On flagged K-theoretic symmetric polynomials

By

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Abstract

We present a fermionic description of flagged skew Grothendieck polynomials, which can be seen as a K-theoretic version of flagged skew Schur polynomial. Our proof depends on the Jacobi-Trudi type formula proved by Matsumura. This result generalizes the author's previous result of a fermionic description for skew Grothendieck polynomials.

§ 1. Introduction

§ 1.1. Overview

Grothendieck polynomials [6] are a family of polynomials which represent the structure sheaf of a Schubert variety in the K-theory of the flag variety. As each Schubert variety is naturally associated with a permutation, Grothendieck polynomials are parametrized by permutations.

A flagged Grothendieck polynomial is a Grothendieck polynomial that associates with a vexillary permutation. As a K-theoretic analog of the flagged Schur polynomials, the flagged Grothendieck polynomials have various interesting combinatorial and algebraic properties. Knuston-Miler-Yong [5] showed that the flagged Grothendieck polynomial can be seen as a generating function of flagged set-valued tableaux. Hudson-Matsumura [2] proved a Jacobi-Trudi type formula for them.

For a permutation $w \in S_n$, the inversion set (see [7, 10]) of w is defined as $I_i(w) = \{j \mid i < j \text{ and } w(i) > w(j)\} \subset \{1, 2, \dots, n\}$. The permutation w is called *vexillary* if the

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family $\{I_i(w)\}_{i=1,2,\dots,n}$ forms a chain by inclusion. For a vexillary permutation w, we associate a partition $\lambda(w)$ by arranging the cardinalities of the inversion sets. A flagging of w is the increasing sequence obtained by arranging $\min I_i(w)-1$ in increasing order. The flagged Grothendieck polynomial $G_w(x)$ is also written as $G_{\lambda,f}(x)$, where $\lambda=\lambda(w)$ and f is the flagging of w.

In the work [8], Matsumura introduced a generalization of these functions associated to a skew shape λ/μ with a flagging f/g, where $f=(f_1,\dots,f_r)$ and $g=(g_1,\dots,g_r)$ are sequences of natural numbers. He proved that the flagged skew Grothendieck polynomials, which are defined as a generating function of flagged skew set-valued tableaux, admits a Jacobi-Trudi type formula. For $n,p,q\in\mathbb{Z}$, define $G_n^{[p/q]}(x)$ by the generating function

$$\sum_{n\in\mathbb{Z}} G_n^{[p/q]}(x)z^n = \begin{cases} \frac{1}{1+\beta u^{-1}}\prod_{k=q}^{p}\frac{1+\beta x_k}{1-x_k u} & (p\geq q)\\ \frac{1}{1+\beta u^{-1}} & (p<q)\end{cases}. \tag{1.1}$$

Matsumura's determinant formula [8, §4] is given as

$$G_{\lambda/\mu,f/g}(x)=\det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s G^{[f_i/g_j]}_{\lambda_i-\mu_j-i+j+s}(x)\right). \tag{1.2}$$

In this paper, however, we adopt the slightly different definition

$$\sum_{n\in\mathbb{Z}} G_n^{[[p/q]]}(x)z^n = \begin{cases} \frac{1}{1+\beta u^{-1}}\prod_{k=q}^{p}\frac{1+\beta x_k}{1-x_k u} & (p\geq q)\\ \frac{1}{1+\beta u^{-1}} & (p=q-1)\\ \frac{1}{1+\beta u^{-1}}\prod_{k=p+1}^{q-1}\frac{1-x_k u}{1+\beta x_k} & (p<q-1)\end{cases} \tag{1.3}$$

and consider the polynomial

$$G_{\lambda/\mu,[[f/g]]}(x)=\det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s G^{[[f_i/g_j]]}_{\lambda_i-\mu_j-i+j+s}(x)\right). \tag{1.4}$$

These polynomials are different in general but coincide with each other if $f_i+\lambda_i-i\geq g_j+\mu_j-j$ whenever $f_i<g_j-1$. In particular, if $g_1=g_2=\cdots=g_r=1$, we have $G_{\lambda/\mu,f/g}(x)=G_{\lambda/\mu,[[f/g]]}(x)$ for any skew shape λ/μ.

Our aim is to construct a new algebraic description of $G_{\lambda/\mu,[[f/g]]}(x)$ by using the vertex operators acting on the fermion Fock space. In the previous work [3, §4], the author of the paper presented a fermionic description of skew Grothendieck polynomials. Generalizing this method, we show the main theorem (Theorem 3.2) that presents a fermionic description for the flagged Grothendieck polynomial.

