Characteristic quasi-polynomials of hyperplane arrangements and Ehrhart quasi-polynomials

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Abstract

This is a short summary of the author's talk "Characteristic quasi-polynomials of hyperplane arrangements and Ehrhart quasi-polynomials" in RIMS joint research "Research on finite groups, algebraic combinatorics, and vertex algebras", 5-8 December 2022.

1 Quasi-polynomial

Definition 1.1. A function $F : \mathbb{Z}$ (or $\mathbb{Z}_{\geq 0}$) $\longrightarrow \mathbb{C}$ is called a quasi-polynomial if there exist a positive integer $\rho > 0$ (called the period) and polynomials $f_1(t), f_2(t), \dots, f_\rho(t) \in \mathbb{C}[t]$ such that $F(q)$ is expressed as follows.

$$F(q) = \begin{cases} f_1(q), & \text{if } q \equiv 1 \mod \rho \\ f_2(q), & \text{if } q \equiv 2 \mod \rho \\ \vdots & \vdots \\ f_\rho(q), & \text{if } q \equiv \rho \mod \rho. \end{cases} \tag{1}$$

Each polynomial $f_1(t), \dots, f_\rho(t)$ is called the constituent.

Example 1.2. $q_{10}(n) = \left\lfloor \frac{n}{10} \right\rfloor$ is a quasi-polynomial. Indeed,

$$q_{10}(n) = \begin{cases} \frac{n}{10}, & \text{if } n \equiv 0 \mod 10 \\ \frac{n-1}{10}, & \text{if } n \equiv 1 \mod 10 \\ \frac{n-2}{10}, & \text{if } n \equiv 2 \mod 10 \\ \vdots & \vdots \\ \frac{n-9}{10}, & \text{if } n \equiv 9 \mod 10 \end{cases}$$

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The quasi-polynomial is one of the well-known classes of counting functions that appear in enumerative problems. ([4, 11]). The definition of a quasi-polynomial is somewhat more complex than that of a polynomial. However, recent research has revealed that the very complexity of quasi-polynomials contain lots of subtle properties of the objects. The purpose of this note is to introduce recent research on the constituents of quasi-polynomials.

2 Characteristic quasi-polynomials of hyperplane arrangements

A subspace of codimension one in vector space, projective space, affine space, etc. is called a hyperplane. A (finite) collection of hyperplanes is called a **hyperplane arrangement**, and it appears in various fields of mathematics [10]. One of the most important invariants is called the **characteristic polynomial**. In this section we introduce the **characteristic quasi-polynomial** which is considered as a refinement of the characteristic polynomial. The characteristic quasi-polynomial is defined for hyperplane arrangement defined over integers, namely, defined by linear forms with integer coefficients.

Let $q \in \mathbb{Z}_{>0}$. For an integer vector $\boldsymbol{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$, consider the linear form

$$H_{\boldsymbol{a}} := \{(x_1, \dots, x_\ell) \mid a_1x_1 + \cdots + a_\ell x_\ell = 0\}.$$

Denote " mod q hyperplane" by

$$\overline{H}_{\boldsymbol{a}} := \{(x_1, \dots, x_\ell) \in (\mathbb{Z}/q\mathbb{Z})^\ell \mid a_1x_1 + \cdots + a_\ell x_\ell \equiv 0 \mod q\}$$

and " mod q hyperplane" of $\mathcal{A} = \{\boldsymbol{a_1}, \dots, \boldsymbol{a_n}\} \subset \mathbb{Z}^\ell$ by

$$M(\mathcal{A}, q) := (\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{i=1}^{n} \overline{H}_{\boldsymbol{a_i}}.$$

Then $M(\mathcal{A}, q)$ is a finite set and it is known that its cardinality is a quasi-polynomial in q.

Theorem 2.1. *(Kamiya-Takemura-Terao [7]) Under the above notation, $\#M(\mathcal{A}, q)$ is a quasi-polynomial in q (the characteristic quasi-polynomial). Furthermore, let $\rho > 0$ be the period and $f_1(t), f_2(t), \dots, f_\rho(t) \in \mathbb{Z}[t]$ be the constituents, The it has the following "**GCD-property**".*

$$(i, \rho) = (j, \rho) \Longrightarrow f_i(t) = f_j(t),$$

The above result shows that the constituents $f_i(t)$ depends only on the *gcd* (i, ρ). It means that the characteristic quasi-polynomials form a very special class of quasi-polynomials. Athanasiadis [2, 3] proved that the "prime constituent $f_1(t)$" is equal to the characteristic polynomial $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ of $\mathcal{A}$. The characteristic polynomial captures lots of information on $\mathcal{A}$. For example, for a hyperplane arrangement in $\mathbb{C}^\ell$, the Poincaré polynomial of is expressed as

$$(-t)^\ell \cdot \chi(\mathcal{A}, -\frac{1}{t}). \tag{2}$$

Namely, the coefficients of the characteristic polynomial are Betti numbers of the complement (Orlik-Solomon).

