A SURVEY ON CATEGORIFIED CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

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1. INTRODUCTION

Let *R* be a *quiver Hecke algebra* associated with a simple Lie algebra 9 and "*R*-gmod" the category of finite-dimensional graded *R*-modules. We set $\mathcal{K}(R\text{-}gmod)$ to be the Grothendieck ring of *R*-gmod. It is well-known that the (unipotent) quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is categorified by $\mathcal{K}(R\text{-}gmod)$. The basic theory of localization for the monoidal category \widetilde{R} -gmod of *R*-gmod is initiated by [5] and its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-}gmod)$ defines the localized (unipotent) quantum coordinate ring $\mathcal{H}_q(\mathfrak{n})$. In [11], Lauda-Vazirani defined certain crystal structure on the family of simple modules of *R*-gmod and they have shown that this crystal is isomorphic to the crystal $B(\infty)$ of the nilpotent half of $U_q(\mathfrak{g})$. In this survey, considering the family of self-dual simple module $\mathbb{B}(\widetilde{R}\text{-}gmod)$ of the localized category $\widetilde{R}\text{-}gmod$, we define a crystal structure of $\mathcal{H}_q(\mathfrak{n})$ and show that it is isomorphic to the cellular crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_N}$, which is defined for a reduced word $\mathbf{i} = i_1 \cdots i_N$ of the longest Weyl group element w_0 . This result can be seen as a localized version of the result by Lauda-Vazirani. The article is a survey of [13]. But, the subsection 2.1 and Example 3.16 are not described in [13], which are new parts added here.

2. Preliminaries

2.1. **Setting.** Let $g = n \oplus t \oplus n_{-} = \langle e_i, h_i, f_i \rangle_{i \in I := \{1, 2, \dots, n\}}$ be a simple Lie algebra associated with a Cartan matrix $A = (a_{ij})_{i,j \in I}$ where $\{e_i, f_i, h_i\}_{i \in I}$ are the standard Chevalley generators and $n = \langle e_i \rangle_{i \in I}$ (resp. $t = \langle h_i \rangle_{i \in I}$, $n_{-} = \langle f_i \rangle_{i \in I}$) is the positive nilpotent subalgebra (resp. the Cartan subalgebra, the negative nilpotent subalgebra).

Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots of g and \langle , \rangle a pairing on $t \times t^*$ satisfying $a_{ij} = (\langle h_i, \alpha_j \rangle)_{i,j \in I}$. We also define a symmetric bilinear form (,) on t^* such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in t^*$.

Let $P := \{\lambda \in t^* | \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for any } i \in I\}$ be the weight lattice and $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I\}$ the set of dominant weights. Set $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ (resp. $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$), which is called the root lattice (resp. positive root lattice). For an element $\beta = \sum_i m_i \alpha_i \in Q_+$ define $|\beta| = \sum_i m_i$, which is called the height of β . Let $W = \langle s_i | s_i \rangle_{i \in I}$ be the Weyl group associated with P, where s_i is the simple reflection defined by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \ (\lambda \in P)$.

We denote the dual weight lattice of *P* by $P^* := \{h \in t \mid \langle h, P \rangle \subset \mathbb{Z}\}$. Let $U_q(\mathfrak{g}) := \langle e_i, f_i, q^h \rangle_{i \in I, h \in P^*}$ be the quantum algebra associated with \mathfrak{g} with the defining relations (see e.g., [1, 2]) and $U_q^-(\mathfrak{g}) := \langle f_i \rangle_{i \in I}$ (resp. $U_q^+(\mathfrak{g}) := \langle e_i \rangle_{i \in I}$) the negative (resp. positive) nilpotent subalgebras of $U_q(\mathfrak{g})$. We also define the \mathbb{Z} -form $U_{\mathbb{Z}[q,q^{-1}]}^-(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ as in [5]. Set $q_i := q^{(\alpha_i,\alpha_i)/2}$, $[n]_i = (q_i^n - q_i^{-1})/(q_i - q_i^{-1})$, $[n]_i! := \prod_{0 \le k \le n} [k]_i$ and $X_i^{(n)} := X_i^n / [n]_i!$ for $X_i = f_i, e_i$ for $i \in I, n \in \mathbb{Z}_{\ge 0}$.

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2.2. Quantum shuffle algebra and quantum coordinate ring. See [12] for this subsection. Let $\mathcal{F} := \langle F_i \mid i \in I \rangle$ be a free associative $\mathbb{Q}(q)$ -algeba. For a multi-index $\nu = (\nu_1, \dots, \nu_m) \in I^m$ let us define a monomial $F_{\nu} := F_{\nu_1} \cdots F_{\nu_m} \in \mathcal{F}$ and its weight wt(F) := $\alpha_{\nu_1} + \cdots + \alpha_{\nu_m}$. For monomials $x, x', y, y' \in \mathcal{F}$, define

$$(x \otimes y)(x' \otimes y') := q^{(\operatorname{wt}(y), \operatorname{wt}(x'))} xx' \otimes yy',$$

it induces an associative multiplication on $\mathcal{F} \otimes_{\mathbb{Q}(q)} \mathcal{F}$ and then $\mathcal{F} \otimes_{\mathbb{Q}(q)} \mathcal{F}$ becomes an associative $\mathbb{Q}(q)$ -algebra. We can also define an comultiplication $\Delta : \mathcal{F} \to \mathcal{F} \otimes_{\mathbb{Q}(q)} \mathcal{F}$ by setting $\Delta(F_i) = 1 \otimes F_i + F_i \otimes 1$. The quantum shuffle algebra \mathcal{F}^* is defined as a dual of \mathcal{F} . For $\beta \in Q_+$, set $\mathcal{F}_{\beta} := \bigoplus_{v \in I^{ht(\beta)}, wt(v) = \beta} \mathbb{Q}(q) F_v$. Define

$$\mathcal{F}^* := \bigoplus_{\beta \in Q_+} \mathcal{F}^*_{\beta}, \qquad \qquad \mathcal{F}^*_{\beta} := \operatorname{Hom}_{\mathbb{Q}(q)}(\mathcal{F}_{\beta}, \mathbb{Q}(q))$$

The comultiplication Δ induces a multiplication on \mathcal{F}^* by

$$\langle y \cdot y', x \rangle = \langle y \otimes y', \Delta(x) \rangle, \qquad (y, y' \in \mathcal{F}^*, \ x \in \mathcal{F}),$$

where \langle , \rangle is a natural paring on $\mathcal{F}^* \times \mathcal{F}$. Now, by this multiplication \mathcal{F}^* becomes an associative $\mathbb{Q}(q)$ -algebra, which is called the quantum shuffle algebra. The following lemma is known as the shuffle lemma:

Lemma 2.1 (shuffle lemma). For $v = (v_1, \dots, v_{m+l}) \in I^{m+l}, v' = (v_1, \dots, v_m) \in I^m$ and $v'' = (v_{m+1}, \dots, v_{m+l}) \in I^l$, we obtain

(2.1)
$$F_{\nu'}^* \cdot F_{\nu''}^* = \sum_{w \in S_{m,l}} \left(\prod_{a < b, w(a) > w(b)} q^{-(\alpha_{\nu_{w(a)}}, \alpha_{\nu_{w(b)}})} \right) F_{w(\nu)}^*$$

where $S_{m,l}$ is a subset of the symmetric group S_{m+l} defined by

 $S_{m,l} := \{ w \in S_{m+l} \mid w(1) < w(2) < \dots < w(m), \quad w(m+1) < w(m+2) < \dots < w(m+l) \},\$

and note that the action of $w \in S_{m+l}$ on a multi-index $v = (v_1, \dots, v_{m+l}) \in I^{m+l}$ is defined by

$$(wv)_k := v_{w^{-1}(k)} \qquad (1 \le k \le m+l).$$

Now, let us define the (unipotent) quantum coordinate ring $\mathcal{R}_{a}(\mathfrak{n})$ a restricted dual of $U_{a}^{+}(\mathfrak{g})$ as

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in \mathcal{Q}_-} \mathcal{A}_q(\mathfrak{n})_{\beta} \qquad \mathcal{A}_q(\mathfrak{n})_{\beta} := \operatorname{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$

As is well-known that there exists a natural projection $\pi : \mathcal{F} \twoheadrightarrow U_q^+(\mathfrak{g})$ and then considering the dual of this map, we obtain the embedding of algebra $\mathcal{A}_q(\mathfrak{n}) \hookrightarrow \mathcal{F}^*$. Note that $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$ as a $\mathbb{Q}(q)$ -algebra. The \mathbb{Z} -form $\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ is defined as in [5].

