An extension of Chapple's formula by Blaschke-like maps

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Abstract

We define the Blaschke-like map on elliptic domains using conformal deformation and study the geometric properties of the maps. In particular, we give an extension of Chapple's theorem and show each Poncelet's triangle for two nested ellipses is constructed from a Blaschke-like map.

1 Introduction

The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R-2r)$, where R and r are the circumradius and inradius, respectively. In particular, if the circumscribed circle is the unit circle, then the distance is given by

$$d^2 = 1 - 2r.$$

This formula is known as Chapple's formula and is independently given by Chapple [Cha46] and Euler (1765). So this formula is also called the Chapple-Euler formula.

This Chapple's formula gives no information about the location of the triangle. But the following Poncelet's theorem [Pon66] guarantees that any point on the outer circle can be a vertex of an inscribed triangle. The following Poncelet's theorem holds for two nested conics, but in this paper, we discuss only the elliptic case.

Theorem 1 (Poncelet [Pon66])
Let E_2 is a conic (ellipse), and E_1 is another conic (ellipse) that contains E_2 in its interior. If there exists an n-sided polygon inscribed in E_1 and simultaneously circumscribed about E_2, then for any point P_0 of E_1, there exists an n-sided polygon with P_0 as a vertex, inscribed in E_1 and circumscribed about E_2.

The n-sided polygon that satisfies the above conditions is called *Poncelet n-polygon with respect to E_1 and E_2*, and E_2 is called an *n-inscribed ellipse in E_1*. The above two theorems are the subject of algebraic geometry, but here we approach this problem analytically.

A *Blaschke product* of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

For a Blaschke product of degree d, set

$$f_1(z) = e^{-\frac{\theta}{d} i} z, \quad \text{and} \quad f_2(z) = \frac{z - (-1)^d a_1 \cdots a_d e^{i\theta}}{1 - (-1)^d \overline{a_1 \cdots a_d e^{i\theta}} z}.$$

Then, the composition $f_2 \circ B \circ f_1$ is a canonical one, and geometric properties with respect to the preimages of B and $f_2 \circ B \circ f_1$ are the same.

Remark that there are d distinct preimages $z_1, \cdots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative of B has no zeros on $\partial\mathbb{D}$.

Let $z_1, \cdots, z_d$ be the d distinct preimages of $\lambda \in \partial\mathbb{D}$ by B, and ℓ_λ the set of lines joining z_j and z_k $(j \neq k)$. Here, we consider the family of lines $\{\ell_\lambda\}_{\lambda \in \mathbb{D}}$, and the envelope I_B of the family. We call the envelope I_B the *interior curve associated with* B.

The interior curve associated with a Blaschke product of degree 3 forms an ellipse and corresponds to the inner ellipse of Poncelet's theorem (see Fig. 1). See also [Fuj13] for the case of degree 4.

Theorem 2 (Daepp, Gorkin, and Mortini [DGM02])
Let B be a canonical Blaschke product of degree 3 with zeros a and b. For $\lambda \in \partial\mathbb{D}$, let z_1, z_2, and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation

$$|z-a| + |z-b| = |1 - \overline{a}b|.$$

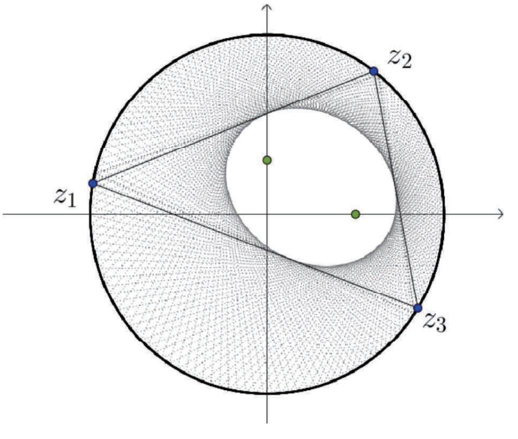

Figure 1: The figure indicates the case of $a = 0.5$ and $b = 0.3i$. The interior curve given by E in Theorem 2 corresponds to "the inner ellipse" in Poncelet's Theorem.

The following result guarantees that any 3-inscribed ellipse in $\partial\mathbb{D}$ can be constructed from a Blaschke product of degree 3.

Theorem 3 (Frantz [Fra04])
For the case of a triangle, the ellipse E is a Poncelet's inner ellipse if and only if E is the interior curve for some Blaschke product of degree three.

The two theorems above allow the result of Chapple's formula to extend the inner circle to the inner ellipse. If we could extend the outer circle to an ellipse in the above theorems, would we obtain a result more similar to Poncelet's theorem? In the next section, we will consider such problems.

