Applications of the strong propagation estimate for Scrödinger operator with sub-quadratic repulsive potentials

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Abstract

In the paper [2], the strong propagation estimate has been shown and, in this note, we see the some applications of this estimate.

1 Introduction

Throughout this paper, we deal with the hamiltonian written by

$$H = p^2 - \sigma |x|^{\alpha} + V,$$

where $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$, $p = -i\nabla$, $\sigma > 0$, $0 < \alpha < 2$, and V is the smooth external potential satisfying

$$\left| \partial^{\beta} V(x) \right| \leq C_{V,\beta} \left\langle x \right\rangle^{-\theta(1-\alpha/2)-|\beta|} \tag{1}$$

for some $\theta > 1$ and for any multi-index $\beta \in \mathbf{N}^n$. We let a conjugate operator $\mathscr{A}$ as follows:

$$\mathscr{A} := \left\langle x \right\rangle^{-\alpha} x \cdot p + p \cdot x \left\langle x \right\rangle^{-\alpha}, \quad \langle \cdot \rangle = (1 + \cdot^2)^{1/2}.$$

One of the main theorem in the paper [2] is the following:

Theorem 1.1. *Let $\alpha_0 = \min\{\alpha\sigma, (2-\alpha)\sigma\}$, $0 < \delta \ll \alpha_0$, and $g \in C^\infty(\mathbf{R})$ be a cut-off function such that $g(x) = 1$ if $x < \delta$ and $g(x) = 0$ if $x > 2\delta$. Then, for any $\kappa \geq 0$, $\varphi \in C_0^\infty(\mathbf{R})$ and $\psi \in L^2(\mathbf{R}^n)$, there exists $C_\kappa > 0$ such that*

$$\left\| g(\mathscr{A}/t)e^{-itH}\varphi(H) \langle \mathscr{A} \rangle^{-\kappa} \psi \right\|_{L^2} \leq C_\kappa |t|^{-\kappa} \|\psi\|_{L^2}$$

holds for $|t| \geq 1$.

Remark 1. *For such $\mathscr{A}$, we can see*

$$\varphi(H)i[H, \mathscr{A}]\varphi(H) \geq \alpha_0 \varphi(H)^2 + C_0$$

and some conditions, they are necessary for considering the Mourre estimate. Hence we can also have the limiting absorption principle

$$\sup_{\lambda \in I \subset \mathbf{R}^n,\ \nu > 0} \left\| \langle \mathscr{A} \rangle^{-s} (H - \lambda \mp i\nu)^{-1} \langle \mathscr{A} \rangle^{-s} \right\|_{\mathscr{B}} < \infty$$

with any $s > 1/2$.

2 Applications

We now consider some applications of Theorem 1.1. Throughout this section, we auxiliary introduce some important lemmas:

Lemma 2.1 ([2], Lemma 3.1.). *For all $\phi \in L^2$,*

$$\sum_{j=1}^{n} \left\| \langle x \rangle^{-\alpha/2} p_j (H_0 + i)^{-1} \phi \right\|_{L^2}^2 \leq C \|\phi\|_{L^2}^2 .$$

Lemma 2.2 ([2], §4). *The operator $i[H_0, \mathscr{A}](H_0 + i)^{-1}$ is bounded on $L^2(\mathbf{R}^n)$.*

Remark 2. *Both Lemma 2.1 and Lemma 2.2 with replacing H_0 to H will be true, if V is bounded and smooth (or relatively bound with respect to H_0).*

2.1 Unitarity of wave operators

We first consider the completeness of wave operators. Owing to the result of [3], we have that $\sigma_{pp}(H) = \emptyset$, and hence the asymptotic completeness of wave operators in our model is equivalent to $\mathrm{Ran} W^\pm = L^2(\mathbf{R}^n)$, where

$$W^\pm = \mathrm{s-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_0}, \quad H_0 = H - V.$$

By the Cook-Kuroda method, it ie enough to show that for $\phi \in \mathscr{S}(\mathbf{R}^n)$ and $\varphi \in C_0^\infty(\mathbf{R})$,

$$\int_{\pm 1}^{\pm\infty} \left\| V\varphi(H)e^{-itH}\phi \right\|_{L^2} dt \le C.$$

We show this for the case where the interval of integral is positive.

