 156-161.

[14] B. Weisfeiler, On construction and identification of graphs, Lecture Notes in Mathematics, Vol. 558, Springer-Verlag, Berlin-New York, 1976.

[15] T.-T. Xia, Y.-Y. Tan, X. Liang and J. Koolen, On association schemes generated by a relation or an idempotent, preprint.

Kanazawa University
Kanazawa 920-1192, Japan
E-mail address: tito@staff.kanazawa-u.ac.jp