COLORED LATTICE MODELS FOR DEMAZURE CHARACTERS IN TYPES ABC

TRAVIS SCRIMSHAW

Abstract. Characters of irreducible representations of $\mathfrak{gl}_n$, the Schur functions are well-known to be described as partition functions of solvable lattice models. One method to prove this uses the Gelfand–Tsetlin interpretation of semistandard tableaux, which are naturally in bijection with states of the lattice model. This approach has been extended to type C in 2012 by Ivanov using Proctor patterns. A recent trend in solvable lattice models has been to use "colored" lattice models that are based on the R-matrices of finite dimensional quantum affine general linear group $U_q(\widehat{\mathfrak{gl}}_n)$ representations instead of $U_q(\widehat{\mathfrak{gl}}_2)$. In this note, we obtain Demazure characters, which are representations of the corresponding Borel subalgebra and can be thought of as "partial" versions of the irreducible simple Lie algebra representations, and their Demazure atoms analogs by using a colored lattice model built from that of Bump–Brubaker–Buciumas–Gustafsson for type A Demazure atoms. A key component of these lattice models is a natural relationship with the wiring diagram of the defining element of the corresponding Weyl group. We then will discuss the colored version of Ivanov's lattice model to obtain Demazure characters in type B and C and the related tableaux.

1. Introduction

Solvable (or integrable) lattice models are those formed on a square grid such that there exists an R-matrix that satisfies the Yang–Baxter equation, also known as the star–triangle equation from its description in electrical networks [Ken99]. This was used by Baxter in his solution of the eight-vertex model [Bax71, Bax72] using what is now known as the standard train argument. For additional history, we refer the reader to [McC10, Ch. 13]. Some notable examples of applications of solvable lattice models include Kuperberg's proof [Kup96] counting the number of alternating sign matrices [Zei96]; Hall–Littlewood polynomials for the BC_n root system [WZJ16]; and characters of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ and the corresponding Tokuyama formula [Gra17, Iva12]. Some other examples are [CdGW15, MS13, WZJ16, WZJ19], but this list is far from exhaustive.

A recent trend has been to replace the underlying $U_q(\widehat{\mathfrak{gl}}_2)$ underlying the R-matrix with other (Drinfel'd–Jimbo) quantum groups for other Lie (super)algebras. In particular, we can produce colored lattice models by using $U_q(\widehat{\mathfrak{gl}}_n)$, where a pioneering example was given by Borodin and Wheeler in [BW18]. For some additional recent applications, some recent examples (but also far from exhaustive) include [AGS19, BBBG21, Bor17, BP18, BSW20, BS22, BW20, MS20]. One feature of the colored lattice model is that it allows us to see further connections with a symmetric group by considering its wiring diagram, which underlies the construction in [BBBG21] and was explicitly noted in [BSW20].

The goal in this note is to combine the colored lattice model techniques with the character computations in [Gra17, Iva12] to produce new colored lattice models on a U-turn grid whose partition function is a Demazure character and atom for $\mathrm{Sp}_{2n}(\mathbb{C})$.

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Our lattice model encodes the wiring diagrams of the correspond Weyl group of signed permutations with the U-turn corresponding to the simple transposition s_n action that negates the last letter (in the window notation). Our general idea of proof follows the usual colored lattice model proofs (*e.g.*, in [BBBG21]): We apply an R-matrix and the train argument to give functional equations on the partition function that exactly matches the application of Demazure (atom) operators that define the Demazure characters and atoms. However, our lattice model has one wrinkle: it only has three R-matrices instead of the four needed for it to be solvable (as we have two different types of rows connected to each U-turn "vertex"). Despite this, there is enough to be able to produce the desired functional relations for π_i and $\overline{\pi}_i$, the Demazure (atom) operators corresponding to s_i, with $i < n$, and so we call our lattice model *quasi-solvable*. For π_n and $\overline{\pi}_n$, we follow [Iva12] to construct the fish equation, but we have to directly analyze each case.

Through our analysis, we are also able to construct a model for Demazure characters and atoms for $\mathrm{SO}_{2n+1}(\mathbb{C})$ by a simple tweak of the K-matrix, the collection of Boltzmann weights for the U-turns. There are natural tableaux called King tableaux [Kin75, Kin76] and Sundaram tableaux [Sun90] to compute characters of $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$, respectively. From [Gra17, Iva12], it is know the (uncolored) lattice model for $\mathrm{Sp}_{2n}(\mathbb{C})$ is naturally in bijection with the Proctor pattern [Pro94] version of the reverse King tableaux, which comes from our ordering of the parameters in parallel to [BBBG21, BSW20]. This also holds in the colored setting and gives an algorithm to compute the (left) key of the King tableaux and can be described in terms of Kashiwara crystals [Kas93] (which we implicitly have through the Sheats bijection [She99] with the corresponding Kashiwara–Nakashima tableaux [KN94]). We also extend all of these results for $\mathrm{SO}_{2n+1}(\mathbb{C})$ using the same "marking" idea, such as in [BSW20].

This note is organized as follows. In Section 2, we give the necessary background on Demazure characters and atoms. In Section 3, we construct our lattice model for Demazure atoms and prove its quasi-solvability to give functional equations. In Section 4, we give a slightly modified quasi-solvable lattice model for Demazure characters. In Section 5, we relate the admissible states in our lattice models to Proctor patterns and the correspond (reverse) King and Sundaram tableaux.

This work is primarily an abridged version joint work [BS22] with Valentin Buciumas. Independent work by Zhong [Zho22] on stochastic type C vertex models was also posted shortly after [BS22] appeared on the arXiv, which uses a colored model with a different quantization than the R-matrix for $U_q\big(\widehat{\mathfrak{gl}}(2n|1)\big)$ we utilize and is possibly a gauge transformation of the atom model when taking $q = 0$.

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2. Background

We will start with a review of the theory of Demazure operators. Let Φ be the root system and Λ be the weight lattice of a complex reductive Lie group G with maximal torus T. Let n be the rank of Φ. We identify Λ with the group $X^*(T)$ of rational characters of T. For $\mathbf{z} \in T$ and $\lambda \in \Lambda$, we denote by $\mathbf{z}^\lambda$ the application of λ to $\mathbf{z}$. Let $\mathcal{O}(T)$ be the set of polynomial functions on T, that is, finite linear combinations of the functions $\mathbf{z}^\lambda$ for $\lambda \in \Lambda$. Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots, and let α_i ($i \in I = \{1, 2, \ldots, n\}$) be the simple positive roots. Let $\alpha_i^\vee \in X_*(T)$ denote the corresponding simple coroots and s_i the corresponding simple reflections generating the Weyl group W. The Weyl group acts on the weight lattice and therefore on the space $\mathcal{O}(T)$. We shall denote this action by $w \cdot f(\mathbf{z}) := f(w\mathbf{z})$. For $w \in W$, let $\ell(w)$ denote the length of w, the smallest number of simple reflections such that $w = s_{i_1} \cdots s_{i_\ell}$, which is called a reduced word for w. Let w_0 be the long element of the Weyl group and $\leq$ denote the (strong) Bruhat order on W. For more information about properties of the Weyl group, we the refer the reader to [Hum90].

2.1. **Demazure characters and atoms.** Given s_i a simple reflection, we can define the associated isobaric Demazure operator acting on $f \in \mathcal{O}(T)$ as

$$\pi_i f(\mathbf{z}) = \frac{f(\mathbf{z}) - \mathbf{z}^{-\alpha_i} f(s_i \mathbf{z})}{1 - \mathbf{z}^{-\alpha_i}}. \tag{2.1}$$

The numerator is divisible by the denominator, so the result is again in $\mathcal{O}(T)$.

