An overview on design theory

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この原稿はRIMS joint research （代表者：三枝崎剛）:
「Research on finite groups, algebraic combinatorics, and vertex algebras」 での講演の slides をほぼそのままの形で記録したものです. この原稿は独自の論文として完成しているものではなく, 私が考えているいくつかのことを preliminary manuscript という形で述べただけのものであることをご了承ください. ここで述べたことは色々の方と色々の形での共同研究に基づいていますので共同研究の皆様に感謝したいと思います. 特にほとんど全ては Da Zhao（京都大学ポスドク）との共同研究の内容です.

In this talk I will give an overview on design theory.

Roughly speaking, the purpose of design theory is, for a given space M, to find good finite subsets X that approximate M well. M may be a continuous topological space, say sphere, or a finite set such as point set of an association scheme, for example, Johnson association scheme, Hamming association scheme, etc. etc.

We start with spherical t-designs as well as combinatorial t-designs. I will not repeat all these definitions, but a spherical t-design is a subset of the unit sphere S^{n-1} and a combinatorial t-design is a subset of $\binom{V}{k}$ (point set of the Johnson association scheme $J(v,k)$), where $|V| = v$.

There are many generalizations of these t-design concepts, say
(i) Changing the original base space,
(ii) Consider weighted designs,
(iii) Consider T-design instead of t-design,
(iv) Consider designs on several concentric spheres (Euclidean desingns), or on the several shells of association schemes (relative t-designs, allow different block sizes),
etc. etc.

I will not repeat the explanations of all these generalizations. For the details, please look at the survey article:

1. Bannai-Bannai-Tanaka-Zhu: Design Theory from a viewpoint of algebraic combinatorics, Graphs and Combinatorics (2017), or

2. 坂内一坂内一伊藤: 代数的組合せ論入門, 共立出版 (2016), or its English version: Bannai-Bannai-Ito-Rie Tanaka: Algebraic Combinatorics, De Gruyter (2021).

The concepts of t-design in more general spaces

• t-designs in polynomial spaces (Godsil, 1993)

• t-designs in DDR graphs (distance degree regular graphs) (Solé and others, around 2021)

Polynomial spaces (Godsil, 1993)

$\Omega =$ a set.
$\rho =$ a function $\Omega \times \Omega \to \mathbb{R}$.
$< \ , \ >=$ an inner product on the space of functions on Ω.
(Usually, $< f, g >= \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)g(x)$.)

A polynomial space is a triple $(\Omega, \rho, < \ , \ >)$ satisfying the following four axioms (I) to (IV).

(I) For $x, y \in \Omega$, $\rho(x, y) = \rho(y, x)$.

(II) The dimension of the vector space $Pol(\Omega, 1)$ is finite.
Here, for an integer $r \geq 0$, $Pol(\Omega, r)$ is defined as follows. For any polynomial $f \in \mathbb{R}[x]$ and $a \in \Omega$, we define $\rho_a(x) = \rho(a, x)$ and define

$$Z(\Omega, r) = \text{the space spanned by } \{f \circ \rho_a \mid f \in \mathbb{R}[x], a \in \Omega, deg(f) \leq r\}.$$

Then we define $Pol(\Omega, 1) = Z(\Omega, 1)$ and we define by induction

$$Pol(\Omega, r+1) = \text{ the space spanned by } \{fg \mid f \in Pol(\Omega, 1), g \in Pol(\Omega, r)\}.$$

We define $Pol(\Omega) = \bigcup_{r=0}^{\infty} Pol(\Omega, r)$.

(III) For $f, g \in Pol(\Omega)$,

$$< f, g >=< 1, fg > .$$

(IV) If f is a non-negative polynomial on Ω then $< 1, f > \geq 0$, with equality if and only if f is identically 0.

Remark. The polynomial spaces (Ω, ρ) and(Ω, ρ') are affinely equivalent, if $\rho' = \alpha\rho + \beta$, for some real constant numbers $\alpha \neq 0$ and β.

Examples of polynomial spaces

(a) The Johnson scheme $J(v,k)$.
$\Omega = \binom{V}{k}, \rho(x,y) = |x \cap y|$.
(There are many other possible ρ, say $\rho(x,y) = k - |x \cap y|$.)

(b) The Hamming scheme $H(n,q)$.
$\Omega = F_q \times F_q \times \cdots \times F_q$ (n times), where $\rho(x,y)$ is the Hamming distance.
(There are many other possible ρ.)