Example 2.2. A_2 -case. Set $I = \{1, 2\}$, $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -1$. By the formula (2.1) we get easily $F_1^* \cdot F_1^* = (1 + q^{-2})F_{11}^*$, $F_1^* \cdot F_2^* = F_{12}^* + qF_{21}^*$, $F_2^* \cdot F_1^* = F_{21}^* + qF_{12}^*$. Here note that

$$S_{1,2} = \left\{ \begin{pmatrix} 123\\123 \end{pmatrix}, \begin{pmatrix} 123\\312 \end{pmatrix}, \begin{pmatrix} 123\\213 \end{pmatrix} \right\} \text{ and their inverses } S_{1,2}^{-1} = \left\{ \begin{pmatrix} 123\\123 \end{pmatrix}, \begin{pmatrix} 123\\231 \end{pmatrix}, \begin{pmatrix} 123\\213 \end{pmatrix} \right\}.$$

Then, we get

$$\begin{split} F_1^* \cdot F_1^* \cdot F_2^* &= F_1^* \cdot (F_{12}^* + qF_{21}^*) = (1 + q^{-2})F_{112}^* + (q^{-1} + q)F_{121}^* + (1 + q^2)F_{211}^*, \\ F_1^* \cdot F_2^* \cdot F_1^* &= F_1^* \cdot (F_{21}^* + qF_{12}^*) = (q + q^{-1})F_{112}^* + 2F_{121}^* + (q + q^{-1})F_{211}^*, \\ F_2^* \cdot F_1^* \cdot F_1^* &= F_2^* \cdot ((1 + q^{-2})F_{11}^*) = (1 + q^{-2})(q^2F_{112}^* + qF_{121}^* + F_{211}^*). \end{split}$$

Finally, we obtain the "q-Serre relation":

$$F_1^* \cdot F_1^* \cdot F_2^* - (q + q^{-1})F_1^* \cdot F_2^* \cdot F_1^* + F_2^* \cdot F_1^* \cdot F_1^* = 0.$$

3. CRYSTAL BASES AND CRYSTALS

3.1. Crystal Base of $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$. Let us define the crystal base $(L(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})([1])$. For $i \in I$ the operator $e'_i \in \text{End}(U_q^-(\mathfrak{g}))$ is defined by the formula

$$e'_{i}(PQ) = e'_{i}(P)Q + q^{\langle h_{i},\beta \rangle}_{i}Pe'_{i}(Q), \quad e'_{i}(f_{j}) = \delta_{i,j}, \quad e'_{i}(1) = 0,$$

for any $P \in U_q(\mathfrak{g})_\beta$, $Q \in U_q(\mathfrak{g})$, $i, j \in I$. By the fact that for $P \in U_q(\mathfrak{g})_\beta$, there exists the following unique decomposition

(3.1)
$$P = \sum_{k \ge 0} f_i^{(k)} P_n,$$

where $P_n \in \text{Ker}(e'_i) \cap U^-_q(\mathfrak{g})_{\beta+k\alpha_i}$. And define the operators $\tilde{e}_i, \tilde{f}_i \in \text{End}(U^-_q(\mathfrak{g}))$ on $P \in U^-_q(\mathfrak{g})_{\beta}$ by using the decomposition (3.1)

$$\tilde{e}_i P = \sum_{k>0} f_i^{(k-1)} P_n, \qquad \tilde{f}_i P = \sum_{k\geq 0} f_i^{(k+1)} P_n,$$

which are called Kashiwara operators. Now, set

$$L(\infty) := \sum_{k \ge 0, i_1, \cdots, i_k \in I} \mathbb{A}\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_{\infty}, \qquad B(\infty) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_{\infty} \mod qL(\infty) \mid k \ge 0, i_1, \cdots, i_k \in I\} \setminus \{0\},$$

$$\varepsilon_i(b) = \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \operatorname{wt}(b) \rangle,$$

where $u_{\infty} = 1 \in U_q(\mathfrak{g})$ and $\mathbb{A} \subset \mathbb{Q}(q)$ is the local subring at q = 0.

Theorem 3.1 ([1]). A pair $(L(\infty), B(\infty))$ is a crystal base of $U_q^-(\mathfrak{g})$. Indeed, we obtain

$$\begin{split} \tilde{e}_i L(\infty) \subset L(\infty), \quad & f_i L(\infty) \subset L(\infty), \\ \tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}, \quad & \tilde{f}_i B(\infty) \subset B(\infty) \sqcup \{0\}, \\ \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i \quad \text{for } b, \tilde{e}_i b \in B(\infty), \quad \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i \quad \text{for } b, \tilde{f}_i b \in B(\infty), \\ \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \quad \varphi_i(\tilde{e}_i b) = \varepsilon_i(b) + 1, \quad \text{for } b, \tilde{e}_i b \in B(\infty), \\ \varepsilon_i(\tilde{f}_i b) = \varphi(b) + 1 \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \quad \text{for } b, \tilde{f}_i b \in B(\infty), \\ \tilde{f}_i b = b' \longleftrightarrow \tilde{e}_i b' = b, \quad \text{for } b, \ b' \in B(\infty) \end{split}$$

3.2. **Crystals.** We shall introduce the notion *crystal* following [2], which is a combinatorial object obtained by abstracting the properties of crystal bases in Theorem 3.1.

Definition 3.2 ([2]). A 6-tuple $(B, wt, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i \in I}$ is a *crystal* if *B* is a set and there exists a certain special element 0 outside of *B* and maps:

(3.2) wt:
$$B \to P$$
, $\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}$, $\varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}$ $(i \in I)$,

$$(3.3) \qquad \qquad \tilde{e}_i: B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i: B \sqcup \{0\} \to B \sqcup \{0\} \ (i \in I),$$

satisfying :

(1)
$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \operatorname{wt}(b) \rangle$$
.

(2) If
$$b, \tilde{e}_i b \in B$$
, then wt $(\tilde{e}_i b) = wt(b) + \alpha_i, \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.

(3) If
$$b, \tilde{f}_i b \in B$$
, then wt $(\tilde{f}_i b) = wt(b) - \alpha_i, \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

(4) For $b, b' \in B$ and $i \in I$, one has $\tilde{f}_i b = b'$ iff $b = \tilde{e}_i b'$.

(5) If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$ and $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Here, a *ccrystal graph* of crystal *B* is a *I*-colored oriented graph defined by $b \xrightarrow{i} b' \Leftrightarrow \tilde{f}_i(b) = b'$ for $b, b' \in B$.

Definition 3.3 ([2]). For crystals B_1 and B_2 , Ψ is a *strict embedding* (resp. *isomorphism*) from B_1 to B_2 if $\Psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is an injective (resp. bijective) map satisfying that $\Psi(0) = 0$, wt($\Psi(b)$) = wt(b), $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for any $b \in B_1$ and Ψ commutes with all \tilde{e}_i 's and \tilde{f}_i 's,

We obtain the tensor structure of crystals as follows([1, 2]):

Proposition 3.4. For crystals B_1 and B_2 , set

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 := (b_1, b_2) \mid b_1 \in B_1, b_2 \in B_2\} (= B_1 \times B_2).$$

Then, $B_1 \otimes B_2$ becomes a crystal by defining:

(3.4)
$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2),$$

(3.5)
$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle),$$

(3.6) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle),$

(3.7)
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

(3.8)
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2). \end{cases}$$

Example 3.5. For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and

$$\begin{aligned} & \text{wt}((n)_i) = n\alpha_i, \ \varepsilon_i((n)_i) = -n, \ \varphi_i((n)_i) = n \\ & \varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty \ (i \neq j), \\ & \tilde{e}_i((n)_i) = (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i, \\ & \tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0 \ (i \neq j). \end{aligned}$$

Then B_i ($i \in I$) possesses a crystal structure. Note that as a set the crystal B_i can be identified with the set of integers \mathbb{Z} .

3.3. Explicit structure of the crystal $B_{i_1} \otimes \cdots \otimes B_{i_m}$. Here we shall describe an explicit structure of tensor product of B_i 's. Fix a sequence of indices $\mathbf{i} = (i_1, \cdots, i_m) \in I^m$ and write

$$(x_1,\cdots,x_m):=\tilde{f}_{i_1}^{x_1}(0)_{i_1}\otimes\cdots\otimes\tilde{f}_{i_m}^{x_m}(0)_{i_m}=(-x_1)_{i_1}\otimes\cdots\otimes(-x_m)_{i_m},$$

where if n < 0, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$. Note that here we do not necessarily assume that **i** is a reduced word though later we will take **i** to be a reduced longest word. By the tensor structure of crystals in Proposition 3.4, for the sequence **i** as above, we can describe the explicit crystal structure on $\mathbb{B}_i := B_{i_1} \otimes \cdots \otimes B_{i_m}$ as follows: For $x = (x_1, \cdots, x_m) \in \mathbb{B}_i$, define

$$\sigma_k(x) := x_k + \sum_{j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j$$

and for $i \in I$ define

$$\begin{aligned} \widetilde{\sigma}^{(i)}(x) &:= \max\{\sigma_k(x) \mid 1 \le k \le m \text{ and } i_k = i\}, \\ \widetilde{M}^{(i)} &= \widetilde{M}^{(i)}(x) := \{k \mid 1 \le k \le m, i_k = i, \sigma_k(x) = \widetilde{\sigma}^{(i)}(x)\}, \\ \widetilde{m}^{(i)}_f &= \widetilde{m}^{(i)}_f(x) := \max \widetilde{M}^{(i)}(x), \quad \widetilde{m}^{(i)}_e = \widetilde{m}^{(i)}_e(x) := \min \widetilde{M}^{(i)}(x). \end{aligned}$$

Now, the actions of the Kashiwara operators \tilde{e}_i, \tilde{f}_i and the functions ε_i, φ_i and wt are written explicitly:

(3.9)
$$\tilde{f}_i(x)_k := x_k + \delta_{k,\widetilde{m}_f^{(i)}}, \qquad \tilde{e}_i(x)_k := x_k - \delta_{k,\widetilde{m}_e^{(i)}},$$

(3.10)
$$\operatorname{wt}(x) := -\sum_{k=1}^{m} x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \widetilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \operatorname{wt}(x) \rangle + \varepsilon_i(x).$$

Define the function $\beta_k^{(i)}$ on \mathbb{B}_i by :

(3.11)
$$\beta_k^{(i)}(x) := \sigma_{k^+}(x) - \sigma_k(x) = x_k + \sum_{k < j < k^+} \langle h_i, \alpha_{i_j} \rangle x_j + x_{k^+}$$

for $x = (x_1, \dots, x_m) \in \mathbb{B}_i$, where for $k \in [1, N]$, k^+ (resp. k^-) is the minimum (resp. maximum) number $j \in [1, N]$ such that k < j (resp. l < k) and $i_k = i_j$ if it exists, otherwise N + 1 (resp. 0). Here one knows that $\widetilde{m}_f^{(i)}(x)$ and $\widetilde{m}_e^{(i)}(x)$ are determined by $\{\beta_k^{(i)}(x) | 1 \le k \le N, i_k = i\}$.

