A Rellich type theorem for the generalized oscillator

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Abstract

In this article, we review the results in [12] on a Rellich type theorem, or characterize the order of growth of eigenfunctions for the generalized oscillator. The proofs are given by extensive use of commutator arguments invented recently by Ito and Skibsted from [8].

This article is organized as follows. In Section 1, we introduce our setting and our results. In Section 2, we discuss the conjugate operator A and commutators with weight inside, which play important roles in the proofs of the main results. In Section 3, we introduce a brief overview of the proof of the main result.

1 Introduction

1.1 Basic Setting

We consider the generalized oscillator

$$H = -\frac{1}{2}\Delta + \frac{a}{2}|x|^{2b} + V \text{ on } \mathcal{H} = L^2(\mathbb{R}^d),$$

where $a, b > 0$, $d \geq 1$ and V is the real-valued function that may grow slightly slower than $\frac{a}{2}|x|^{2b}$.

Let us impose on V a more precise condition. We choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \quad \chi' \leq 0, \tag{1.1}$$

and set $r \in C^\infty(\mathbb{R}^d)$ as

$$r(x) = \chi(|x|) + |x|(1 - \chi(|x|)). \tag{1.2}$$

Condition 1.1. The perturbation V is a real-valued function. Moreover, there exists a splitting by real-valued functions:

$$V = V_1 + V_2;\ V_1 \in C^1(\mathbb{R}^d; \mathbb{R}),$$

such that for some $\mu, C > 0$ the following bounds hold globally on $\mathbb{R}^d$:

$$|V_1| \le Cr^{2b-\mu},\ |\partial_r V_1| \le Cr^{2b-1-\mu},\ |V_2| \le Cr^{b-1-\mu}. \tag{1.3}$$

Here $\partial_r = (\partial_i r)\partial_i$ denotes the radial differential operator. Note that we use the Einstein summation convention and throughout the paper we will use this notation.

In this article we assume Condition 1.1. From this, we note that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ by [9, Theorem X.28].

Next we introduce the weighted Hilbert space $\mathcal{H}_s$ for $s \in \mathbb{R}$ by

$$\mathcal{H}_s = r^{-s}\mathcal{H}.$$

We also denote the locally L^2-space by

$$\mathcal{H}_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\mathbb{R}^d).$$

1.2 Results

Our main results are the following three theorems.

Theorem 1.2. Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{H}_{\mathrm{loc}}$ satisfies that

- $(H - \lambda)\phi = 0$ in the distributional sense,
- there exists an $\alpha > \frac{\sqrt{a}}{1+b}$ such that $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$,

then $\phi = 0$ in $\mathbb{R}^d$.

Theorem 1.3. Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{H}_{\mathrm{loc}}$ satisfies that

- $(H - \lambda)\phi = 0$ in the distributional sense,
- there exists an $|\alpha| < \frac{\sqrt{a}}{1+b}$ such that $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$,

then $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$ for any $|\alpha| < \frac{\sqrt{a}}{1+b}$.

To state the third main theorem, we introduce a differential operator

$$L = p_i \ell^{ij} p_j \text{ with } \ell^{ij} = \delta^{ij} - (\partial_{x_i} r)(\partial_{x_j} r),$$

which may be considered the spherical part of $-\Delta$ on $\{x \in \mathbb{R}^d \mid r(x) \ge 2\}$. We also use the notation $\langle T \rangle_\phi = \langle \phi, T\phi \rangle$.

Theorem 1.4. Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{H}_{\mathrm{loc}}$ satisfies that

- $(H - \lambda)\phi = 0$ in the distributional sense,
- there exists an $\alpha < -\frac{\sqrt{a}}{1+b}$ such that $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$,
- there exist $C, \rho > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^d)$ the following property holds

$$\langle p_i \chi \ell^{ij} p_j \rangle_\phi \leq C \langle \chi r^{2b-\rho} \rangle_\phi,$$

then $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$ for any $\alpha < -\frac{\sqrt{a}}{1+b}$.