(c) The unit sphere S^{n-1} in $\mathbb{R}^n$.
$\Omega = S^{n-1}, \rho(x,y) = x \cdot y$, where $x \cdot y$ is a usual dot product in $\mathbb{R}^n$.
The inner product $<\ ,\ >$ is defined by

$$< f, g >= \int_{\Omega} f(x)g(x)d\mu,$$

where μ is the Haar measure with $\mu(\Omega) = 1$.
(There are many other possible ρ.)

(d) The symmetric group $Sym(n) = S_n$.
$\Omega = Sym(n), \rho(x,y) = |\mathrm{fix}(x^{-1}y)|$ is the number of points left fixed by the permutation $x^{-1}y$.
(Or we can define $\rho = n - |\mathrm{fix}(x^{-1}y)|$. Both definitions are affinely equivalent and basically the same.)

(e) The orthogonal group $O(n)$.
$\Omega =$ the set of all $n \times n$ orthogonal matrices, and $\rho(x,y) =$trace $x^T y$.
(The polynomial space $Sym(n)$ in (e) is embedded in $O(n)$ naturally.)

(f) The q-Johnson scheme $J_q(n,k)$.
$\Omega =$ the set of all k-dim subspaces of a v-dim vector space over F_q, $\rho(x,y) =$ the number of 1-dim subspaces in $x \cap y$. (Or we can take $\rho(x,y) = \dim(x \cap y)$.) (There are many other choices of ρ.)

For any Q-polynomial association scheme, let $\theta_0^*, \theta_1^*, ..., \theta_d^*$ the elements in the first column (starting 0) of the second eigen matrix Q. Taking $\rho(x,y) = \theta_i^*$, if $(x,y) \in R_i$, is the natural choice of ρ (from the viewpoint of association schemes and Delsarte theory).

There are many other good examples of polynomial spaces.

• The perfect matchings in K_{2n}. $\rho(x,y) =$ the number of edges they have in common. This is essentially $S_{2n}/(W(B_n))$.
(The perfect matching in $K_{n,n}$ gives essentially the example (d) again. This is essentially $(S_n \times S_n)/S_n$.)

• Real n-space $\mathbb{R}^n$?
There are many choices. Say, $\rho(x,y) = x \cdot y$ (or $(x-y)\cdot(x-y)$) and define the inner product

$$< f, g >= \int f(x)g(x)d\mu$$

for any rotationally invariant measure $d\mu$ on $\mathbb{R}^n$ with respect to which all polynomials in n variables are integrable.

There are many many other examples of polynomial spaces. (See Godsil (1993).)

t-design in polynomial space

Definition. A subset $Y \subset \Omega$ is called a t-design, if for all $f \in Pol(\Omega, t)$,

$$< 1, f >= \frac{1}{|Y|}\sum_{x \in Y} f(x) \quad (=< 1, \rho >_Y).$$

• A polynomial space $(\Omega, \rho, < \ , \ >)$ is called spherical, if there is an injection τ say, of Ω into a sphere centered at the origin in some real vector space for any x, y in Ω,

$$\rho(x, y) = \tau(x) \cdot \tau(y).$$

Theorem (Theorem 4.1 and Theorem 4.3 in Godsil's book,1993).
(i) If (Ω, ρ) is a spherical polynomial space, then $Z(\Omega, r) = Pol(\Omega, r)$ for all non-negative integer r.
(ii) Let (Ω, ρ) be a spherical polynomial space with $\dim(Z(\Omega, 1)) = n$. If we view Ω as a subset of a sphere centered at the origin in $\mathbb{R}^n$, then $Pol(\Omega, r)$ is the set of all polynomials with degree at most r in n variables, restricted to Ω.

Another approach by Solé. (This is an ambitious approach, but perhaps not very appropriate.) See, e.g., arXiv:2105.07979.

Let (X, d) be a metric space. It is called a DDR (distance degree regular) space, if (X, Γ_i),

where $\Gamma_i = \{(x, y) \mid x, y \in X, d(x, y) = i\}$ is a regular graph for $i = 0, 1, ..., d$. (Some Γ_i may be empty.) (There are some mistakes there. He implicitly assumes $d(x, y)$ are all integers. Let n be the diameter of the metric space. Then a t-design Y is defined as: Y is a subset of X, and satisfy

$$\frac{1}{|X|^2} \sum_{j=0}^{n} |\Gamma_j| j^i = \frac{1}{|Y|^2} \sum_{j=0}^{n} |\Gamma_j \cap (Y \times Y)| j^i$$

for $i = 0, 1, ..., t$.