One can check that $\pi_i^2 = \pi_i = s_i \pi_i$. Given any $\mu \in \Lambda$, set $k = \langle \mu, \alpha_i^\vee \rangle$ so $s_i(\mu) = \mu - k\alpha_i$. Then the action on the monomial $\mathbf{z}^\mu$ is given explicitly by

$$\pi_i \mathbf{z}^\mu = \begin{cases} \mathbf{z}^\mu + \mathbf{z}^{\mu - \alpha_i} + \cdots + \mathbf{z}^{s_i(\mu)} & \text{if } k \geqslant 0, \\ 0 & \text{if } k = -1, \\ -(\mathbf{z}^{\mu + \alpha_i} + \mathbf{z}^{\mu + 2\alpha_i} + \cdots + \mathbf{z}^{s_i(\mu + \alpha_i)}) & \text{if } k < -1. \end{cases} \tag{2.2}$$

Define $\overline{\pi}_i := \pi_i - 1$. Explicitly, we have

$$\overline{\pi}_i f(\mathbf{z}) := \frac{f(\mathbf{z}) - f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}. \tag{2.3}$$

Both π_i and $\overline{\pi}_i$ satisfy the braid relations. Thus, for any $w \in W$, we can choose any reduced word $w = s_{i_1} \cdots s_{i_k}$ to define $\pi_w = \pi_{i_1} \cdots \pi_{i_k}$ and $\overline{\pi}_w = \overline{\pi}_{i_1} \cdots \overline{\pi}_{i_k}$ by Matsumoto's theorem [Mat64]. For $w = 1$, we set $\pi_1 = \overline{\pi}_1 = 1$.

For λ a dominant weight, let χ_λ denote the character of the irreducible representation V_λ with highest weight λ. The *Demazure character formula* is the identity, for $\mathbf{z} \in T$:

$$\chi_\lambda(\mathbf{z}) = \pi_{w_0} \mathbf{z}^\lambda.$$

For a proof, see [Bum13, Thm. 25.3]. More generally for any Weyl group element w, we may consider $\pi_w \mathbf{z}^\lambda$ and $\overline{\pi}_w \mathbf{z}^\lambda$. These polynomials are called *Demazure characters* and *Demazure atoms*, respectively. The following relation between the two is well-known.

Theorem 2.1 ([LS90]; see also [Pun16, Lemma 2.5])**.** *Let $f \in \mathcal{O}(T)$. Then*

$$\pi_w f(\mathbf{z}) = \sum_{y \leqslant w} \overline{\pi}_y f(\mathbf{z}), \qquad \operatorname{ch} V_\lambda = \pi_{w_0} \mathbf{z}^\lambda = \sum_y \overline{\pi}_y \mathbf{z}^\lambda.$$

2.2. **Signed permutations and the Weyl group action.** For the remainder of the paper, we will only consider Cartan types BC. We identify the maximal torus T with the space $(\mathbb{C}^*)^n$, where n is the rank of the group. The Weyl group W of type B_n is isomorphic to the Weyl of type C_n, and it is known as the hyperoctohedral group. It is generated by the simple reflections s_i for $i \in \{1, \dots, n\}$ subject to the relations:

$$\begin{aligned} s_n s_{n-1} s_n s_{n-1} &= s_{n-1} s_n s_{n-1} s_n, && \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && \text{if } i < n-1 \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2 \\ s_i^2 &= 1. && \end{aligned} \tag{2.4}$$

The Weyl group acts on elements $\mathbf{z} \in (\mathbb{C}^*)^n$ as follows:

$$s_i(\dots, z_i, z_{i+1}, \dots) = (\dots, z_{i+1}, z_i, \dots), \qquad \text{if } i < n, \tag{2.5a}$$

$$s_n(\dots, z_{n-1}, z_n) = (\dots, z_{n-1}, z_n^{-1}), \qquad \text{if } i = n. \tag{2.5b}$$

The elements of W can be explicitly described using *signed permutations* of n, which are permutations of

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}$$

such that $w(i) = \overline{w(\overline{\imath})}$. Here we use the convention that $\overline{\overline{\imath}} = i$. Thus we can determine a signed permutation by the image of $1 \leq i \leq n$. The simple transposition for $i < n$ is given by $s_i = (i \; i+1)$, and s_n sends $n \leftrightarrow \overline{n}$. An *inversion* is a pair $1 \leq i < j \leq n$ such that $w(i) > w(j)$. The longest element w_0 is the signed permutation $[\overline{1}, \overline{2}, \dots, \overline{n}]$.

The subgroup of W generated by s_i for $i < n$ is a subgroup isomorphic to the Weyl group of type A. We shall denote the subgroup by W^A.

2.3. **Functional equations.** We now discuss the explicit functional equations for the Demazure characters and atoms that will be used in this paper.

We first consider Demazure characters. Equation (2.1) can be written explicitly as

$$\pi_i f(\mathbf{z}) := \frac{f(\mathbf{z}) - z_i^{-1} z_{i+1} f(s_i \mathbf{z})}{1 - z_i^{-1} z_{i+1}}, \qquad \text{if } i < n, \tag{2.6a}$$

$$\pi_n f(\mathbf{z}) := \frac{f(\mathbf{z}) - z_n^{-L_X} f(s_n \mathbf{z})}{1 - z_n^{-L_X}}, \qquad \text{in type X}, \tag{2.6b}$$

where $X \in \{B, C\}$ and $L_B := 1$ and $L_C := 2$. Let us denote $D_w(\mathbf{z}, \lambda) := \pi_w z^\lambda$. Let s_i be a simple reflection and $w \in W$ such that $\ell(s_i w) > \ell(w)$. From Equations (2.6), we deduce

$$(z_i - z_{i+1}) D_{s_i w}(\mathbf{z}, \lambda) = z_i D_w(\mathbf{z}, \lambda) - z_{i+1} D_w(s_i \mathbf{z}, \lambda), \qquad \text{if } i < n, \tag{2.7a}$$

$$(z_n^{L_X} - 1) D_{s_n w}(\mathbf{z}, \lambda) = z_n^{L_X} D_w(\mathbf{z}, \lambda) - D_w(s_n \mathbf{z}, \lambda), \qquad \text{in type } X. \tag{2.7b}$$

Next, for the Demazure atoms, Equation (2.3) can be rewritten as

$$\overline{\pi}_i f(\mathbf{z}) := \frac{f(\mathbf{z}) - f(s_i \mathbf{z})}{z_i z_{i+1}^{-1} - 1}, \qquad \text{if } i < n, \tag{2.8a}$$

$$\overline{\pi}_n f(\mathbf{z}) := \frac{f(\mathbf{z}) - f(s_n \mathbf{z})}{z_n^{L_X} - 1}, \qquad \text{in type } X. \tag{2.8b}$$

Let us denote $A_w(\mathbf{z}, \lambda) := \overline{\pi}_w z^\lambda$. Let s_i be a simple reflection and $w \in W$ such that $\ell(s_i w) > \ell(w)$. We rewrite the equation above as

$$(z_i - z_{i+1}) A_{s_i w}(\mathbf{z}, \lambda) = z_{i+1} \big(A_w(\mathbf{z}, \lambda) - A_w(s_i \mathbf{z}, \lambda) \big), \qquad \text{if } i < n, \tag{2.9a}$$

$$(z_n^{L_X} - 1) A_{s_n w}(\mathbf{z}, \lambda) = A_w(\mathbf{z}, \lambda) - A_w(s_n \mathbf{z}, \lambda), \qquad \text{in type } X. \tag{2.9b}$$

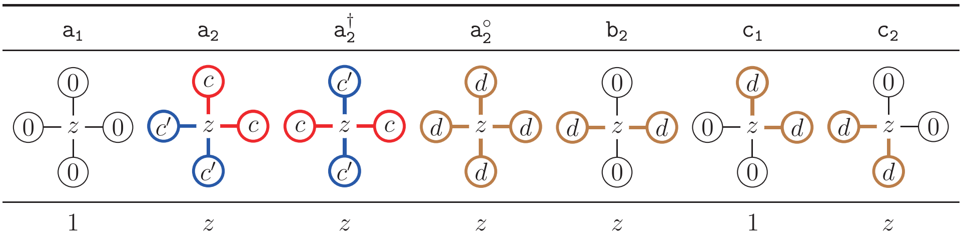


FIGURE 1. The colored Boltzmann Γ-weights with $c > c'$ and d being any color.

3. Colored lattice models and Demazure atoms

We will construct colored lattice models that represent Demazure atoms in Cartan type B and C. These models generalize the work in [BBBG21], where type A Demazure atoms have been represented as partition functions of lattice models. The current paper and [BBBG21] produce colored models that are a refinement of the $q = 0$ uncolored models in [BBF11] (Schur polynomials) and [Iva12] ($\mathrm{Sp}_{2n}(\mathbb{C})$ characters), respectively. Our odd orthogonal model does not refine any pre-existing model.