Thanks to the theorem 1.1 with $\kappa > 1$, it holds that

$$\int_1^\infty \left\| V g(\mathscr{A}/t)\varphi(H)e^{-itH}\phi \right\|_{L^2} dt \le C_\kappa \int_1^\infty |t|^{-\kappa} dt \le C.$$

Hence we show

$$\int_1^\infty \left\| V \left(1 - g(\mathscr{A}/t)\right) \varphi(H)e^{-itH}\phi \right\|_{L^2} dt \le C. \tag{2}$$

In order to show this, we let $V_\theta = |V|^{2/\theta}$ and show that

$$\left\| V_\theta \left(1 - g(\mathscr{A}/t)\right) |\varphi(H)|^{2/\theta} \right\|_{\mathscr{B}} \le C_2 t^{-2}. \tag{3}$$

Then the interpolation together with $\| |V|^0 \left(1 - g(\mathscr{A}/t)\right) |\varphi(H)|^0 \|_{\mathscr{B}} \le 1$ tells us

$$\| V \left(1 - g(\mathscr{A}/t)\right) |\varphi(H)| \|_{\mathscr{B}} = \left\| V_\theta^{\theta/2} \left(1 - g(\mathscr{A}/t)\right) |\varphi(H)|^{2/\theta \times \theta/2} \right\|_{\mathscr{B}} \le C_2^{\theta/2} t^{-\theta}, \tag{4}$$

and which immediately shows (2).

Here the Helffer-Sjöstrand formula and the commutator expansion, see e.g. §C.2. – §C.4. of [1], tell us

$$\begin{aligned} [V_\theta, g(\mathscr{A}/t)] &= -t^{-1} g'(\mathscr{A}/t)[\mathscr{A}, V_\theta] + \mathcal{O}(t^{-2}) \\ &= -2t^{-1} g'(\mathscr{A}/t)(\langle x \rangle^{-\alpha} x \cdot \nabla V_\theta(x)) + \mathcal{O}(t^{-2}) \end{aligned}$$

and hence

$$\begin{aligned} &V_\theta \left(1 - g(\mathscr{A}/t)\right) \varphi(H)\phi \\ &= \left(1 - g(\mathscr{A}/t)\right) V_\theta \varphi(H)\phi + 2t^{-1} g'(\mathscr{A}/t)(\langle x \rangle^{-\alpha} x \cdot \nabla V_\theta(x))\phi + \mathcal{O}(t^{-2})\phi. \end{aligned}$$

Using that $\mathscr{A} > \delta t$ holds on the support of $1 - g(\mathscr{A}/t)$ and $g'(\mathscr{A}/t)$, we deduce the integrability in t for the L^2 norm of first and second term of r.h.s of the above equation. For simplicity, we only consider the first term. To show this, we first state the important lemma:

Lemma 2.3. *For any* $0 \leq \theta \leq 2$,

$$\left\| \left| \langle x \rangle^{-1-\alpha/2} x \cdot p \right|^{\theta} \varphi(H) \right\| \leq C \tag{5}$$

Proof. For $\theta = 0$, (5) is obviously holds, and for $\theta = 1$, (5) can be shown by using Lemma 2.1. Interpolating between $\theta = 0$ and $\theta = 1$, (5) for $0 \leq \theta \leq 1$ can be shown. By the same rule, it is enough to show (5) for the case where $\theta = 2$. Simple commutator calculation tells us

$$\begin{aligned}
&\left| \langle x \rangle^{-1-\alpha/2} x \cdot p \right|^2 \\
&\sim \langle x \rangle^{-2-\alpha} \sum_{i,j} x_i x_j p_i p_j + (\text{ easier to handle }) \\
&\sim \langle x \rangle^{-2-\alpha} \sum_{i,j} x_i x_j p_i p_j (p^2 + i)^{-1} (H_0 + \sigma |x|^{\alpha} + i) + (\text{ easier to handle }).
\end{aligned}$$

Clearly operators

$$\langle x \rangle^{-2-\alpha} x_i x_j p_i p_j (p^2 + i)^{-1} H_0, \quad \langle x \rangle^{-2-\alpha} x_i x_j p_i p_j (p^2 + i)^{-1} |x|^{\alpha}$$

are bounded on $\mathscr{D}(H)$, and hence the desired result can be obtained. $\square$