3.4. **Braid-type isomorphism.** We shall introduce some isomorphism of crystals, called "braid-type isomorphism".

Set $c_{ij} := \langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle$, $c_1 := -\langle h_i, \alpha_j \rangle$ and $c_2 := -\langle h_j, \alpha_i \rangle$. In the sequel, for $x \in \mathbb{Z}$, put

$$x_+ := \begin{cases} x & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proposition 3.6 ([14]). There exist the following isomorphisms of crystals $\phi_{ii}^{(k)}$ (k = 0, 1, 2, 3)

(1) If $c_{ij} = 0$,

$$(3.12) \qquad \qquad \phi_{ij}^{(0)}: B_i \otimes B_i \xrightarrow{\sim} B_j \otimes B_i$$

where
$$\phi_{ij}^{(0)}((x)_i \otimes (y)_j) = (y)_j \otimes (x)_i$$
.
(2) If $c_{ij} = 1$,

$$\phi_{ii}^{(1)}: B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j,$$

where

$$\phi_{ij}^{(1)}((x)_i \otimes (y)_j \otimes (z)_i) = (z + (-x + y - z)_+)_j \otimes (x + z)_i \otimes (y - z - (-x + y - z)_+)_j.$$

(3) If
$$c_{ij} = 2$$
,

(3.13)

$$(3.14) \qquad \qquad \phi_{ij}^{(2)}: B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i,$$

where $\phi_{ij}^{(2)}$ is given by the following: for $(x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j$ we set $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i := \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$.

 $(3.15) X = w + (-c_2x + y - w + c_2(x - c_1y + z)_+)_+,$

(3.16)
$$Y = x + c_1 w + (-x + z - c_1 w + (x - c_1 y + z)_+)_+,$$

$$(3.17) Z = y - (-c_2x + y - w + c_2(x - c_1y + z)_+)_+,$$

(3.18) $W = z - c_1 w - (-x + z - c_1 w + (x - c_1 y + z)_+)_+.$

(4) If $c_{ij} = 3$, the map

$$(3.19) \qquad \qquad \phi_{ii}^{(3)}: B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i,$$

is defined by the following: for $(x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j$ we set $A := -x + c_1y - z$, $B := -y + c_2z - u$, $C := -z + c_1u - v$ and $D := -u + c_2v - w$. Then $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (U)_i \otimes (V)_j \otimes (V)_j \otimes (W)_i := \phi_{ij}^{(3)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j)$ is given by

$$X = w + (D + (c_2C + (2B + A_+)_+)_+)_+,$$

$$Y = x + c_1w + (c_1D + (3C + (2c_1B + 2A_+)_+)_+)_+,$$

$$Z = y + u + w - X - V,$$

$$U = x + z + v - Y - W,$$

$$V = u - w - (2D + (2c_2C + (3B + c_2A_+)_+)_+)_+,$$

$$W = v - c_1w - (c_1D + (2C + (c_1B + A_+)_+)_+)_+,$$

They also satisfy $\phi_{ii}^{(k)} \circ \phi_{ii}^{(k)} = \text{id.}$

We call such isomorphisms of crystals braid-type isomorphisms.

We also define a *braid-move* on the set of reduced words of $w \in W$ to be a composition of the following transformations induced from braid relations:

$$\cdots ij \cdots \rightarrow \cdots ji \cdots (c_{ij} = 0), \quad \cdots iji \cdots \rightarrow \cdots jij \cdots (c_{ij} = 1),$$

$$\cdots ijij \cdots \rightarrow \cdots jijii \cdots (c_{ii} = 2), \quad \cdots ijijij \cdots \rightarrow \cdots jijijij \cdots (c_{ii} = 3).$$

which are called by 2-move, 3-move, 4-move, 6-move respectively.

3.5. **Cellular Crystal** $\mathbb{B}_{\mathbf{i}} = \mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes \cdots \otimes B_{i_k}$. For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_k$ of some Weyl group element, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a *cellular crystal* associated with a reduced word \mathbf{i} . Indeed, it is obtained by applying the tropicalization functor to the geometric crystal on the Langlands-dual Schubert cell ${}^L X_w$, where $w = s_{i_1} \cdots s_{i_k}$ is an element of the Well group W ([15]). It is immediate from the braid-type isomorphisms that for any $w \in W$ and its reduced words $i_1 \cdots i_l$ and $j_1 \cdots j_l$, we get the following isomorphism of crystals:

$$(3.20) B_{i_1} \otimes \cdots \otimes B_{i_l} \cong B_{j_1} \otimes \cdots \otimes B_{j_l}.$$

3.6. Half potential and the crystal $B(\infty)$. For a Laurent polynomial $\phi(x_1, \dots, x_n)$ with positive coefficients, the tropicalization of ϕ is denoted by ϕ := Trop(ϕ), which is given by the rule: Trop(ax + by) = min(x, y) with a, b > 0, Trop(xy) = x + y and Trop(x/y) = x - y and Trop(c) = 0 for c > 0. In [10], the crystal $B(\infty)$ has been realized as a certain subset of \mathbb{B}_i defined as follows:

Theorem 3.7 ([10, Theorem 5.11]). Define the subset of \mathbb{B}_i :

$$(\mathbb{B}_{w_0}^-)_{\Phi^{(+)},\Theta_{\mathbf{i}}} = \{x = (x_1, \cdots, x_N) \in \mathbb{B}_{\mathbf{i}} \mid \overline{\Phi}^{(+)}(x) \ge 0\},\$$

where $\mathbb{B}_{w_0}^-$ is a certain geometric crystal, $\widetilde{\Phi}^{(+)}$ is a tropicalization of the half potential $\Phi^{(+)}$ which is a Laurent polynomial with positive coefficients in N variables and Θ_i is a certain positive structure on the geometric crystal $\mathbb{B}_{w_0}^-$. Then, $(\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)},\Theta_i} \cong B(\infty)$.

Remark 3.8. To define the crystal structure on $(\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)},\Theta_i}$, it is supposed that if $\tilde{e}_i x \notin (\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)},\Theta_i}$, then $\tilde{e}_i x = 0$. Thus, in this sense, the embedding $B(\infty) \cong (\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)},\Theta_i} \hookrightarrow \mathbb{B}_i$ is not a strict embedding. In [16, 15], it has been given the strict embedding of $B(\infty) \hookrightarrow \mathbb{B}_i$, which is called "Kashiwara embedding" and the method to describe the image of this embedding, called "polyhedral realization".

3.7. **Subspace** $\mathcal{H}_{\mathbf{i}}$. The object $\mathcal{H}_{\mathbf{i}}$ will play a significant role for this article. Fix a reduced longest word $\mathbf{i} = i_1 \cdots i_N$ and take the function $\beta_k^{(\mathbf{i})}(x) = x_k + \sum_{k < j < k^+} \langle h_{i_k} \alpha_{i_j} \rangle x_j + x_{k^+}$ $(1 \le k \le N)$ as in (3.11). In what follows, let us identify the \mathbb{Z} -lattice \mathbb{Z}^N with $B_{\mathbf{i}}$ and then we define the summation of elements $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$ by $x + y = (x_1 + y_1, \cdots, x_N + y_N)$ as a standard one in \mathbb{Z}^N . Here, we define the subspace $\mathcal{H}_{\mathbf{i}} \subset \mathbb{Z}^N$ by

(3.21)
$$\mathcal{H}_{\mathbf{i}} := \{ x \in \mathbb{Z}^{N} (= \mathbb{B}_{\mathbf{i}}) | \beta_{k}^{(\mathbf{i})}(x) = 0 \text{ for any } k \text{ such that } k^{+} \leq N \} \subset \mathbb{B}_{\mathbf{i}}$$

The following result was presented in [10]:

Proposition 3.9 ([10]). For $\mathbf{i} = i_1 i_2 \cdots i_N$, $k = 1, 2, \cdots, N$ and a fundamental weight Λ_i , set

(3.22)
$$h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle$$
 and $\mathbf{h}_i := (h_i^{(1)}, h_i^{(2)}, \dots, h_i^{(N)}) \in \mathbb{B}_{\mathbf{i}}$

Then, we obtain that $\{\mathbf{h}_1, \cdots, \mathbf{h}_N\}$ is a \mathbb{Z} -basis of \mathcal{H}_i , namely,

$$(3.23) \qquad \qquad \mathcal{H}_{\mathbf{i}} = \mathbb{Z}\mathbf{h}_1 \oplus \mathbb{Z}\mathbf{h}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{h}_n$$

Proof. Let $\{\alpha'_i\}_i, \{h'_i\}_i$ and $\{s'_i\}_i$ be the simple roots, the simple co-roots and the simple reflections of the Langlands dual Lie algebra \mathfrak{g}^{\vee} respectively. Define $m_i^{(k)} \in \mathbb{Z}_{\geq 0}$ $(k \in [1, N], i \in I)$ by

$$\alpha^{\prime(k)} := s_{i_N}' s_{i_N-1}' \cdots s_{k+1}' (\alpha_{i_k}') = \sum_{i \in I} m_i^{(k)} \alpha_i'.$$

By [10, Lemma 9.1], one has that $\{\mathbf{m}_i := (m_i^{(1)}, m_i^{(2)}, \cdots, m_i^{(N)}) \mid i \in I\}$ is a \mathbb{Z} -basis of \mathcal{H}_i . Thus, it suffices to show that $h_i^{(k)} = m_i^{(k)}$ for any $k \in [1, N]$ and $i \in I$.

Let us define the set of paths from *a* to b ($a, b \in \mathbb{Z}$, $a \ge b$) by

$$\mathcal{P}(a,b) := \{(a, j_1, j_2, \dots, j_l, b) \mid a > j_1 > j_2 > \dots > j_l > b, \ l \ge 0\},\$$

where set $\mathcal{P}(a, a) = \emptyset$ and l = -1. By the following lemma, we can complete the proof of the proposition.