Remark 3.1. Our model is in fact a refinement of a semidual version of the model in [Iva12] obtained by interchanging $0 \leftrightarrow 1$ on each of the horizontal components. This choice allows us to have a more natural description of our colored lattice model and helps with visualizing admissible states in the model by using colored paths.

We work with fixed set $\mathbf{c} = \{c_1 < c_2 < \cdots < c_n < \overline{c}_n < \cdots < \overline{c}_1\}$ of ordered colors. We use the conventions $\overline{\overline{c}}_i := c_i$ and $c_{\overline{i}} := \overline{c}_i$. For $w \in W$ we define $w\mathbf{c} = (c_{w(1)}, c_{w(2)}, \ldots, c_{w(n)}, c_{w(\overline{n})}, \ldots, c_{w(\overline{1})})$ to be the set of colors permuted by w. Explicitly, s_i permutes the colors $c_i \leftrightarrow c_{i+1}$ and $\overline{c}_i \leftrightarrow \overline{c}_{i+1}$ and s_n permutes the colors $c_n \leftrightarrow \overline{c}_n$. The set $\{w\mathbf{c} \mid w \in W\}$ will index the left boundary conditions of our model.

3.1. **Rectangular lattice model.** Let us now briefly explain the model for type A given by Bump–Brubaker–Buciumas–Gustafsson [BBBG21]. For this subsection only, we will assume $W = S_n = \langle s_1, \ldots, s_{n-1}\rangle$, the symmetric group on n letters. We consider a grid of n horizontal lines numbered from top to bottom with a parameter $z_i \in \mathbb{C}^*$, which we call Γ. One can think of $\mathbf{z} := (z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})$ as living in the torus of $\mathrm{GL}_n(\mathbb{C})$. We also consider m vertical lines, and the intersection of a vertical and horizontal line is called a *vertex*. An *interior edge* connects two vertices in the model, while an *outer edge* (or a boundary edge) is attached to one vertex alone. The *Boltzmann weight* of a vertex is a function that assigns a complex number to each assignment of spins to the edges of a vertex that depends on the assigned parameter. The collection of vertices and their Boltzmann weights is called an *L-matrix*. The Boltzmann weights for Γ are given in Figure 1, and each weight that is not portrayed in the L-matrix is considered to be 0.

An assignment of spins to the inner edges is called a *state* of the system. The *weight* of a state is the product over all vertices of the weights of each vertex. A state is called *admissible* if its weight is non-zero. The *partition function* is the sum of the weights of the states over all states of the system with boundary conditions determined by w, λ, and the parameters $\mathbf{z}$. Our system has fixed spins on the boundary that depend on $w \in W$ and dominant $\lambda \in \Lambda$. The bottom and right edges are labeled by 0. The left edges are

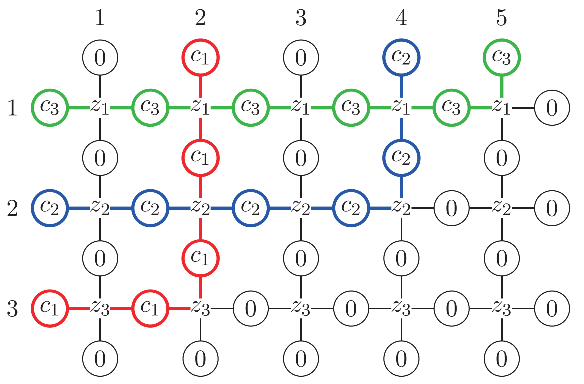


FIGURE 2. The unique "ground" state for the colored system for $m = 5$, $n = 3$, and $\lambda = (2, 2, 1)$. We use colors $c_1 > c_2 > c_3$. The Boltzmann weight of this state is $z_1^2 z_2^2 z_3$.

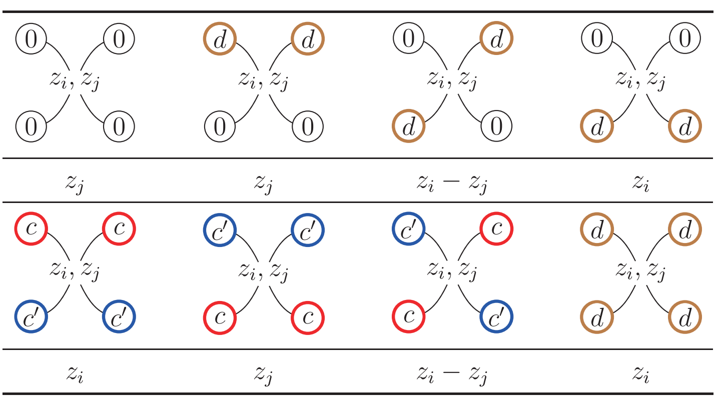


FIGURE 3. The colored R_{Γ}^{Γ}-matrix with $c > c'$ and d being any color.

labeled by $ww_0\mathbf{c}$ from top to bottom, and the top edges are labeled by $c_n, \dots, c_1$ (from right to left) in places $\lambda + \rho$, where $\rho = (n-1, n-2, \dots, 0)$.

There is an R-matrix for this model given in Figure 3 that satisfies the *Yang–Baxter equation*; see Proposition 3.3 below for the precise details. In particular, this means we can apply the *standard train argument* depicted in Figure 4 to produce functional equations for the partition function. These functional equations are precisely those for the Demazure atoms in type A, which means the partition function is equal to a Demazure atom as they satisfy the same initial term given by the ground state in Figure 2. This was the proof given in [BBBG21].

3.2. **U-turn lattice model.** Now we describe the lattice model for types BC. Consider a rectangular grid with $2n$ horizontal lines which we number from top to bottom and m vertical lines numbered from right to left as in Figure 7. We call the odd numbered lines Γ and the even numbered ones Δ. The L-matrix for Δ is given by Figure 6. On the right, we connect the Γ line $2i-1$ to the Δ line $2i$ by a *U-turn*.

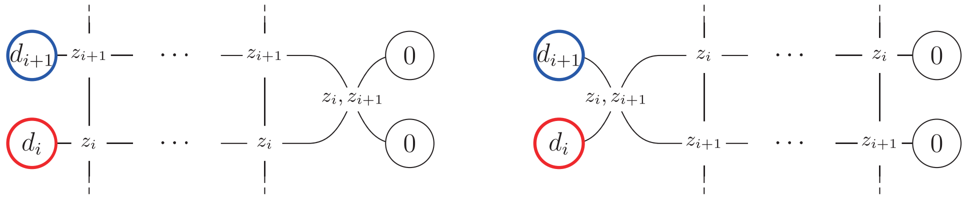


FIGURE 4. Left: The model $\overline{\mathfrak{S}}_{\lambda,w}$ with an R-matrix attached on the right. Right: The model after using the Yang–Baxter equation in the same model.

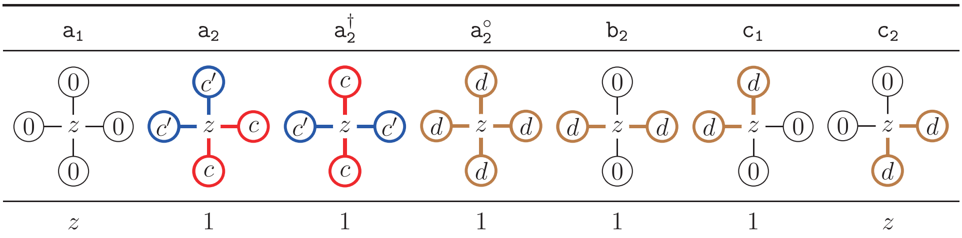


FIGURE 5. The colored Boltzmann Δ-weights with $c > c'$ and d being any color.

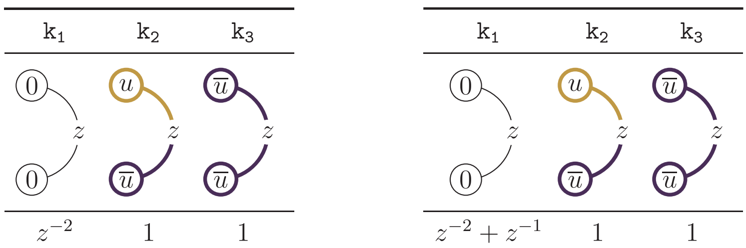


FIGURE 6. On the left (resp. right) we have the colored $\substack{\Gamma\\\Delta}$ (resp. $\substack{\Delta\\\Gamma}$) K-matrix weights for type C (resp. B) with $u > \overline{u}$.