Remark. Solé says that his definition is a special case of Godsil's t-designs in polynomial space, but that is not correct. If the metric space is a spherical polynomial space, that definition might work (under the assumption that all distances are integers), but this definition does not coincide with Godsil's definition for non-spherical polynomial spaces.

Now we want to return to the discussion of what are the natural definitions of t-design on association schemes (and on finite groups)?

What are the reasonable choices of $\rho = \rho(x, y)$, for a Q-polynomial association scheme, for example for $J_q(v, k)$?

In the case of Q-polynomial association scheme, the most(?) natural choice is as follows, as we mentioned already.

Let $\theta_0^*, \theta_1^*, ..., \theta_d^*$ be the elements in the first column (starting the 0th column) of the second eigenmatrix Q, i.e., eigenvalues of the dual intersection matrix B_1^*. Then define $\rho(x, y) = \theta_i^*$ if $(x, y) \in R_i$.

Then use the definition of t-design for the polynomial space (by Godsil). Then this concept of t-design is equivalent to the usual definition of t-design of the Q-polynomial association scheme (due to Delsarte). This polynomial space is spherical. This space is embedded in $\mathbb{R}^{m_1}$, and the Gram matrix is E_1.

More generally, let $\mathfrak{X} = (X, \{R_i(i = 0, 1, ..., d)\})$ be a symmetric association scheme (that may not be Q-polynomial). Let $A_0, A_1, ..., A_d$ be the adjacency matrices and $E_0, E_1, ..., E_d$ be the primitive idempotents. For a fixed E_1, we define the concept of t-design (w.r.t. E_1) as follows. Here let $z_0^*, z_1^*, ..., z_d^*$ be the entries of the 1st column of Q (they are not necessarily distinct), then a subset Y in X is a t-design (with respect to E_1, if

$$\frac{1}{|X|} \sum_{(x,y) \in X \times X} (z^*(x, y))^j = \frac{1}{|Y|} \sum_{(x,y) \in Y \times Y} (z^*(x, y))^j,$$

for j $(1 \le j \le t)$, where $z^*(x, y) = z_i^*$, for $(x, y) \in R_i$.

As it is mentioned before, for $J_q(v,k)$, Godsil takes $\rho(x,y) =$ the number of 1-dimensional subspace in $x \cap y$. Then this concept of t-design is not exactly the same as the definition using the Q-polynomial structure, but essentially equivalent to the definition of t-design of the Q-polynomial association scheme. Namely, they are affinely equivalent. This polynomial space (defined by Godsil) is spherical, and is embedded in $\mathbb{R}^{m_0+m_1}$, and the Gram matrix is $E_0 + E_1$.

On the other hand, for $J_q(v,k)$, if we take $\rho(x,y) =$ the dimension of $x \cap y$. Then this polynomial space is not spherical. In fact, in most cases, say in $J_2(6,3)$, $Z(\Omega,1) = Pol(\Omega)$. So, we cannot have good reasonable concept of t-design for this (non-spherical) polynomial space. Actually, there are no non-trivial t-design in this polynomial space! (Also the definition of t-design in the sense of Solé is certainly not meaningful.)

Let us consider the case of a finite group.

Let G be a finite group, and let ρ be an irreducible representation of G (just for simplicity). For simplicity, we also assume that $\rho(G) \subset O(n)$. Let χ_ρ be the character of ρ. Then, we say a subset Y in G is a t-design of G with respect to ρ, if the following condition is satisfied.

$$\frac{1}{|G|^2}\sum_{x,y\in G}(\chi_\rho(x^{-1}y))^j (= \frac{1}{|G|}\sum_{x\in G}(\chi_\rho(x))^j) = \frac{1}{|Y|^2}\sum_{x,y\in Y}(\chi_\rho(x^{-1}y))^j.$$

for $j = 0, 1, \ldots, t$.

This definition of t-design in G is essentially equivalent to the previous definition of t-design on the group association scheme $\mathfrak{X}(G)$ with respect to E_1 (corresponding to ρ.)

Now let us consider the systems of orthogonal polynomials for a finite group G. We consider the following point-measure on $\mathbb{R}$.

(i) the points are real numbers $\{\chi_\rho(x), x \in G\}$, and the weight $= 1$ for each $x \in G$, or ,
(ii) the points are real numbers $\{z_0^*, z_1^*, ..., z_d^*\}$ which are the elements of the first column (corresponding to E_1) of Q of the group association scheme $\mathfrak{X}(G)$, (these z_i may not be distinct each other), and the weight $= k_i$ for each z_i^*.