Lemma 3.10. We obtain the following explicit formulas:

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$$(3.24) \qquad \langle h_{i_{k}}, s_{i_{k+1}} \cdots s_{i_{p-1}}(\alpha_{i_{p}}) \rangle \\ = \sum_{(p,j_{1},\cdots,j_{l},k) \in \mathcal{P}(p,k)} (-1)^{l} \langle h_{i_{k}}, \alpha_{i_{j_{l}}} \rangle \langle h_{i_{j_{l}}}, \alpha_{i_{j_{l-1}}} \rangle \cdots \langle h_{i_{j_{2}}}, \alpha_{i_{j_{1}}} \rangle \langle h_{i_{j_{1}}}, \alpha_{i_{p}} \rangle \quad (p > k),$$

$$(3.25) \qquad s_{i_{N}}' s_{i_{N-1}}' \cdots s_{i_{k+1}}' (\alpha_{i_{k}}') \\ = \sum_{L=k}^{N} \sum_{(L,j_{1},\cdots,j_{l},k) \in \mathcal{P}(L,k)} (-1)^{l+1} \langle h_{i_{L}}', \alpha_{i_{j_{1}}}' \rangle \langle h_{i_{j_{1}}}', \alpha_{i_{j_{2}}}' \rangle \cdots \langle h_{i_{j_{l-1}}}', \alpha_{i_{j_{l}}}' \rangle \langle h_{i_{j_{l}}}', \alpha_{i_{k}}' \rangle \alpha_{i_{L}}',$$

where note that in (3.25) if k = L, namely $\mathcal{P}(L, k) = \emptyset$, then the corresponding term is α'_{i_k} .

Example 3.11. In $g = G_2$ -case. Set $a_{12} = -1$ and $a_{21} = -3$. Taking a reduced longest word **i** = 121212, one has

 $\beta_1^{(i)}(x) = x_1 - x_2 + x_3, \quad \beta_2^{(i)}(x) = x_2 - 3x_3 + x_4, \quad \beta_3^{(i)}(x) = x_3 - x_4 + x_5, \quad \beta_4^{(i)}(x) = x_4 - 3x_5 + x_6.$ By the formula (3.22), one gets

 $\mathbf{h}_1 = (1, 3, 2, 3, 1, 0), \qquad \mathbf{h}_2 = (0, 1, 1, 2, 1, 1).$

Then the solution space $\mathcal{H}_{\mathbf{i}}$ of $\beta_1^{(\mathbf{i})}(x) = \beta_2^{(\mathbf{i})}(x) = \beta_3^{(\mathbf{i})}(x) = \beta_4^{(\mathbf{i})}(x) = 0$ is given by

$$\mathcal{H}_{\mathbf{i}} = \{c_1\mathbf{h}_1 + c_2\mathbf{h}_2 = (c_1, c_2 + 3c_1, c_2 + 2c_1, 2c_2 + 3c_1, c_2 + c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}\}.$$

Lemma 3.12. The braid-type isomorphisms are well-defined on $\mathcal{H}_{\mathbf{i}}$, that is, $\phi_{ij}^{(k)}(\mathcal{H}_{\mathbf{i}}) = \mathcal{H}_{\mathbf{i}'}$, where \mathbf{i}' is the reduced word obtained by applying the corresponding braid-moves. We also obtain the following formula:

(1) For any $h = (\dots, x, y, \dots) = \dots \otimes (-x)_i \otimes (-y)_j \otimes \dots \in \mathcal{H}_i$, assume that $a_{ij} = a_{ji} = 0$. Applying the braid-type isomorphism $\phi_{ij}^{(0)}$ on (x, y) in h, we have

(3.26)
$$\phi_{ij}^{(0)}(h) = (\cdots, y, x, \cdots) = \cdots \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{\mathbf{i}'}$$

(2) For any $h = (\dots, x, y, z, \dots) = \dots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes \dots \in \mathcal{H}_i$, assume that $a_{ij} = a_{ji} = -1$. Applying the braid-type isomorphism $\phi_{ij}^{(1)}$ on (x, y, z) in h, we have

(3.27)
$$\phi_{ij}^{(1)}(h) = (\cdots, z, y, x, \cdots) = \cdots \otimes (-z)_j \otimes (-y)_i \otimes (-x)_j \otimes \cdots \in \mathcal{H}_{\mathbf{i}'}$$

(3) For $h = (\cdots, x, y, z, w, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-w)_j \cdots \in \mathcal{H}_i$, assume that $a_{ij} \cdot a_{ji} = 2$. Applying the braid-type isomorphism $\phi_{ij}^{(2)}$ on (x, y, z, w) in h, we have

$$(3.28) \qquad \qquad \phi_{ij}^{(2)}(h) = (\cdots, w, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{i'}$$

(4) For $h = (\dots, x, y, z, u, v, w, \dots) = \dots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-u)_j \otimes (-v)_i \otimes (-w)_j \dots \in \mathcal{H}_i$, assume that $a_{ij} \cdot a_{ji} = 3$. Applying the braid-type isomorphism $\phi_{ij}^{(3)}$ on (x, y, z, u, v, w) in h, we have

$$(3.29) \qquad \phi_{ij}^{(3)}(h) = (\cdots, w, v, u, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{i'}$$

In [10, Sect.8], we have shown the following statements under the condition " \mathbf{H}_i ", where we omit the explicit form of \mathbf{H}_i since we do not need it here. But, we succeed in showing the following proposition without the condition \mathbf{H}_i since in [10] we have shown that there exists a specific reduced longest word \mathbf{i}_0 satisfying the condition \mathbf{H}_{i_0} for each simple Lie algebra g and we got Lemma 3.12.

Proposition 3.13. Let $\mathbf{i} = i_1 i_2 \cdots i_N$ be an arbitrary reduced longest word. Here if the crystal $B(\infty)$ is realized in $\mathbb{B}_{\mathbf{i}}$ as in 3.6, we shall denote it by $B(\infty)_{\mathbf{i}}$ to emphasize the word \mathbf{i} . For $h \in \mathcal{H}_{\mathbf{i}}$, define

$$B^{h}(\infty)_{\mathbf{i}} := \{x + h \in \mathbb{Z}^{N}(=\mathbb{B}_{\mathbf{i}}) \mid x \in B(\infty)_{\mathbf{i}}\} \subset \mathbb{B}_{\mathbf{i}}.$$

(1) For any $x + h \in B^h(\infty)_i$ and $i \in I$, we obtain

(3.30)
$$\tilde{e}_i(x+h) = \tilde{e}_i(x) + h, \qquad \tilde{f}_i(x+h) = \tilde{f}_i(x) + h.$$

(2) For any $h \in \mathcal{H}_{\mathbf{i}}$, we have $B(\infty)_{\mathbf{i}} \cap B^{h}(\infty)_{\mathbf{i}} \neq \emptyset$.

(3)

$$\mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^{h}(\infty)_{\mathbf{i}}$$

Remark 3.14. In the setting of the half-potential method in [10], as mentioned in Remark 3.8, the crystal $B(\infty)$ is realized as a subset of \mathbb{B}_i and it is supposed that $\tilde{e}_i x = 0$ if $\tilde{e}_i x \notin (\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)},\Theta_i} \cong B(\infty)$. At the statement (2), since $x \in B(\infty)_i$ is considered as an element of \mathbb{B}_i , $\tilde{e}_i x$ is also considered as an element in \mathbb{B}_i . That is, even if $\tilde{e}_i x \notin B(\infty)$, we consider that $\tilde{e}_i x \in \mathbb{B}_i$ and then it never vanishes.

It is immediate from this proposition that one has the following theorem:

Theorem 3.15 ([10]). For any simple Lie algebra g and any reduced word $i_1i_2\cdots i_k$, the cellular crystal $\mathbb{B}_{i_1i_2\cdots i_k} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ is connected as a crystal graph.

Example 3.16. For G₂-case, by the polyhedral realization method, we obtain

$$B(\infty) = \{ (x_1, x_2, x_3, x_4, x_5, x_6) \mid x_6 \ge 0, x_5 \ge \frac{x_4}{3} \ge \frac{x_3}{2} \ge \frac{x_2}{3} \ge x_1 \ge 0 \},\$$

where $(x_1, x_2, x_3, x_4, x_5, x_6)$ stands for $(-x_1)_1 \otimes (-x_2)_2 \otimes (-x_3)_1 \otimes (-x_4)_2 \otimes (-x_5)_1 \otimes (-x_6)_2$. As has seen in Example 3.11, we get

$$\mathbf{h}_1 = (1, 3, 2, 3, 1, 0), \qquad \mathbf{h}_2 = (0, 1, 1, 2, 1, 1).$$

Thus, let us see that for any $\mathbf{v} = (a, b, c, d, e, f) \in \mathbb{B}_{121212}$, the conditions on c_1, c_2 such that $\mathbf{v} + c_1 \mathbf{h}_1 + c_2 \mathbf{h}_2 \in B(\infty)$. Indeed, they are given by

$$c_1 \ge \max(3a - b, 2b - 3c, 3c - 2d, d - 3e, -f), \quad c_2 \ge -a.$$

4. QUIVER HECKE ALGEBRA AND ITS MODULES

In this section, we shall introduce the quiver Hecke algebra and its basic properties (see [4, 5, 7, 17]).

4.1. **Definition of Quiver Hecke Algebra.** For a finite index set *I* and a field **k**, let $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I} \in \mathbf{k}[u, v]$ be polynomials satisfying:

- (1) $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$ for any $i, j \in I$.
- (2) $\mathcal{Q}_{i,j}(u, v)$ is in the form:

$$\mathcal{Q}_{i,j}(u,v) = \begin{cases} \sum_{a(\alpha_i,\alpha_i)+b(\alpha_j,\alpha_j)=-2(\alpha_i,\alpha_j)} t_{i,j;a,b} u^a v^b & \text{ if } i \neq j, \\ 0 & \text{ if } i = j, \end{cases}$$

where $t_{i,j;-a_{ii},0} \in \mathbf{k}^{\times}$.