§ 2. Preliminaries

§ 2.1. Fermion Fock space

Let $\mathcal{A}$ be the $\mathbb{C}$-algebra generated by the *free fermions* ψ_n, ψ_n^* ($n \in \mathbb{Z}$) with anti-commutation relations

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \qquad [\psi_m, \psi_n^*]_+ = \delta_{m,n},$$

where $[A, B]_+ = AB + BA$ is the anti-commutator.

Let $\mathcal{F} = \mathcal{A} \cdot |0\rangle$ be the *Fock space*, the left $\mathcal{A}$-module generated by the *vacuum vector*

$$\psi_m|0\rangle = \psi_n^*|0\rangle = 0, \quad m < 0,\ n \geq 0.$$

We also use the *dual Fock space* $\mathcal{F}^* := \langle 0| \cdot \mathcal{A}$, the right $\mathcal{A}$-module generated by the *dual vacuum vector*

$$\langle 0|\psi_n = \langle 0|\psi_m^* = 0, \quad m < 0,\ n \geq 0.$$

There uniquely exists an anti-algebra involution on $\mathcal{A}$

$$^* : \mathcal{A} \to \mathcal{A}; \quad \psi_n \leftrightarrow \psi_n^*,$$

satisfying $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$, which induces the $\mathbb{C}$-linear involution

$$\omega : \mathcal{F} \to \mathcal{F}^*, \quad X|0\rangle \mapsto \langle 0|X^*.$$

The *vacuum expectation value* [9, §4.5] is the unique $\mathbb{C}$-bilinear map

$$\mathcal{F}^* \otimes_k \mathcal{F} \to k, \quad \langle w| \otimes |v\rangle \mapsto \langle w|v\rangle, \tag{2.1}$$

satisfying $\langle 0|0\rangle = 1$, $(\langle w|\psi_n)|v\rangle = \langle w|(\psi_n|v\rangle)$, and $(\langle w|\psi_n^*)|v\rangle = \langle w|(\psi_n^*|v\rangle)$. For any expression X, we write $\langle w|X|v\rangle := (\langle w|X)|v\rangle = \langle w|(|X|v\rangle)$. The expectation value $\langle 0|X|0\rangle$ is often abbreviated as $\langle X\rangle$.

Theorem 2.1 (Wick's theorem (see [1, §2], [9, Exercise 4.2])). *Let $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_r\}$ be sets of integers. Then we have*

$$\langle \psi_{m_1} \cdots \psi_{m_r} \psi_{n_r}^* \cdots \psi_{n_1}^* \rangle = \det(\langle \psi_{m_i} \psi_{n_j}^* \rangle)_{1 \leq i,j \leq r}.$$

For an integer m, we define the *shifted vacuum vectors* $|m\rangle \in \mathcal{F}$ and $\langle m| \in \mathcal{F}^*$ by

$$|m\rangle = \begin{cases} \psi_{m-1}\psi_{m-2} \cdots \psi_0|0\rangle, & m \geq 0, \\ \psi_m^* \cdots \psi_{-2}^* \psi_{-1}^*|0\rangle, & m < 0, \end{cases} \qquad \langle m| = \begin{cases} \langle 0|\psi_0^* \psi_1^* \ldots \psi_{m-1}^*, & m \geq 0, \\ \langle 0|\psi_{-1}\psi_{-2} \ldots \psi_m, & m < 0. \end{cases}$$

§ 2.2. Vertex operators and commutation relations

For any monomial expression M in ψ_n and ψ_n^*, the *normal ordering*

$$:M: \quad \in \mathcal{A}$$

is defined by moving the *annihilation operators*

$$\psi_m, \quad \psi_n^*, \qquad m<0,\, n\geq 0$$

to the right, and multiplying -1 for each move (See [1, §2], [9, §5.2]). For example, we have $:\psi_1\psi_1^*:=\psi_1\psi_1^*$ and $:\psi_1^*\psi_1:=-\psi_1\psi_1^*$. The normal ordering can extend naturally to the $\mathbb{C}$-linear map

$$\{\text{polynomial expressions in } \psi_n \text{ and } \psi_n^* \text{ with coefficients in } \mathbb{C}\} \to \mathcal{A}; \quad X \mapsto :X:$$

Let a_m $(m\in\mathbb{Z})$ be the *current operator* $a_m=\sum_{k\in\mathbb{Z}}:\psi_k\psi_{k+m}^*:$, which satisfies

$$[a_m,a_n]=m\delta_{m+n,0}, \qquad [a_m,\psi_n]=\psi_{n-m}, \qquad [a_m,\psi_n^*]=-\psi_{n+m}^*, \tag{2.2}$$

where $[A,B]=AB-BA$. (see [9, §5.3].) If $|v\rangle)=\langle v^*|$, we have $\omega(a_i|v\rangle)=\langle v^*|a_{-i}$ for any $n\in\mathbb{Z}$.