The Schrödinger operator corresponding to the usual harmonic oscillator has L^2-eigenfunctions of the form $a(x)\exp(-|x|^2/2)$ and generalized eigenfunctions of the form $b(x)\exp(|x|^2/2)$, where $a(x)$ and $b(x)$ are certain polynomials. Our main results describe the asymptotic behavior of eigenfunctions for the generalized oscillator. The first theorem states the non-existence of eigenfunctions that decay stronglier than $\exp(-\frac{\sqrt{a}}{1+b}|x|^{b+1})$. The second theorem states the non-existence of eigenfunctions with increasing rates between $\exp(-\frac{\sqrt{a}}{1+b}|x|^{b+1})$ and $\exp(\frac{\sqrt{a}}{1+b}|x|^{b+1})$. The third theorem states the non-existence of eigenfunctions with increasing rates greater than $\exp(\frac{\sqrt{a}}{1+b}|x|^{b+1})$ under the assumption on the angular momentum L. We note that the constants $\pm\frac{\sqrt{a}}{1+b}$ in the main theorems are optimal. This can be seen from the following. We set

$$\phi_\pm(x) = \exp(\pm\frac{\sqrt{a}}{1+b} r^{b+1} - r^b).$$

Then by taking some appropriate $V_\pm$, we have $H\phi_\pm = 0$. We also note that the condition on L in the third theorem cannot be removed. By giving an example we show this. Now we consider

$$H = -\frac{1}{2}\Delta + \frac{1}{2}(x^2 + y^2) \text{ on } \mathbb{R}^2.$$

We also set

$$\phi(x, y) = \exp(x^2 + i\sqrt{3}xy - y^2).$$

Then we can verify the fact. Of course, this condition holds automatically in one dimension, since $L = 0$.

To prove our results we apply the commutator argument invented recently by Ito and Skibsted from [8]. We are directly motivated by their result in which they studied spectral properties of the Schrödinger operator on a manifold with ends. They consider potentials decaying at infinity. In [7], Itakura proved the non-existence of B_0^*-eigenfunctions for the Schrödinger operators with potentials diverging to $-\infty$ at infinity by using the method of the commutator argument. However, they do not consider the Schrödinger operator with growing potentials, which are considered in this paper.

In case $b = 0$, there is an extensive amount of literature on eigenvalue problems, e.g. [1], [2], [3], [4], [5], [8]. As for the case $a, b > 0$, Simon studied for the asymptotic behavior of the L^2-eigenfunctions under a smooth potential in [10], but he does not consider the generalized eigenfunctions. Recently Steinerberger discussed a sharp pointwise Agmon estimate in [11]. We also mention a result [6] by Isozaki and Morioka that studies Rellich's theorem for the discrete Schrödinger operator.

2 Preliminaries

In this section we prepare some tools to prove our theorems.

2.1 Unitary group and generator

Let

$$y : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, (t, x) \mapsto y(t, x) = \exp(t\nabla r)(x),$$

be the maximal flow generated by the gradient vector field ∇r. Note that by definition it satisfies

$$\partial_t y^i(t, x) = (\nabla r)^i(y(t, x)),\ \ y(0, x) = x.$$

We define $T(t) : \mathcal{H} \to \mathcal{H}, t \in \mathbb{R}$, by

$$(T(t)\psi)(x) = J(t, x)^{1/2}\psi(y(t, x))$$

where $J(t, \cdot)$ is the Jacobian of the mapping $y(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$. By definition $T(t), t \in \mathbb{R}$ forms a strongly continuous one-parameter unitary group. Hence by the Stone theorem the generator A of group $T(t), t \in \mathbb{R}$ is self-adjoint on $\mathcal{H}$. It is easy to verify that $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, and that $T(t)$ preserves $C_0^\infty(\mathbb{R}^d)$. Hence by [9, Theorem X.49] the space $C_0^\infty(\mathbb{R}^d)$ is a core for A. Therefore, the following facts hold:

$$\mathcal{D}(A) = \left\{\psi \in \mathcal{H} \ \middle|\ \left(p_r - \frac{i}{2}\Delta r\right)\psi \in \mathcal{H}\right\},$$

$$A\psi = \left(p_r - \frac{i}{2}\Delta r\right)\psi\ (\psi \in \mathcal{D}(A));\ p_r = -i\partial_r.$$