Each Γ line $2i - 1$ is assigned the parameter $z_i \in \mathbb{C}^*$ and Δ line $2i$ is assigned the parameter $z_i^{-1} \in \mathbb{C}^*$. The Boltzmann weights for the Δ vertices are given by Figure 5. We also assign the U-turn from line $2i-1$ to $2i$ the parameter z_i, which we also assign Boltzmann weights to by Figure 6. The collection of such Boltzmann weights for a U-turn is called a *K-matrix*. One can think of $\mathbf{z} := (z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})$ as living in the torus of $\mathrm{Sp}_{2n}(\mathbb{C})$ or $\mathrm{SO}_{2n+1}(\mathbb{C})$. These models only differ in the U-turn weights.

Our system has fixed spins on the boundary that depend on $w \in W$ and dominant $\lambda \in \Lambda$. The bottom edges are labeled by 0, the left Γ edges are labeled by 0. The left Δ edges are labeled by the first n colors of $ww_0\mathbf{c}$ (the others are determined from these) from top to bottom, and the top edges are labeled by $c_n, \ldots, c_1$ (from right to left) in places $\lambda + \rho$, where $\rho = (n-1, n-2, \ldots, 0)$. The rest of the boundary edges are assigned spin 0. See Figure 7 for an example, where $w = 1$, $\lambda = (3,1)$ and $\lambda + \rho = (4,1)$. We denote this model by $\overline{\mathfrak{S}}^X_{\lambda,w}$, for $X \in \{B, C\}$. The partition function is denoted by $Z(\overline{\mathfrak{S}}^X_{\lambda,w}; \mathbf{z})$.

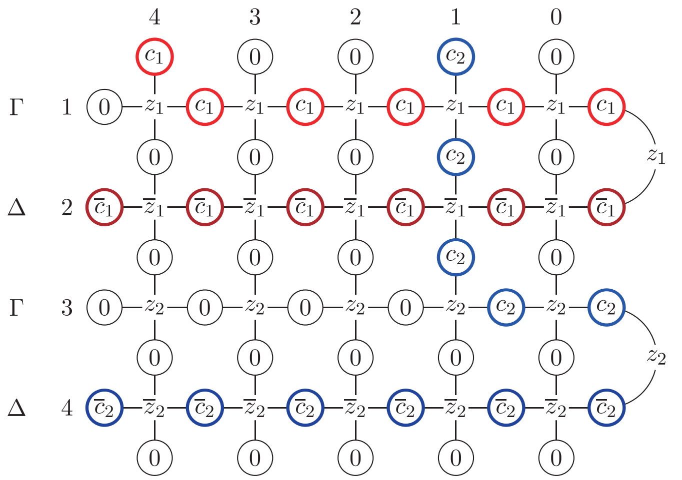


FIGURE 7. The unique admissible state in $\overline{\mathfrak{S}}_{\lambda,w}$ for $\lambda = (3,1)$ and $w = 1$. We use the convention $\overline{z} = z^{-1}$. The top boundary condition will consist of colors on columns $(4,1) = \lambda + \rho$.

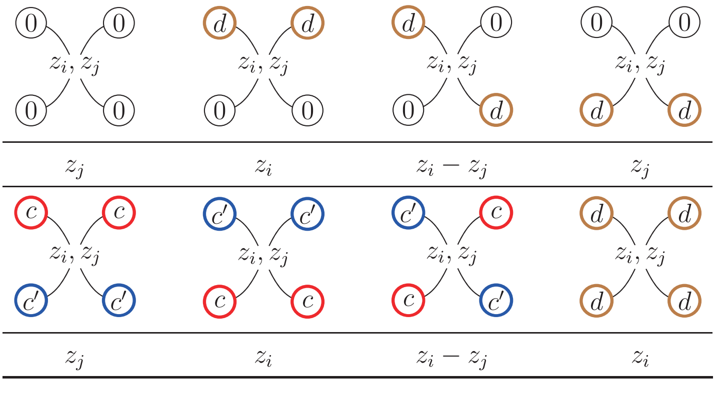


FIGURE 8. The colored R^{Δ}_{Δ}-matrix with $c > c'$ and d being any color.

We often simply write $\overline{\mathfrak{S}}_{\lambda,w} = \overline{\mathfrak{S}}^{X}_{\lambda,w}$ since the states of the two models are the same and the Boltzmann weights only differ in $\mathtt{k}_1$ (see Figure 6). See Figure 7 for an example.

Remark 3.2. The $z+1$ ratio between the $\mathtt{k}_1$ K-matrix entry in types B and C is exactly the ratio between the type C_n and B_n characters in rank $n = 1$.

A lattice model is called *solvable* or *integrable* if it there exists a full set of solutions of the Yang–Baxter equation and its generalizations that enable one to derive functional equations for the partition function that can be used to characterize it. For example, the model in [Iva12] is integrable because of the existence of four R-matrices R^{Γ}_{Γ}, R^{Δ}_{Γ}, R^{Γ}_{Δ}, and R^{Δ}_{Δ} that satisfy the appropriate Yang–Baxter and reflection equations.

Our model is not integrable in this sense, but it is close. We produce three R-matrices R^{Γ}_{Γ}, R^{Δ}_{Δ}, and R^{Δ}_{Γ} that are given in Figures 3, 8, and 9, respectively. These R-matrices

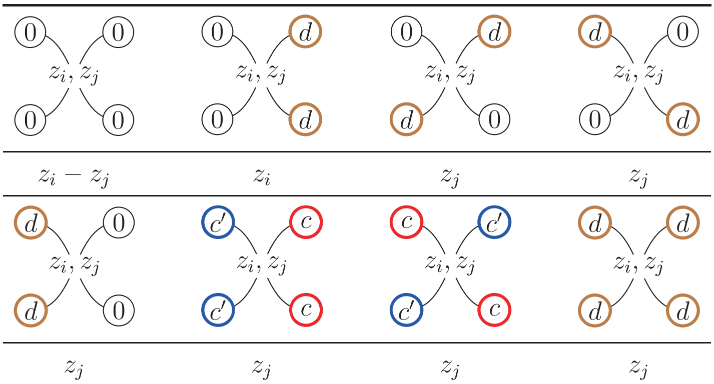


FIGURE 9. The colored R_{Γ}^{Δ}-matrix with $c > c'$ and d being any color.

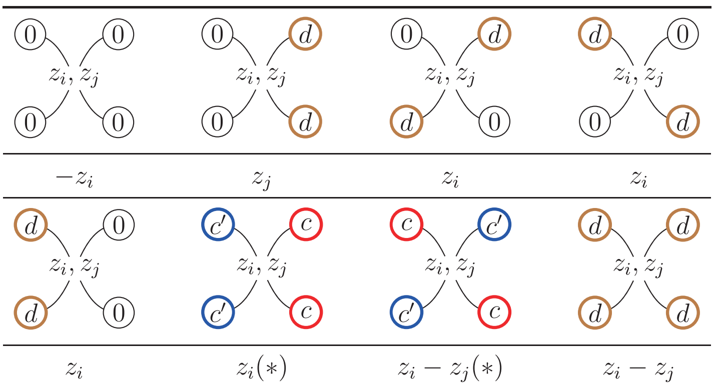


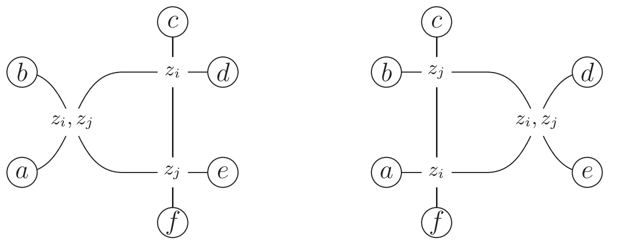
FIGURE 10. The colored R_{Δ}^{Γ}-matrix with $c > c'$ and d being any color.

satisfy the Yang–Baxter equation with the corresponding L-matrices as explained in Proposition 3.3. However, it can be shown, computationally, that there is no solution for the Yang–Baxter equation corresponding to ${}^{\Gamma}_{\Delta}$. The problem, compared to the uncolored setting discussed in [Iva12] where such a solution exists, is that certain colored loops can be formed inside one side of Equation (3.1). This then ends up multiplying that side's partition function by the total number of colors, which is $2n$, whereas the other side does not depend on n. Hence, the two partition functions cannot be equal.