Let $X=(X_1,X_2,\dots)$ be a set of (commutative) variables. We define the *Hamiltonian operator*

$$H(X)=\sum_{n>0}\frac{p_n(X)}{n}a_n, \qquad p_n(X)=X_1^n+X_2^n+\cdots$$

and its dual

$$H^*(X)=\sum_{n>0}\frac{p_n(X)}{n}a_{-n}.$$

We define the *vertex operators* by

$$e^{H(X)}=\sum_{n=0}^{\infty}\frac{H(X)^n}{n!}, \qquad e^{H^*(X)}=\sum_{n=0}^{\infty}\frac{H^*(X)^n}{n!}$$

Let $\psi(z)=\sum_{n\in\mathbb{Z}}\psi_n z^n$ and $\psi^*(z)=\sum_{n\in\mathbb{Z}}\psi_n^* z^n$ be the fermion fields. Here, we enumerate some important commutation relations:

$$e^{H(X)}\psi(z)e^{-H(X)}=\left(\prod_i\frac{1}{1-X_iz}\right)\psi(z), \tag{2.3}$$

$$e^{H^*(X)}\psi(z)e^{-H^*(X)}=\left(\prod_i\frac{1}{1-X_iz^{-1}}\right)\psi(z), \tag{2.4}$$

$$e^{H(X)}e^{H^*(Y)}=\left(\prod_{i,j}\frac{1}{1-X_iY_j}\right)e^{H^*(Y)}e^{H(X)}, \tag{2.5}$$

$$\langle -r|\psi^*(w)\psi(z)|-r\rangle=\frac{z^{-r}w^{-r}}{1-zw}, \tag{2.6}$$

where $\frac{z^{-r}w^{-r}}{1-zw} = \sum_{p=-r}^{\infty} z^p w^p$.

§ 3. Flagged Skew Grothendieck polynomial

§ 3.1. $G_n^{[[f/g]]}(x)$

We can relate the commutation relations of the vertex operators defined in the previous section with the generating function of $G_n^{[[f/g]]}(x)$ (Eq. (1.3)). For brevity, we adopt the convention

$$\prod_{k=q}^{p} X_k = \begin{cases} X_q X_{q+1} \cdots X_p & (p \geq q) \\ 1 & (p = q-1) \\ X_{p+1}^{-1} X_{p+2}^{-1} \cdots X_{q-1}^{-1} & (p < q-1) \end{cases}$$

for a sequence $X_1, X_2, \dots$ of (commutative) functions.

Let $x^{[f]} = (x_1, x_2, \dots, x_f)$ and

$$H(x^{[f/g]}) = H(x^{[f]}) - H(x^{[g-1]}).$$

If $f \geq g$, $H(x^{[f/g]})$ coincides with $H(x_g, x_{g+1}, \dots, x_f)$. From (2.3–2.5), we have

$$e^{H(x^{[f/g]})}\psi(z)e^{-H^*(-\beta)} = \left(\frac{1}{1+\beta z^{-1}} \prod_{j=g}^{f} \frac{1+\beta x_i}{1-x_i z} \right) e^{-H^*(-\beta)}\psi(z)e^{H(x^{[f/g]})},$$

where the rational function on the right hand side expands in the ring[1]

$$\mathbb{C}[x_1, x_2, \dots]((z))[[\beta]].$$

Comparing this equation to (1.3), we obtain

$$e^{H(x^{[f/g]})}\psi(z)e^{-H^*(-\beta)} = \left(\sum_{n \in \mathbb{Z}} G_n^{[[f/g]]}(x) z^n \right) e^{-H^*(-\beta)}\psi(z)e^{H(x^{[f/g]})}. \tag{3.1}$$

A similar calculation leads

$$e^{H^*(-\beta)}\psi^*(w)e^{-H(x^{[f/g]})} = \left(\sum_{n \in \mathbb{Z}} G_n^{[[f/g]]}(x) w^{-n} \right)^{-1} e^{-H(x^{[f/g]})}\psi^*(w)e^{H^*(-\beta)}. \tag{3.2}$$

[1]Note that the two rings $\mathbb{C}((z))[[\beta]]$ and $\mathbb{C}[[\beta]]((z))$ are different. In fact, $\mathbb{C}((z))[[\beta]]$ contains

$$1 + \frac{\beta}{z} + \frac{\beta^2}{z^2} + \cdots,$$

while $\mathbb{C}[[\beta]]((z))$ does not.

