A REVIEW OF RANK ONE BISPECTRAL CORRESPONDENCE OF QUANTUM AFFINE KZ EQUATIONS AND MACDONALD-TYPE EIGENVALUE PROBLEMS

KOHEI YAMAGUCHI, SHINTAROU YANAGIDA

ABSTRACT. This note consists of two parts. The first part (§1 and §2) is a partial review of the works by van Meer and Stokman (2010), van Meer (2011) and Stokman (2014) which established a bispectral analogue of the Cherednik correspondence between quantum affine Knizhnik-Zamolodchikov equations and the eigenvalue problems of Macdonald type. In this review we focus on the rank one cases, i.e., on the reduced type A_1 and the non-reduced type $(C_1^\vee, C_1)$, to which the associated Macdonald-Koornwinder polynomials are the Rogers polynomials and the Askey-Wilson polynomials, respectively. We give detailed computations and formulas that may be difficult to find in the literature. The second part (§3) is a complement of the first part, and is also a continuation of our previous study (Y.-Y., 2022) on the parameter specialization of Macdonald-Koornwinder polynomials, where we found four types of specialization of the type $(C_1^\vee, C_1)$ parameters (which could be called the Askey-Wilson parameters) to recover the type A_1. In this note, we show that among the four specializations there is only one which is compatible with the bispectral correspondence discussed in the first part.

CONTENTS

0. INTRODUCTION

This note is written for two purposes. The first purpose is to give a partial review of the bispectral correspondence [vMS09, vM11, St14] between quantum affine Knizhnik-Zamolodchikov equations and the eigenvalue problems of Macdonald type, and the review forms the major part of this text (§§ 1 and 2). The second purpose is to study the relationship between the bispectral correspondence and the parameter specialization investigated in the authors' previous study [YY22], and it is fulfilled in § 3. We summarize these two contents in the following § 0.1 and § 0.2, respectively.

Date: 2023.02.12.

Key words and phrases. Macdonald-Koornwinder polynomials, Askey-Wilson polynomials, (double) affine Hecke algebras, quantum affine Knizhnik-Zamolodchikov equations, bispectral problems.

K.Y. is supported by JSPS Fellowships for Young Scientists (No. 22J11816). S.Y. is supported by JSPS KAKENHI Grant Number 19K03399.

0.1. **Rank one review of bispectral correspondence.** The first part (§1, §2) is devoted to the review of the bispectral correspondence between QAKZ solutions and Macdonald-type eigenvalue problems, established by the works [vM11, vM11, St14].

Let us begin by recalling on the original Cherednik's correspondence. We refer to [C05, §1.3] for an exposition of this correspondence. In [C92a], Cherednik introduced his QAKZ equations for arbitrary reduced root systems and for the type GL_n. Let $H = H(k)$ be the affine Hecke algebra of type GL_n with complex parameter k. Hereafter we will call k the Hecke parameter. Also, let $T := \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$ be the algebraic torus associated with the weight lattice Λ. Then the QAKZ equations are q-difference equations for functions of the torus variable $t \in T$ valued in a (left) H-module M satisfying certain conditions. In [C92b], Cherednik constructed a correspondence between solutions of the QAKZ equations for the principal series representation M_γ with central character $\gamma \in T$, and eigenfunctions of the q-difference operators of Macdonald type.

Below we explain the correspondence for the type GL_n. In this case, we can identify $\Lambda = \mathbb{Z}^n$ and put $t = (t_1, \dots, t_n), \gamma = (\gamma_1, \dots, \gamma_n) \in T$. For a nonzero complex parameter $q \in \mathbb{C}$, which will be called the quantum parameter, let $\mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma$ be the eigenspace of the Macdonald-Ruijsenaars q-difference operators of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma := \left\{ f(t) \in \mathcal{M}(T) \mid L_p^t f(t) = p(\gamma) f(t), \ \forall p \in \mathbb{C}[T]^{\mathfrak{S}_n} \right\},$$

where $\mathcal{M}(T)$ is the set of meromorphic functions on T, and L_p^t denotes the Macdonald-Ruijsenaars q-difference operator [R87, M95] associated to each symmetric polynomial p which acts on the functions of t. For example, to the first elementary symmetric polynomial $e(z) = z_1 + \cdots + z_n$, the operator L_e^t is given by

$$L_e^t := \sum_{i=1}^n \prod_{j \neq i} \frac{k t_i - k^{-1} t_j}{t_i - t_j} T_{q, t_i}. \tag{0.1.1}$$

Here we used the q-shift operator T_{q,t_i} for $i = 1, \dots, n$:

$$(T_{q,t_i} f)(t_1, \dots, t_n) = f(t_1, \dots, q t_i, \dots, t_n), \quad f(t) \in \mathcal{M}(T).$$

Moreover, let $\mathrm{SOL}_{\mathrm{qKZ}}(k, q)_\gamma$ be the QAKZ equations of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{qKZ}}(k, q)_\gamma := \left\{ f(t) \in H_0^{\mathcal{M}(T)} \mid C_{\mathrm{t}(\lambda)}^\gamma (t) f(q^{-\lambda} t) = f(t), \ \lambda \in \Lambda \right\},$$

where $H_0 = H_0(k)$ is the finite Hecke algebra of type A_{n-1} and $H_0^{\mathcal{M}(T)} := \mathcal{M}(T) \otimes_{\mathbb{C}} H_0$. We omit the precise definition of the q-difference operators $C_{\mathrm{t}(\lambda)}^\gamma (t)$. We will explain in detail the case of type A_1 and $(C_1^\vee, C_1)$ in §1 and §2, respectively. Cherednik's correspondence for the type GL_n is now described as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{qKZ}}(k, q)_\gamma \longrightarrow \mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma. \tag{0.1.2}$$

A bispectral analogue of Cherednik's correspondence is investigated by van Meer and Stokman [vMS09] for type GL, who introduced the bispectral QAKZ equations using Cherednik's duality anti-involution $* \colon \mathbb{H} \to \mathbb{H}$ of the double affine Hecke algebra $\mathbb{H}$ (see Definition 1.1.4). The bispectral QAKZ equations are consistent systems of q-difference equations for functions on the product torus $T \times T$, and splits up into two subsystems. Denoting by $(t, \gamma) \in T \times T$ the variable, we have:

- The first subsystem only acts on t, and for a fixed γ, the equations in t are Cherednik's QAKZ equations for the principal series representation M_γ of the affine Hecke algebra $H \subset \mathbb{H}$.
- For a fixed $t \in T$, the equations in γ are essentially the QAKZ equations for $M_{t^{-1}}$ of the image $H^* \subset \mathbb{H}$.

This argument can be extended to arbitrary reduced and non-reduced root systems, as done by van Meer [vM11] for reduced types and by Takeyama [T10] for the non-reduced type $(C_n^\vee, C_n)$.

After the build-up of bispectral QAKZ equations, it is rather straightforward, except for one issue, to make an analogue of Cherednik's construction of correspondence to the bispectral eigenvalue problems of Macdonald-type. Below we explain the case of type GL_n again. Let $\mathrm{SOL}_{\mathrm{bMac}}(k, q)$ be the bispectral eigenspace of the Macdonald-Ruijsenaars q-difference operators of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{bMac}}(k, q) := \left\{ f(t, \gamma) \in \mathcal{M}(T \times T) \ \middle| \ \begin{matrix} L_p^t f(t, \gamma) = p(\gamma) f(t, \gamma) \\ L_e^\gamma f(t, \gamma) = p(t) f(t, \gamma) \end{matrix} \quad \forall p \in \mathbb{C}[T]^{\mathfrak{S}_n} \right\}$$

where $\mathcal{M}(T \times T)$ is the set of meromorphic function on $T \times T$, and L_p^t, L_p^γ denote the Macdonald-Ruijsenaars q-difference operators attached to each symmetric polynomial p, acting on functions of t and

type	Dynkin	orbits	Hecke parameters			
$(C_1^\vee, C_1)$ Askey-Wilson	$\overset{*}{\underset{0}{\circ}}\!=\!=\!\overset{*}{\underset{1}{\circ}}$	$O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$	k_0	k_1	l_0	l_1
A_1		O_1	1	t	1	t
	$\underset{0}{\circ}\!=\!=\!\underset{1}{\circ}$	O_3	t	1	t	1
Rogers		O_2	1	t^2	1	1
		O_4	t^2	1	1	1

TABLE 0.1. Type A_1 subsystems in $(C_1^\vee, C_1)$ and parameter specializations

γ, respectively. For the first elementary symmetric polynomial $e(z) = z_1 + \cdots + z_n$, they are given by

$$L_e^t := \sum_{i=1}^n \prod_{j \neq i} \frac{kt_i - k^{-1}t_j}{t_i - t_j} T_{q,t_i}, \quad L_e^\gamma := \sum_{i=1}^n \prod_{j \neq i} \frac{k^{-1}\gamma_i - k\gamma_j}{\gamma_i - \gamma_j} T_{q,\gamma_i}^{-1}.$$

Note that L_e^t is the same as (0.1.1), and the parameters q^{-1}, k^{-1} in L_p^γ are the reciprocal of those in L_p^t.

Next, let $\mathrm{SOL}_{\mathrm{bqKZ}}(k,q)$ be the solution space of the bispectral QAKZ equations of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{bqKZ}}(k,q) := \left\{ f(t,\gamma) \in H_0^{\mathcal{M}(T\times T)} \;\middle|\; \begin{array}{l} C_{(\mathrm{t}(\lambda),e)}(t,\gamma) f(q^{-\lambda}t,\gamma) = f(t,\gamma) \\ C_{(e,\mathrm{t}(\mu))}(t,\gamma) f(t,q^{\mu}\gamma) = f(t,\gamma) \end{array} \quad \forall \lambda,\mu \in \Lambda \right\},$$

where $H_0^{\mathcal{M}(T\times T)} := \mathcal{M}(T \times T) \otimes_{\mathbb{C}} H_0$. We omit the exact definitions of the q-difference operators $C_{(\mathrm{t}(\lambda),e)}(t,\gamma)$ and $C_{(e,\mathrm{t}(\mu))}(t,\gamma)$, and refer to § 1 and § 2 for the explanation for type A_1 and $(C_1^\vee, C_1)$.

Mimicking (0.1.2), the resulting bispectral correspondence is written as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(k,q) \longrightarrow \mathrm{SOL}_{\mathrm{bMac}}(k,q).$$

The issue here is the existence of (some nice) asymptotically free solutions of the bispectral QAKZ equations, i.e., the non-emptiness of the source, which was carefully proved for type GL_n in [vM11, §5, Appendix]. The same argument works with minor modifications for reduced and non-reduced root types (see [St14, §3]).

In this note, we give a review of the bispectral correspondence explained so far. Since the correspondence itself looks rather abstract, we decided to concentrate on the rank one cases and give detailed computations.

- In § 1, we consider the reduced root system of type A_1. The associated Macdonald-Koornwinder polynomials are the Rogers polynomials.
- In § 2, we consider the non-reduced root system of type $(C_1^\vee, C_1)$. The associated polynomials are the Askey-Wilson polynomials.

The GL_2 case could be included, but it is essentially the same with A_1, and we will not treat it.

0.2. **Specialization of parameters in the rank one bispectral problems.** The second part (§ 3) is a complement of the first part, and is also a continuation of the paper [YY22] on the parameter specializations of Macdonald-Koornwinder polynomials. There we classify all the specializations based on the affine root systems, which appear as subsystems of the type $(C_n^\vee, C_n)$ system. The obtained parameter specializations are compatible with degeneracies of the Macdonald-Koornwinder inner product to the subsystem inner products.

In the rank one case [YY22, §2.6], where the polynomials in question are Askey-Wilson polynomials, we discovered four ways of specializing the type $(C_1^\vee, C_1)$ parameters to recover the type A_1. Table 0.1 is an excerpt from [YY22, §2.6, Table 2].