2 Blaschke-like maps

Let φ_t be a *Joukowski transformation* of the form

$$z = \varphi_t(w) = \frac{1}{1+t^2}\left(t^2 w + \frac{1}{w}\right) \qquad (0 < t < 1).$$

Note that for $t = 0$, φ_0 maps the unit disk $\mathbb{D}$ to $\widehat{\mathbb{C}} \setminus \mathbb{D}$, and for $t = 1$, φ_1 maps $\mathbb{D}$ to $\widehat{\mathbb{C}} \setminus \{[-1, 1]\}$.

Let $\mathbb{E}_t$ be the elliptic disk

$$\mathbb{E}_t = \left\{ \left| z - \frac{2t}{1+t^2} \right| + \left| z + \frac{2t}{1+t^2} \right| < 2 \right\}.$$

Then, φ_t conformally maps $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$ and continuously maps $\overline{\mathbb{D}}$ to $\widehat{\mathbb{C}} \setminus \mathbb{E}_t$ (see Fig. 2).

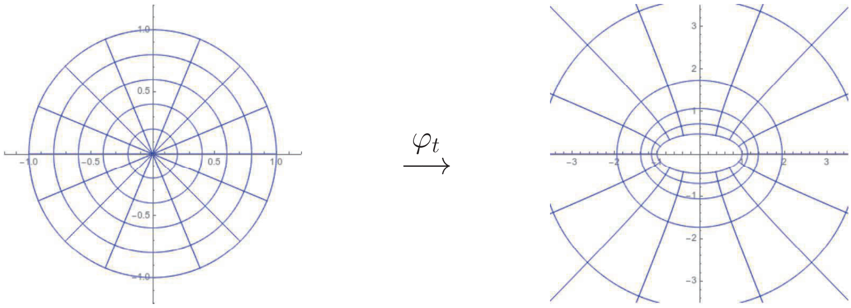

Figure 2: The map φ_t conformally maps $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$.

For any eccentricity e $(0 < e < 1)$, the unit disk can be mapped to an ellipse of eccentricity e by φ_t with a suitable value of t.

For a canonical Blaschke product B, let $B_{\varphi_t} = \varphi_t \circ B \circ \varphi_t^{-1}$.

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "B"] \arrow[d, "\varphi_t"'] & \mathbb{D} \arrow[d, "\varphi_t"] \\
\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t \arrow[r, "B_{\varphi_t}"'] & \widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t
\end{tikzcd}$$

As the equality $\varphi_t\left(\dfrac{1}{t^2 w}\right) = \varphi_t(w)$ holds, φ_t conformally maps $\left\{\dfrac{1}{t^2} < |w| \leq \infty\right\}$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$. Since $\varphi_t : \mathbb{D} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$ (onto, conformal), we can choose a unique branch w of $\varphi_t^{-1}(z)$ with $|w| < 1$ for each $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$. Then, B_{φ_t} maps the exterior of the elliptic disk $\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$ onto itself.

We call B_{φ_t} a *Blaschke-like map associated with B and φ_t.*

For a Blaschke-like map associated with a canonical Blaschke product of degree 3, we have the following results (see also Fig.3, and see [FG22] for the proof).

Theorem 4
Let B_{φ_t} be a Blaschke-like map associated with a Blaschke product B of degree 3 and φ_t. Then, the interior curve with respect to B_{φ_t} is an ellipse.

In fact, the interior curve is given by

$$\overline{U} z^2 + P z\overline{z} + U\overline{z}^2 + \overline{V} z + V\overline{z} + Q = 0,$$

where

$$\begin{aligned}
U &= (a-b)^2 t^4 + 2(2|ab|^2 - |a+b|^2 + 2)t^2 + (\overline{a} - \overline{b})^2, \\
P &= -2\Big((2|ab|^2 - |a+b|^2 + 2)(t^4+1) + \big((a-b)^2 + (\overline{a}-\overline{b})^2\big)t^2\Big), \\
V &= -2(1-t^2)\Big(\big((|ab|^2+1)(a+b) - (a^2+b^2)(\overline{a}+\overline{b})\big)t^2 \\
&\qquad + (a+b)(\overline{a}^2 + \overline{b}^2) - (|ab|^2+1)(\overline{a}+\overline{b})\Big), \\
Q &= (1-t^2)^2\big((|ab|^2 - |a+b|^2 - 1)^2 - 4|a+b|^2\big).
\end{aligned}$$