Lemma 3.1. $G_n^{[[f/g]]}(x)$ *admits the fermionic description*

$$G_n^{[[f/g]]}(x) = \langle 0|e^{H(x^{[f/g]})}\psi_{n-1}e^{-H^*(-\beta)}|-1\rangle.$$

Proof. Let $F_n = \langle 0|e^{H(x^{[f/g]})}\psi_{n-1}e^{-H^*(-\beta)}|-1\rangle$. Since $\langle 0|e^{H^*(-\beta)} = \langle 0|$ and $e^{H(x^{[f/g]})}|-1\rangle = |-1\rangle$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} F_n z^{n-1} &= \langle 0|e^{H(x^{[f/g]})}\psi(z)e^{-H^*(-\beta)}|-1\rangle \\ &= \left(\sum_{m\in\mathbb{Z}} G_m^{[f/g]}(x)z^m\right)\langle 0|e^{-H^*(-\beta)}\psi(z)e^{H(x^{[f/g]})}|-1\rangle \\ &= \left(\sum_{m\in\mathbb{Z}} G_m^{[f/g]}(x)z^m\right)\langle 0|\psi(z)|-1\rangle.\end{aligned}$$

As $\langle 0|\psi(z)|-1\rangle = \langle 0|\psi(z)\psi^*_{-1}|0\rangle = z^{-1}$, we conclude $F_n = G_n^{[[f/g]]}(x)$. □

§ 3.2. Fermionic description

We introduce a fermionic presentation of skew Flagged Grothendieck polynomial in this section. For a sequence of noncommutative elements $P_1, P_2, \ldots,$ denote

$$\prod_{i:1\to N} P_i := P_1P_2\cdots P_N, \qquad \prod_{i:N\to 1} P_i := P_N\cdots P_2P_1.$$

For any X, Y, we use the notation

$$\mathrm{Ad}_{e^X}(Y) = e^X Y e^{-X}.$$

Let $f = (f_1, f_2, \ldots, f_r)$ and $g = (g_1, g_2, \ldots, g_r)$ be sequences of positive integers. Let $G_{\lambda/\mu,[[f/g]]}(x)$ be the polynomial defined by the determinantal formula (1.4). The following is the main theorem of the paper:

Theorem 3.2. *Let λ/μ be a skew partition. Then, the flagged skew Grothendieck polynomial $G_{\lambda/\mu,[[f/g]]}(x)$ is expressed as*

(3.3)
$$\langle -r|\left(\prod_{j:r\to 1}\psi^*_{\mu_j-j}e^{H^*(-\beta)}e^{-H(x^{[g_j-1/g_{j-1}]})}\right)\left(\prod_{i:1\to r}e^{H(x^{[f_i/f_{i-1}+1]})}\psi_{\lambda_i-i}e^{-H^*(-\beta)}\right)|-r\rangle.$$

Proof. By using the equations

$$e^{rX}\prod_{i:1\to r}P_i = \prod_{i:1\to r}(e^{(r-i+1)X}P_ie^{-(r-i)X}) = \prod_{i:1\to r}\mathrm{Ad}_{e^{(r-i)X}}(e^XP_i),$$

$$\left(\prod_{j:r\to 1}P_j\right)e^{-rX} = \prod_{j:r\to 1}(e^{(r-j)X}P_je^{-(r-j+1)X}) = \prod_{j:r\to 1}\mathrm{Ad}_{e^{(r-j)X}}(P_je^{-X}),$$

the expectation value (3.3) is rewritten as

(3.4)

$$\langle -r|e^{-H(x^{[g_r-1]})}\left(\prod_{j:r\to 1}\mathrm{Ad}_{e^{(r-j-1)H^*(-\beta)}}\left(e^{H(x^{[g_j-1]})}\psi^*_{\mu_j-j}e^{H^*(-\beta)}e^{-H(x^{[g_j-1]})}e^{-H^*(-\beta)}\right)\right)$$
$$\cdot\left(\prod_{i:1\to r}\mathrm{Ad}_{e^{(r-i)H^*(-\beta)}}\left(e^{H^*(-\beta)}e^{H(x^{[f_i]})}\psi_{\lambda_i-i}e^{-H^*(-\beta)}e^{-H(x^{[f_i]})}\right)\right)|-r\rangle.$$