In § 3, we study the relation between our parameter specializations and the bispectral correspondence. First, let us recall that the bispectral correspondence is constructed using the duality anti-involution $*$ of the DAHA $\mathbb{H}$. As discussed in § 2.1 (2.1.16), the duality anti-involution $*$ of $\mathbb{H}$ acts on the Hecke parameters in the way

$$(k_1^*, k_0^*, l_1^*, l_0^*) = (k_1, l_1, k_0, l_0).$$

Then, we see from Table 0.1 that the specialization corresponding to the orbit O_2 is the only one which is compatible with the bispectral correspondence reviewed in the first part. Under this specialization, we obtain the following commutative diagram (Theorem 3.1.2).

$$\begin{array}{ccc} \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)} & \xhookrightarrow{\chi_+^{(C_1^\vee,C_1)}} & \mathrm{SOL}_{\mathrm{bAW}} \\ {\scriptstyle \mathrm{sp}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{sp}} \\ \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1} & \xhookrightarrow[\chi_+^{A_1}]{} & \mathrm{SOL}_{\mathrm{bMR}} \end{array}$$

Notation and terminology. The following is the notation and terminology used throughout this note.

- We denote by $\mathbb{N} = \mathbb{Z}_{\ge 0} := \{0, 1, 2, \dots\}$ the set of non-negative integers.
- We denote by $\delta_{i,j}$ the Kronecker delta on a set $I \ni i, j$.
- We denote the unit of a group by e or 1.
- Linear spaces are those over the complex number field $\mathbb{C}$ unless otherwise stated, and we denote by $\mathrm{Hom}(V, W)$ and $\mathrm{End}(V)$ the linear spaces of $\mathbb{C}$-linear homomorphisms $V \to W$ and of endomorphisms $V \to V$. We also denote by $\otimes$ the standard tensor product $\otimes_{\mathbb{C}}$ over $\mathbb{C}$.
- A ring or an algebra means a unital associative one unless otherwise stated.
- We denote $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, regarded as the multiplicative group.
- We use the Gasper-Rahman basic hypergeometric notation [GR04] for q-shifted factorials
$$(x; q)_\infty := \prod_{n=0}^{\infty} (1 - xq^n), \quad (x_1, \dots, x_r; q)_\infty := \prod_{i=1}^{r} (x_i; q)_\infty,$$
which are understood as complex numbers if they converge (e.g., if $x, x_i, q \in \mathbb{C}$ and $|q| < 1$), and as formal series of q otherwise. For $n \in \mathbb{N}$, we set
$$(x; q)_n := \frac{(x; q)_\infty}{(xq^{n+1}; q)_\infty}, \quad (x_1, \dots, x_r; q)_n := \prod_{i=1}^{r} (x_i; q)_n. \tag{0.2.1}$$
- We also use the symbol in [GR04] of the basic hypergeometric series
$${}_{r+1}\phi_r \left[\begin{matrix} a_1, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix} ; q, \ z \right] := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. \tag{0.2.2}$$
- We will also use the q-binomial coefficient
$$\begin{bmatrix} \beta \\ n \end{bmatrix}_q := \frac{(q^{\beta-n+1}; q)_n}{(q; q)_n} \tag{0.2.3}$$
for $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$. Note that we have $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q = \frac{(q;q)_m}{(q;q)_n \, (q;q)_{m-n}}$ for $m, n \in \mathbb{N}$ with $m \ge n$.

1. Type A_1

1.1. **Extended affine Hecke algebra.**

Here we recall the extended affine Hecke algebra of type A_1 and the basic representation.

1.1.1. *The extended affine Weyl group of type A_1.* We begin by recalling the extended affine Weyl group of the affine root system of type A_1. For the details, see [M03, §1, §2, §6.1], [vMS09, §2.1] and [vM11, §2.1].

Remark 1.1.1. Let us note beforehand that we are working in the untwisted affine root system [M03, (1.4.1)], although [vM11] works in the twisted affine system [M03, (1.4.2)]. Since we only consider the type A_1, there is no essential difference, but there are some notational differences. For example, we define the extended affine Weyl group W as the semi-direct product $W_0 \ltimes \mathrm{t}(P)$ using the weight lattice P, although in [vM11] it is defined as $W_0 \ltimes \mathrm{t}(P^\vee)$ using the coweight lattice $P^\vee$.

We consider the one-dimensional real Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with
$$V = \mathbb{R}\alpha, \quad \langle \alpha, \alpha \rangle = 2. \tag{1.1.1}$$
Let F be the space of affine real functions on V, which is identified with the real vector space $V \oplus \mathbb{R}c$ by the map $(u \mapsto \langle v, u \rangle + r) \mapsto v + rc$ for $u, v \in V$ and $r \in \mathbb{R}$. Using the gradient map $D \colon F \to V$, $v + rc \mapsto v$, we extend the inner product $\langle \cdot, \cdot \rangle$ on V to a positive semi-definite bilinear form on F by $\langle f, g \rangle := \langle D(f), D(g) \rangle$ for $f, g \in F$.

Let $S(A_1) := \{\pm\alpha+nc \mid n \in \mathbb{Z}\} \subset F$ be the affine root system $S(A_1)$ in the sense of Macdonald [M03]. A basis of $S(A_1)$ is given by $\{a_1 := \alpha, a_0 := c-\alpha\}$, and the associated simple reflections $s_i \colon V \to V$ for $i = 0,1$ are given by

$$s_i(v) := v - a_i(v)D(a_i^\vee) \quad (v \in V), \tag{1.1.2}$$

where $a_i^\vee := 2a_i/\langle a_i, a_i\rangle = a_i \in F$. Explicitly, we have

$$s_1(r\alpha) = -r\alpha, \quad s_0(r\alpha) = (1-r)\alpha \qquad (r \in \mathbb{R}). \tag{1.1.3}$$

We denote by $W_0 \subset O(V, \langle\cdot,\cdot\rangle)$ the subgroup generated by s_1. It is the Weyl group of the irreducible root system $R(A_1) = \{\pm\alpha\}$ of type A_1 in the sense of Bourbaki, and as an abstract group, we have $W_0 = \langle s_1 \mid s_1^2\rangle \cong \mathfrak{S}_2$, the symmetric group of degree 2. Let us also denote the fundamental weight ϖ and the weight lattice Λ of the root system $R(A_1)$ by

$$\varpi := \tfrac{1}{2}\alpha, \quad \Lambda := \mathbb{Z}\varpi \subset V.$$

Then the W_0-action (1.1.3) preserves Λ.

We denote by $\mathrm{t}(\Lambda) := \{\mathrm{t}(\lambda) \mid \lambda \in \Lambda\}$ the abelian group with relations $\mathrm{t}(\lambda)\,\mathrm{t}(\mu) = \mathrm{t}(\lambda+\mu)$ for $\lambda, \mu \in \Lambda$. The group $\mathrm{t}(\Lambda)$ acts on V by translation:

$$\mathrm{t}(\lambda)v = v + \lambda \quad (\lambda \in L,\ v \in V). \tag{1.1.4}$$

Then *the extended affine Weyl group W of $S(A_1)$* is defined to be the semi-direct product group

$$W := W_0 \ltimes \mathrm{t}(\Lambda) \tag{1.1.5}$$

which acts on V faithfully. In other words, the group W is determined by W_0 and $\mathrm{t}(\Lambda)$, and by the additional relations

$$s_1\,\mathrm{t}(\lambda)s_1 = \mathrm{t}(s_1(\lambda)) \quad (\lambda \in \Lambda) \tag{1.1.6}$$

with $s_1(\lambda)$ given by (1.1.3).

The group W is generated by s_1, s_0 and $\mathrm{t}(\varpi)$. It is convenient to introduce $u := \mathrm{t}(\varpi)s_1$. By (1.1.6), we have $u^2 = \mathrm{t}(\varpi)\,\mathrm{t}(s_1(\varpi)) = \mathrm{t}(\varpi)\,\mathrm{t}(-\varpi) = e$. Also, by (1.1.6) and (1.1.3), we can check $s_0(v) = us_1u(v)$ for any $v \in V$. Thus, as an abstract group, W is generated by s_1, s_0, u with defining relations

$$s_1^2 = s_0^2 = u^2 = e, \quad us_1 = s_0u. \tag{1.1.7}$$

For later use, we write down a few relations in W.

$$\mathrm{t}(\varpi) = us_1 = s_0u, \quad \mathrm{t}(-\varpi) = s_1u = us_0. \tag{1.1.8}$$

$$\mathrm{t}(\alpha) = \mathrm{t}(2\varpi) = us_1us_1 = s_0s_1. \tag{1.1.9}$$

1.1.2. *The extended affine Hecke algebra of type A_1.* Here we recall the extended affine Hecke algebra H associated to the affine root system $S(A_1)$. For the detail, see [M03, §4, §6.1] and [vM11, §2.2, §2.3]. Hereafter we fix nonzero complex numbers $k \in \mathbb{C}^\times$.

Remark 1.1.2. Our parameter k correspond to τ in [M03].

Definition 1.1.3. *The extended affine Hecke algebra of type A_1*, denoted by

$$H = H(k) = H^{A_1}(k),$$

is the $\mathbb{C}$-algebra generated by T_1, T_0 and U with fundamental relations

$$(T_i - k)(T_i + k^{-1}) = 0 \quad (i = 1,0), \quad U^2 = 1, \quad UT_1 = T_0U. \tag{1.1.10}$$

By comparing (1.1.7) and (1.1.10), we see that H is a deformation of the group ring $\mathbb{C}[W]$ of the extended affine Weyl group W of $S(A_1)$ explained above.

In order to attach an element $T_w \in H$ to each $w \in W$, let us recall from [M03, §2.2] that we have the length function and reduced expressions in W. The group W is an extension of the affine Weyl group $W_S := \langle s_1, s_0 \mid s_1^2, s_0^2\rangle$ of $S(A_1)$ by the automorphism u of the Dynkin diagram of $S(A_1)$, so that any element $w \in W$ can be written as $w = w'u^r$ with $w' \in W_S$ and $r \in \{0,1\}$. The group W_S is a Coxeter group, so that it has the length function $\ell(\cdot)$ and reduced expression of each element. Now, let $w' = s_{i_1}\cdots s_{i_l}$ be a reduced expression in W_S with $l = \ell(w')$. Then we define the length of $w \in W$ to be $\ell(w) := \ell(w') = l$, and call the expression $w = s_{i_1}\cdots s_{i_l}u^r \in W$ a reduced expression of w.

Now, for $w \in W$, take a reduced expression $w = s_{i_1}\cdots s_{i_l}u^r$ and define

$$T_w := T_{i_1}\cdots T_{i_l}U^r \in H.$$

Then T_w is independent of the choice of reduced expression. By convention we have $T_e = 1$, the unit of the ring H.

Next we introduce *the Dunkl operator* to be

$$Y := UT_1 \in H. \tag{1.1.11}$$

By (1.1.10), Y is invertible and

$$Y^{-1} = T_1^{-1}U = (T_1 - k + k^{-1})U.$$

Also note that these can be regarded as deformations of the translations $\mathrm{t}(\pm\varpi) \in W$ given in (1.1.8). Let us also define

$$Y^\lambda := Y^l \in H \quad (\lambda = l\varpi \in \Lambda,\ l \in \mathbb{Z}).$$

In particular, we have

$$Y^\alpha = Y^{2\varpi} = Y^2 = UT_1UT_1 = T_0T_1, \tag{1.1.12}$$

which corresponds to (1.1.9). We denote by $\mathbb{C}[Y^{\pm 1}] \subset H$ the ring of Laurent polynomials in Y. We have an isomorphism of $\mathbb{C}$-linear spaces

$$H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}], \tag{1.1.13}$$

where

$$H_0 = H_0(k) := \mathbb{C}T_e + \mathbb{C}T_{s_1} = \mathbb{C} + \mathbb{C}T_1 \tag{1.1.14}$$

is the subalgebra of H generated by T_1. We call H_0 *the finite Hecke algebra of type A_1*.