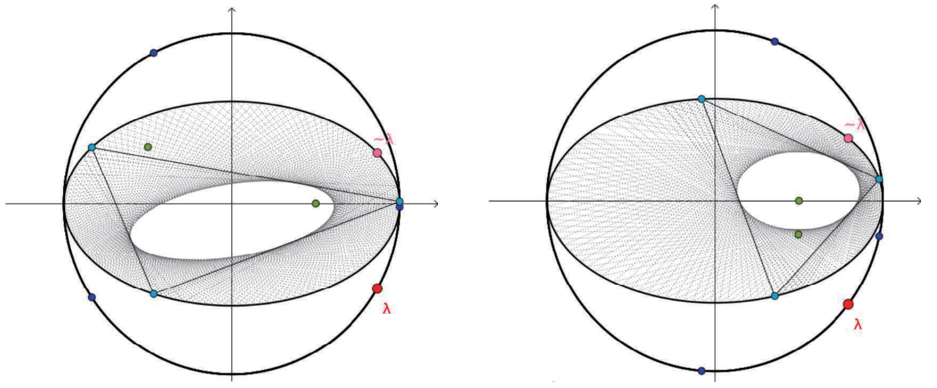

Figure 3: Each envelope indicates the interior curve of a Blaschke-like map associated with the canonical Blaschke product B and $\varphi_{1/2}$. The zeros of B are $0, \frac{1}{2}, -\frac{1}{2}+\frac{1}{3}i$ (the left figure), and $0, \frac{1}{2}, \frac{1}{2}-\frac{1}{5}i$ (the right figure), respectively.

Moreover, we can check that the above equation gives a non-degenerate ellipse.

Proposition 5
For each ellipse E_t $(0 < t < 1)$, C_2 is the 3-inscribed ellipses in E_t if and only if C_2 is the interior curve with respect to a Blaschke-like map B_{φ_t} for some Blaschke product B of degree 3.

The above results allow the result of Chapple's formula to extend the outer circle to the outer ellipse.

3 Another type of Blaschke-like map

In section 2, we could extend Chapple's formula by constructing a Blaschke-like map on an elliptic domain using Joukowski transformation. Since there are other conformal maps from the unit disk onto a elliptic domain, the following question arises naturally. Does the conformal transformation that maps a disk onto an elliptic domain always induce an elliptic interior curve? The answer to this question is no.

Here, we consider a conformal map that maps the unit disk to an elliptic disk (see, for example, Nehari [Neh75], Schwarz [Sch69] for details).

Let

$$\mathrm{sn}^{-1}(w,k) = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \quad K(k) = \int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}},$$

$$\text{and} \qquad K'(k) = \int_1^{\frac{1}{k}} \frac{dw}{\sqrt{(w^2-1)(1-k^2w^2)}}.$$

For $0 < p < 1$, we consider the following transformations

$$u(w) = u = \frac{w-1}{w+1}, \quad v(u) = v = c \cdot \mathrm{sn}^{-1}(u,k),$$

$$x(v) = x = \sqrt{\frac{1+p}{1-p}}e^v, \quad \text{and} \quad z(x) = z = \frac{\sqrt{1-p^2}}{2}\left(x + \frac{1}{x}\right),$$

where k and c are chosen such that $\log\sqrt{\frac{1+p}{1-p}} = \pi\frac{K(k)}{K'(k)}$, $\log\sqrt{\frac{1+p}{1-p}} = cK(k)$. Then, u, v, x, and z are conformal maps of D_w to D_u, D_u to D_v, D_v to D_x, and D_x to D_z, respectively.

Let $\mathcal{E}_p$ be the elliptic disk

$$\mathcal{E}_p = \left\{ \left|z - \sqrt{1-p^2}\right| + \left|z + \sqrt{1-p^2}\right| < 2 \right\}.$$

Setting the composition γ_p as

$$z = \gamma_p(w) = z \circ x \circ v \circ u(w),$$

γ_p is a conformal map that maps the upper half the unit disk to the upper half of the elliptic disk $\mathcal{E}_p$. Using the Schwarz reflection principle, there exists a conformal map $\widetilde{\gamma}_p$ from $\mathbb{D}$ onto $\mathcal{E}_p$ (see Fig. 4). Note that for $p = 1$, $\widetilde{\gamma}_1$ maps the unit circle to itself.

Then, the map $B_{\widetilde{\gamma_p}} = \widetilde{\gamma}_p \circ B \circ \widetilde{\gamma}_p^{-1}$ is a Blaschke-like map on the elliptic disk $\mathcal{E}_p$.