Let

$$A_i(z_i) := \mathrm{Ad}_{e^{(r-i)H^*(-\beta)}}\left(e^{H^*(-\beta)}e^{H(x^{[f_i]})}\psi(z_i)e^{-H^*(-\beta)}e^{-H(x^{[f_i]})}\right)$$
$$B_j(w_j) := \mathrm{Ad}_{e^{(r-j-1)H^*(-\beta)}}\left(e^{H(x^{[g_j-1]})}\psi^*(w_j)e^{H^*(-\beta)}e^{-H(x^{[g_j-1]})}e^{-H^*(-\beta)}\right)$$

Then, by Wick's theorem (Theorem 2.1), the expectation value (3.4) equals to the coefficient of $z_1^{\lambda_1-1}\cdots z_r^{\lambda_r-r}\cdot w_1^{\mu_1-1}\cdots w_r^{\mu_r-r}$ of the determinant

$$\det\left(\langle -r|e^{-H(x^{[g_r-1]})}B_j(w_j)A_i(z_i)|-r\rangle\right)_{i,j}.$$

From (3.1), $A_i(z_i)$ satisfies

$$\begin{aligned}A_i(z_i) &= \left(\sum_{n\in\mathbb{Z}}G_n^{[f_i]}(x)z_i^n\right)\cdot\mathrm{Ad}_{e^{(r-i)H^*(-\beta)}}(\psi(z_i))\\ &= \left(\sum_{n\in\mathbb{Z}}G_n^{[f_i]}(x)z_i^n\right)\cdot(1+\beta z_i^{-1})^{-(r-i)}\cdot\psi(z_i).\end{aligned}$$

We also have

$$\begin{aligned}B_j(w_j) &= \left(\sum_{n\in\mathbb{Z}}G_n^{[g_j-1]}(x)w_j^{-n}\right)^{-1}\cdot(1+\beta w_j)^{-1}\mathrm{Ad}_{e^{-(r-j)H^*(-\beta)}}(\psi^*(w_j))\\ &= \left(\sum_{n\in\mathbb{Z}}G_n^{[g_j-1]}(x)w_j^{-n}\right)^{-1}\cdot(1+\beta w_j)^{r-j-1}\cdot\psi^*(w_j)\end{aligned}$$

by (3.2). Therefore, we have

$$
\begin{aligned}
&\langle -r|e^{-H(x^{[g_r-1]})}B_j(w_j)A_i(z_i)|-r\rangle \\
&= \frac{\sum_n G_n^{[f_i]}(x)z_i^n}{\sum_n G_n^{[g_j-1]}(x)w_j^{-n}} \frac{(1+\beta w_j)^{r-j-1}}{(1+\beta z_i^{-1})^{r-i}} \langle -r|e^{-H(x^{[g_r-1]})}\psi^*(w_j)\psi(z_i)e^{H(x^{[g_r-1]})}|-r\rangle \\
&= \frac{\sum_n G_n^{[f_i]}(x)z_i^n}{\sum_n G_n^{[g_j-1]}(x)w_j^{-n}} \frac{(1+\beta w_j)^{r-j-1}}{(1+\beta z_i^{-1})^{r-i}} \frac{\prod_{k=1}^{g_r-1}(1-x_kz_i)}{\prod_{k=1}^{g_r-1}(1-x_kw_j^{-1})} \langle -r|\psi^*(w_j)\psi(z_i)|-r\rangle \\
&= \frac{\sum_n G_n^{[f_i]}(x)z_i^n}{\sum_n G_n^{[g_j-1]}(x)w_j^{-n}} \frac{(1+\beta w_j)^{r-j-1}}{(1+\beta z_i^{-1})^{r-i}} \frac{\prod_{k=1}^{g_r-1}(1-x_kz_i)}{\prod_{k=1}^{g_r-1}(1-x_kw_j^{-1})} \frac{z_i^{-r}w_j^{-r}}{1-z_iw_j} \\
&= \frac{\prod_{k=1}^{f_i}(1+\beta x_k)}{\prod_{k=1}^{g_j-1}(1+\beta x_k)} \frac{(1+\beta w_j)^{r-j}}{(1+\beta z_i^{-1})^{r-i+1}} \frac{\prod_{k=f_i+1}^{g_r-1}(1-x_kz_i)}{\prod_{k=g_j}^{g_r-1}(1-x_kw_j^{-1})} \frac{z_i^{-r}w_j^{-r}}{1-z_iw_j} =: F(z_i,w_j).
\end{aligned}
$$