1.1.3. *The basic representation and the double affine Hecke algebra of type A_1.* Next, we review the basic representation of the extended affine Hecke algebra $H = H(k)$, mainly following [M03, §6.1]. See also [C05, Theorem 3.2.1] and references therein.

Below we choose and fix a parameter $q^{1/2} \in \mathbb{C}^\times$. The extended affine Weyl group W acts on the ring of Laurent polynomials

$$\mathbb{C}[x^{\pm 1}], \quad x := e^{\varpi} = e^{\alpha/2} \tag{1.1.15}$$

by letting the generators s_1, s_0, u operate as

$$(s_{1,q}f)(x) = f(x^{-1}), \quad (s_{0,q}f)(x) = f(qx^{-1}), \quad (u_q f)(x) = f(q^{1/2}x^{-1}), \tag{1.1.16}$$

where we indicated the dependence on q explicitly.

Now, using the parameter $k \in \mathbb{C}^\times$, and define $b(x;k), c(x;k) \in \mathbb{C}(x)$ by

$$c(x;k) := \frac{k^{-1} - kx}{1-x}, \quad b(x;k) := k - c(x;k) = \frac{k - k^{-1}}{1-x}. \tag{1.1.17}$$

Then, denoting $x_1 := x^2$ and $x_0 := qx^{-2}$, we have an algebra embedding

$$\rho_{k,q} \colon H(k) \longrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}]), \tag{1.1.18}$$

$$\rho_{k,q}(T_i) := c(x_i;k)s_{i,q} + b(x_i;k) = k + c(x_i;k)(s_{i,q} - 1), \quad \rho_{k,q}(U) := u_q. \tag{1.1.19}$$

Note that the image is in $\mathrm{End}(\mathbb{C}[x^{\pm 1}]) \subsetneq \mathrm{End}(\mathbb{C}(x))$. We call $\rho_{k,q}$ *the basic representation of $H(k)$*.

Using the basic representation $\rho_{k,q}$, we introduce:

Definition 1.1.4. *The double affine Hecke algebra (DAHA) of type A_1*, denoted as

$$\mathbb{H} = \mathbb{H}(k,q) = \mathbb{H}^{A_1}(k,q),$$

is defined to be the subalgebra of $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ generated by $X^{\pm 1} :=$ (the multiplication operator by $x^{\pm 1}$) and the image $\rho_{k,q}\big(H(k)\big)$.

As an abstract algebra, the DAHA $\mathbb{H}$ of type A_1 is presented with generators T_1, U, X and relations

$$(T_1 - k)(T_1 + k^{-1}) = 0, \quad U^2 = 1, \quad T_1XT_1 = X^{-1}, \quad UXU = q^{1/2}U^{-1}. \tag{1.1.20}$$

See [M03, §4.7] and [C05] for the detail. The map $\rho_{k,q}$ of (1.1.18) extends to the embedding $\rho_{k,q} \colon \mathbb{H} \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$.

We have the Poincaré-Birkhoff-Witt type decomposition of $\mathbb{H}$ as a $\mathbb{C}$-linear space:

$$\mathbb{H} \cong \mathbb{C}[X^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}]. \tag{1.1.21}$$

This decomposition is compatible with $H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}]$ in (1.1.13) under the identification of $H = H(k)$ with the faithful image $\rho_{k,q}(H) \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$. Below we often identify $X^{\pm 1}$ and $x^{\pm 1}$, and denote the decomposition (1.1.21) as $\mathbb{H} = \mathbb{C}[x^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}]$.

Let us also recall *the duality anti-involution* introduced by Cherednik ([C95], [M03, (4.7.6)]). It is the unique $\mathbb{C}$-algebra anti-involution

$$*\colon \mathbb{H}(k,q) \longrightarrow \mathbb{H}(k^*,q), \quad h \longmapsto h^* \tag{1.1.22}$$

such that, denoting by $X^\lambda :=$ (the multiplication operator by x^l) for $\lambda = l\varpi \in \Lambda$, $l \in \mathbb{Z}$, we have

$$T_1^* = T_1, \quad (Y^\lambda)^* = X^{-\lambda}, \quad (X^\lambda)^* = Y^{-\lambda} \quad (\lambda \in \Lambda), \quad k^* = k.$$

Here and hereafter we use the redundant symbol k^* for the comparison with type $(C_1^\vee, C_1)$ (see (2.1.15)).

Finally, we denote by

$$H(k)^* \subset \mathbb{H}(k^*,q) = \mathbb{H}(k,q) \tag{1.1.23}$$

the image of $H(k) \subset \mathbb{H}(k,q)$ under the duality anti-involution $*$. Then $H(k)^*$ is equal to the subalgebra of $\mathbb{H}(k,q)$ generated by the finite Hecke algebra $H_0(k)$ (see (1.1.14)) and $X^{\pm 1} = x^{\pm 1}$.

1.2. **Bispectral quantum Knizhnik-Zamolodchikov equation.** Let us explain the bispectral qKZ equation of the affine root system $S(A_1)$, mainly following [vM11, §3.2]. Hereafter we fix the parameters $q^{1/2}, k \in \mathbb{C}^\times$, and consider the basic representation $\rho_{k,q}\colon H(k) \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ of the affine Hecke algebra $H(k)$ in (1.1.18) and the DAHA $\mathbb{H}(k,q)$ in Definition 1.1.4.

1.2.1. *The affine intertwiners of type* A_1. Following [C05, §1.3], [vMS09, §2.3] and [vM11, Proposition 3.3], we introduce the affine intertwines of type A_1. Corresponding to the generators s_1, s_0, u of the extended Weyl group W (and T_1, T_0, U of $H(k)$), we define $\widetilde{S}_1, \widetilde{S}_0, \widetilde{S}_u \in \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ by

$$\widetilde{S}_i = \widetilde{S}_i(k,q) := d_i(x;k,q) s_{i,q} \quad (i = 1,0), \quad \widetilde{S}_u = \widetilde{S}_u(q) := u_q, \tag{1.2.1}$$

where $s_{i,q}$ and u_q are the operators in (1.1.16), and the function $d_i(x)$ is given by

$$d_i(x) = d(x_i;k,q) := k^{-1} - kx_i, \quad x_1 := x^2, \ x_0 := qx^{-2}. \tag{1.2.2}$$

The elements $\widetilde{S}_1$, $\widetilde{S}_0$ and $\widetilde{S}_u$ belong to the subalgebra $\mathbb{H} \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ since

$$\widetilde{S}_i = (1 - x_i)(\rho_{k,q}(T_i) - k) + k^{-1} - kx_i, \quad \widetilde{S}_u = \rho_{k,q}(U) \tag{1.2.3}$$

More generally, for each $w \in W$, taking a reduced expression $w = s_{j_1} \cdots s_{j_l} u^r$ with $j_1, \ldots, j_l, r \in \{0,1\}$, we define the element $\widetilde{S}_w \in \mathbb{H}$ by

$$\widetilde{S}_w := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{l-1}} d_{j_l})(x) \cdot w_q. \tag{1.2.4}$$

Here we used the action of s_i's on functions in x and the operator w_q, both given in (1.1.16). Note that this definition includes (1.2.1) by setting $\widetilde{S}_1 = \widetilde{S}_{s_1}$ and $\widetilde{S}_0 = \widetilde{S}_{s_0}$. The element $\widetilde{S}_w \in \mathbb{H}$ is independent of the choice of reduced expression $w = s_{j_1} \cdots s_{j_l} u^r$, since

$$d_w(x) := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{l-1}} d_{j_l})(x) \tag{1.2.5}$$

depends only on w [M03, (2.2.9)]. Moreover, by [vM11, Proposition 3.3 (ii)], we have

$$\widetilde{S}_w = \widetilde{S}_{j_1} \cdots \widetilde{S}_{j_l} \widetilde{S}_u^r. \tag{1.2.6}$$

We call the elements $\widetilde{S}_w$ in (1.2.4) *the affine intertwiners of type* A_1.

Remark 1.2.1. Our affine intertwines are obtained from those in [vM11] by replacing k, x with k^{-1}, x^{-1}. We made this replacement to simplify the comparison with the type $(C_1^\vee, C_1)$ discussed in § 3.

1.2.2. *The double extended Weyl group.* Extending the representation space $\mathbb{C}[x^{\pm 1}]$ of the basic representation $\rho_{k,q}$ (see (1.1.15) and (1.1.18)), we introduce

$$\mathbb{L} := \mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[\xi^{\pm 1}] = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]. \tag{1.2.7}$$

We sometimes call x *the geometric variable* and ξ *the spectral variable*.

Remark 1.2.2. The papers [vMS09, vM11, St14] considered (for a root system of arbitrary type) the ring $\mathbb{L}' := \mathbb{C}[T \times T] \cong \mathbb{C}[T] \otimes \mathbb{C}[T]$ of regular functions on the product $T \times T$, where $T := \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$ is the algebraic torus associated to the lattice Λ. In loc. cit., the value of $t \in T$ at $\lambda \in \Lambda$ is written as $t^\lambda \in \mathbb{C}^\times$, and a point of $T \times T$ is denoted by $(t, \gamma) \in T \times T$. For the type A_1 we are considering, the lattice is $\Lambda = \mathbb{Z}\varpi$, and there is a natural identification $\mathbb{L}' \cong \mathbb{L}$ given by $(t \mapsto t^\varpi) \mapsto x$ and $(\gamma \mapsto \gamma^\varpi) \mapsto \xi$. The geometric and spectral variables x, ξ are called the coordinate (functions) of $T \times T$ in loc. cit. The formulas and arguments given in the following text are obtained from those in loc. cit. by replacing $f(t, \gamma) \in \mathbb{L}'$ with $f(x, \xi) \in \mathbb{L}$.

Then the DADA $\mathbb{H} = \mathbb{H}(k, q)$ in Definition 1.1.4 has a structure of an $\mathbb{L}$-module by

$$(f \otimes g)h := f(X) \cdot h \cdot g(Y) \tag{1.2.8}$$

for $f = f(x) \in \mathbb{C}[x^{\pm 1}] \subset \mathbb{L}$, $g = g(\xi) \in \mathbb{C}[\xi^{\pm 1}] \subset \mathbb{L}$ and $h \in \mathbb{H}$. Here $X \in \mathbb{H}$ denotes the multiplication operator by x (see Definition 1.1.4), and $Y \in H = \rho_{k,q}(H) \subset \mathbb{H}$ denotes the Dunkl operator (1.1.11). The $\cdot$ in the right hand side means to take the multiplication of the ring $\mathbb{H}$. Note that the PBW type decomposition (1.1.21) yields the natural $\mathbb{L}$-module isomorphism

$$\mathbb{H} \cong H_0^{\mathbb{L}} := \mathbb{L} \otimes H_0, \tag{1.2.9}$$

where in the right hand side $\mathbb{L}$ acts on the first tensor component $\mathbb{L}$ by ring multiplication.

We turn to the introduction of *the double extended Weyl group* $\mathbb{W}$, following [vMS09, §3.1] and [vM11, §3.2]. Let ι denote the nontrivial element of the group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. We define the group $\mathbb{W}$ as the semi-direct product

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W), \tag{1.2.10}$$

where $\iota \in \mathbb{Z}_2$ acts on the product $W \times W$ of the extended affine Weyl group W by

$$\iota(w, w') = (w', w)\iota \quad (w, w' \in W).$$

The group $\mathbb{W}$ acts on $\mathbb{L}$ as follows. We define an involution $\diamond \colon W \to W$ by

$$w^\diamond := w, \quad \mathrm{t}(\lambda)^\diamond := \mathrm{t}(-\lambda) \tag{1.2.11}$$

for $w \in W_0$ and $\lambda \in \Lambda$. Then the $\mathbb{W}$-action on $\mathbb{L}$ is given by

$$(wf)(x) := (w_q f)(x), \quad (w'g)(\xi) := ((w'^\diamond)_q g)(\xi), \quad (\iota F)(x, \xi) = F(\xi^{-1}, x^{-1}) \tag{1.2.12}$$

for $w \in W = W \times \{e\} \subset \mathbb{W}$, $w' \in W = \{e\} \times W \subset \mathbb{W}$ and $f = f(x), g = g(\xi), F = F(x, \xi) \in \mathbb{L}$. Here w_q denotes the W-action in (1.1.16).