We also note $\mathcal{D}(H) \subset \mathcal{D}(A)$. Next, we consider describing H using A.

Lemma 2.1. One has a decompotision on $C_0^\infty(\mathbb{R}^d)$

$$H = \frac{1}{2}A^2 + \frac{1}{2}L + q + \frac{a}{2}|x|^{2b} + V \text{ with } q = \frac{1}{8}(\Delta r)^2 + \frac{1}{4}\partial_r(\Delta r).$$

This formula is used to compute the commutator of the next Lemma.

2.2 Commutators with weight inside

Next we introduce a commutator with a weight Θ inside:

$$[H, iA]_\Theta = i\,(H\Theta A - A\Theta H)\,.$$

We assume a weight $\Theta = \Theta(r)$ satisfies the following properties:

- Θ is a smooth function with compact support,
- $r \geq 2$ on supp Θ,
- $\Theta \geq 0$ on $\mathbb{R}$,
- $|\Theta^{(k)}| \leq C_k$, $k = 0, 1, 2, \ldots$

where $\Theta^{(k)}$ denotes the k-th derivative of Θ in r. We first define the quadratic form $[H, iA]_\Theta$ on $C_0^\infty(\mathbb{R}^d)$, and then extend it to $H^1(\mathbb{R}^d)$ according the following lemma.

Lemma 2.2. As a form on $C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}[H, iA]_\Theta = {}& A\Theta' A + r^{-1}\Theta L - \frac{1}{4}\Theta''' - (\partial_r q)\Theta - abr^{2b-1}\Theta - (\partial_r V_1)\Theta + V_2\Theta' \\ & - 2\mathrm{Im}(V_2\Theta p_r) + (\Delta r)V_2\Theta - \mathrm{Re}(\Theta' H).\end{aligned}$$

Therefore by the Cauchy–Schwarz inequality $[H, iA]_\Theta$ extends as a bounded form on $H^1(\mathbb{R}^d)$.

In the above argument, we defined the weighted commutator $[H, iA]_\Theta$ as a quadratic form on $H^1(\mathbb{R}^d)$ as an extension from $C_0^\infty(\mathbb{R}^d)$. On the other hand, throughout the paper, we shall use the notation

$$\mathrm{Im}(A\Theta H) = \frac{1}{2i}(A\Theta H - H\Theta A)$$

as a quadratic form defined on $\mathcal{D}(H)$, i.e. for $\psi \in \mathcal{D}(H)$

$$\langle \mathrm{Im}(A\Theta H)\rangle_\psi = \frac{1}{2i}\left(\langle A\psi, \Theta H\psi\rangle - \langle H\psi, \Theta A\psi\rangle\right).$$

Note that by $\mathcal{D}(H) \subset \mathcal{D}(A)$ the above quadratic form is well-defined. Obviously, the quadratic forms $[H, iA]_\Theta$ and $2\mathrm{Im}(A\Theta H)$ coincide on $C_0^\infty(\mathbb{R}^d)$, and since H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ we obtain

$$[H, iA]_\Theta = 2\mathrm{Im}(A\Theta H) \text{ on } \mathcal{D}(H).$$

3 Outline of proof

In this section we only introduce the proof of Theorem 1.2. Note, however, that the proof of Theorem 1.3 and Theorem1.4 can be done by a similar computation to that of Theorem 1.2.