Let $\phi_0, \phi_1, \phi_2, ...$ be the set of orthogonal polynomials with respect to the point measure (i) (up to some degree), and let $\phi'_0, \phi'_1, \phi'_2, ...$ be the set of orthogonal polynomials with respect to the point measure (ii)(up to some degree). How are they related? They are affinely equivalent.

In the case of $G = S_n$, the ϕ_i are related to Charlier polynomials (up to certain degree, i.e., $i \leq [\frac{n}{2}]$). See Tarnanen (1999). Hence ϕ'_i will be related to Charlier polynomials as well, with some affine transformation.

Another possible definition of designs on graphs.

Here is a sample example of one such attempt.

Definition. (A modified version of Steinberger:
Generalized designs on graphs, sampling, spectra, symmetries, J. Graph Theory, 2020).
Let G be a finite regular graph and let $\theta_0, \theta_1, \cdots, \theta_\ell$ be all the distinct eigenvalues of the adjacency matrix of G suh that $|\theta_0| \geq |\theta_1| \geq \cdots \geq |\theta_\ell|$. Let V_i be the space of eigenvectors for the eigenvalue θ_i. (V_i is identified as a subspace of functions on G.) Then a subset Y of the vertex set $V(G)$ of G is called a k-design

if

$$\frac{1}{|Y|}\sum_{x\in Y} f(x) = \frac{1}{|V(G)|}\sum_{x\in V(G)} f(x),$$

for any $f \in V_i$ with $i = 0, 1, \ldots, k$.

Charlier polynomials and the polynomial space S_n

Let S_n be the symmetric group, and let $\pi = 1 + \chi$ be the natural permutation representation of S_n on n letters. Then χ is the irreducible representation of degree $n-1$ corresponding to the Young diagram (partition) of type $(n-1, 1)$. Then the character of χ is given by $\chi(x) = \pi(x) - 1 = |\text{fix}(x)| - 1$.

An irreducible representation ρ of S_n is called of depth i if the number of squares in the Young diagram which are not in the first row is equal to i. (Namely, for $(n_1, n_2, \ldots, n_\ell), n_2 + n_3 + \cdots + n_\ell = i$.)

It is known that a subset $Y \subset S_n$ is a t-transitive set, if and only if Y is a ρ-design for all ρ of depth $\leq t$, namely

$$\sum_{(x,y)\in Y\times Y} \rho(x^{-1}y) = 0$$

for all ρ of depth $1 \leq i \leq t$.

Let w_i be the number of element $x \in S_n$ with $|\text{fix}(x)| = i$. Set

$$v_i = w_{n-i}.$$

(Note that $v_0 = 1$.) For $f, g \in \mathbb{R}[x]$, we define the inner product

$$< f, g >_{S_n} = \frac{1}{n!}\sum_{i=0}^{n} v_i f(i) g(i)\ .$$

If we consider the set of orthogonal polynomials w.r.t. this inner product, then we get Charlier polynomials:

$C_0(x) = 1,$
$C_1(x) = x - 1,$
$C_2(x) = x^2 - 3x + 1,$
$C_3(x) = x^3 - 6x^2 + 8x - 1,$
$C_4(x) = x^4 - 10x^3 + 29x^2 - 24x + 1,$
etc.

More explicitly, we get

$$C_k(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} x(x-1)\cdots(x-i+1).$$

Also, we have:

$$e^t(1-t)^x = \sum_{k=0}^{\infty} C_k(x)\frac{t^k}{k!}.$$

Then, it is known that

$$< C_r(n-x), C_s(n-x) >_{S_n} = r!\delta_{rs}$$

for $r, s \leq \frac{n}{2}$. (See Tarnanen (Europ. J. Comb.,1999).)

The above presentation was originally obtained using the value of the permutation character π (hence of the irreducible character χ) of the group S_n. If we define the orthogonal polynomials using $z_0^*, z_1^*, ..., z_d^*$, the 1st column of the second eigenmatrix Q of the group association scheme of S_n, then the orthogonal polynomials are obtained from the point measure: the points are z_i^* in $\mathbb{R}$ and the weight on each z_i^* is k_i. Then the set of orthogonal polynomials $\phi_0, \phi_1, \cdots$ (up to degree $n/2$) are described by using Charlier polynomials (with certain modification). Interestingly, this happens only up to degree $n/2$. Here are some questions.