For $\beta = \sum_i m_i \alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$, set $I^\beta := \{\nu = (\nu_1, \cdots, \nu_m) \in I^m \mid \sum_{k=1}^m \alpha_{\nu_k} = \beta\}$.

Definition 4.1. For $\beta \in Q_+$, the *quiver Hecke algebra* $R(\beta)$ associated with a Cartan matrix A and polynomials $\mathcal{Q}_{i,j}(u, v)$ is the **k**-algebra generated by

$$\{e(v)|v \in I^{\beta}\}, \quad \{x_k|1 \le k \le n\}, \quad \{\tau_i|1 \le i \le n-1\}$$

with the following relations:

$$e(v)e(v') = \delta_{v,v'}e(v), \qquad \sum_{v \in l^{\beta}} e(v) = 1, \qquad e(v)x_k = x_k e(v), \qquad x_k x_l = x_l x_k,$$

$$\tau_l e(v) = e(s_l(v))\tau_l, \qquad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k-l| > 1,$$

$$\tau_k^2 e(v) = \mathcal{Q}_{v_k, v_{k+1}}(x_k, x_{k+1})e(v),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(v) = \begin{cases} -e(v) \quad \text{if } l = k, \quad v_k = v_{k+1}, \\ e(v) \quad \text{if } l = k+1, \quad v_k = v_{k+1}, \\ 0 \quad \text{otherwise}, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(v) = \begin{cases} \overline{\mathscr{Q}}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(v) & \text{if } \nu_k = \nu_{k+2} \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathcal{Q}}_{i,j}(u, v, w) = \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u - w} \in \mathbf{k}[u, v, w].$

(1) The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a \mathbb{Z} -graded algebra. Here we define the *weight* of $R(\beta)$ -module M as $wt(M) = -\beta$.

(2) Let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a \mathbb{Z} -graded $R(\beta)$ -module. Define a *grading shift functor* q on the category of graded $R(\beta)$ -modules $R(\beta)$ -Mod by

$$qM := \bigoplus_{k \in \mathbb{Z}} (qM)_k$$
, where $(qM)_k = M_{k-1}$.

- (3) For $M, N \in R(\beta)$ -Mod, let $\operatorname{Hom}_{R(\beta)}(M, N)$ be the space of degree preserving morphisms and define $\operatorname{Hom}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}_{R(\beta)}(q^k M, N)$, which is a space of morphisms up to grading shift. We define $\operatorname{deg}(f) = k$ for $f \in \operatorname{Hom}_{R(\beta)}(q^k M, N)$.
- (4) Let ψ be the anti-automorphism of $R(\beta)$ preserving all generators. For $M \in R(\beta)$ -Mod, define $M^* := \text{Hom}_{\mathbf{k}}(M, \mathbf{k})$ with the $R(\beta)$ module structure by $(r \cdot f)(u) := f(\psi(r)u)$ for $r \in R(\beta), u \in M$ and $f \in M^*$, which is called a *dual module* of M. In particular, if $M \cong M^*$ we call M is *self-dual*.
- (5) For $\beta, \gamma \in Q_+$, set $e(\beta, \gamma) = \sum_{\nu \in l^{\beta}, \nu' \in l^{\gamma}} e(\nu, \nu')$. We define an injective homomorphism $\xi_{\beta,\gamma} : R(\beta) \otimes R(\gamma) \to e(\beta, \gamma)R(\beta + \gamma)e(\beta, \gamma)$ by $\xi(\beta, \gamma)(e(\nu) \otimes e(\nu')) = e(\nu, \nu'), \ \xi(\beta, \gamma)(x_k e(\beta) \otimes 1) = x_k e(\beta, \gamma), \ \xi(\beta, \gamma)(1 \otimes x_k e(\gamma)) = x_{k+|\beta|}e(\beta, \gamma), \ \xi(\beta, \gamma)(\tau_k e(\beta) \otimes 1) = \tau_k e(\beta, \gamma), \ \xi(\beta, \gamma)(1 \otimes \tau_k e(\gamma)) = x_{k+|\beta|}e(\beta, \gamma).$
- (6) For $M \in R(\beta)$ -Mod and $N \in R(\gamma)$ -Mod, define the *convolution product* \circ by

 $M \circ N := R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N)$

For simple $M \in R(\beta)$ -Mod and simple $N \in R(\gamma)$ -Mod, we say M and N strongly commutes if $M \circ N$ is simple and M is *real* if $M \circ M$ is simple.

(7) For $M \in R(\beta)$ -Mod and $N \in R(\gamma)$ -Mod, denote by $M\nabla N := hd(M \circ N)$ the head of $M \circ N$ and $M\Delta N := soc(M \circ N)$ the socle of $M \circ N$, where the head of module *M* is the quotient by its radical and the socle of module *M* is the summation of all simple submodules.

4.2. Categorification of quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$. Let $R(\beta)$ -gmod be the full subcategory of $R(\beta)$ -Mod whose objects are finite-dimensional graded $R(\beta)$ -modules and set R-gmod= $\bigoplus_{\beta \in Q_1} R(\beta)$ -gmod. Define the functors

$$E_i: R(\beta) \operatorname{-gmod} \to R(\beta - \alpha_i) \operatorname{-gmod}, \qquad F_i: R(\beta) \operatorname{-gmod} \to R(\beta + \alpha_i) \operatorname{-gmod},$$

by $E_i(M) := e(\alpha_i, \beta - \alpha_i)M$, $F_i(M) = L(i) \circ M$, where $e(\alpha_i, \beta - \alpha_i) := \sum_{\nu \in l^\beta, \nu_1 = i} e(\nu)$ and $L(i) := R(\alpha_i)/R(\alpha_i)x_1$ is a 1-dimensional simple $R(\alpha_i)$ -module. Let $\mathcal{K}(R$ -gmod) be the Grothendieck ring of *R*-gmod and then $\mathcal{K}(R$ -gmod) becomes a $\mathbb{Z}[q, q^{-1}]$ -algebra with the multiplication induced by the convolution product and $\mathbb{Z}[q, q^{-1}]$ -action induced by the grading shift functor q. Here, one obtain the following:

Theorem 4.2 ([4, 17]). As a $\mathbb{Z}[q, q^{-1}]$ -algebra there exists an isomorphism

$$\mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}.$$

4.3. Categorification of the crystal $B(\infty)$ by Lauda and Vazirani [11]. The following lemma is given in [4]:

Lemma 4.3 ([4]). For any simple $R(\beta)$ -module M, soc (E_iM) , hd (E_iM) and hd (F_iM) are all simple modules. Here we also have that soc $(E_iM) \cong hd(E_iM)$ up to grading shift.

For $M \in R(\beta)$ -gmod, define

(4.1)
$$\operatorname{wt}(M) = -\beta, \quad \varepsilon_i(M) = \max\{n \in \mathbb{Z} \mid E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \operatorname{wt}(M) \rangle$$

(4.2)
$$\widetilde{E}_i M := q_i^{1-\varepsilon_i(M)} \operatorname{soc}(E_i M) \cong q_i^{\varepsilon_i(M)-1} \operatorname{hd}(E_i M), \qquad \widetilde{F}_i M := q_i^{\varepsilon_i(M)} \operatorname{hd}(F_i M).$$

Set $\mathbb{B}(R\text{-gmod}) := \{S \mid S \text{ is a self-dual simple module in } R\text{-gmod}\}$. Then, it follows from Lemma 4.3 that \widetilde{E}_i and \widetilde{F}_i are well-defined on $\mathbb{B}(R\text{-gmod})$.

Theorem 4.4 ([11]). The 6-tuple ($\mathbb{B}(R\text{-gmod}), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \text{wt}, \{\varepsilon_i\}, \{\varphi_i\}\}_{i \in I}$ holds a crystal structure and there exists the following isomorphism of crystals:

$$I': \mathbb{B}(R\operatorname{-gmod}) \xrightarrow{\sim} B(\infty).$$

Remark 4.5. Note that Lauda and Vasirani showed this theorem under more general setting that g is arbitrary symmetrizable Kac-Moody Lie algebra. Here we assume that g is a simple Lie algebra. The definition of \tilde{E}_i and \tilde{F}_i in (4.2) differs from the one in [11], which follows the one in [7].

5. LOCALIZATION OF MONOIDAL CATEGORY

Here we shall review the theory of localization for monoidal category following [5].

5.1. **Braiders and Real Commuting Family.** Let Λ be \mathbb{Z} -lattice and $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} \mathcal{T}_{\lambda}$ be a **k**-linear Λ -graded monoidal category with a data consisting of a bifunctor $\otimes : \mathcal{T}_{\lambda} \times \mathcal{T}_{\mu} \to \mathcal{T}_{\lambda+\mu}$, an isomorphism $a(X, Y, Z) : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$ satisfying $a(X, Y, Z \otimes W) \circ a(X \otimes Y, Z, W) = \operatorname{id}_X \otimes a(Y, Z, W) \circ a(X, Y \otimes Z, W) \circ a(X, Y, Z) \otimes \operatorname{id}_W$ and an object $\mathbf{1} \in \mathcal{T}_0$ endowed with an isomorphism $\epsilon : \mathbf{1} \otimes \mathbf{1} \longrightarrow \mathbf{1}$ such that the functor $X \mapsto X \otimes \mathbf{1}$ and $X \mapsto \mathbf{1} \otimes X$ are fully-faithful.