We do however produce a fourth R-matrix called R_{Δ}^{Γ} in Figure 10 that is partly determined. This means that the weights marked with $(*)$ in Figure 10 are free, so they can be changed and this does not affect our results. Yet, we do stress that no matter how you change them, the corresponding Yang–Baxter equation will still not be satisfied, including changing the allowed colorings (such that the colors are preserved). Given these four R-matrices satisfying a total of three Yang–Baxter equations, we prove in Section 3.3 a functional equation for the partition function for each of the simple reflections s_i, for $i < n$. The method of proof is by a modified version of the train argument applied to U-turn lattice model; the modification is technical and needed as the fourth Yang–Baxter equation does not have a solution. In Section 3.4, we then prove certain modified fish equations which are used to show a functional equation for the last remaining simple reflection. Our model therefore lacks a solution for the Yang–Baxter equations, but can still be studied via modified versions of the original tools used to study solvable lattice models. We shall call such a model *quasi-solvable*.

Proposition 3.3. *The R^{Γ}_{Γ}-matrix, R^{Δ}_{Δ}-matrix, or R^{Δ}_{Γ}-matrix satisfy the corresponding Yang–Baxter equation: The partition function of the following two models are equal for any boundary conditions $a, b, c, d, e, f \in \mathbf{c} \sqcup \{0\}$:*

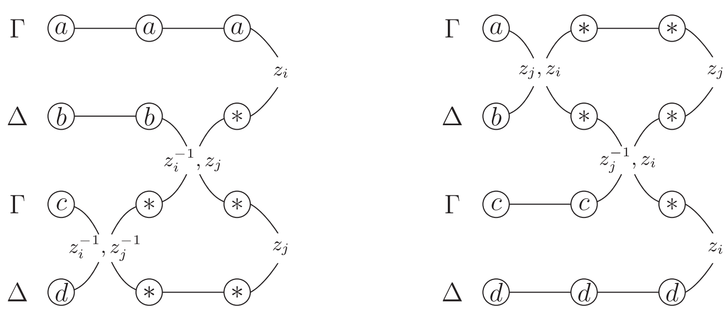
(3.1)



where the z_i, z_j weights are R^{Φ}_{Θ}-weights, the z_i-weights are Φ-weights and the z_j-weights are Θ-weights for $\Phi\Theta \in \{\Gamma\Gamma, \Delta\Gamma, \Delta\Gamma\}$.

This model generally satisfies the *reflection equation*.

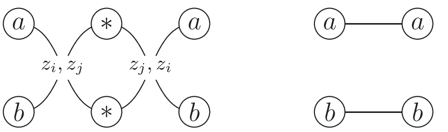
Proposition 3.4. *For any fixed boundary condition $a, b, c, d \in \mathbf{c} \sqcup \{0\}$, the partition function of the model on left*



equals to the partition function on the right times $\alpha = z_i^{-2}$.

We will use the so-called *unitary equation* to describe what happens when we uncross two strands. More precisely, we show that the partition function of the model on the left

(3.2)



is simply a fixed scalar value β independent of the boundary condition $a, b \in \{0, c_1, \ldots, c_k\}$ times the partition function on the right, which we set to be 1 by definition.

Proposition 3.5. *The partition function of the model on the left in (3.2) with both of the R-matrices being either R^{Γ}_{Γ} or R^{Δ}_{Δ} is equal to $\beta = z_i z_j$.*

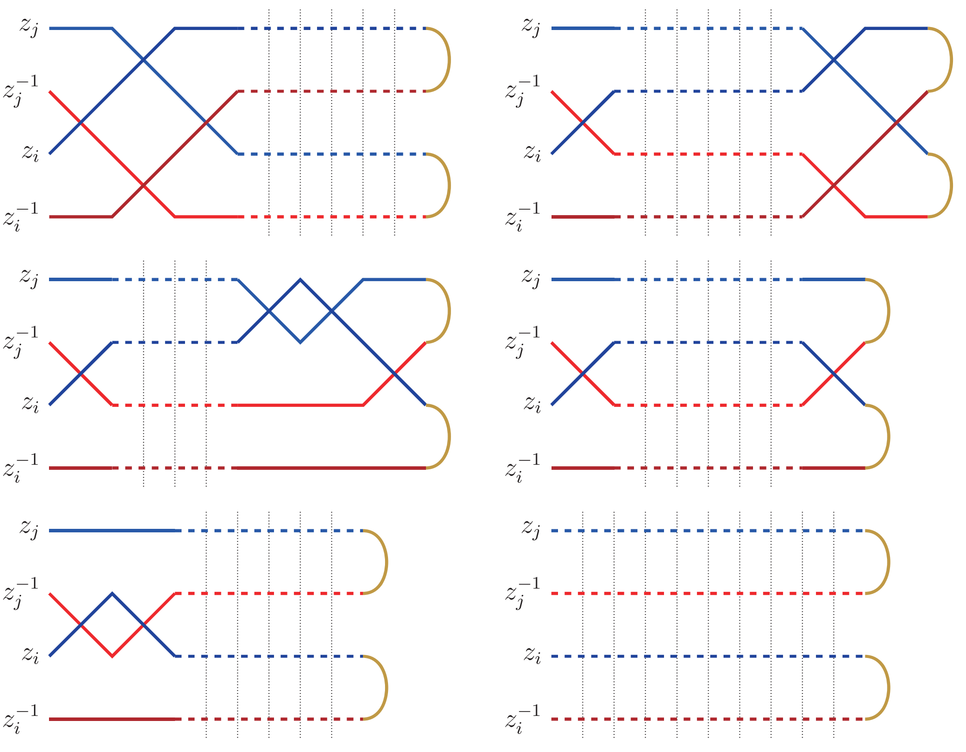


FIGURE 11. Pictorial description of the sequence of steps to compute the action of the i-th atom operator with the Γ (resp. Δ) rows colored in blue (resp. red). The R-matrices are determined by the color on the row; *e.g.*, the left most R-matrix in the first step is the R^{Γ}_{Δ} R-matrix.

Proposition 3.6. *For a state in $\overline{\mathfrak{S}}_{\lambda,w}$, the vertices $\mathtt{a}_2^{\dagger}$ and $\mathtt{k}_2$ correspond to inversions in w_0w, the number of which equals $\ell(w_0w)$.*

For examples of Proposition 3.6, see Example 3.10 below.

3.3. **Billiards.** Our first goal is to prove the following "type A" functional equation for the partition function.

Lemma 3.7. *Choose $i < n$, $j = i+1$ and $w \in W$ such that $\ell(s_iw) = \ell(w)+1$. Then*

$$(z_i - z_j)Z(\overline{\mathfrak{S}}_{\lambda,s_iw};\mathbf{z}) = z_j\big(Z(\overline{\mathfrak{S}}_{\lambda,w};\mathbf{z}) - z_iz_j^{-1}Z(\overline{\mathfrak{S}}_{\lambda,w};s_i\mathbf{z})\big). \tag{3.3}$$

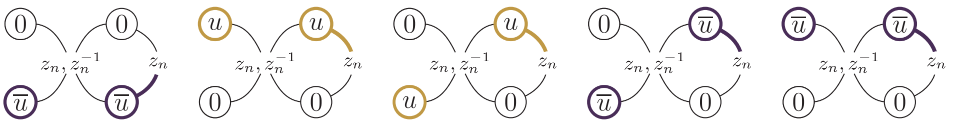
The functional equation in Lemma 3.7 is shown by following the sequence of steps pictured in Figure 11.

3.4. **Ichthyology.** We now study a version of the fish equation that we use to show a functional equation corresponding to the last simple reflection s_n. We do not actually prove the usual fish equation, but instead dissect it to its component pieces to obtain the desired functional equation.