3.1 Proof of Theorem 1.2

The proof of Theorem 1.2 consists of two steps. Obviously, Theorem 1.2 follows immediately as a combination of the following propositions.

Proposition 3.1. Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{H}_{\mathrm{loc}}$ satisfies that

- $(H-\lambda)\phi = 0$ in the distributional sense,
- there exists an $\alpha > \frac{\sqrt{a}}{1+b}$ such that $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$,

then $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$ for any $\alpha > \frac{\sqrt{a}}{1+b}$.

Proposition 3.2. Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{H}_{\mathrm{loc}}$ satisfies that

- $(H-\lambda)\phi = 0$ in the distributional sense,
- $\exp(\alpha|x|^{b+1})\phi \in \mathcal{H}$ for any $\alpha > \frac{\sqrt{a}}{1+b}$,

then $\phi = 0$ in $\mathbb{R}^d$.

Now, using the function $\chi \in C^\infty(\mathbb{R}^d)$ of (1.1), we define $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(\mathbb{R}^d)$ for $n > m \geq 1$ by

$$\chi_m(r) = \chi\left(\frac{r}{2^m}\right), \ \bar{\chi}_n = 1-\chi_n, \ \chi_{m,n} = \bar{\chi}_m\chi_n. \tag{3.1}$$

To prove Proposition 3.1 we introduce an explicit weight Θ with parameters α, β, R and $n > m \geq 1$:

$$\Theta = \Theta^{\alpha,\beta}_{m,n,R} = \chi_{m,n}\mathrm{e}^{\theta}.$$

Here the exponent θ is given by

$$\theta = 2\alpha r^{b+1} + 2(\beta-\alpha)r^{b+1}\left(1+\frac{r^{b+1}}{R}\right)^{-1}; \ \beta > \alpha > \frac{\sqrt{a}}{1+b}, \ \beta-\alpha \leq 1, \ R > 0.$$

Note that the following commutator estimate is established from Lemma 2.2 and is the key to our main result.

Lemma 3.3. Let $\lambda \in \mathbb{R}$, and fix any $\alpha_0 > \frac{\sqrt{a}}{1+b}$. Then there exist $c, C > 0$, $n_0 \geq 1$, $\beta > \alpha_0$, $\alpha_0 > \tilde{\alpha}$ such that for any $n > m \geq n_0$, $R > 0$, $\tilde{\alpha} < \alpha < \alpha_0$,

$$\mathrm{Im}(A\Theta(H-\lambda)) \geq cr^{2b-1}\Theta - C\left(\chi^2_{m-1,m+1} + \chi^2_{n-1,n+1}\right) r^{2b-1}\mathrm{e}^{\theta} + \mathrm{Re}(\gamma(H-\lambda)) \tag{3.2}$$

as forms on $\mathcal{D}(H)$, where $\gamma = \gamma_{m,n,R}$ is a function satisfying

$$\mathrm{supp}\ \gamma \subset \mathrm{supp}\ \chi_{m,n}, \ \ |\gamma| \leq C_{m,n}\mathrm{e}^{\theta}.$$

Let us prove Proposition 3.1 using Lemma 3.3.

Proof of Proposition 3.1. Let $\lambda \in \mathbb{R}$, $\phi \in \mathcal{H}_{\mathrm{loc}}$ be as in the assertion, and set

$$\alpha_0 = \sup\left\{\alpha \geq \frac{\sqrt{a}}{1+b}\ \middle|\ \exp(\alpha|x|^{b+1})\phi \in \mathcal{H}\right\}.$$