Definition 5.1 ([5]). Let *q* be the grading shift functor on \mathcal{T} . A graded braider is a triple (C, R_C, ϕ) , where $C \in \mathcal{T}$, \mathbb{Z} -linear map $\phi : \Lambda \to \mathbb{Z}$ and a morphism:

$$R_C: C \otimes X \to q^{\phi(\lambda)} X \otimes C \quad (X \in \mathcal{T}_{\lambda}),$$

satisfying the following commutative diagram:

and being functorial, that is, for any $X, Y \in \mathcal{T}$ and $f \in \text{Hom}_{\mathcal{T}}(X, Y)$ it satisfies the following commutative diagram:

$$\begin{array}{c|c} C \otimes X & \xrightarrow{\mathrm{id} \otimes f} & C \otimes Y \\ \hline R_C(X) & & & \\ X \otimes C & \xrightarrow{f \otimes \mathrm{id}} & Y \otimes C \end{array}$$

Definition 5.2 ([5]). Let *I* be an index set and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ a family of graded braiders in \mathcal{T} . We say that $(C_i, R_{C_i}, \phi_i)_{i \in I}$ is a *real commuting family of graded braiders* in \mathcal{T} if

- (1) $C_i \in \mathcal{T}_{\lambda_i}$ for some $\lambda_i \in \Lambda$, and $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ for any $i, j \in I$.
- (2) $R_{C_i}(C_i) \in \mathbf{k}^{\times} \mathrm{id}_{C_i \otimes C_i}$ for any $i \in I$.
- (3) $R_{C_i}(C_j) \otimes R_{C_i}(C_i) \in \mathbf{k}^{\times} \mathrm{id}_{C_i \otimes C_i}$ for any $i, j \in I$.

Note that R_{C_i} 's satisfy so-called "Yang-Baxter equation", such as,

$$R_{C_i}(C_j) \circ R_{C_i}(C_k) \circ R_{C_i}(C_k) = R_{C_i}(C_k) \circ R_{C_i}(C_k) \circ R_{C_i}(C_j) \text{ on } C_i \circ C_j \circ C_k.$$

For a finite index set *I*, set $\Gamma := \bigoplus_{i \in I} \mathbb{Z} e_i$ and $\Gamma_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} e_i$.

Lemma 5.3 ([5]). Suppose that we have a real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}$. We can choose a bilinear map $H : \Gamma \times \Gamma \to \mathbb{Z}$ such that $\phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i)$ and there exist

(1) an object C^{α} for any $\alpha \in \Gamma_+$.

(2) an isomorphism $\xi_{\alpha,\beta} : C^{\alpha} \otimes C^{\beta} \xrightarrow{\sim} q^{H(\alpha,\beta)} C^{\alpha+\beta}$ for any $\alpha, \beta \in \Gamma_+$ such that $C^0 = 1$ and $C^{e_i} = C_i$. 5.2. Localization. Let \mathcal{T} and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ be as above and $\{C^{\alpha}\}_{\alpha \in \Gamma_+}$ objects as in the previous lemma. We define a partial order \leq on Γ by

$$\alpha \leq \beta \Longleftrightarrow \beta - \alpha \in \Gamma_+$$

For $\alpha_1, \alpha_2, \dots \in \Gamma$, define

$$\mathcal{D}_{\alpha_1,\alpha_2,\cdots} := \{ \delta \in \Gamma \mid \alpha_j + \delta \in \Gamma_+ \text{ for any } j = 1, 2, \cdots \}$$

For $X \in \mathcal{T}_{\lambda}$, $Y \in \mathcal{T}_{\mu}$ and $\delta \in \mathcal{D}_{\alpha,\beta}$, set

$$H_{\delta}((X,\alpha),(Y,\beta)) := \operatorname{Hom}_{\mathcal{T}}(C^{\delta+\alpha} \otimes X, q^{P(\alpha,\beta,\delta,\mu)}Y \otimes C^{\beta+\delta})$$

where a \mathbb{Z} -valued function $P(\alpha, \beta, \delta, \mu) := H(\delta, \beta - \alpha) + \phi(\delta + \beta, \mu)$ and the map $\phi : \Gamma \times \Lambda \to \mathbb{Z}$ is defined by $\phi(\alpha, L(\beta)) = H(\alpha, \beta) - H(\beta, \alpha)$ and $L : \Gamma \to \Lambda$ is defined by $L(e_i) = \lambda_i$ ([5]).

Lemma 5.4 ([5]). For $\delta \leq \delta'$ there exists the map

$$\zeta_{\delta,\delta'}: H_{\delta}((X,\alpha),(Y,\beta)) \to H_{\delta'}((X,\alpha),(Y,\beta))$$

satisfying

$$\zeta_{\delta,\delta'} \circ \zeta_{\delta',\delta''} = \zeta_{\delta,\delta''} \text{ for } \delta \leq \delta' \leq \delta''$$

Therefore, we find that $\{H_{\delta}((X, \alpha), (Y, \beta))\}_{\delta \in \mathcal{D}_{\alpha\beta}}$ becomes an inductive system.

Definition 5.5 (Localization [5]). We define the category $\widetilde{\mathcal{T}}$ by

$$\begin{aligned} \operatorname{Ob}(\tilde{\mathcal{T}}) &:= \operatorname{Ob}(\mathcal{T}) \times \Gamma, \\ \operatorname{Hom}_{\widetilde{\mathcal{T}}}((X, \alpha), (Y, \beta)) &:= \underbrace{\lim_{\delta \in \mathcal{D}(\alpha, \beta), \\ \lambda + L(\alpha) = \mu + L(\beta)}} H_{\delta}((X, \alpha), (Y, \beta)), \end{aligned}$$

where $X \in \mathcal{T}_{\lambda}, Y \in \mathcal{T}_{\mu}$ and the function $L : \Gamma \to \Lambda$ $(e_i \mapsto \lambda_i)$ is as above. We call this $\widetilde{\mathcal{T}}$ a *localization* of \mathcal{T} by $(C_i, R_{C_i}, \phi_i)_{i \in I}$ and denote it by $\mathcal{T}[C_i^{\otimes -1} | i \in I]$ when we emphasize $\{C_i | i \in I\}$.

Theorem 5.6 ([5]). $\widetilde{\mathcal{T}}$ becomes a monoidal category. Moreover, there exists a monoidal functor $\Upsilon: \mathcal{T} \to \widetilde{\mathcal{T}}$ such that

- (1) $\Upsilon(C_i)$ is *invertible* in $\widetilde{\mathcal{T}}$ for any $i \in I$, namely, the functors $X \mapsto X \otimes \Upsilon(C_i)$ and $X \mapsto \Upsilon(C_i) \otimes X$ are equivalence of categories.
- (2) For any $i \in I$ and $X \in \mathcal{T}$, $\Upsilon(R_{C_i}(X)) : \Upsilon(C_i \otimes X) \to \Upsilon(X \otimes C_i)$ is an isomorphism.
- (3) The functor Υ holds the following universality: If there exists another monoidal category \mathcal{T}' and a monoidal functor $\Upsilon' : \mathcal{T} \to \mathcal{T}'$ satisfying the above statements (1) and (2), then there exists a monoidal functor $F : \widetilde{\mathcal{T}} \to \mathcal{T}'$ (unique up to iso.) such that $\Upsilon' = F \circ \Upsilon$.

Proposition 5.7 ([5]). Under the setting above, we obtain

(1)
$$(X, \alpha + \beta) \cong q^{-H(\beta, \alpha)}(C^{\alpha} \otimes X, \beta), (1, \beta) \otimes (1, -\beta) \cong q^{-H(\beta, \beta)}(1, 0)$$
 for $\alpha \in \Gamma_+, \beta \in \Gamma$ and $X \in \mathcal{T}$.

- (2) If \mathcal{T} is an abelian category, then so is $\widetilde{\mathcal{T}}$.
- (3) The functors $\Upsilon : \mathcal{T} \to \widetilde{\mathcal{T}}$ is exact.
- (4) If the functor $-\otimes Y$ and $Y \otimes -$ are exact for any Y in \mathcal{T} , then the functors $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} (X \mapsto X \otimes Y$ (resp. $X \to Y \otimes X$)) are exact for any Y in $\widetilde{\mathcal{T}}$.

6. LOCALIZATION OF THE CATEGORY *R*-gmod

In this section, we shall apply the method of localization to the category *R*-gmod.

6.1. **Determinantial Modules.** Here we just go back to the setting as in Sect.4. Let $L(i^n) := q_i^{\frac{n(n-1)}{2}}L(i)^{\circ n}$ be a simple $R(n\alpha_i)$ -module satisfying $\operatorname{qdim}(L(i^n)) = [n]_i! := \prod_{k=1}^n \frac{q_k^i - q_i^{-k}}{q_i - q_i^{-1}}$ $(q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}).$

Definition 6.1 ([5, 7]). For $M \in R$ -gmod, define

$$\widetilde{F}_i^n(M) := L(i^n) \nabla M.$$

For a Weyl group element w, let $s_{i_1} \cdots s_{i_l}$ be its reduced expression. For a dominant weight $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l} \Lambda \rangle \qquad (k = 1, \cdots, l).$$

We define the *determinantial module* associated with w and Λ by

$$\mathbf{M}(w\Lambda,\Lambda):=\widetilde{F}_{i_1}^{m_1}\cdots\widetilde{F}_{i_l}^{m_l}\mathbf{1},$$

where $\mathbf{1}$ is a trivial R(0)-module.

Note that in general, one can define determinantial modules $\mathbf{M}(w\Lambda, u\Lambda)$ ($w, u \in W$) which corresponds to the generalized minor $\Delta_{w\Lambda, u\Lambda}$.