Lemma 3.8. *Choose $w \in W$ such that $\ell(s_n w) = \ell(w) + 1$. Then we have*

$$\begin{aligned}
(z_n^2 - 1) Z(\overline{\mathfrak{S}}_{\lambda, s_n w}; \mathbf{z}) &= Z(\overline{\mathfrak{S}}_{\lambda, w}; \mathbf{z}) - Z(\overline{\mathfrak{S}}_{\lambda, w}; s_n \mathbf{z}) && (\textit{type } C), \\
(z_n - 1) Z(\overline{\mathfrak{S}}_{\lambda, s_n w}; \mathbf{z}) &= Z(\overline{\mathfrak{S}}_{\lambda, w}; \mathbf{z}) - Z(\overline{\mathfrak{S}}_{\lambda, w}; s_n \mathbf{z}) && (\textit{type } B).
\end{aligned}$$

Proof sketch. We first change the last Δ row into a Γ row and apply the standard train argument. This leads us to require analyzing all of the following possible *fish*, or local configurations of an R_Γ^Γ-matrix and K_Γ^Γ-matrix:



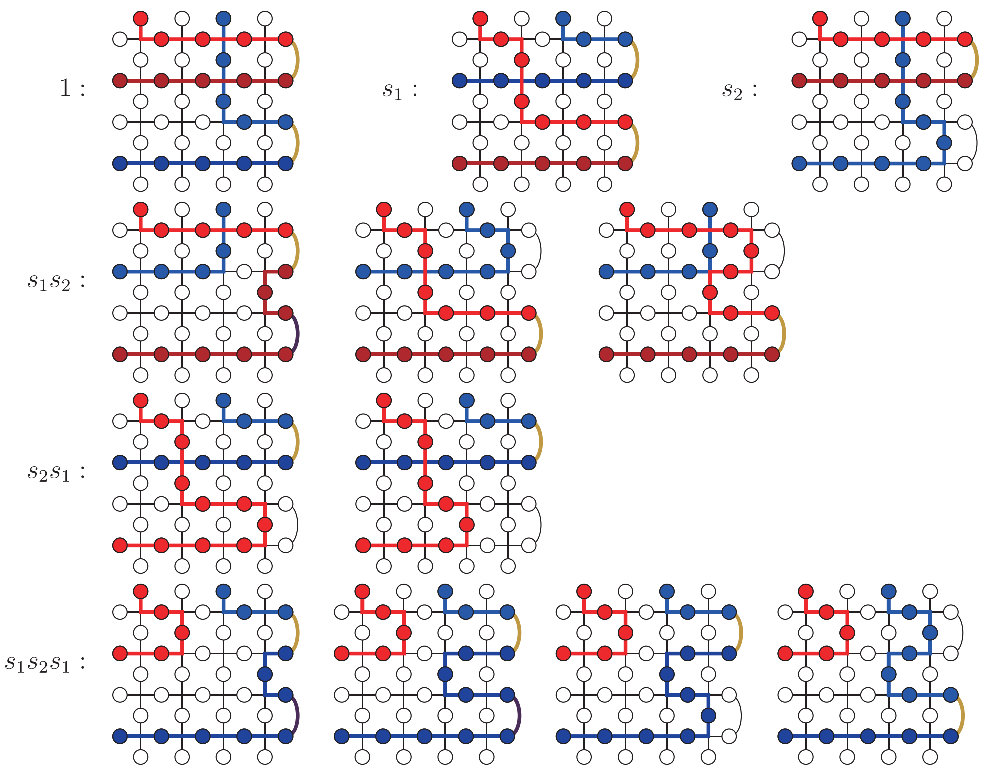
Then by some algebraic manipulations, we obtain the functional equations. □

3.5. **The first main theorem.** Using the functional equations we have shown above, we obtain our first main result.

Theorem 3.9. *For Cartan type $X \in \{B, C\}$, we have*

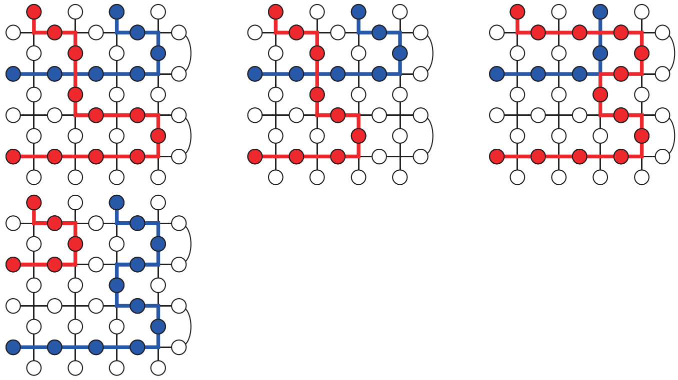
$$Z(\overline{\mathfrak{S}}^X_{\lambda, w}; \mathbf{z}) = \mathbf{z}^\rho A_w(\mathbf{z}, \lambda).$$

Example 3.10. Let $n = 2$ and $\lambda = (2, 1)$. Then we have the following states in $\overline{\mathfrak{S}}_w$:



$s_2s_1s_2$:

w_0 :



Next, we compute the partition functions for each model in type C for $\overline{Z}_w := Z(\overline{\mathfrak{S}}^C_{\lambda,w};\mathbf{z})$:

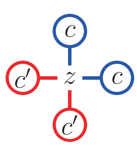
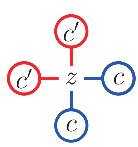
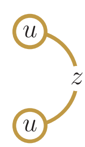
$$\overline{Z}_1 = z_1^3 z_2, \qquad \overline{Z}_{s_1} = z_1^2 z_2^2, \qquad \overline{Z}_{s_2} = z_1^3 z_2^{-1},$$
$$\overline{Z}_{s_1s_2} = z_1^2 + z_2^2 + z_1z_2, \qquad \overline{Z}_{s_2s_1} = z_1^2 + z_1^2 z_2^{-2},$$
$$\overline{Z}_{s_1s_2s_1} = 1 + z_1z_2 + z_1z_2^{-1} + z_1^{-1}z_2, \qquad \overline{Z}_{s_2s_1s_2} = 1 + z_2^{-2} + z_1z_2^{-1},$$
$$\overline{Z}_{w_0} = z_1^{-1}z_2^{-1}.$$

We can see that these differ by $\mathbf{z}^\rho = z_1$ from the atoms $A_w := A_w(\mathbf{z},\lambda)$:

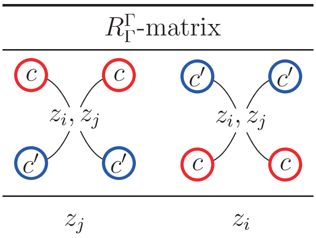
$$A_1 = z_1^2 z_2, \qquad A_{s_1} = z_1 z_2^2, \qquad A_{s_2} = z_1^2 z_2^{-1},$$
$$A_{s_1s_2} = z_1 + z_1^{-1}z_2^2 + z_2, \qquad A_{s_2s_1} = z_1 + z_1z_2^{-2},$$
$$A_{s_1s_2s_1} = z_1^{-1} + z_2 + z_2^{-1} + z_1^{-2}z_2, \qquad A_{s_2s_1s_2} = z_1^{-1} + z_1^{-1}z_2^{-2} + z_2^{-1},$$
$$A_{w_0} = z_1^{-2}z_2^{-1}.$$

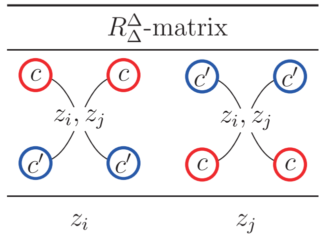
4. Colored lattice models and Demazure characters

In this section we construct a colored lattice model for a Demazure character by modifying our previous lattice model. Consider our previous model, but replace the $\mathtt{a}_2$ vertices in both the Γ L-matrix (Figure 1) and the Δ L-matrix (Figure 5) and replacing $\mathtt{k}_3$ in K-matrix with

$\mathtt{a}_2'$ for Γ	$\mathtt{a}_2'$ for Δ	$\mathtt{k}_3$
		
z	1	1

Let $\mathfrak{S}^X_{\lambda,w}$ denote the new model using these new L-matrices and K-matrix. Analogous to $\overline{\mathfrak{S}}^X_{\lambda,w}$, we will often write this simply as $\mathfrak{S}_{\lambda,w}$. This causes the lower left two values in the R^Γ_Γ-matrix and R^Δ_Δ-matrix to swap values. We use the same R^Γ_Δ-matrix and R^Δ_Γ-matrix as before. Therefore, the R-matrices that satisfy the Yang–Baxter equation in the new model will be





with all weights that are not listed above remaining the same as in Figures 3, 8, 9 and 10. A direct check shows that these modified R-matrices and K-matrix still satisfy the corresponding reflection and unitary equations.