By the assumption, we have $\alpha_0 > \frac{\sqrt{a}}{1+b}$. Assume $\alpha_0 < \infty$, and we choose β, $\tilde{\alpha}$ and $n_0 \geq 1$ as in Lemma 3.3. For any function ϕ obeying the assumptions of Proposition 3.1 we have $\chi_{m,n}\phi \in H^2_{\mathrm{comp}}(\mathbb{R}^d) \subset \mathcal{D}(H)$ for all $n > m \geq 1$. Note that we may assume $n_0 \geq 3$, so that for all $n > m \geq n_0$

$$\chi_{m-2,n+2}\phi \in \mathcal{D}(H).$$

We let $\alpha \in (\tilde{\alpha}, \alpha_0)$. With these parameters fixed evaluate the form inequality from Lemma 3.3 on the state $\chi_{m-2,n+2}\phi \in \mathcal{D}(H)$. Then for any $n > m \geq n_0$ and $R > 0$

$$\|(r^{2b-1}\Theta)^{1/2}\phi\|^2 \leq C_m\|\chi_{m-1,m+1}\phi\|^2 + C_R\|\chi_{n-1,n+1}r^{b-1/2}\exp(\alpha r^{b+1})\phi\|^2.$$

Here we let $\alpha_1 \in (\alpha, \alpha_0)$. Then we have

$$\begin{aligned}&\|\chi_{n-1,n+1}r^{b-1/2}\exp(\alpha r^{b+1})\phi\|^2\\ &\leq \sup|r^{2b-1}\exp(2(\alpha-\alpha_1)r^{b+1})|\ \|\chi_{n-1,n+1}\exp(\alpha_1 r^{b+1})\phi\|^2.\end{aligned}$$

From this the above second term vanishes as $n \to \infty$, and consequently by Lebesgue's monotone convergence theorem

$$\|(r^{2b-1}\bar{\chi}_m\mathrm{e}^{\theta})^{1/2}\phi\|^2 \leq C_m\|\chi_{m-1,m+1}\phi\|^2.$$

Next, we let $R \to \infty$. Again by Lebesgue's monotone convergence theorem it follows that

$$r^{b-1/2}\bar{\chi}_m^{1/2}\exp(\beta r^{b+1})\phi \in \mathcal{H}.$$

Thus $\exp(\beta r^{b+1})\phi \in \mathcal{H}$, but this is a contradiction, since $\beta > \alpha_0$. Hence we have $\alpha_0 = \infty$. This implies the assertion. $\square$

To prove Proposition 3.2 we choose a weight

$$\Theta = \Theta^{\alpha}_{m,n} = \chi_{m,n}\mathrm{e}^{\theta};\ n > m \geq 1.$$

Here the exponent θ is given by

$$\theta = \theta^{\alpha} = 2\alpha r^{b+1};\ \alpha > \frac{\sqrt{a}}{1+b}.$$

Using this weight, we show the following Lemma.

Lemma 3.4. Let $\lambda \in \mathbb{R}$ and $\alpha_0 > \frac{\sqrt{a}}{1+b}$. Then there exist $c, C > 0$ and $n_0 \geq 1$ such that for any $\alpha > \alpha_0$ and $n > m \geq n_0$, as quadratic forms on $\mathcal{D}(H)$,

$$\begin{aligned}\mathrm{Im}(A\Theta(H-\lambda)) \geq c\alpha^2 r^{2b-1}\Theta - C(1+\alpha^2)\left(\chi^2_{m-1,m+1} + \chi^2_{n-1,n+1}\right) r^{2b-1}\mathrm{e}^{\theta} \\ + \mathrm{Re}(\gamma(H-\lambda)),\end{aligned}$$

where $\gamma = \gamma_{m,n,\alpha}$ is a certain function satisfying

$$\mathrm{supp}\ \gamma \subset \mathrm{supp}\ \chi_{m,n},\ \ |\gamma| \leq C_{m,n,\alpha}\mathrm{e}^{\theta}.$$

Using the above inequality in the same way as in the proof of Proposition 3.1, we obtain the conclusion.

Acknowledgement

The author would like to thank his supervisor Kenichi Ito for discussions and encouragement. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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