Remark 1.2.3. The element $\iota \in \mathbb{W}$ is designed to be consistent with the duality anti-involution $*$ (1.1.22) and the actions of $\mathbb{W}$ and $\mathbb{H}$ on $\mathbb{L}$.

Now, following [vMS09, §3.1] and [vM11, §3.2], we define $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_\iota \in \mathrm{End}(\mathbb{H})$ by

$$\widetilde{\sigma}_{(w,w')}(h) := \widetilde{S}_w \cdot h \cdot (\widetilde{S}_{w'})^*, \quad \widetilde{\sigma}_\iota(h) := h^* \quad (h \in \mathbb{H}). \tag{1.2.13}$$

Here $*$ denotes the anti-involution (1.1.22), and $\cdot$ denotes the multiplication of the ring $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ (or the composition of operators on $\mathbb{C}[x^{\pm 1}]$). The action is well defined since $\widetilde{S}_w \in \mathbb{H}$.

Fact 1.2.4 ([vMS09, Lemma 3.2], [vM11, Lemma 3.5])**.** For $h \in \mathbb{H}$, $f \in \mathbb{L}$ and $w, w' \in W$, we have

$$\widetilde{\sigma}_{(w,w')}(fh) = ((w, w')f)\widetilde{\sigma}_{(w,w')}(h), \quad \widetilde{\sigma}_\iota(fh) = (\iota f)\widetilde{\sigma}_\iota(h). \tag{1.2.14}$$

1.2.3. *The cocycles.* Below we denote the field of meromorphic functions of variables x and ξ by

$$\mathbb{K} := \mathcal{M}(x, \xi),$$

and set

$$H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0. \tag{1.2.15}$$

An element $f \in H_0^{\mathbb{K}}$ is regarded as a meromorphic function of x, ξ valued in $H_0 \subset \operatorname{End}_{\mathbb{C}}(\mathbb{C}[x^{\pm 1}])$. Also, we have a $\mathbb{C}$-linear isomorphism $H_0^{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$ by (1.2.9), and $f \in H_0^{\mathbb{K}}$ can be expressed as

$$f = \sum_{w \in W_0} f_w T_w, \quad f_w \in \mathbb{K}. \tag{1.2.16}$$

The $\mathbb{W}$-action on $\mathbb{L}$ given by (1.2.12) naturally extends to that on $\mathbb{K}$. Now the group $\mathbb{W}$ acts on $H_0^{\mathbb{K}}$ by

$$\mathbf{w} f := \sum_{w \in W_0} (\mathbf{w} f_w) T_w \tag{1.2.17}$$

for $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

By Fact 1.2.4, we can extend the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ uniquely to $\mathbb{C}$-linear endomorphisms of $H_0^{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$ such that the formulas (1.2.14) are valid for $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. We denote them by the same symbols $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_\iota \in \operatorname{End}_{\mathbb{C}}(H_0^{\mathbb{K}})$.

Fact 1.2.5 ([vMS09, Theorem 3.3], [vM11, Theorem 3.6])**.** There is a unique group homomorphism

$$\tau \colon \mathbb{W} \longrightarrow \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$$

satisfying

$$\tau(w, w')(f) = d_w(x)^{-1} d_{w'}(\xi^{-1})^{-1} \cdot \widetilde{\sigma}_{(w,w')}(f), \quad \tau(\iota)(f) = \widetilde{\sigma}_\iota(f) \tag{1.2.18}$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. Here we used the function d_w given by (1.2.5), and $\cdot$ denotes the $\mathbb{K}$-action given by (1.2.8). Moreover, we have

$$\tau(\mathbf{w})(gf) = wg\tau(\mathbf{w})(f)$$

for $g \in \mathbb{K}$, $f \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

Remark 1.2.6. In [vM11, Theorem 3.6], the action of $\tau(w, w')$ is written using $d_{w'}^{\diamond}(Y)$, which is equal to $d_{w'}(Y^{-1})$ according to [vMS09, Proof of Lemma 3.2].

Now we recall a terminology of non-abelian group cohomology. Let G be a group, and M be a G-group. We denote by $m^g \in M$ the action of $g \in G$ on $m \in M$. Then, a (1-)cocycle means a map $z \colon G \to M$ such that $z(g_1 g_2) = z(g_1) z(g_2)^{g_1}$ for any $g_1, g_2 \in G$.

Recall that $\mathbb{W}$ acts on $H_0^{\mathbb{K}}$ by (1.2.17). This action makes the group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ into a $\mathbb{W}$-group by

$$(\mathbf{w}, A) \longmapsto \mathbf{w} A \mathbf{w}^{-1} \quad (\mathbf{w} \in \mathbb{W},\ A \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})).$$

Fact 1.2.7 ([vMS09, Corollary 3.4], [vM11, Corollary 3.8])**.** The map

$$\mathbf{w} \longmapsto C_{\mathbf{w}} := \tau(\mathbf{w}) \mathbf{w}^{-1} \tag{1.2.19}$$

is a cocycle of $\mathbb{W}$ with values in the $\mathbb{W}$-group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$. In other words, for any $\mathbf{w}, \mathbf{w}' \in \mathbb{W}$, we have $C_{\mathbf{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ and

$$C_{\mathbf{w}\mathbf{w}'} = C_{\mathbf{w}} \mathbf{w} C_{\mathbf{w}'} \mathbf{w}^{-1}. \tag{1.2.20}$$

Note that the cocycles $C_{\mathbf{w}}$ depend on the parameters (k, q). Also note that, by the natural isomorphism

$$\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K} \otimes \mathrm{GL}_{\mathbb{C}}(H_0), \tag{1.2.21}$$

we can regard an element $C_{\mathbf{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as a meromorphic function of x, ξ valued in $\mathrm{GL}_{\mathbb{C}}(H_0)$. To stress this point, we denote it as

$$C_{\mathbf{w}}(x, \xi). \tag{1.2.22}$$

1.2.4. *The bispectral bispectral quantum KZ equations of type A_1.* Let us focus on the cocycles associated to the translations in $\mathbb{W}$, i.e., the elements in the subgroup

$$\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda) \subset W \times W \subset \mathbb{W}.$$

Recalling $\Lambda = \mathbb{Z}\varpi$, we denote

$$C_{l,m} := C_{(\mathrm{t}(l\varpi),\mathrm{t}(m\varpi))} \quad (l, m \in \mathbb{Z}). \tag{1.2.23}$$

Definition 1.2.8 ([vMS09, Dfn. 3.7], [vM11, Dfn. 3.9], c.f. [St14, Dfn. 3.2])**.** The system of q-difference equations

$$C_{l,m}(x, \xi) f(q^{-l}x, q^m \xi) = f(x, \xi) \quad (l, m \in \mathbb{Z})$$

for $f \in H_0^{\mathbb{K}}$ *the bispectral quantum KZ equations* (*the bqKZ equations* for short) *of type* A_1. The solution space is denote by

$$\mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k, q) := \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } A_1\}.$$

Remark 1.2.9. The solution space is denoted by SOL in [vMS09, vM11], and by $\mathcal{K}_{k,q}$ in [St14]. Our symbol is a modification of the notation Sol_{QAKZ} in [C05, Theorem 1.3.8].

1.2.5. *The cocycle values.* As before, let $H = H(k)$ be the affine Hecke algebra of type A_1, $H_0 = H_0(k)$ be the subalgebra of H generated by T_1, and $H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0$. We can write down the cocycles $C_{1,0}$ and $C_{0,1}$ by the following representations of the affine Hecke algebra H and its duality anti-involution image H^* (see (1.1.23)).

Definition 1.2.10. $H_0^{\mathbb{K}}$ has the following left H-module structure and the right H^*-module structure: We define an algebra homomorphism $\eta_L \colon H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_L(A)\Big(\sum_{w \in W_0} f_w T_w\Big) := \sum_{w \in W_0} f_w (A T_w) \quad (A \in H), \tag{1.2.24}$$

using the expression (1.2.16) of an element of $H_0^{\mathbb{K}}$. We also define an algebra anti-homomorphism $\eta_R \colon H^* \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_R(A)\Big(\sum_{w \in W_0} f_w T_w\Big) := \sum_{w \in W_0} f_w (T_w A) \quad (A \in H^*). \tag{1.2.25}$$

Remark 1.2.11. The map η_L was introduced in [vMS09, §4.1] and [vM11, §4.1], denoted by η, under the name of *the formal principal series representation of* H, since it is a formal version of the principal series representation used in [C92b, C94]. We borrowed the symbol η_R from [T10, §4.2].

Lemma 1.2.12 (c.f. [vM11, (5.3)])**.** Regarding the cocycles $C_{1,0}, C_{0,1}$ as $\mathrm{GL}(H_0)$-valued meromorphic functions of x, ξ (see (1.2.22)), we have

$$C_{1,0}(x, \xi) = R_0^L(x_0)\eta_L(U), \tag{1.2.26}$$

$$C_{0,1}(x, \xi) = R_0^R(\xi_0')\eta_R(U^*), \tag{1.2.27}$$

where we denoted $x_0 := qx^{-2}$, $\xi_0' := q\xi^2$ and

$$R_i^L(z) := c(z, k)^{-1}\big(\eta_L(T_i) - b(z; k)\big) \quad = c(z; k)^{-1}\big(\eta_L(T_i) - k\big) + 1,$$
$$R_i^R(z) := c(z, k^*)^{-1}\big(\eta_R(T_i^*) - b(z; k^*)\big) = c(z; k^*)^{-1}\big(\eta_R(T_i^*) - (k^*)^{-1}\big) + 1,$$

using $c(z; k), b(z; k)$ in (1.1.17) and the duality anti-involution $*$ in (1.1.22). We also used the redundant notation $k^* = k$.

Proof. We first calculate $C_{1,0} = C_{(\mathrm{t}(\varpi),e)} = \tau(\mathrm{t}(\varpi), e)\,(\mathrm{t}(\varpi), e)^{-1}$. We have $\mathrm{t}(\varpi) = us_1 = s_0 u$ by (1.1.8). Then, using (1.2.19) and (1.2.18), for any element $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$ ($f_w \in \mathbb{K}$), we have

$$\begin{aligned} C_{1,0} f &= \tau(s_0 u, e)\,(s_0 u, e)^{-1}\Big(\sum_{w \in W_0} f_w T_w\Big) = \tau(s_0 u, e)\Big(\sum_{w \in W_0} ((s_0 u, e)^{-1} f_w) T_w\Big) \\ &= d_{s_0 u}(x)^{-1}\widetilde{\sigma}_{(s_0 u, e)}\Big(\sum_{w \in W_0} (s_0 u, e)^{-1} f_w T_w\Big) \\ &= d_{s_0 u}(x)^{-1}\Big(\sum_{w \in W_0} \big((s_0 u, e)(s_0 u, e)^{-1} f_w\big)\widetilde{S}_{s_0 u} T_w\Big) = d_{s_0 u}(x)^{-1}\Big(\sum_{w \in W_0} f_w \widetilde{S}_{s_0 u} T_w\Big). \end{aligned}$$

Now, by (1.2.3), we have

$$\widetilde{S}_{s_0u} = \widetilde{S}_0\widetilde{S}_u = \big((1-x_0)(\rho_{k,q}(T_0)-k)+k^{-1}-kx_0\big)\rho_{k,q}(U).$$

On the other hand, (1.2.5) and (1.2.2) yield $d_{s_0u}(x) = k^{-1}-kx_0$, and by (1.1.17), we have

$$d_{s_0u}(x)^{-1}(1-x_0) = c(x_0;k)^{-1}. \tag{1.2.28}$$

Then, using Definition 1.2.10, we have

$$C_{1,0}f = \big(c(x_0;k)^{-1}(\eta_L(T_0)-k^{-1})+1\big)\eta_L(U)(f),$$

which yields (1.2.26).