By the same argument as in Lemma 3.7, we can prove the following functional equation for the Demazure character model.

Lemma 4.1. *Choose $i < n$, $j = i + 1$ and $w \in W$ such that $\ell(s_i w) = \ell(w) + 1$. Then*

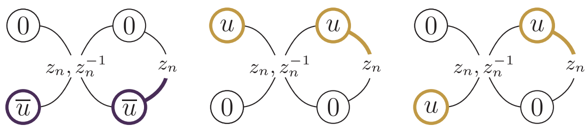
$$(z_i - z_j) Z(\mathfrak{S}_{\lambda, s_i w}; \mathbf{z}) = z_i \big(Z(\mathfrak{S}_{\lambda, w}; \mathbf{z}) - Z(\mathfrak{S}_{\lambda, w}; s_i \mathbf{z}) \big). \tag{4.1}$$

Now we look at the corresponding version of the fish equation.

Lemma 4.2. *Choose $w \in W$ such that $\ell(s_n w) = \ell(w) + 1$. Then we have*

$$(z_n^2 - 1) Z(\mathfrak{S}_{\lambda, s_n w}; \mathbf{z}) = z_n^2 Z(\mathfrak{S}_{\lambda, w}; \mathbf{z}) - Z(\mathfrak{S}_{\lambda, w}; s_n \mathbf{z}) \qquad (\textit{type } C),$$
$$(z_n - 1) Z(\mathfrak{S}_{\lambda, s_n w}; \mathbf{z}) = z_n Z(\mathfrak{S}_{\lambda, w}; \mathbf{z}) - Z(\mathfrak{S}_{\lambda, w}; s_n \mathbf{z}) \qquad (\textit{type } B).$$

Proof sketch. We need to consider only three possible fish:

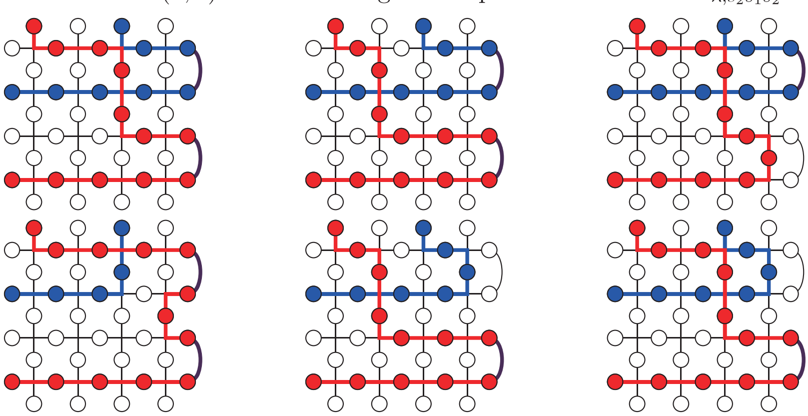


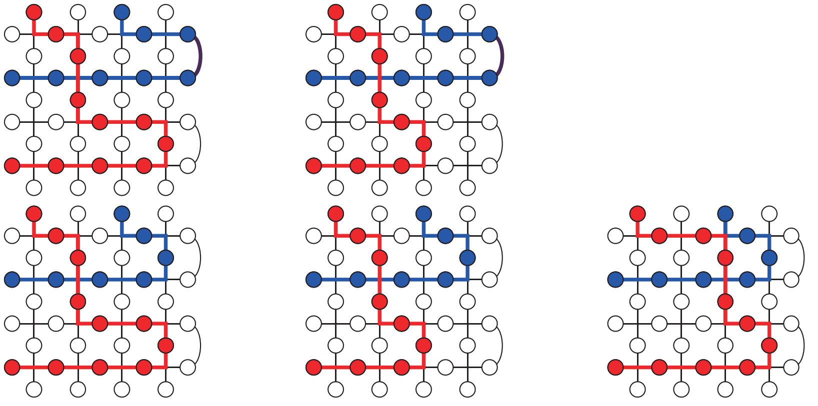
where u is an unbarred color. Then by some algebraic manipulations, we obtain the desired functional equations for types C and B. □

Theorem 4.3. *For Cartan type $X \in \{B, C\}$, we have*

$$Z(\mathfrak{S}^X_{\lambda, w}; \mathbf{z}) = \mathbf{z}^{\rho} D_w(\mathbf{z}, \lambda).$$

Example 4.4. Let $\lambda = (2, 1)$. The following are all possible states for $\mathfrak{S}_{\lambda, s_2 s_1 s_2}$:





Compare against the atoms for

$$\overline{\mathfrak{S}}_{\lambda,1} \sqcup \overline{\mathfrak{S}}_{\lambda,s_1} \sqcup \overline{\mathfrak{S}}_{\lambda,s_2} \sqcup \overline{\mathfrak{S}}_{\lambda,s_1s_2} \sqcup \overline{\mathfrak{S}}_{\lambda,s_2s_1} \sqcup \overline{\mathfrak{S}}_{\lambda,s_2s_1s_2}$$

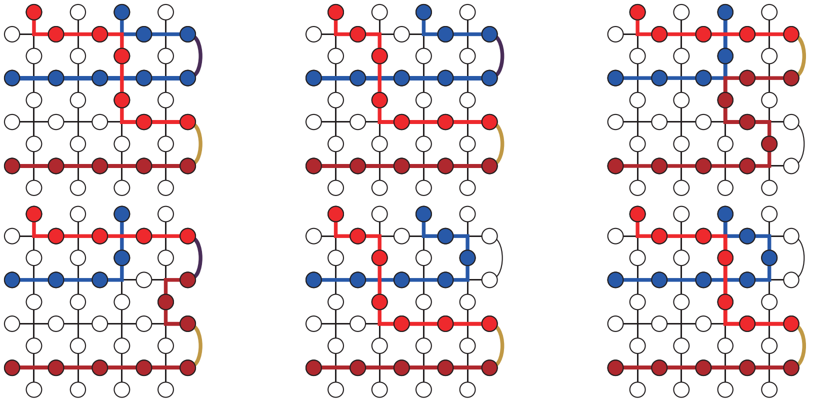
given in Example 3.10. Note that each inversion that was present in the atom is now either an inversion for w_0w, an L-matrix $\mathtt{a}_2'$, or a K-matrix $\mathtt{k}_3'$.

Theorem 4.5. *We have*

$$Z(\mathfrak{S}_{\lambda,w};\mathbf{z}) = \sum_{y\leqslant w} Z(\overline{\mathfrak{S}}_{\lambda,y};\mathbf{z}).$$

An immediate corollary of Theorem 4.5 is $D_w(\mathbf{z},\lambda) = \sum_{y\leqslant w} A_y(\mathbf{z},\lambda)$ (see Theorem 2.1).

Example 4.6. Let $\lambda = (2,1)$. We compute all admissible states in $\mathfrak{S}_{\lambda,s_1s_2}$:



5. Proctor patterns and tableaux

In this section, we will use our model to construct a key algorithm on *reverse* King tableaux [Kin75, Kin76] for $G = \mathrm{Sp}_{2n}$ or Sundaram tableaux [Sun90] for $G = \mathrm{SO}_{2n+1}$. We begin by recalling the weight preserving bijection between states of the uncolored type C model and symplectic Proctor patterns [Pro94, Thm. 4.2] given in [Iva12, Ch. 1]. We then give the analogous bijection between the states of the uncolored type B model and odd orthogonal Proctor patterns [Pro94, Thm. 7.1]. Similar to [BSW20], the order of our variables is different by $i \leftrightarrow \overline{n+1-i}$, which is the reason we naturally work with reverse King tableaux. As a consequence, these bijections with our model provides a new proof of [Pro94, Thm. 4.2, Thm. 7.1].