Remark 3. *By repeating the similar calculations, we may remove the restriction* $\theta \leq 2$. *Here, we omit to discuss about this.*

By the condition of g, it follows that

$$\| (1 - g(\mathscr{A}/t)) V_{\theta} \varphi(H) \phi \|_{L^2} \leq C t^{-2} \left\| \mathscr{A}^2 V_{\theta} \varphi(H) \phi \right\|_{L^2}.$$

By the rough calculation, we have

$$\mathscr{A}^2 \sim \langle x \rangle^{-2\alpha} (x \cdot p)^2 + (\text{ easier to handle }),$$

and then it follows that

$$\begin{aligned}
\left\| \mathscr{A}^2 V_{\theta} \varphi(H) \right\| &\leq \left\| V_{\theta} \langle x \rangle^{-2\alpha} (x \cdot p)^2 \varphi(H) \right\| + (\text{ bdd }) \\
&\leq C \left\| \langle x \rangle^{-2-\alpha} (x \cdot p)^2 \varphi(H) \right\| + (\text{ bdd }) \\
&\leq C \left\| \left(\langle x \rangle^{-1-\alpha/2} (x \cdot p) \right)^2 \varphi(H) \right\| + (\text{ bdd })
\end{aligned}$$

is bounded.

2.2 Existence of wave operators with the abstract settings

We let

$$H = H_0 + V$$

with $V = V_{sing} + V_r$, where $V_r(x)$ satisfies (1), and V_{sing} is the singular part. The typical example of $V_{sing}(x)$ is something like $C_n|x|^{-\theta}$ with some $\theta > 0$. For simplicity we do not give the precise condition on V_{sing}, while give the abstract condition:

Assumption 2.4. *Let $a \in \mathbf{R}^n$. Then the followings hold:*
(I): *H is a selfadjoint operator on certain domain $\mathscr{D}(H)$.*
(II): *unitary propagator e^{-itH} is differentiable in t on $\mathscr{D}(H) \cap L^2(\mathbf{R}^n \backslash B_a(1))$.*
(III): *$V_{sing} \in L^\infty(\mathbf{R}^n \backslash B_a(1))$ and, as $|x| \to \infty$, V_{sing} decays faster than V_r.*

Remark 4. *We assume neither the boundedness of $V(H_0 + i)^{-1}$ nor the domain invariant $\mathscr{D}(H) = \mathscr{D}(H_0)$.*

The aim inhere is to show the existence of $W^\pm$ under these conditions. At first, we state the important lemma:

Lemma 2.5. *Let $R > |a| + 2$. Then for all $\phi \in \mathscr{S}(\mathbf{R}^n)$,*

$$\left\| F(|x| \leq R)\,(1 - g(\mathscr{A}/t))\,\varphi(H_0)e^{-itH_0}\phi \right\|_{L^2} \leq Ct^{-2},$$

where $F(|x| \leq R)$ is the smooth cut-off function so that $|x| \leq R$ on the support of $F(|x| \leq R)$ and $|x| \geq R + 1$ on the support of $1 - F(|x| \leq R)$.

Proof. This lemma can be proven by replacing V_θ with $F(|x| \leq R)$ in (3). □

We denote $1 - F(|x| \leq R)$ as $F(|x| \geq R+1)$. Using this lemma, we notice that it is enough to show the existence of

$$\text{s-}\lim_{t \to \pm\infty} e^{itH} F(|x| \geq R + 1)\,(1 - g(\mathscr{A}/t))\,\varphi(H_0)e^{-itH_0}$$

for showing the existence of wave operators. Straightforward calculation shows that

$$\frac{d}{dt} e^{itH} F(|x| \geq R + 1)\,(1 - g(\mathscr{A}/t))\,\varphi(H_0)e^{-itH_0}\phi = \mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t)$$

with

$$\mathcal{E}_1(t) = e^{itH} V F(|x| \geq R+1) \,(1 - g(\mathscr{A}/t))\, \varphi(H_0) e^{-itH_0}\phi,$$
$$\mathcal{E}_2(t) = e^{itH} i[p^2, F(|x| \geq R+1)] \,(1 - g(\mathscr{A}/t))\, \varphi(H_0) e^{-itH_0}\phi$$

and

$$\mathcal{E}_3(t) = e^{itH} F(|x| \geq R+1) i[H_0, g(\mathscr{A}/t)]\varphi(H_0) e^{-itH_0}\phi.$$