Question 1. Instead of using Q, let us consider the 1st column of the eigenmatrix P of the group association scheme $\mathfrak{X}(S_n)$. (Namely, 1st column corresponds to the irreducible representation χ of degree $n-1$ of S_n.) Let z_i be the elements of the first column of $\boldsymbol{P}$. Take the point measure: the points are z_i in $\mathbb{R}$ and with the weight m_i on each z_i.

What are the set of orthogonal polynomials $\psi_0, \psi_1, \ldots$ (up to certain degree) for this point measure?

Unfortunately, this set of orthogonal polynomials are not well described yet. Originally I expected this is something to do with the duality in Charlier polynomials. (See the duality: if we write $c_k(x) = (-1)^k C_k(x)$, then $c_k(x) = c_x(k)$ (see T. S. Chihara's book: Introduction of Orthogonal Polynomials, Chapter IV, for the details.)
I originally hoped that some duality between $\{\phi_0, \phi_1, \ldots\}$ and $\{\psi_0, \ \psi_1, \ldots\}$ coming from $c_k(x) = c_x(k)$ may exist, but so far unsuccessful.

Question 2. How about the orthogonal polynomials coming from the first column of $\boldsymbol{Q}$ for the group association scheme $W(B_n)$ and the symmetric association scheme $S_{2n}/W(B_n)$. We expect this set of orthogonal polynomials should be related to the Charlier polynomials.) Also, it would be interesting to consider the orthogonal polynomials coning from the first column of $\boldsymbol{P}$ for the group association scheme of $W(B_n)$ and the symmetric association scheme $S_{2n}/W(B_n)$.

Question 3.

Let $\mathfrak{X}$ be a symmetric association scheme that are polynomial and also co-polynomial (in the sense of Tatsuro Ito's talk). Let us consider the case of $A = A_1$ and $E = E_1$ for simplicity. (The general case is treated similarly.) Then using the 1st column of $\boldsymbol{Q}$, we get the system of orthogonal polynomials $\phi_0, \phi_1, \cdots, \phi_d$, and the system of orthogonal polynomials $\psi_0, \psi_1, \cdots, \psi_d$. Note that $z_0^*, z_1^*, ..., z_d^*$ are all distinct, and $z_0, z_1, ..., z_d$ are all distinct. So, $\phi_0, \phi_1, \cdots, \phi_d$ and $\psi_0, \psi_1, \cdots, \psi_d$ are defined up to degree d.

Is there any case that these polynomials satisfy the condition

$$\phi_i(z_j^*)/m_i = \psi_j(z_i)/k_j$$

for all i, j with $0 \leq i, j \leq d$, other than the case of P-and Q-polynomial association schemes?

Question 4.

If some of z_i and z_j are equal, or z_i^* and z_j^* are equal, namely, here $z_0^*, z_1^*, \cdots, z_r^*$ be the distinct numbers that appear in the first column of Q and let $z_0, z_1, \cdots, z_s$ be the distinct numbers that appear in the first column of P. Then we get $\phi_0, \phi_1, \cdots, \phi_r$ and $\psi_0, \psi_1, \cdots, \psi_s$ up to degrees for r and s. Then, is there any interesting examples that $\phi_0, \phi_1, \cdots, \phi_r$ and $\psi_0, \psi_1, \cdots, \psi_d$ satisfy the following local duality condition for appropriate constants K_i and M_j.

$$\phi_i(z_j^*)/M_i = \psi_j(z_i)/K_j,$$

(with i, j up to a certain relatively large common upper bound depending on r and s), where K_j is the sum of those k_j corresponding to those whose entry in the1st column in Q is z_i, and M_i is the sum of those m_i corresponding to those whose entry in the first column in Q is z_j^*.

Question 5.

Is there any association scheme where Meixner polynomials are related to the system of orthogonal polynomials $\phi_0, \phi_1, \cdots, \phi_r$ or $\psi_0, \psi_1, \cdots, \psi_s$?

Some special unitary t-designs in $U(d)$.

We consider the inclusions of polynomial spaces.
For the symmetric group $Sym(n) = S_n$, the (S_n, ρ) with

$\rho(x,y) = \chi(x^{-1}y)(= \pi(x^{-1}y)^1 - 1) = |\text{fix}(x^{-1}y)| - 1$
is a spherical polynomial space. Since $\chi(S_n)$ is a subgroup of $O(n-1)$, S_n is embedded in $(O(n-1),\rho)$ with $\rho(x,y) = x^T y$ as a spherical polynomial space. (S_n is also considered as embedded in the polynomial space $S^{(n-1)^2-1}$.)