Example 2.2. Let $\mathcal{A} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subset \mathbb{Z}^2$. Namely, consider three lines defined by $y = 0, x + y = 0, 3x + y = 0$. The characteristic quasi-polynomial has a period $\rho = 6$ and constituents are.

$$\#M(\mathcal{A}, q) = \begin{cases} q^2 - 3q + 2, & \text{if } q \equiv 1, 5 \mod 6 \\ q^2 - 3q + 3, & \text{if } q \equiv 2, 4 \mod 6 \\ q^2 - 3q + 4, & \text{if } q \equiv 3 \mod 6 \\ q^2 - 3q + 5, & \text{if } q \equiv 0 \mod 6. \end{cases}$$

As we noticed, for i which is coprime to ρ, the i-th constituent is $f_i(t) = f_1(t)$, which is equal to the characteristic polynomial $\chi(\mathcal{A}, t)$. It is a natural question to ask what are the other constituents. Recently, in a joint work with Y. Liu, T. N. Tran, it has been shown that the constituents of the characteristic quasi-polynomial are closely related to the topology of toric arrangements. Here, "considering toric arrangement" means that taking tensor product with $\otimes\mathbb{C}^\times$. In other words, interpret the integer linear forms through $\otimes_\mathbb{Z}\mathbb{C}^\times$. For example, using Example 2.2, consider subtori of $(\mathbb{C}^\times)^2 \ni (t_1, t_2)$, defined by $t = 1, t_1t_2 = 1, t_1^3t_2 = 1$. Denote the complement of associated toric arrangement by $M(\mathcal{A}, \mathbb{C}^\times)$. Then on the contrary to the characteristic polynomial, the most degenerate constituent $f_\rho(t)$ is shown to be related to the Poincarè polynomial of $M(\mathcal{A}, \mathbb{C}^\times)$.

Theorem 2.3. *([8, 13])*

1. *The Poincarè polynomial of $M(\mathcal{A}, \mathbb{C}^\times)$ is equal to*

$$(-t)^\ell \cdot f_\rho(-\frac{1+t}{t}). \tag{3}$$

2. *The constituent $f_k(t)$ is equal to the characteristic polynomial of the poset of layers (connected components of subtori of intersections) which contain k-torsion points.*

Remark 2.4. Observe that only the difference between formulae (2) and (3) is the numerator, one is1 the other is $1+t$. These are exactly equal to the Poincarè polynomial of $\mathbb{C}$ and $\mathbb{C}^\times$. This is not just by chance, but more generally, for any abelian Lie group G, one can define the G-characteristic polynomial, and when G is non-compact, we have a similar formula. Then $f_1(t)$, $f_\rho(t)$ are nothing but the G-characterisctic polynomials for $G=\mathbb{C},\mathbb{C}^\times$ respectively. See [8] for details.

3 Ehrhart quasi-polynomials of translated lattice polytopes

For a rational polytope $P\subset\mathbb{R}^n$, let

$$L_P(t) := \#(tP\cap\mathbb{Z}^n).$$

It is known that $L_P(t)$ is a quasi-polynomial in $t\in\mathbb{Z}_{\geq 0}$ (Ehrhart quasi-polynomial [4]). It is also known that the gcd of denominators of coordinates of vertices gives a period (not necessarily the minimum period). In particular, when P is a lattice polytope, $L_P(t)$ is a polynomial in t.

As was shown in the previous section, the characteristic quasi-polynomial of a hyperplane arrangement has GCD-property. On the other hand, the Ehrhart quasi-polynomials does not have GCD-property. However, there are some classes of polytopes which Ehrhart quasi-polylnomials have GCD-property.

For example, Suter ([12]) computed Ehrhart quasi-polynomials for the fundamental alcoves of root systems, and observed that the Ehrhart quasi-polylnomials have GCD-property[1].

To formulate the main result, we need the following.

Definition 3.1. A quasi-polynomial (1) is **symmetric** if the constituents satisfy $f_i(t)=f_{\rho-i}(t)$ $(i=1,\ldots,\rho-1)$.

Clearly, quasi-polynomial with GCD-property is symmetric.

Example 3.2. Let P_1,P_2,P_3 be

- $P_1=\frac{1}{9}\cdot[0,1]^3$ (the cube with edge length $\frac{1}{9}$).

[1]At the moment of Suter's computation, the reason of GCD-property was not clear. However, later it was revealed that it comes from the GCD-property of the characteristic quasi-polynomial of arrangements [14].

- $P_2 = (\frac{5}{9}, \frac{5}{9}, \frac{2}{3})^t + \mathrm{Conv}\{\pm e_i \mid i = 1, 2, 3\}$ (A rational translation of the regular octahedron $\mathrm{Conv}\{\pm e_i \mid i = 1, 2, 3\}$).
- $P_3 = (\frac{1}{9}, \frac{2}{9}, \frac{1}{3})^t + [0, 1]^3$ (Rational translation of the unit cube).