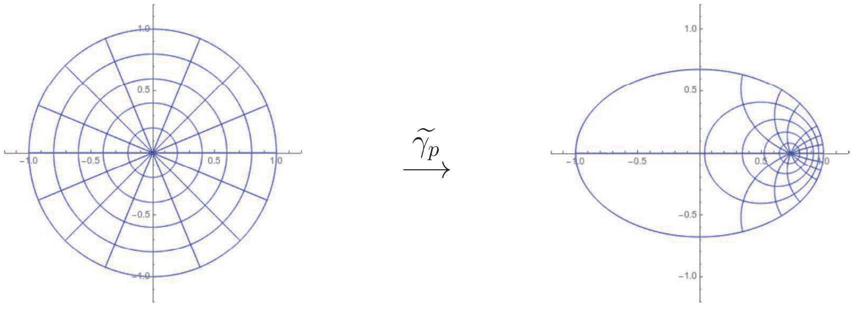

Figure 4: By using the Schwarz reflection principle, we can construct a conformal map that maps the unit disk $\mathbb{D}$ onto an elliptic disk $\mathcal{E}_p$.

3.1 computer experiments

Let $B(z) = z \dfrac{z-a}{1-\overline{a}z} \dfrac{z-b}{1-\overline{b}z}$. Here, we consider the Blaschke-like map associated with B and $\widetilde{\gamma}$.

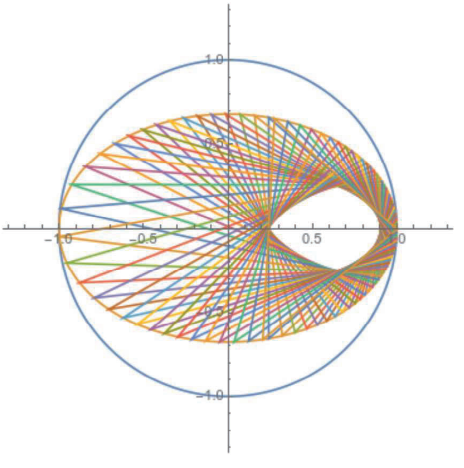

Figure 5: The envelope indicates the interior curve of $B_{\widetilde{\gamma}}$ with $a = 0,\ b = -0.9,\ k = \frac{1}{100},\ c \approx 0.524$. Clearly, the envelope is not an ellipse.

Fig. 5 indicates the family of lines connecting the inverse images of points on the ellipse $\partial\mathcal{E}_p$ with $p \approx 0.677$, using Mathematica. The envelope gives the interior curve with respect to $B_{\widetilde{\gamma}}$, and we can see that it is not an ellipse.

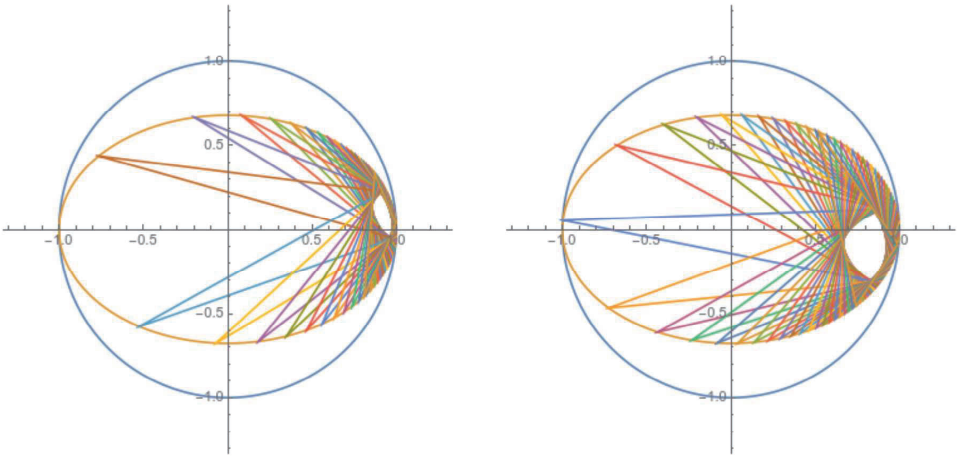

Figure 6: Each envelope indicates the interior curve of $B_{\widetilde{\gamma}}$ with $a = \frac{1}{2}i,\ b = \frac{1}{2},\ k = \frac{1}{100},\ c \approx 0.524$ (the left figure) and $B_{\widetilde{\gamma}}$ with $a = -\frac{1}{2}i,\ b = -\frac{1}{2},\ k = \frac{1}{100},\ c \approx 0.524$ (the right figure), respectively. It is clear to see that neither of them is conic.

The zero points of the Blaschke products corresponding to the two figures in Fig. 6 are symmetric with respect to the origin. But we can see that this symmetry is not inherited by the interior curves. This property of $B_{\widetilde{\gamma}}$ is different from the fact that symmetry is inherited in B_{φ_t}.

Acknowledgements

This work was partially supported by JSPS KAKENHI Grant Number JP19K03531.

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