Similarly, the action of $C_{0,1}$ on $f=\sum_{w\in W_0} f_wT_w \in H_0^{\mathbb{K}}$ is computed as

$$C_{0,1}f = \tau(e,s_0u)(e,s_0u)^{-1}\Big(\sum_{w\in W_0} f_wT_w\Big) = d_{s_0u}(\xi^{-1})^{-1}\cdot\Big(\sum_{w\in W_0} f_wT_w\widetilde{S}^*_{s_0u}\Big),$$

where $\cdot$ denotes the $\mathbb{K}$-action (see (1.2.8)). By (1.2.1) and (1.2.3), we have

$$\widetilde{S}^*_{s_0u} = \widetilde{S}^*_u\widetilde{S}^*_0 = \rho_{k,q}(U)^*\big((\rho_{k,q}(T_0)^*-k)(1-q^{-1}Y^{-2})+k^{-1}-kq^{-1}Y^{-2}\big).$$

Now recall that a function $g(\xi)$ acts on H_0 by the right multiplication of $g(Y)$ (see (1.2.8)). Then, by (1.2.28) and Definition 1.2.10, we have

$$C_{0,1}f = \big((\eta_R(T_0^*)-k)c(qY^2;k)^{-1}+1\big)\eta_R(U^*)(f),$$

which yields (1.2.27). □

Remark 1.2.13. A few comments on Lemma 1.2.12 are in order.

(1) By [vM11, Remark 4.4], we have

$$C_{(e,w)}(x,\xi) = C_\iota C_{(w,e)}(\xi^{-1},x^{-1})C_\iota \tag{1.2.29}$$

for any $w\in W$, where we used the notation (1.2.22). The result of Lemma 1.2.12 is consistent with this equality.

(2) As shown in [vMS09, Lemma 4.3], the rational function

$$R_i(z) := c(z,k)^{-1}\big(\eta_L(T_i)-b(z;k)\big)$$

valued in $\mathrm{End}(H_0)$ satisfies the Yang-Baxter equation $R_0(z)R_1(zz')R_0(z') = R_1(z')R_0(zz')R_1(z)$. In the terminology [C05, §1.3.6], $R_i(z)$ is called the baxterization of T_i.

For later use, let us cite the following two facts.

Fact 1.2.14 ([vM11, Lemma 5.1])**.** Let $\mathcal{A} := \mathbb{C}[x^{-1}] \subset \mathbb{L} = \mathbb{C}[x^{\pm1},\xi^{\pm1}]$, and $\mathcal{Q}_0(\mathcal{A})$ be the subring of the quotient field $\mathcal{Q}(\mathcal{A}) = \mathbb{C}(x)$ consisting of rational functions which are regular at $x^{-1}=0$. Considering $\mathcal{Q}_0(\mathcal{A})\otimes\mathbb{C}[\xi^{\pm1}]$ as a subring of $\mathbb{C}(x,\xi)$, we have

$$C_{1,0} \in (\mathcal{Q}_0(\mathcal{A})\otimes\mathbb{C}[\xi^{\pm1}])\otimes\mathrm{End}(H_0). \tag{1.2.30}$$

Moreover, setting $C^{(0)}_{1,0} := C_{1,0}|_{x^{-1}=0} \in \mathbb{C}[\xi^{\pm1}]\otimes\mathrm{End}(H_0)$, we have

$$C^{(0)}_{1,0} = k\eta_L(T_1Y^{-1}T_1^{-1}). \tag{1.2.31}$$

Similarly, defining $\mathcal{B} := \mathbb{C}[\xi] \subset \mathbb{L}$, and $\mathcal{Q}_0(\mathcal{B}) \subset \mathcal{Q}(\mathcal{B})$ to be the subring consisting of rational functions which are regular at $\xi=0$, we have

$$C_{0,1} \in (\mathbb{C}[x^{\pm1}]\otimes\mathcal{Q}_0(\mathcal{B}))\otimes\mathrm{End}(H_0).$$

Moreover, setting $C^{(0)}_{0,1} := C_{0,1}|_{\xi=0} \in \mathbb{C}[x^{\pm1}]\otimes\mathrm{End}(H_0)$, we have

$$C^{(0)}_{0,1} = k^*\eta_R(T_1Y^{-1}T_1^{-1}).$$

Proof. We only show the statements for $C_{1,0}$ using Lemma 1.2.12. Let us denote $A(x)\approx A_0$ if $A(x) = A_0+O(x^{-1})$ by expansion in terms of x^{-1}. Then we have $c(x_0;k) = c(qx^{-2};k)\approx k$, and the expression (1.2.26) yields

$$C_{1,0}\approx C^{(0)}_{1,0} := \big(k(\eta_L(T_0)-k)+1\big)\eta_L(U) = k\eta_L(T_1Y^{-1}T_1^{-1}),$$

where we used $T_0U = UT_1$ and $T_1^{-1} = T_1-k+k^{-1}$ in $H=H(k)$ from (1.1.10), and $Y^{-1} = T_1^{-1}U$ from (1.1.11). Thus we have (1.2.30) and (1.2.31). □

For the next fact, note that we have $\widetilde{S}^*_w \in H \subset \mathbb{H}$ for all $w\in W_0$.

Fact 1.2.15 ([vMS09, Lemma 4.2])**.** For $w \in W_0$, we set

$$\tau_w := \eta_L(\widetilde{S}^*_{w^{-1}})T_e \in \mathbb{C}[\{e\} \times T] \otimes H_0 \subset H_0^{\mathbb{K}}.$$

Then the following statements hold.

(1) $\{\tau_w \mid w \in W_0\}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of simultaneous eigenfunctions for the η_L-action of $\mathbb{C}[Y^{\pm 1}] \subset \mathbb{H}$ on $H_0^{\mathbb{K}}$.

(2) For $p \in \mathbb{C}[T]$ and $w \in W_0$, we have

$$\eta_L(p(Y))(\gamma)\, \tau_w(\gamma) = (w^{-1}p)(\gamma)\, \tau_w(\gamma)$$

as H_0-valued regular functions in $\gamma \in T$.

We close this subsection with:

Lemma 1.2.16. The cocycles $C_{2,0}$ and $C_{0,2}$ are given by

$$C_{2,0} = R_0^L(x_0)R_1^L(x_1'), \quad C_{0,2} = R_0^R(\xi_0')R_1^R(\xi_1')$$

Here we used the notation of Lemma 1.2.12: $x_0 := qx^{-2}$, $\xi_0' := q\xi^2$ and

$$\begin{aligned} R_i^L(z) &:= c(x_i,k)^{-1}\big(\eta_L(T_i) - b(x_i;k)\big) &&= c(x_i;k)^{-1}\big(\eta_L(T_i) - k\big) + 1, \\ R_i^R(z) &:= c(\xi_i,k^*)^{-1}\big(\eta_R(T_i^*) - b(\xi_i;k^*)\big) &&= c(\xi_i;k^*)^{-1}\big(\eta_R(T_i^*) - (k^*)^{-1}\big) + 1. \end{aligned}$$

We further used $x_1' := q^2x^{-2}$ and $\xi_1' := q^2\xi^2$.

Proof. It is a consequence of the cocycle relation (1.2.20) and a similar calculation of Lemma 1.2.12. We omit the detail. □

1.3. **Bispectral Macdonald-Ruijsenaars equations.** As in the previous §1.2, we fix generic complex numbers $q^{1/2}$ and k.

We consider the crossed product algebra (the smash product algebra)

$$\mathbb{D}_q^{\mathbb{W}} := \mathbb{W} \ltimes \mathbb{C}(x,\xi),$$

where $\mathbb{W}$ acts as field automorphisms on $\mathbb{C}(x,\xi)$ by (1.2.12), and also the subalgebra $\mathbb{D}_q$ of $\mathbb{D}_q^{\mathbb{W}}$ defined by

$$\mathbb{D}_q := \big(\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)\big) \ltimes \mathbb{C}(x,\xi) \subset \mathbb{D}_q^{\mathbb{W}},$$

where $\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)$ is regarded as a subgroup of $W \times W \subset \mathbb{W}$. The subalgebra $\mathbb{D}_q$ is identified with the algebra of q-difference operators on $\mathbb{C}(x,\xi)$. We can expand each $D \in \mathbb{D}_q^{\mathbb{W}}$ as

$$D = \sum_{\mathbf{w} \in \mathbb{W}} f_{\mathbf{w}} \mathbf{w} = \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}} \mathbf{s} \tag{1.3.1}$$

with $f_{\mathbf{w}} \in \mathbb{C}(x,\xi)$ and $D_{\mathbf{s}} = \sum_{\mathbf{t} \in \mathrm{t}(\Lambda)\times\mathrm{t}(\Lambda)} g_{\mathbf{ts}}\mathbf{t} \in \mathbb{D}_q$. Then we define the restriction map $\mathrm{Res}\colon \mathbb{D}_q^{\mathbb{W}} \to \mathbb{D}_q$ to be the $\mathbb{C}(x,\xi)$-linear map

$$\mathrm{Res}(D) := \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}}. \tag{1.3.2}$$

Next, we introduce two realizations of the basic representation ρ of H. One is given by

$$\rho^x_{1/k,q}\colon H(1/k) \longrightarrow \mathbb{D}_q^{\mathbb{W}} \tag{1.3.3}$$

which is the map $\rho_{1/k,q}$ from (1.1.18), regarded as an algebra homomorphism from $H(1/k)$ to the subalgebra $\mathbb{C}(x)[W \times \{e\}]$ of $\mathbb{D}_q^{\mathbb{W}}$. The other is given by

$$\rho^\xi_{k,1/q}\colon H(k) \longrightarrow \mathbb{D}_q^{\mathbb{W}} \tag{1.3.4}$$

defined as the map $\rho_{k,1/q}$ from (1.1.18), regarded as an algebra homomorphism from $H(1/k)$ to the subalgebra $\mathbb{C}(\xi)[\{e\} \times W]$ of $\mathbb{D}_q^{\mathbb{W}}$.

Definition 1.3.1. For $h \in H(1/k)$, we define

$$D_h^x := \rho^x_{1/k,q}(h) \in \mathbb{D}_q^{\mathbb{W}}.$$

Also, for $h' \in H(k)$, we define

$$D_{h'}^\xi := \rho^\xi_{k,1/q}(h') \in \mathbb{D}_q^{\mathbb{W}}.$$

Remark 1.3.2. Our choice (1.3.3) and (1.3.4) of the basic representations affects the parameters in the bispectral correspondence (1.4.3) of quantum Knizhnik-Zamolodchikov and Macdonald-Ruijsenaars equations. Our argument is equivalent to [vMS09, §6.2] and [vM11, §6.1], and opposite to [St14, Definition 2.17]. See Definition 2.3.1 for the $(C_1^\vee, C_1)$ case.

Let $\mathbb{C}[z^{\pm 1}]^{W_0}$ denote the ring of Laurent polynomials of variable z which are invariant under the W_0-action $s_1(z) \coloneqq z^{-1}$. Using the restriction map Res in (1.3.2), we introduce:

Definition 1.3.3. For $p \in \mathbb{C}[z^{\pm 1}]^{W_0}$, we define $L_p^x, L_p^\xi \in \mathbb{D}_q$ by

$$L_p^x = L_p^x(k,q) \coloneqq \mathrm{Res}(D_{p(Y)}^x), \quad L_p^\xi = L_p^\xi(k,q) \coloneqq \mathrm{Res}(D_{p(Y)}^\xi), \tag{1.3.5}$$

where we regard $p(Y) \in H(1/k)$ for L_p^x, and $p(Y) \in H(k)$ for L_p^ξ.

Since we have $\mathbb{C}[z^{\pm 1}]^{W_0} \cong \mathbb{C}[z + z^{-1}]$, it is natural to introduce:

Definition 1.3.4. We denote $p_1 \coloneqq z + z^{-1}$, the generator of the invariant ring $\mathbb{C}[z^{\pm 1}]^{W_0}$.