To take the coefficient of $z_i^{\lambda_i-i}w_j^{\mu_j-j}$ on the both side, we use the complex line integral. Note that the expansion of the rational function $F(z_i,w_j)$ in the field

$$\mathbb{C}[x_1,x_2,\dots]((w_j^{-1}))((z_i))[[\beta]]$$

coincides with the Laurent expansion on the domain $\{|\beta|<|z_i|<|w_j^{-1}|<|x_k^{-1}|\}$. Then, we have

$$
\begin{aligned}
&[w_j^{\mu_j-j}]\left\langle e^{-H(x^{[g_r-1]})}B_j(w_j)A_i(z_i)e^{H(x^{[g_r-1]})}\right\rangle \\
&= \frac{1}{2\pi\sqrt{-1}}\oint_{|\beta|<|z_i|<|w^{-1}|<|x_k^{-1}|} F(z_i,w)\cdot(w^{-1})^{\mu_j-j}\frac{d(w^{-1})}{(w^{-1})} \\
&= \frac{1}{2\pi\sqrt{-1}}\oint_{|\beta|<|z_i|<|t|<|x_k^{-1}|} F(z_i,t^{-1})\cdot t^{\mu_j-j-1}dt. \qquad (3.5)
\end{aligned}
$$

Since $F(z_i,t^{-1})\cdot t^{\mu_j-j-1}dt$ expands as

$$\frac{\prod_{k=1}^{f_i}(1+\beta x_k)}{\prod_{k=1}^{g_j-1}(1+\beta x_k)} \frac{(t+\beta)^{r-j}}{(1+\beta z_i^{-1})^{r-i+1}} \frac{\prod_{k=f_i+1}^{g_r-1}(1-x_kz_i)}{\prod_{k=g_j}^{g_r-1}(1-x_kt)} \frac{z_i^{-r}}{t-z_i}\cdot t^{\mu_j}dt,$$

the contour integral (3.5) should equal to the residue of the differential form at $t=z_i$. Finally, we obtain

$$
\begin{aligned}
&[w_j^{\mu_j-j}]\left\langle e^{-H(x^{[g_r-1]})}B_j(w_j)A_i(z_i)e^{H(x^{[f_r]})}\right\rangle \\
&= \frac{1}{1+\beta z_i^{-1}}\prod_{k=g_j}^{f_i}\frac{1+\beta x_k}{1-x_kz_i}\cdot(z_i+\beta)^{i-j}z_i^{\mu_j-i} \\
&= \left(\sum_{n\in\mathbb{Z}} G_n^{[[f_i/g_j]]}(x)z_i^n\right)(1+\beta z_i^{-1})^{i-j}z_i^{\mu_j-j}.
\end{aligned}
$$

Since the coefficient of $z_i^{\lambda_i-i}$ is

$$\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s G^{[[f_i/g_j]]}_{\lambda_i-\mu_j-i+j+s}(x),$$

we conclude the theorem. $\square$

§ 3.3. Remarks

If $g_1=g_2=\cdots=g_r=1$, $G_{\lambda/\mu,[[f/g]]}(x)$ reduces to the (usual) flagged Grothendieck polynomial $G_{\lambda/\mu,f}(x)$. In this case, our main Theorem 3.2 reduces to

$$\begin{aligned}G_{\lambda/\mu,f}(x)=&\langle -r|\psi^*_{\mu_r-r}e^{H^*(-\beta)}\dots\psi^*_{\mu_2-2}e^{H^*(-\beta)}\psi^*_{\mu_1-1}e^{H^*(-\beta)}\\&\cdot(e^{H(x^{[f_1]})}\psi_{\lambda_1-1}e^{-H^*(-\beta)})\cdots(e^{H(x^{[f_r/f_{r-1}]})}\psi_{\lambda_r-r}e^{-H^*(-\beta)})|-r\rangle.\end{aligned}$$

By taking $f_1=f_2=\cdots=f_r=n$, we recover the fermionic presentation of the symmetric Grothendieck polynomial given in [3, §4.2]. This expression is *not* included in the fermionic presentation of the multi-Schur function [4].

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