By the conditions on V_r and V_{sing}, and the same calculations in deducing (4), we have

$$\begin{aligned}
&\|\mathcal{E}_1(t)\|_{L^2} \\
&\leq \left\| V F(|x| \geq R+1) \langle x \rangle^{\theta(1-\alpha/2)} \right\|_{\mathscr{B}} \left\| \langle x \rangle^{-\theta(1-\alpha/2)} (1 - g(\mathscr{A}/t))\, \varphi(H_0) e^{-itH_0}\phi \right\|_{L^2} \\
&\leq Ct^{-\theta}. \qquad (6)
\end{aligned}$$

Next divide $\mathcal{E}_2(t) = \mathcal{E}_2^1(t) + \mathcal{E}_2^2(t)$ by

$$\mathcal{E}_2^1(t) = e^{itH} i[p^2, F(|x| \geq R+1)] \,[\tilde{\varphi}(H_0), g(\mathscr{A}/t)]\, \varphi(H_0) e^{-itH_0}\phi$$

and

$$\mathcal{E}_2^2(t) = e^{itH} i[p^2, F(|x| \geq R+1)] \tilde{\varphi}(H_0) \,(1 - g(\mathscr{A}/t))\, \varphi(H_0) e^{-itH_0}\phi,$$

where $\tilde{\varphi} \in C_0^\infty(\mathbf{R})$ is the one satisfying $\varphi\tilde{\varphi} = \varphi$. Commutator expansion yields

$$i[H_0, g(\mathscr{A}/t)] = t^{-1} i[H_0, \mathscr{A}] g'(\mathscr{A}/t) + \mathcal{O}(t^{-2})$$

and which tells us, by the Helffer-Sjöstrand formula, that

$$\begin{aligned}
\mathcal{E}_2^1(t)\phi = \int_{\mathbf{C}} &\overline{\partial_z} \tilde{\varphi}_0(z) e^{itH} i[p^2, F(|x| \geq R+1)](z - H_0)^{-1} \\
&\times B_0(z) g'(\mathscr{A}/t) \varphi(H_0) e^{-itH_0}\phi \frac{dz d\bar{z}}{t} + \mathcal{O}(t^{-2}),
\end{aligned}$$

where $\tilde{\varphi}_0$ is the almost analytic extension of $\tilde{\varphi}$ and $B_0(z)$ is the some bounded operator. Noting that the support of $F'(|x| \geq R+1)$ is compact and Lemma 2.1 with $\theta = 1$, we have

$$\begin{aligned}
&\left\|\mathcal{E}_2^1(t)\phi\right\|_{L^2} \\
&\leq Ct^{-1} \left\| i[p^2, F(|x| \geq R+1)](H_0 + i)^{-1} \right\|_{\mathscr{B}} \left\| g'(\mathscr{A}/t) \varphi(H_0) e^{-itH_0}\phi \right\|_{L^2} + Ct^{-2} \\
&\leq Ct^{-2}, \qquad (7)
\end{aligned}$$

where we apply Theorem 1.1 with replacing g to g'. Since the support of $F'(|x| \geq R+1)$ is compact, we can find

$$i[p^2, F(|x| \geq R+1)]\tilde{\varphi}(H_0)\left\langle x\right\rangle^{2-\alpha}$$

is bounded, while by the similar calculations in proving (3), it also holds that

$$\left\langle x\right\rangle^{-2+\alpha}(1-g(\mathscr{A}/t))\,\varphi(H_0)e^{-itH_0}\phi = \mathcal{O}(t^{-2}).$$

Combining them, we get

$$\left\|\mathcal{E}_2^2(t)\phi\right\|_{L^2} \leq Ct^{-2}. \tag{8}$$

By (2.1) and the theorem 1.1 with g', we immediately get

$$\left\|\mathcal{E}_3(t)\phi\right\|_{L^2} \leq Ct^{-2}. \tag{9}$$

Each estimates (6)–(9) and the Cook-Kuroda method show the existence of $W^{\pm}$.

Acknowledgement.
This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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