Using the function $c(\cdot; k)$ in (1.1.17), we can write down

$$L_{p_1}^x, L_{p_1}^\xi \in \mathbb{D}_q \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big).$$

Let us denote the action of $w \in W$ on functions of x given in (1.1.16) as

$$w^x \in \mathrm{End}(\mathbb{C}(x)) \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big).$$

Explicitly, for $f = f(x) \in \mathbb{C}(x)$, we have

$$(s_0^x f)(x) \coloneqq f(qx^{-1}), \quad (s_1^x f)(x) = f(x^{-1}), \quad (u^x f)(x) = f(q^{1/2}x^{-1}), \quad (\mathrm{t}(\varpi)^x f)(x) = f(q^{1/2}x). \tag{1.3.6}$$

Recall that it is compatible with $\rho_{1/k,q}^x$ in (1.3.3). We also denote by

$$w^\xi \in \mathrm{End}(\mathbb{C}(\xi)) \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big)$$

the action on functions $g = g(\xi) \in \mathbb{C}(\xi)$. It is given by

$$(s_0^\xi g)(\xi) \coloneqq g(q^{-1}\xi^{-1}), \quad (s_1^\xi g)(\xi) = g(\xi^{-1}), \quad (u^\xi g)(\xi) = g(q^{-1/2}\xi^{-1}), \quad (\mathrm{t}(\varpi)^\xi g)(\xi) = g(q^{-1/2}\xi), \tag{1.3.7}$$

and is compatible with $\rho_{k,1/q}^\xi$ in (1.3.4).

Proposition 1.3.5. We have

$$L_{p_1}^x(k,q) = A(x)T_{q^{1/2},x} + A(x^{-1})T_{q^{-1/2},x}, \quad L_{p_1}^\xi(k,q) = A^*(\xi^{-1})T_{q^{1/2},\xi} + A^*(\xi)T_{q^{-1/2},\xi} \tag{1.3.8}$$

with

$$A(z) \coloneqq c(z^2;k) = \frac{k^{-1} - kz^2}{1 - z^2}, \quad A^*(z) \coloneqq c(z^2;k^*) = A(z).$$

Here we used the redundant notation $k^* = k$ for the comparison with $(C_1^\vee, C_1)$ case (Proposition 2.3.2).

Proof. Let us compute $L_{p_1}^x = \mathrm{Res}(D_{Y+Y^{-1}}^x)$. Since $Y = UT_1$ and $u = \mathrm{t}(\varpi)s_1$, using (1.1.10) and (1.1.19), we have

$$\begin{aligned} D_{Y+Y^{-1}}^x &= \rho_{1/k,q}^x(UT_1 + T_1^{-1}U) \\ &= \big(\mathrm{t}(\varpi)^x s_1^x\big)\big(k^{-1} + c(x^2;k^{-1})(s_1^x - 1)\big) + \big(k + c(x^2;k^{-1})(s_1^x - 1)\big)\big(\mathrm{t}(\varpi)^x s_1^x\big). \end{aligned}$$

Then, using

$$\begin{aligned} &\mathrm{Res}\big(\mathrm{t}(\varpi)^x s_1^x\big) = \mathrm{t}(\varpi)^x, \quad \mathrm{Res}\big(\mathrm{t}(\varpi)^x s_1^x(s_1^x - 1)\big) = 0, \\ &\mathrm{Res}\big((s_1^x - 1)\,\mathrm{t}(\varpi)^x s_1^x\big) = \mathrm{t}(-\varpi)^x - \mathrm{t}(\varpi)^x, \end{aligned}$$

$k + k^{-1} - c(x^2;k^{-1}) = c(x^2;k)$ and $c(x^2;k^{-1}) = c(x^{-2};k)$, we have

$$\begin{aligned} \mathrm{Res}(D_{Y+Y^{-1}}^x) &= k^{-1}\,\mathrm{t}(\varpi)^x + k\,\mathrm{t}(\varpi)^x + c(x^2;k^{-1})(\mathrm{t}(-\varpi)^x - \mathrm{t}(\varpi)^x) \\ &= \big(k + k^{-1} - c(x^2;k^{-1})\big)\,\mathrm{t}(\varpi)^x + c(x^2;k^{-1})\,\mathrm{t}(-\varpi)^x \\ &= c(x^2;k)\,\mathrm{t}(\varpi)^x + c(x^{-2};k)\,\mathrm{t}(-\varpi)^x. \end{aligned}$$

By (1.3.6), we obtain the first half of (1.3.8).

For $L^{\xi}_{p_1}$, we replace (x,k,q) in $L^{x}_{p_1}$ with (ξ,k^{-1},q^{-1}) and calculate

$$L^{\xi}_{p_1}(k,q)=c(\xi^2;k^{-1})\,\mathrm{t}(-\varpi)^{\xi}+c(\xi^{-2};k^{-1})\,\mathrm{t}(\varpi)^{\xi}=c(\xi^{-2};k)\,\mathrm{t}(-\varpi)^{\xi}+c(\xi^{2};k)\,\mathrm{t}(\varpi)^{\xi}.$$

Then, by (1.3.7), we obtain the second half of (1.3.8). □

Remark 1.3.6. By the expression (1.1.17) of $c(\cdot;k)$ and (1.3.6), the formula of $L^{x}_{p_1}\in\mathbb{D}_q$ in (1.3.8) can be rewritten by

$$L^{x}_{p_1}(k,q)=\frac{kx-k^{-1}x^{-1}}{x-x^{-1}}T_{q^{1/2},x}+\frac{k^{-1}x-kx^{-1}}{x-x^{-1}}T_{q^{-1/2},x},$$

where $T_{q,x}$ denotes the q-shift operator acting on a function f in x as $(T_{q,x}f)(x)=f(qx)$. Similarly, for $L^{\xi}_{p_1}$, recalling $\mathrm{t}(\varpi)^{\xi}=T^{-1}_{q^{1/2},\xi}$ from (1.3.7), we have

$$L^{\xi}_{p_1}(k,q)=\frac{k^{-1}\xi-k\xi^{-1}}{\xi-\xi^{-1}}T_{q^{1/2},\xi}+\frac{k\xi-k^{-1}\xi^{-1}}{\xi-\xi^{-1}}T_{q^{-1/2},\xi}.$$

Now let us recall the Macdonald q-difference operator of type GL_2 [M87, Chap. VI], or the two-variable trigonometric Ruijsenaars operator [R87]:

$$D_{\mathrm{MR}}(x_1,x_2;q,t):=\frac{tx_1-x_2}{x_1-x_2}T_{q,x_1}+\frac{tx_2-x_1}{x_2-x_1}T_{q,x_2}$$

The specialization $D_{\mathrm{MR}}(x,x^{-1};q,t)$ is essentially equal to the Macdonald q-difference operator of type A_1 (see [M87, (9.13)] and [M03, §6.3]). Comparing these operators, we have

$$\begin{aligned}L^{x}_{p_1}(k,q)&=k^{-1}D_{\mathrm{MR}}(x,x^{-1};q^{1/2},k^2),\\ L^{\xi}_{p_1}(k,q)&=k\,D_{\mathrm{MR}}(\xi,\xi^{-1};q^{1/2},k^{-2})=k^{-1}D_{\mathrm{MR}}(\xi^{-1},\xi;q^{1/2},k^2).\end{aligned}$$

Lem42 In particular, using the action (1.2.12) of ι and noting $\iota T_{q,x}\iota=T_{q,\xi}$, we have

$$L^{\xi}_{p_1}=\iota L^{x}_{p_1}\iota.$$

See [vM11, Lemma 6.2] for a generalization of this relation.

Now we reach the main object in this § 1.3.

Definition 1.3.7. The following system of eigen-equations for $f=f(x,\xi)\in\mathbb{K}=\mathcal{M}(x,\xi)$ is called *the bispectral Macdonald-Ruijsenaars equation of type* A_1, and *the bMR equation* for short.

$$\begin{cases}(L^{x}_{p_1}(k,q)f)(x,\xi)&=p_1(\xi^{-1})f(x,\xi)\\ (L^{\xi}_{p_1}(k,q)f)(x,\xi)&=p_1(x)f(x,\xi)\end{cases}. \tag{1.3.9}$$

The solution space is denoted as

$$\mathrm{SOL}_{\mathrm{bMR}}(k,q):=\{f\in\mathbb{K}\mid f\text{ satisfies (1.3.9)}\}.$$

Remark 1.3.8. Continuing Remark 1.2.9, the solution space is denoted as BiSP in [vMS09, vM11]. Our symbol is a modification of Sol_{Mac} in [C05, Theorem 1.3.8].

1.4. **Bispectral qKZ/MR correspondence.** The works [vMS09, vM11] established the following correspondence between the two solution spaces $\mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k,q)$ (Definition 1.2.8) and $\mathrm{SOL}_{\mathrm{bMR}}(k,q)$ (Definition 1.3.7).

Definition 1.4.1. We define a $\mathbb{K}$-linear function $\chi_+\colon H_0\to\mathbb{C}$ by

$$\chi_+(T_w):=k^{\ell(w)} \tag{1.4.1}$$

for the basis element $T_w\in H_0$ ($w\in W_0$). It is extended to $H_0^{\mathbb{K}}$ as

$$\chi_+\colon H_0^{\mathbb{K}}\longrightarrow\mathbb{K},\quad \sum_{w\in W_0}f_wT_w\longmapsto\sum_{w\in W_0}f_w\chi_+(T_w), \tag{1.4.2}$$

where we used the expression (1.2.16).

Remark 1.4.2. This is a bispectral analogue of the map tr in [C05, §1.3.4, Theorem 1.3.8].

Fact 1.4.3 ([vMS09, Theorem 6.16, Corollary 6.21], [vM11, Theorem 6.6])**.** Assume $0 < q < 1$. Then the map χ_+ restricts to an injective $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map

$$\chi_+ \colon \mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k,q) \longrightarrow \mathrm{SOL}_{\mathrm{bMR}}(k,q), \tag{1.4.3}$$

where $\mathbb{F}$ is the subspace of $\mathbb{K} = \mathcal{M}(x,\xi)$ defined by

$$\mathbb{F} := \bigl\{ f(x,\xi) \in \mathbb{K} \mid \bigl((\mathrm{t}(\lambda),\mathrm{t}(\mu))f\bigr)(x,\xi) = f(x,\xi),\ \forall\,(\lambda,\mu) \in \Lambda \times \Lambda \bigr\},$$

and $\mathbb{W}_0$ is the subgroup of $\mathbb{W}$ defined by

$$\mathbb{W}_0 := \mathbb{Z}_2 \ltimes (W_0 \times W_0) \subset \mathbb{W}.$$

Remark 1.4.4. As mentioned in Remark 1.3.2, we follow the arguments in [vMS09, vM11] giving the bispectral correspondence $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(k,q) \to \mathrm{SOL}_{\mathrm{bMR}}(k,q)$. The claim in [St14, Theorem 3.1] is based on the correspondence $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(1/k,q) \to \mathrm{SOL}_{\mathrm{bMR}}(k,q)$, $\chi_+(T_w) = k^{-\ell(w)}$.

Let us explain the outline of the proof. We abbreviate $\mathrm{SOL}_{\mathrm{bqKZ}} := \mathrm{SOL}_{\mathrm{bqKZ}}(k,q)$ and $\mathrm{SOL}_{\mathrm{bMR}} := \mathrm{SOL}_{\mathrm{bMR}}(k,q)$. The proof is divided into three parts.

(i) χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$.

(ii) The image $\chi_+(\mathrm{SOL}_{\mathrm{bqKZ}})$ is contained in $\mathrm{SOL}_{\mathrm{bMR}}$.

(iii) $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathrm{SOL}_{\mathrm{bMR}}$ is injective

We omit the part (iii), and refer to [vMS09, Corollary 6.21] for the detail. For the part (i), we give a preliminary lemma.