Now, let us see some similarity between the family of determinantial modules $\{\mathbf{M}(w_0\Lambda,\Lambda)\}_{\Lambda\in P_+}$ and the subspace \mathcal{H}_i . As has seen above that for a reduced longest word $\mathbf{i} = i_1 \cdots i_N$, the subspace $\mathcal{H}_i \subset \mathbb{B}_i$ is presented by

$$\mathcal{H}_{\mathbf{i}} = \bigoplus_{i \in I} \mathbb{Z} \mathbf{h}_{i}, \quad \mathbf{h}_{i} = ((h_{i}^{(k)} := \langle h_{i_{k}}, s_{i_{k+1}} \cdots s_{i_{N}} \Lambda_{i} \rangle)_{k=1,\cdots,N}.$$

Furthermore, we also get

Proposition 6.2. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$ and $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} s_{i_{k+2}} \cdots s_{i_l} \Lambda \rangle$$
 $(k = 1, 2, \cdots, N)$ and $\mathbf{h}_{\Lambda} := (m_1, \cdots, m_N).$

Then we obtain

$$\mathbf{h}_{\Lambda} = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N} ((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) = \tilde{f}_{i_1}^{m_1} (0)_{i_1} \otimes \tilde{f}_{i_2}^{m_2} (0)_{i_2} \otimes \cdots \otimes \tilde{f}_{i_N}^{m_N} (0)_{i_N} \in \mathcal{H}_{\mathbf{i}},$$

where note that for $\Lambda = \sum_i a_i \Lambda_i$, one has $\mathbf{h}_{\Lambda} = \sum_i a_i \mathbf{h}_{\Lambda_i}$.

By this proposition, one observes that there would exist a certain correspondence

(6.1)
$$\mathbf{M}(w_0\Lambda,\Lambda) = \widetilde{F}_{i_1}^{m_1}\cdots\widetilde{F}_{i_l}^{m_l}\mathbf{1} \quad \longleftrightarrow \quad \mathbf{h}_{\Lambda} = \widetilde{f}_{i_1}^{m_1}\cdots\widetilde{f}_{i_N}^{m_N}((0)_{i_1}\otimes(0)_{i_2}\otimes\cdots\otimes(0)_{i_N}).$$

Definition 6.3 ([5]). For $\beta \in Q_+$, define a central element in $R(\beta)$ by

 $\mathfrak{p}_i := \sum_{\nu \in I^\beta} \left(\prod_{a \in \{1, 2, \cdots, \operatorname{ht}(\beta)\}, \nu_a = i} x_a \right) e(\nu) \in R(\beta).$ For a simple $M \in R(\beta)$ -gmod, define an *affinization* \widehat{M} of M with degree d:

- There is an endomorphism z : M → M of degree d > 0 such that M is finitely generated free module of k[z] and M/zM ≅ M.
- (2) $\mathfrak{p}_i M \neq 0$ for any $i \in I$.

Theorem 6.4 ([5, Theorem 3.26]). For any $\Lambda \in P_+$ and $w \in W$, the determinantial module $\mathbf{M}(w\Lambda, \Lambda)$ is a real simple module and admits an affinization $\widehat{\mathbf{M}}(w\Lambda, \Lambda)$.

Note that indeed, if g is simply-laced, then the affinization \widehat{M} always exists for any simple $M \in R(\beta)$ -gmod as ([3]),

$$M = \mathbf{k}[z] \otimes_{\mathbf{k}} M.$$

6.2. Localization.

Definition 6.5 ([5]). Let *M* be a simple *R*-module. A graded braider (M, R_M, ϕ) is *non-degenerate* if $R_M(L(i)) : M \circ L(i) \to L(i) \circ M$ is a non-zero homomorphism.

For *R*-gmod, there exists a non-degenerate real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}([5])$. Set $C_{\Lambda} := \mathbf{M}(w_0 \Lambda, \Lambda)$ and denote C_{Λ_i} by C_i .

Proposition 6.6 ([8]). For $\Lambda = \sum_{i} m_i \Lambda_i \in P_+$, we obtain the following isomorphism up to grading shift:

(6.2)
$$C_{\Lambda} := \mathbf{M}(w_0\Lambda, \Lambda) \cong C_1^{\circ m_1} \circ \cdots \circ C_n^{\circ m_n}$$

Theorem 6.7 ([5, Proposition 5.1]). Define the function $\phi_i : Q \to \mathbb{Z}$ by

$$\phi_i(\beta) := -(\beta, w_0 \Lambda_i + \Lambda_i).$$

Then there exists $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$ a non-degenerate real commuting family of graded braiders of the monoidal category *R*-gmod.

Now, we take $\Gamma = P = \bigoplus_i \mathbb{Z}\Lambda_i$ and $\Gamma_+ = P_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\Lambda_i$. Here, we obtain the localization R-gmod $[C_i^{\circ-1} | i \in I]$ by $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$, which will be denoted by \widetilde{R} -gmod.

By the above Proposition, it holds the following properties:

Proposition 6.8 ([5]). Let Φ : *R*-gmod $\rightarrow \widetilde{R}$ -gmod be the canonical functor. Then,

- (1) \widetilde{R} -gmod is an abelian category and the functor Φ is exact.
- (2) For any simple object $S \in R$ -gmod, $\Phi(S)$ is simple in \widetilde{R} -gmod.
- (3) $\widetilde{C}_i := \Phi(C_i) \ (i \in I)$ is invertible central graded braider in \widetilde{R} -gmod.

For $\mu \in P$, define \widetilde{C}_{μ} such that $\widetilde{C}_{\mu} := \Phi(C_{\mu})$ for $\mu \in P_+$, $\widetilde{C}_{-\Lambda_i} = C_i^{\circ-1}$ and $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_{\lambda} \circ \widetilde{C}_{\mu}$ for $\lambda, \mu \in P$ up to grading shift.

(4) Any simple object in \widetilde{R} -gmod is isomorphic to $\widetilde{C}_{\Lambda} \circ \Phi(S)$ for some simple module $S \in R$ -gmod and $\Lambda \in P$.

Note that in (4) $\Lambda \in P$ and $S \in R$ -gmod are not necessarily unique.

Remark 6.9. In [5], the localization is applied to more general category \mathscr{C}_w , which is the full subcategory of *R*-gmod associated with a Weyl group element *w*. The category *R*-gmod here coincides with \mathscr{C}_{w_0} associated with the longest element w_0 in *W*.

Definition 6.10. The category \widetilde{R} -gmod is abelian and monoidal. Therefore, its Grothendieck ring $\mathcal{K}(\widetilde{R}$ -gmod) holds a natural $\mathbb{Z}[q, q^{-1}]$ -algebra structure, which defines a *localized quantum coordinate ring* $\widetilde{\mathcal{A}}_{q}(\widetilde{\mathfrak{n}}) := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{K}(\widetilde{R}$ -gmod).

Indeed, the Grothendieck ring $\mathcal{K}(\widetilde{R}$ -gmod) is described as follows:

Proposition 6.11 ([5, Corollary 5.4]). The Grothendieck ring $\mathcal{K}(\widetilde{R}$ -gmod) is isomorphic to the left ring of quotients of the ring $\mathcal{K}(R$ -gmod) with respect to the multiplicative set

$$\mathcal{S} := \{ q^k \prod_{i \in I} [C_i]^{a_i} \mid k \in \mathbb{Z}, \ (a_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I \},\$$

that is, $\mathcal{K}(\widetilde{R}$ -gmod) $\cong S^{-1}\mathcal{K}(R$ -gmod).

CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

7. CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

We shall mention the main theorem, crystal structure on localized quantum coordinate ring $\hat{\mathcal{A}}_{q}(\mathfrak{n})$. More precisely, we shall define a crystal structure on a family of self-dual simple objects in the category \tilde{R} -gmod (Theorem 7.4) and mention that it is isomorphic to the cellular crystal \mathbb{B}_{i} (Theorem 7.5), where **i** is a reduced word for the longest Weyl group element w_0 .

Lemma 7.1 ([4, Proposition 2.18]). For any $i \in I$, $\beta, \gamma \in Q_+$, any modules $M \in R(\beta)$ -gmod and $N \in R(\gamma)$ -gmod, one has the following exact sequence in $R(\beta + \gamma - \alpha_i)$ -gmod:

(7.1)
$$0 \longrightarrow E_i M \circ N \longrightarrow E_i (M \circ N) \longrightarrow q^{-(\alpha,\beta)} M \circ E_i N \longrightarrow 0.$$

For $i \in I$, let $i^* \in I$ be a unique index satisfying $\Lambda_{i^*} = -w_0\Lambda_i$.

- **Lemma 7.2.** (1) For $S \in R$ -gmod and $i \in I$, if $E_i S = 0$, then the module $E_i C_{\Lambda_{i^*}} \circ S$ is a simple module.
 - (2) If $E_i S = 0$ for $S \in R$ -gmod, then we get for $\Lambda \in P_+$ with $\langle h_{i^*}, \Lambda \rangle > 0$,

(7.2)
$$\operatorname{soc}(E_i(C_{\Lambda} \circ S)) \cong C_{\Lambda - \Lambda_{i^*}} \circ (E_iC_{i^*} \circ S),$$

up to grading shift.

We set

 $\mathbb{B}(\widetilde{R}\text{-gmod}) := \{L \mid L \text{ is a self-dual simple module in } \widetilde{R}\text{-gmod}\}.$

Lemma 7.3 ([5]). For any simple $L \in \widetilde{R}$ -gmod, there exists a unique $n \in \mathbb{Z}$ such that $q^n L$ is self-dual simple. For a simple module $L \in \widetilde{R}$ -gmod we define $\delta(L)$ to be this integer n.

Then by this lemma, we find that $\mathbb{B}(\widetilde{R}\operatorname{-gmod})$ includes all simple modules in $\widetilde{R}\operatorname{-gmod}$ up to grading shift. For a simple object $\widetilde{C}_{\Lambda} \circ \Phi(S) \in \widetilde{R}\operatorname{-gmod}$ we write simply $C_{\Lambda} \circ S$ if there is no confusion.