All Ehrhart quasi-polynomials $L_{P_1}(t)$, $L_{P_2}(t)$, and $L_{P_3}(t)$ have the minimum period 9. The constituents are as follows.

$$L_{P_2}(t) = \begin{cases} \frac{4}{3}t^3 - \frac{4}{3}t, & (t \equiv 1, 8 \mod 9), \\ \frac{4}{3}t^3 + \frac{2}{3}t, & (t \equiv 2, 7 \mod 9), \\ \frac{4}{3}t^3 + t^2 + \frac{2}{3}t, & (t \equiv 3, 6 \mod 9), \\ \frac{4}{3}t^3 - \frac{1}{3}t, & (t \equiv 4, 5 \mod 9), \\ \frac{4}{3}t^3 + 2t^2 + \frac{8}{3}t + 1, & (t \equiv 9 \mod 9), \end{cases}$$

$$L_{P_3}(t) = \begin{cases} t^3 & (t \equiv 1, 2, 4, 5, 7, 8 \mod 9), \\ t^3 + t & (t \equiv 3, 6 \mod 9), \\ (t+1)^3 & (t \equiv 9 \mod 9). \end{cases}$$

Note that $L_{P_1}(t)$ has 9 mutually distinct constituents. $L_{P_2}(t)$ is symmetric, and $L_{P_3}(t)$ has GCD-property.

The purpose of this section is to formulate a relationship between these properties and the shape of polytopes.

Theorem 3.3. ([6]) Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytopes. Then the following are equivalent.

(1a) P is centrally symmetric.

(1b) For any rational vector $v \in \mathbb{Q}^d$, $L_{v+P}(t)$ is symmetric quasi-polynomial.

Theorem 3.4. ([6]) Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytopes. Then the following are equivalent.

(2a) P is a zonotope.

(2b) For any rational vector $v \in \mathbb{Q}^d$, $L_{v+P}(t)$ has GCD-property.

Simply speaking, symmetry or GCD-property of the Ehrhart quasi-polynomials are closely related to central symmetry and being zonotope.

Here we give a sketch of the proof. The implication (2a)$\Longrightarrow$(2b) is proved by using a recent formula Ardila-Beck-McWhirter [1], which is a generalization of Stanley's formula of the Ehrhart polynomial of a lattice zonotope.

(1a)$\Longrightarrow$(1b) needs a description of the constituents of rationally translated lattice polytope. We consider a variant of Ehrhart quasi-polynomial as follows. Let P be a lattice polytope, $\boldsymbol{v}$ be a rational vector. Define $L_{(P,\boldsymbol{v})}(t)$ to be

$$L_{(P,\boldsymbol{v})}(t) := \#(\boldsymbol{v} + tP) \cap \mathbb{Z}^n.$$

Then we have the following.

Lemma 3.5. *([6]) $L_{(P,\boldsymbol{v})}(t)$ is a polynomial in t.*

Using this formula, we can describe constituents.

Lemma 3.6. *([6]) Let P be a lattice polytope, $\boldsymbol{v}$ be a rational vector. Then, the k-th constituent of the Ehrhart quasi-polynomial of $P + \boldsymbol{v}$ is equal to $L_{(P,k\boldsymbol{v})}(t)$.*

Using this lemma, (1a)$\Longrightarrow$(1b) is easily obtained.

The converse implications (b)$\Longrightarrow$(a) for (1), (2) are not easy. It needs characterizations of centrally symmetric polytopes and zonotopes by Minkowski and McMullen. More explicitly, for non centrally symmetric (or non zonotope) poytope P, we need to find a rational vector $\boldsymbol{v}$ such that $L_{(P,\boldsymbol{v})}(t) \neq L_{(P,-\boldsymbol{v})}(t)$ ($L_{(P,\boldsymbol{v})}(t) \neq L_{(P,2\boldsymbol{v})}(t)$ (with the odd denominator $\boldsymbol{v}$)). See [6] for details.

The results may be summarized as in the table.

Polytopes	Zonotoes	$\subset$	Centrally symmetric	$\subset$	General
Quasi-polynomial	GCD-property	$\subset$	Symmetry	$\subset$	General

4 Future problems

So far, extensive research has been done on the Ehrhart (quasi-)polynomials of rational polytopes. One of the well studied subjects is the so-called the period collapse phenomenon. The present research may suggest a different direction, namely, "rational polytopes obtained by translating lattice polytopes with rational vectors" may be a noteworthy class.

It is also a natural to ask for what kind of rational polytopes, the Ehrhart quasi-polynomials have GCD-property. There are few known examples.

- The rational polytope P such that $2P$ is a lattice polytope (period is 2).
- The fundamental alcoves of root systems.

- A polytope obtained from a lattice polytope by translation with a rational vector (see below).

At this moment, the above are only known classes of rational polytopes whose Ehrhart quasi-polynomials have GCD-property.

Since GCD-property gives a very special type of quasi-polynomials. It would be also a natural question to ask the relationship between the Ehrhart quasi-polynomials of the translated lattice zonotopes and the characteristic quasi-polynomials of hyperplane arrangements.

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