Lemma 1.4.5 ([vMS09, Lemma 6.6])**.** For each $\mathbf{w} \in \mathbb{W}_0$ and $F \in H_0^{\mathbb{K}}$, we have

$$\chi_+(C_{\mathbf{w}}F) = \chi_+(F).$$

Proof. First, we have $\chi_+ \circ C_\iota = \chi_+$ since, for any $w \in W_0$, the element $T_w \in H_0 \subset H_0^{\mathbb{K}}$ satisfies $C_\iota(T_w) = T_{w^{-1}}$. Second, since $C_{(e,s_1)} = C_\iota C_{(s_1,e)} C_\iota$ by Remark 1.2.13, (1.2.29), it is sufficient to show $\chi_+ \circ C_{(s_1,e)} = \chi_+$. But it is a consequence of

$$C_{(s_1,e)}h = c(x_1;k,q)^{-1}(\eta_L(T_1) - k)h + h, \quad \chi_+(T_1) = k, \quad \chi_+ \circ \eta_L = \eta_L \circ \chi_+ \tag{1.4.4}$$

for any $h \in H_0$. □

Part (i) *of the proof of Fact 1.4.3.* We first show that χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$. By (1.2.19), Lemma 1.4.5 and (1.2.12), for any $f \in H_0^{\mathbb{K}}$ and $w \in \mathbb{W}_0$, we have

$$\chi_+(\tau(w)f) = \chi_+(C_w w f) = \chi_+(wf) = w\bigl(\chi_+(f)\bigr).$$

Hence χ_+ is $\mathbb{W}_0$-equivariant. Then, by Definition 1.2.8, (1.4.1) and (1.4.2), we obtain the $\mathbb{W}_0$-equivariant and $\mathbb{F}$-linear map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$ by restriction. □

The part (ii) of the proof consists of several arguments, and we may say that this part is one of the main body of [vMS09]. It is further divided into the following steps.

- Describe of $\mathrm{SOL}_{\mathrm{bqKZ}}$ in terms of the basic asymptotically free solution Φ.
- Analyze the map χ_+ using Φ.

The first step requires the following Fact 1.4.6 and Fact 1.4.8.

Fact 1.4.6 ([vMS09, §§5.1–5.2], [vM11, §5.2], [St14, §3.2])**.** Denote $w_0 := s_1 \in W_0$. Let

$$\mathcal{W}(x,\xi) = \mathcal{W}(x,\xi;k,q) \in \mathbb{K} = \mathcal{M}(x,\xi) \tag{1.4.5}$$

be a meromorphic function satisfying the q-difference equations (quasi-periodicity)

$$\mathcal{W}(q^{l/2}x,\xi) = (k/\xi)^l \mathcal{W}(x,\xi) \quad (l \in \mathbb{Z}) \tag{1.4.6}$$

and the self-duality

$$\mathcal{W}(\xi^{-1},x^{-1};k^*,q) = \mathcal{W}(x,\xi;k,q). \tag{1.4.7}$$

Here we used the redundant notation $k^* = k$ for the comparison with the $(C_1^\vee, C_1)$ case (2.4.5). Then, there is a unique element $\Psi \in H_0^{\mathbb{K}}$ satisfying the following conditions (i)–(iii).

(i) We have the self-dual solution

$$\Phi := \mathcal{W}\Psi \in \mathrm{SOL}_{\mathrm{bqKZ}}(k,q), \quad \iota(\Phi) = \Phi.$$

(ii) We have a series expansion

$$\Psi(t,\gamma) = \sum_{m,n\in\mathbb{N}} K_{m,n} x^{-2m} \xi^{2n} \quad (K_{\alpha,\beta} \in H_0)$$

for $(x,\xi) \in B_\varepsilon^{-1} \times B$ with B_ε being some open ball of radius $\varepsilon > 0$, which is normally convergent on compact subsets of $B_\varepsilon^{-1} \times B_\varepsilon$.

(iii) $K_{0,0} = T_{w_0}$.

The solution Φ is called *the basic asymptotically free solution of the bqKZ equation* in [vMS09, Definition 5.5], [vM11, Definition 5.5] and *the self-dual basic Harish-Chandra series* in [St14, Definition 3.8].

Remark 1.4.7. The function $\mathcal{W}$ is designed so that the element $\mathcal{W}(x,\xi)T_{w_0} = \mathcal{W}(x,\xi)T_1$ is a solution of the formal asymptotic form of the quantum KZ equation $C_{(l\varpi,e)}(x,\xi)f(q^{-l/2}x,\xi) = f(x,\xi)$ in the region $|x| \gg 0$. Indeed, noting that we are working in $H(1/k)$, recall from (1.2.31) the asymptotic form of $C_{(\varpi,e)} = C_{1,0}$ in this region:

$$C_{1,0} \approx C_{1,0}^{(0)} = k\eta_L(T_1 Y^{-1} T_1^{-1}).$$

The definition (1.2.24) of the map η_L and the $\mathbb{K}$-module structure (1.2.8) yield $\eta_L(T_1Y^{-1}T_1^{-1})T_1 = Y^{-1}T_1 = \xi^{-1}T_1$. Thus we have

$$\begin{aligned} C_{1,0}^{(0)}(x,\xi)\big(\mathcal{W}(q^{-1/2}x,\xi)T_1\big) = \mathcal{W}(x,\xi)T_1 &\iff k\xi^{-1}\mathcal{W}(q^{-1/2}x,\xi)T_1 = \mathcal{W}(x,\xi)T_1 \\ &\iff \mathcal{W}(q^{-1/2}x,\xi) = k^{-1}\xi\mathcal{W}(t,\gamma), \end{aligned}$$

which holds by (1.4.6). See also the argument in [vMS09, §5.1]. We give an example of such $\mathcal{W}$ in Example 1.4.12.

Fact 1.4.8 ([vMS09, (5.18), Lem. 5.12, Prop. 5.13], [vM11, Prop. 5.12])**.** Denoting $w_0 := s_1 \in W_0$, we define $U \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big) = \mathbb{K} \otimes \mathrm{End}\big(H_0\big)$ by

$$U(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) := \tau(e,w)\Phi \quad (w \in W_0).$$

Then the following statements hold.

(1) U is an invertible $\mathrm{End}(H_0)$-valued solution of the bqKZ equation. In particular, under the natural isomorphism $\mathbb{K} \otimes \mathrm{End}(H_0) \cong \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$, we have $U \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$.

(2) $U' \in \mathbb{K} \otimes \mathrm{End}\big(H_0\big)$ is an $\mathrm{End}\big(H_0\big)$-valued meromorphic solution of the bqKZ equation if and only if $U' = UF$ for some $F \in \mathbb{F} \otimes \mathrm{End}\big(H_0\big)$.

(3) $U \in \mathrm{GL}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$ restricts to an $\mathbb{F}$-linear isomorphism $U\colon H_0^{\mathbb{F}} \to \mathrm{SOL}_{\mathrm{bqKZ}}$.

(4) $\{\tau(e,w)\Phi \mid w \in W_0\}$ is an $\mathbb{F}$-basis of $\mathrm{SOL}_{\mathrm{bqKZ}}$.

We turn to the second step, which requires the following Fact 1.4.9–Fact 1.4.11.

Fact 1.4.9 ([vMS09, Lemma 6.5 (ii), (6.3)])**.** For $F \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$, we denote by

$$\phi_{\chi,v}^F := \chi(Fv) \in \mathbb{K} \tag{1.4.8}$$

the matrix coefficient of F with respect to $\chi \in H_0^*$ and $v \in H_0$. Also, using U in Fact 1.4.8, we define a twisted algebra homomorphism $\vartheta'\colon D_q \to \mathrm{End}\big(\mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})\big)$ by

$$\vartheta'(f)F = fF, \quad \vartheta'(\mathbf{w})F = \mathbf{w}(F)U^{-1}(\tau(\mathbf{w})U)$$

for $f \in \mathbb{C}(x,\xi)$, $\mathbf{w} \in \mathbb{W}$ and $F \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$. Then we have the following.

(1) ϑ' is an algebra homomorphism.

(2) For $D = \sum_{\mathbf{s}\in\mathbb{W}_0} D_{\mathbf{s}}\mathbf{s} \in D_q^{\mathbb{W}}$ (see (1.3.1)), we have

$$\phi_{\chi,v}^{\vartheta'(D)U} = \sum_{\mathbf{s}\in\mathbb{W}_0} D_{\mathbf{s}}(\phi_{\chi,v}^{C_{\mathbf{s}}^{-1}U}). \tag{1.4.9}$$

(3) If $\chi \in H_0^*$ satisfies $\chi(C_{\mathbf{s}}U) = \chi(U)$ for all $\mathbf{s} \in \mathbb{W}_0$, then we have

$$\mathrm{Res}(D)(\phi_{\chi,v}^U) = \phi_{\chi,v}^{\vartheta'(D)U}$$

for any $D \in D_q^{\mathbb{W}}$ and $v \in H_0$.

Fact 1.4.10 ([vMS09, Proposition 6.9])**.** For $h \in H(1/k)$, we have

$$\vartheta'(D^x_h)U = \eta_L(h^\dagger)U, \tag{1.4.10}$$

where $\dagger\colon H(1/k) \to H(k)$ is the unique algebra anti-isomorphism satisfying

$$T_1^\dagger = T_1^{-1}, \quad \pi^\dagger = \pi^{-1}.$$

Similarly, for $h' \in H(k)$, we have

$$\vartheta'(D^\xi_{h'})U = C_\iota \iota(\eta_L(h'^\ddagger))C_\iota U, \tag{1.4.11}$$

where $\ddagger\colon H(k) \to H(k)$ is the unique algebra anti-involution satisfying

$$T_1^\ddagger = T_1, \quad \pi^\ddagger = \pi^{-1}.$$

Fact 1.4.11 ([vMS09, Lemma 6.10])**.** For $p \in \mathbb{C}[z^{\pm 1}]^{W_0}$, we have

$$p(Y)^\dagger = p(Y)^\ddagger = p(Y^{-1}).$$

Now we can explain:

Part (ii) *of the proof of Fact 1.4.3.* We want to show $\chi_+(f) \in \mathrm{SOL}_{\mathrm{bMR}}(k,q)$ for $f \in \mathrm{SOL}_{\mathrm{bqKZ}}(1/k,q)$. By Fact 1.4.8 (2) and the $\mathbb{F}$-linearity of χ_+, it is enough to consider the case $f = Uv$ with $v \in H_0(1/k)$. Then $\chi_+(f) = \phi^U_{\chi_+,v}$ by (1.4.8).

Let us check the first equality of (1.3.9), extending it to general $p \in \mathbb{C}[T]^{W_0}$. By (1.3.5), we have

$$(L^x_p \phi^U_{\chi_+,v})(t,\gamma) = \bigl(\mathrm{Res}(D^x_{p(Y)})(\phi^U_{\chi_+,v})\bigr)(t,\gamma).$$

Now, by Lemma 1.4.5, χ_+ satisfies the condition of Fact 1.4.9 (3). Then we have

$$\bigl(\mathrm{Res}(D^x_{p(Y)})(\phi^U_{\chi_+,v})\bigr)(t,\gamma) = \phi^{\vartheta'(D^x_{p(Y)})U}_{\chi_+,v}(t,\gamma),$$

Then, by (1.4.10) in Fact 1.4.10 and by Fact 1.4.11, we have

$$\phi^{\vartheta'(D^x_{p(Y)})U}_{\chi_+,v}(t,\gamma) = \phi^{\eta_L(p(Y)^\dagger)U}_{\chi_+,v}(t,\gamma) = \phi^{\eta_L(p(Y^{-1}))U}_{\chi_+,v}(t,\gamma).$$

Finally, by Fact 1.2.15 and that p is W_0-invariant, we have

$$\phi^{\eta_L(p(Y^{-1}))U}_{\chi_+,v}(t,\gamma) = p(\gamma^{-1})\phi^U_{\chi_+,v}(t,\gamma).$$

Hence we have the desired equality $(L^x_p\chi_+(f))(t,\gamma) = p(\gamma^{-1})\chi_+(f)(t,\gamma)$.