Now let us define the Kashiwara operators \widetilde{F}_i and \widetilde{E}_i $(i \in I)$ on $\mathbb{B}(\widetilde{R}$ -gmod) by

(7.3)
$$\widetilde{F}_i(C_{\Lambda} \circ S) = q^{\delta(C_{\Lambda} \circ \overline{F}_i S)} C_{\Lambda} \circ \widetilde{F}_i S$$

(7.4)
$$\widetilde{E}_{i}(C_{\Lambda} \circ S) = \begin{cases} q^{\delta(C_{\Lambda} \circ \widetilde{E}_{i}S)} C_{\Lambda} \circ \widetilde{E}_{i}S & \text{if } E_{i}S \neq 0, \\ q^{\delta(C_{\Lambda - \Lambda_{i^{*}}} \circ (\widetilde{E}_{i}C_{\Lambda_{i^{*}}} \circ S))} C_{\Lambda - \Lambda_{i^{*}}} \circ (\widetilde{E}_{i}C_{\Lambda_{i^{*}}} \circ S) & \text{if } E_{i}S = 0, \end{cases}$$

where $C_{\Lambda} \circ S$ is a self-dual simple module in \widetilde{R} -gmod, the actions $\widetilde{E}_i S$ and $\widetilde{F}_i S$ are given in (4.2), which is defined on the family of all self-dual simple modules in *R*-gmod and in (7.4) the module $\widetilde{E}_i C_{\Lambda_i^*} \circ S$ is simple by Lemma 7.2. Note that for any m > 0, $\widetilde{E}_i^m (C_{\Lambda} \circ S) \neq 0$, $\widetilde{F}_i^m (C_{\Lambda} \circ S) \neq 0$. Let $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ be as in Theorem 4.4. For $C_{\Lambda} \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, we also define

(7.5)
$$\begin{aligned} \varepsilon_i(C_\Lambda \circ S) &= \varepsilon_i(\Psi(S)) - \langle h_i, w_0 \Lambda \rangle, \quad \operatorname{wt}(C_\Lambda \circ S) = \operatorname{wt}(\Psi(S)) + w_0 \Lambda - \Lambda, \\ \varphi_i(C_\Lambda \circ S) &= \varepsilon_i(\Psi(C_\Lambda \circ S)) + \langle h_i, \operatorname{wt}(C_\Lambda \circ S) \rangle. \end{aligned}$$

Theorem 7.4. The 6-tuple $(\mathbb{B}(\widetilde{R}$ -gmod), wt, $\{\varepsilon_i\}, \{\varphi_i\}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\})_{i \in I}$ is a crystal.

Here, by Proposition 6.2 we observe that there seems to exist a certain correspondence:

$$\{C_{\Lambda} \mid \Lambda \in P_{+}\} \subset R\text{-gmod} \quad \longleftrightarrow \quad \mathcal{H}_{\mathbf{i}}$$
$$C_{\Lambda} = \widetilde{F}_{i_{1}}^{m_{1}} \cdots \widetilde{F}_{i_{N}}^{m_{N}} \mathbf{1} \quad \longleftrightarrow \quad \mathbf{h}_{\Lambda} = \widetilde{f}_{i_{1}}^{m_{1}} \widetilde{f}_{i_{2}}^{m_{2}} \cdots \widetilde{f}_{i_{N}}^{m_{N}} ((0)_{i_{1}} \otimes (0)_{i_{2}} \otimes \cdots \otimes (0)_{i_{N}})$$

Together with the result of Proposition 3.13, we obtain the following:

Theorem 7.5. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$, there exists an isomorphism of crystals:

$$\widetilde{\Psi} : \mathbb{B}(\widetilde{R}\text{-gmod}) \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^{h}(\infty)$$
$$C_{\Lambda} \circ S \longmapsto \mathbf{h}_{\Lambda} + \Psi(S) \in B^{\mathbf{h}_{\Lambda}}(\infty),$$

where $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ is the isomorphism of crystals given in Theorem 4.4, *S* is simple in $\mathbb{B}(R\text{-gmod})$ and for $\Lambda = \sum_i a_i \Lambda_i$ set $\mathbf{h}_{\Lambda} = \sum_i a_i \mathbf{h}_i$.

8. Application and further problems

8.1. **Operator** \tilde{a} . Define the $\mathbb{Q}(q)$ -linear anti-automorphism \star of $U_q(\mathfrak{g})$ by

$$(q^h)^\star = q^{-h}, \quad e_i^\star = e_i, \quad f_i^\star = f_i.$$

Theorem 8.1 ([2]). Set $L^{*}(\infty) := \{u^{*} \mid u \in L(\infty)\}, B^{*}(\infty) := \{b^{*} \mid b \in B(\infty)\}$. Then we have

$$L^{\star}(\infty) = L(\infty), \quad B^{\star}(\infty) = B(\infty).$$

From the proof of Theorem 5.13 in [5] we get

Proposition 8.2 ([5]). For $v = (v_1, v_2, \dots, v_{m-1}, v_m) \in I^{\beta}$ $(m := |\beta|)$ set $\overline{v} = (v_m, v_{m-1}, \dots, v_2, v_1)$. Define the automorphism \mathfrak{a} on $R(\beta)$ by

$$\mathfrak{a}(e(v)) = e(\overline{v}), \quad \mathfrak{a}(x_i e(v)) = x_{m-i+1} e(\overline{v}), \quad \mathfrak{a}(\tau_j e(v)) = -\tau_{m-j} e(\overline{v})$$

Then, there exists the functor $\mathfrak{a} : R$ -gmod $\to R$ -gmod such that $\mathfrak{a}(C_i) = C_{i^*}$ ($\forall i \in I$), $\mathfrak{a}^2 \cong i\mathfrak{d}$ and $\mathfrak{a}(X \circ Y) \cong \mathfrak{a}(Y) \circ \mathfrak{a}(X)$ for $X, Y \in R$ -gmod. Furthermore, it is extended to the functor $\tilde{\mathfrak{a}} : \widetilde{R}$ -gmod $\to \widetilde{R}$ -gmod which satisfies

(8.1) $\tilde{\mathfrak{a}}^2 \cong \mathrm{id}, \quad \mathrm{and} \quad \tilde{\mathfrak{a}}(X \circ Y) \cong \tilde{\mathfrak{a}}(Y) \circ \tilde{\mathfrak{a}}(X) \quad \mathrm{for} \ X, Y \in \widetilde{R}\text{-gmod}.$

Note that $\mathfrak{a}(\text{resp.} \tilde{\mathfrak{a}})$ induces the operation \star on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}}_q(\mathfrak{n})$) since $\mathfrak{a}(L(i)) = L(i)$ and then one has $\mathfrak{a}(f_i) = f_i$ (resp. $\tilde{\mathfrak{a}}(f_i) = f_i$) on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}}_q(\mathfrak{n})$). Now, we obtain the following:

Proposition 8.3. Let $\tilde{a} : \widetilde{R}$ -gmod $\rightarrow \widetilde{R}$ -gmod be the functor as above. It yields

(8.2)
$$\tilde{\mathfrak{a}}(\mathbb{B}(\widetilde{R}\operatorname{-gmod})) = \mathbb{B}(\widetilde{R}\operatorname{-gmod}).$$

Here note that Proposition 8.3 can be seen as a generalization of Theorem 8.1.

Since as crystals $\mathbb{B}(\overline{R}$ -gmod) $\cong \mathbb{B}_i$ for any reduced longest word i, the proposition above gives rise to the following problem.

Problem 1. Can we describe \tilde{a} -operation on $\mathbb{B}_{i} = B_{i_1} \otimes \cdots \otimes B_{i_N}$ explicitly?

Of course, this problem is non-trivial since even for the case $B(\infty)$ the explicit description has not yet been done before in \mathbb{B}_i .

8.2. **Category** \mathcal{C}_w . In [5], it has been shown that for an arbitrary symmetrizable Kac-Moody Lie algebra and any Weyl group element $w \in W$, there exists a subcategory $\mathcal{C}_w \subset R$ -gmod and it admits a localization

$$\mathscr{C}_{w} = \mathscr{C}_{w}[C_{i}^{\circ-1} | i \in I], \quad (C_{i} = M(w\Lambda_{i}, \Lambda_{i}))$$

Indeed, note that for finite type Lie algebra setting, $\mathscr{C}_{w_0} = R$ -gmod.

Problem 2. We conjecture that the localization \mathscr{C}_w possess a crystal $\mathbb{B}(\mathscr{C}_w)$. If so, we also conjecture that there is an isomorphism of crystals

$$\mathbb{B}(\widetilde{\mathscr{C}_w}) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m},$$

where $i_1 \cdots i_m$ is a reduced word of w.

8.3. Rigidity.

Definition 8.4. Let *X*, *Y* be objects in a monoidal category \mathcal{T} , and $\varepsilon : X \otimes Y \to 1$ and $\eta : 1 \to Y \otimes X$ morphisms in \mathcal{T} . We say that a pair (*X*, *Y*) is *dual pair* or *X* is a *left dual* to *Y* or *Y* is a *right dual* to *X* if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\operatorname{Id} \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes \operatorname{Id}} 1 \otimes X \simeq X, \ Y \simeq 1 \otimes Y \xrightarrow{\eta \otimes \operatorname{Id}} Y \otimes X \otimes Y \xrightarrow{\operatorname{Id} \otimes \varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathcal{D}(X)$ and a left dual to X by $\mathcal{D}^{-1}(X)$.

Theorem 8.5 ([5]). For any finite type R, \tilde{R} -gmod is rigid, i.e., every object in \tilde{R} -gmod has left and right duals.

Note that in [6], it is shown that for any symmetrizable Kac-Moody setting the localized category $\widetilde{\mathscr{C}}_{w}$ is rigid.

Problem 3. For a simple object $C_{\Lambda} \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, describe the right and left duals explicitly: $\widetilde{\Psi}(\mathcal{D}(C_{\Lambda} \circ S)), \quad \widetilde{\Psi}(\mathcal{D}^{-1}(C_{\Lambda} \circ S)) \in \mathbb{B}_{i}.$

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