5.1. **Symplectic patterns and King tableaux.** We consider the case for $G = \mathrm{Sp}_{2n}$, which is the Lie group of Cartan type C_n. We note that these patterns were first given by Želobenko [Žel62]. A *symplectic Proctor pattern* is a pattern of non-negative integers of the form

$$\begin{array}{ccccccccccc}
a_{1,1} & & a_{1,2} & & a_{1,3} & & \cdots & & a_{1,n} & & \\
& b_{1,1} & & b_{1,2} & & b_{1,3} & & \cdots & & b_{1,n} & \\
& & a_{2,2} & & a_{2,3} & & \cdots & & a_{2,n} & & \\
& & & b_{2,2} & & b_{2,3} & & \cdots & & b_{2,n} & \\
& & & & \ddots & & \ddots & & \vdots & \vdots & \\
& & & & & & a_{n-1,n-1} & & a_{n-1,n} & & \\
& & & & & & & b_{n-1,n-1} & & b_{n-1,n} & \\
& & & & & & & & a_{n,n} & & \\
& & & & & & & & & b_{n,n} &
\end{array}$$

that satisfies the interlacing conditions

$$\min\{a_{i,j}, a_{i+1,j}\} \geq b_{i,j} \geq \max\{a_{i,j+1}, a_{i+1,j+1}\},$$
$$\min\{b_{i-1,j-1}, b_{i,j-1}\} \geq a_{i,j} \geq \max\{b_{i-1,j}, b_{i,j}\}.$$

The weight of a symplectic Proctor pattern P is given by

$$\mathrm{wt}(P) := \prod_{i=1}^{n} z_i^{A_i - 2B_i + A_{i+1}}, \qquad \text{where } A_i = \sum_{j=i}^{n} a_{i,j},\ A_{n+1} = 0, \text{ and } B_i = \sum_{j=i}^{n} b_{i,j}.$$

Let $\mathcal{P}_\lambda^C$ denote the set of symplectic Proctor patterns with top row λ.

A *King tableau* [Kin75, Kin76] is a semistandard tableaux in $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$ such that the smallest entry in row i is i, and its weight is

$$\mathrm{wt}(T) = \prod_{i=1}^{n} x_i^{m_i - m_{\overline{\imath}}},$$

where m_k is the number of times k appears in T. Let $\mathcal{K}_\lambda$ denote the set of King tableaux of shape λ. A *reverse* King tableau is a King tableau with respect to the alphabet in the reverse order or alternatively the entries in rows (resp. columns) are weakly (resp. strictly) decreasing and the largest entry in row i is $\overline{n+1-i}$.

As discussed in [Pro94], there is a natural bijection $\Theta^C \colon \mathcal{P}_\lambda^C \to \mathcal{K}_\lambda$ by extending the usual bijection between *Gelfand–Tsetlin (GT) patterns* and semistandard tableaux. Indeed, the partition $\lambda^{(k)}$ of the k-th row indicates the subtableau consisting of all of the letters greater than the k-th letter in the alphabet. For instance, if $k = 3$, then we restrict to the letters $\overline{n-1} < n < \overline{n}$.

Proposition 5.1 ([Iva12, Ch. 1])**.** *Let $\mathfrak{S}_\lambda^C$ denote the uncolored model for $G = \mathrm{Sp}_{2n}$. There exists a weight-preserving bijection*

$$\Psi^C \colon \mathfrak{S}_\lambda^C \to \mathcal{P}_\lambda^C.$$

Ivanov constructed the bijection Ψ^C in Proposition 5.1 explicitly by extending the usual bijection between the five-vertex model and GT patterns (see, *e.g.*, [BSW20, Sec. 2.1]) and using the 1 edges between the $\genfrac{}{}{0pt}{}{\Delta}{\Gamma}$ (resp. $\genfrac{}{}{0pt}{}{\Gamma}{\Delta}$) rows to define the $\{a_{ij}\}_{i,j}$ (resp. $\{b_{ij}\}_{i,j}$) values by the GT pattern bijection. More precisely, the i-th row of vertical edges in the model is the 01-sequence of the partition in the i-th row of the symplectic Proctor pattern read from right-to-left. We can make the analogous injection on the colored model

$\overline{\mathfrak{S}}_{\lambda,w}$ or $\mathfrak{S}_{\lambda,w}$ by considering the positions of the colored vertical edges or equivalently by forgetting about the colors in our model as an intermediate step. Note that this does not see the difference with [Iva12] (see Remark 3.1).

Example 5.2. Consider the states given in Example 3.10 for $\overline{\mathfrak{S}}_{\lambda,w}$. Then under the bijection Ψ^C, the states correspond to the following symplectic Proctor patterns:

$$1: \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 0 \end{array} \qquad s_1: \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 0 \end{array} \qquad s_2: \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 1 \end{array}$$

$$s_1 s_2: \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 0 \\ & & 0 & \\ & & & 0 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 0 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 0 \end{array}$$

$$s_2 s_1: \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 1 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 2 \end{array}$$

$$s_1 s_2 s_1: \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 0 & \\ & & & 0 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 0 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 1 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 0 \end{array}$$

$$s_2 s_1 s_2: \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 1 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 2 \end{array} \quad \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{array}$$

$$w_0: \begin{array}{cccc} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 1 \end{array}$$

5.2. **Odd orthogonal patterns and Sundaram tableaux.** Here we instead assume $G = \mathrm{SO}_{2n+1}$, which is the Lie group of Cartan type B_n. An *odd orthogonal Proctor pattern* is a symplectic Proctor pattern such that the values $b_{i,n}$, for all $1 \leq i \leq n$, at the right ends are also allowed to be half integers. We remark that these patterns were first announced by Gelfand and Tsetlin without proof in [GT50]. Let $\mathcal{P}_\lambda^B$ denote the set of odd orthogonal Proctor patterns with top row λ.

A Sundaram tableau [Sun90] is a King tableau with an additional letter $\infty > \overline{n}$ that is allowed to repeat down columns but can only appear once in a row. The weight of a Sundaram tableau is the same as for a King tableau; in particular, we ignore ∞ in the weight computation. We denote the set of Sundaram tableau of shape λ by $\mathcal{S}_\lambda$. A reverse Sundaram tableau is defined analogously to a reverse King tableau. Likewise, we have a natural bijection $\Theta^B\colon \mathcal{P}_\lambda^B \to \mathcal{S}_\lambda$, as noted in [Pro94], by the same description as Θ^C except if the rightmost entry in the odd orthogonal Proctor pattern is a half integer, we replace the leftmost entry in the corresponding row with an ∞.

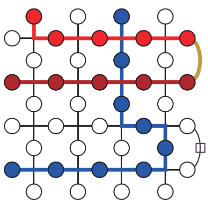
Recall that the model for both type B and C Demazure atoms (as well as for Demazure characters) have the same states, but the $\mathtt{k}_1$ entry of the K-matrix has a binomial weight $z^{-2} + z^{-1}$ in type B as opposed to the monomial weight z^{-2} for type C. Thus, following [BSW20], we can introduce a marking to the states for this K-matrix entry, where if the bend is marked, then it has a Boltzmann weight of z^{-1} and otherwise the Boltzmann weight is z^{-2}. This yields a bijection between the marked states and the monomials of the partition function as opposed to a product of binomials.

Proposition 5.3. *Let $\widehat{\mathfrak{S}}_\lambda$ denote the set of marked states for the uncolored type B model. There exists a weight-preserving bijection*

$$\Psi^B \colon \widehat{\mathfrak{S}}_\lambda \to \mathcal{P}^B_\lambda.$$

Similar to the case when $X = C$, we can extend Ψ^B to an injection with the domain the colored model $\overline{\mathfrak{S}}^B_{\lambda,w}$ or $\mathfrak{S}^B_{\lambda,w}$.

Example 5.4. Let us take $w = s_2$. Then we have the one state in the model $\overline{\mathfrak{S}}^B_{\lambda,w}$ with one $\mathtt{k}_1$ U-turn that we can mark, which corresponds to the following pair of odd orthogonal Proctor patterns and reverse Sundaram tableaux:



$$\longleftrightarrow \quad \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 1 \end{array}, \qquad \begin{array}{cccc} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & \frac{1}{2} \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|}\hline \overline{2} & 1 \\ \hline 1 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|}\hline \infty & 1 \\ \hline 1 \\ \cline{1-1} \end{array},$$

where the box in the unique state of $\overline{\mathfrak{S}}^B_{s_2,\lambda}$ denotes the possible marking.

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(T. Scrimshaw) Department of Mathematics, Hokkaido University, 5 Chōme Kita 8 Jōnishi, Kita Ward, Sapporo, Hokkaidō 060–0808, Japan

Email address: tcscrims@gmail.com

URL: https://tscrim.github.io