We will also consider the space $U(d)$ with $\rho(x,y) = |trace(\bar{x}^T y)|^2$, as a kind of polynomial space.

We want to study some special good subsets of these polynomial spaces, in particular of $U(d)$.

Let (Ω,ρ) be any spherical polynomial space, and let Y be a subset of Ω. We define:

$$s = \text{ the degree of } Y = |\{\rho(x,y) \mid x,y \in Y, x \neq y\}|,$$

$t =$ the strength of $Y =$ the maximal t such that Y is a t-design.

Then it is known that (Cf. Lemma 5.2. Godsil's book.)

$$t \leq 2s.$$

Also, we have:

- Y is of strength $t \implies |Y| \geq \dim(Pol(\Omega,[\frac{t}{2}]))$.
- Y is of degree $s \implies |Y| \leq \dim(Pol(\Omega,s))$.

It is an interesting problem classifying such Y with inequality holds in one of the above inequalities. (Tight designs, and tight codes.)

On the sphere, or on a P-and Q-polynomial association scheme, it is known that if Y is of degree s and strength t, and if $t \geq 2s-2$, then Y has the structure of (Q-polynomial) association scheme of class s.

Does this property holds for other general spherical polynomial spaces? How much does this property hold? I think it is not known whether this hold or not. On the other hand, this property holds for subset Y of $(U(d), \rho)$ if $t \geq 2$ and $s = 2$ by Sho Suda (personal communication). So, it would be interesting to find such strongly regular graphs that are embedded in $(U(d), \rho)$.

The strongly regular graphs that are embedded in $(U(d), \rho)$ as degree $s = 2$ and strength $t = 2$ are very much limited. We have strong restrictions on the parameter of such strongly regular graphs.

For example, m_1 or m_2 must be equal to $(d^2 - 1)^2$. So, if d is small and fixed, then all the possible parameters of such (primitive) strongly regular graphs are explicitly listed (there are only finitely many such possibility). Although we have not yet completely solved this question, it seems possible to solve this problem completely for $d = 2$. One of the most likely case was when the parameters are those of $H(2, 4)$. (So, m_1 or m_2 must be 9.) Actually we have succeeded in showing that $H(2, 4)$ as well as the Shrikhande graph cannot be embedded in $U(2)$ (Bannai-Zhao).

For general d, the parameter of the strongly regular graph $H(2, d^2)$ gives a possibility of such existence. It is still open to show that $H(2, d^2)$ cannot be embedded in $U(d)$ in such a way. (But we expect the non-existence, as the case $d = 2$ was so.)

Actually, there are a lot of interesting problems in this direction of research, as we are very much interested in finding unitary t-design on $U(d)$, smaller sizes as possible, for given t and d.

Finally, I would like to discuss the possible constructions of smaller unitary t-designs in $U(d)$. (Tomorrow, Da Zhao will talk about explicit constructions for general t and d.)

Unitary t-groups in $U(d)$ are completely classified for all $t \geq 2$ and $d \geq 2$ (Cf. Guralnick-Tiep, 1995; Bannai-Navarro-Noelia-

Tiep, 2020).

Let $t = 2$. What are the smallest size of unitary 2-design Y in $U(d)$? (It is well known that $|Y| \geq (d^2-1)^2+1$, but the existence of one with $|Y| = (d^2-1)^2+1$, namely, a tight 2-design, is still open.)

Let $d = q = p^a$, and let $Sp(2a, p)$ act naturally on the vector space W of dimension $2a$ over $\mathbb{F}_p$. It is known (cf. Theorem 3 in BNNT(2020)) that if there is a subgroup of $Sp(2a, p)$ that acts transitively on $W - \{0\}$, then a unitary 2-group is obtained. ($SL(2, q)$ is such a subgroup and of order roughly q^3. (So, the obtained unitary 2-group is roughly of order q^5.) We think we have just obtained the following result.

Theorem (Bannai-Zhao). If there exists a transitive subset of $Sp(2a, p)$ acting on $W - \{0\}$, then we get a unitary 2-design on $U(p^a)$.

We think there is a good possibility of the existence of smaller sized transitive set of $Sp(2a, p)$ on $W - \{0\}$. It is known that no such sharply transitive subset exists (by Müller-Nagy, 2011). While, it seems transitive sets are not much studied in general, we hope the existence of such good transitive sets, and then unitary 2-designs.

Thank you very much