Similarly, we can prove the second equality of (1.3.9), using (1.4.11) instead of (1.4.10). □

Example 1.4.12. We cite from [vMS09, vM11, St14] two examples of the function $\mathcal{W}$ in (1.4.5).

(1) We denote the Jacobi theta function with elliptic nome q by

$$\theta(z;q) := (q, z, q/z; q)_\infty = \prod_{n\in\mathbb{N}} (1-q^{n+1})(1-q^n z)(1-q^{n+1}/z),$$

using the q-shifted factorial (0.2.1). It enjoys the properties

$$\theta(qx;q) = \theta(x^{-1};q) = -x^{-1}\theta(x;q), \quad \theta(qx^{-1};q) = \theta(x;q), \tag{1.4.12}$$

Then, denoting

$$\theta(z,z';q) := \theta(z;q)\theta(z';q), \tag{1.4.13}$$

we define the meromorphic function $\mathcal{W}^{A_1}$ of x, ξ by

$$\mathcal{W}^{A_1}(x,\xi) = \mathcal{W}^{A_1}(x,\xi;k,q) := \frac{\theta(-q^{1/4}x\xi;q^{1/2})}{\theta(-q^{1/4}kx, -q^{1/4}k^{-1}\xi; q^{1/2})}. \tag{1.4.14}$$

By the above identities, it satisfies the properties (1.4.6) and (1.4.7). Let us write them again:

$$\mathcal{W}^{A_1}(q^{\pm 1/2}x,\xi;k,q) = (k/\xi)^{\pm 1}\mathcal{W}^{A_1}(x,\xi;k,q), \tag{1.4.15}$$

$$\mathcal{W}^{A_1}(\xi^{-1},x^{-1};k,q) = \mathcal{W}^{A_1}(x,\xi;k^*,q). \tag{1.4.16}$$

We used the redundant notation $k^* = k$ again for the comparison with the $(C_1^\vee, C_1)$ case (2.4.10).

(2) For later use, let us cite another function $\widehat{\mathcal{W}} \in \mathbb{K} = \mathcal{M}(x, \xi)$ from [St14, p.279]:

$$\widehat{\mathcal{W}}^{A_1}(x, \xi) = \widehat{\mathcal{W}}^{A_1}(x, \xi; k, q) := \frac{\theta(-q^{1/4}k^{-1}x\xi; q^{1/2})}{\theta(-q^{1/4}x; q^{1/2})}. \tag{1.4.17}$$

This function satisfies the q-difference equation

$$\widehat{\mathcal{W}}^{A_1}(q^{\pm 1/2}x, \xi; k, q) = (k/\xi)^{\pm 1}\widehat{\mathcal{W}}^{A_1}(x, \xi; k, q), \tag{1.4.18}$$

but does not satisfy the self-duality.

Remark 1.4.13. We give a few comments on the function $\mathcal{W}^{A_1}$ in Example 1.4.12 (1).

(1) The function $\mathcal{W}^{A_1}$ is equivalent to $G(t, \gamma)$ in [vM11, (5.8)], and equivalent to the function $\mathcal{W}$ [St14, §3.2] with k replaced by k^{-1}. This parameter difference comes from the choice of the basic representation $\rho^x_{k^{-1},q}$ in [vMS09, vM11] and $\rho^x_{k,q}$ in [St14] (see Remark 1.3.2).

(2) Let us explain the function $G(t, \gamma)$ in [vM11], and how to obtain the function $\mathcal{W}^{A_1}(x, \xi)$ from it. We use the torus $T = \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$, the notation t^λ of the value of $t \in T$ at $\lambda \in \Lambda$, the notation of a point $(t, \gamma) \in T \times T$, the ring $\mathbb{L}' = \mathbb{C}[T \times T]$ and the isomorphism $\mathbb{L}' \cong \mathbb{L} = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]$ explained in Remark 1.2.2. The outline is that $G(t, \gamma)$ is defined to be an element of $\mathcal{M}(T \times T)$, i.e., a meromorphic function on $T \times T$, and the function $\mathcal{W}^{A_1}(x, \xi)$ is obtained from $G(t, \gamma)$ under the isomorphism $\mathcal{M}(T \times T) \cong \mathcal{M}(x, \xi)$ induced by $\mathbb{L}' \cong \mathbb{L}$.

Let $\vartheta = \vartheta^{A_1}$ be the theta function associated to the weight lattice $\Lambda = \mathbb{Z}\varpi$ of type A_1 in the sense of Looijenga [L76]. It is a meromorphic function on the torus $T := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^\times)$, and the value at a point $t \in T$ is given by

$$\vartheta(t) = \vartheta^{A_1}(t) := \sum_{\lambda \in \Lambda} q^{1/2\langle \lambda, \lambda \rangle} t^\lambda \tag{1.4.19}$$

Let us also denote $w_0 := s_1 \in W_0$ and

$$\gamma_0 = \gamma_0^* := k^\alpha \in T,$$

which are borrowed from [St14, (2.3),(2.4)]. There the general types are treated in a uniform way under the notation $\gamma_{0,d}$ for our γ_0^*. The symbol $*$ indicates the duality anti-involution (1.1.22). Then, the meromorphic function G on $T \times T$ is defined to be

$$G(t, \gamma) := \frac{\vartheta(\mathrm{t}(w_0\gamma)^{-1})}{\vartheta(\gamma_0 t)\, \vartheta((\gamma_0^*)^{-1}\gamma)}. \tag{1.4.20}$$

Next we explain how to obtain $\mathcal{W}^{A_1}(x, \xi)$ from $G(t, \gamma)$. Using the coordinate $x = (t \mapsto t^\varpi)$, we can rewrite the lattice theta function as

$$\vartheta(t) = \sum_{n \in \mathbb{Z}} q^{l^2/4} x^n = \theta(-q^{1/4}x; q^{1/2}).$$

Using the other coordinate $\xi = (\gamma \mapsto \gamma^\varpi)$, we can also rewrite $tw_0(\gamma)^{-1}$ as $(tw_0(\gamma)^{-1})^\varpi = (t\gamma)^\varpi = x\xi$, $\gamma_0 t$ as $(\gamma_0 t)^\varpi = k^{\langle \alpha, \varpi \rangle} t^\varpi = kx$, and $(\gamma_0^*)^{-1}\gamma$ as $((\gamma_0^*)^{-1}\gamma)^\varpi = k^{-\langle \alpha, \varpi \rangle}\gamma^\varpi = k^{-1}\xi$. Hence, we obtain the function $\mathcal{W}^{A_1}(x, \xi)$.

1.5. **Bispectral Macdonald-Ruijsenaars function of type** A_1**.** In this subsection, we give an explicit solution of the bispectral Macdonald-Ruijsenaars q-difference equation of type A_1, following [NSh] and [St14, §5.3]. One caution is that we work on

$$\mathrm{SOL}_{\mathrm{bMR}}(1/k, q),$$

so that the reciprocal parameter k^{-1} is used in this subsection. As in the previous Fact 1.4.3, we assume $0 < q < 1$. Let us denote $\nu := q^{1/2}$.

Let us write again the bispectral Macdonald-Ruijsenaars equation (1.3.9):

$$\begin{cases} (L^x_{p_1} f)(x, \xi) & = (\xi + \xi^{-1}) f(x, \xi) \\ (L^\xi_{p_1} f)(x, \xi) & = (x + x^{-1}) f(x, \xi) \end{cases}. \tag{1.5.1}$$

By Proposition 1.3.5 and Remark 1.3.6, the operators can be written as

$$L^x_{p_1} = L(x; k, q), \quad L^\xi_{p_1} = L(\xi; k^{-1}, q^{-1}), \tag{1.5.2}$$

$$L(x; k, q) := \frac{k - k^{-1}x^{-2}}{1 - x^{-2}} T_{\nu,x} + \frac{k^{-1} - kx^{-2}}{1 - x^{-2}} T^{-1}_{\nu,x}. \tag{1.5.3}$$

First, we consider the asymptotic form of the x-side q-difference equation

$$\big(L^x_{p_1} - (\xi + \xi^{-1})\big) f(x) = 0$$

in the region $|x| \gg 1$. From (1.5.3) (also recall Remark 1.3.6), the asymptotic form is

$$L^x_{p_1} \approx L^x_{(\infty)} := kT_{\nu,x} + k^{-1}T^{-1}_{\nu,x}.$$

Similarly, in the region $|\xi| \ll 1$, we have

$$L^\xi_{p_1} \approx L^\xi_{(0)} := k^{-1}T_{\nu,\xi} + kT^{-1}_{\nu,\xi},$$

Now recall the functions $\mathcal{W}^{A_1}(x, \xi; 1/k, q)$ and $\widehat{\mathcal{W}}^{A_1}(x, \xi; 1/k, q)$:

$$\mathcal{W}^{A_1}(x, \xi; 1/k, q) = \frac{\theta(-\nu^{1/2}x\xi; \nu)}{\theta(-\nu^{1/2}k^{-1}x, -\nu^{1/2}k\xi; \nu)}, \quad \widehat{\mathcal{W}}^{A_1}(x, \xi; 1/k, q) := \frac{\theta(-\nu^{1/2}kx\xi; \nu)}{\theta(-\nu^{1/2}x; \nu)}. \tag{1.5.4}$$

Lemma 1.5.1. The sets $\{\mathcal{W}^{A_1}(x, \xi^{\pm 1}; 1/k, q)\}$ and $\{\widehat{\mathcal{W}}^{A_1}(x, \xi^{\pm 1}; 1/k, q)\}$ are bases of solutions of the asymptotic q-difference equation

$$\big(L^x_{(\infty)} - (\xi + \xi^{-1})\big) f(x) = 0.$$

Similarly, the sets $\{\mathcal{W}^{A_1}(x^{\pm 1}, \xi; 1/k, q)\}$ and $\{\widehat{\mathcal{W}}^{A_1}(x^{\pm 1}, \xi; k1/, q)\}$ are bases of solutions of

$$\big(L^\xi_{(0)} - (x + x^{-1})\big) g(\xi) = 0.$$

Proof. As seen before, we have $T^{\pm 1}_{\nu,x} f(x) = (k\xi)^{\mp 1} f(x)$ for $f(x) := \mathcal{W}^{A_1}(x^{\pm 1}, \xi; 1/k, q)$, so that these functions are solutions of the x-side equation. Since the equation is second-order and these functions are linear independent by the property of the Jacobi theta function $\theta(x; q)$, we have the x-side statement. The ξ-side is shown similarly using $T^{\pm 1}_{\nu,\xi}\mathcal{W}^{A_1}(x, \xi; 1/k, q) = (x/k)^{\mp 1}\mathcal{W}^{A_1}(x, \xi; 1/k, q)$. The same argument works for $\widehat{\mathcal{W}}^{A_1}$. □

Next, let us recall Heine's basic hypergeometric q-difference equation [GR04, Chap. 1, Exercise 1.13]:

$$\big(D^z_H(a, b, c; q)u\big)(z) = 0, \tag{1.5.5}$$

where the operator D^z_H is given by

$$D^z_H(a, b, c; q) := z(c - abqz)\partial_q^2 + \Big(\frac{1-c}{1-q} + \frac{(1-a)(1-b) - (1-abq)}{1-q}z\Big)\partial_q + \frac{(1-a)(1-b)}{(1-q)^2} \tag{1.5.6}$$

with $(\partial_q u)(z) := \big(u(z) - u(qz)\big)/\big((1-q)z\big)$. A solution of (1.5.5) is given by Heine's basic hypergeometric function

$$u(z) = {}_2\phi_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; q,\ z\right], \tag{1.5.7}$$

where we used the notation (0.2.2).

The following relation between the Macdonald q-difference operator of type A_1 and Heine's basic hypergeometric q-difference equation is well known.

Lemma 1.5.2 (c.f. [St14, Lemma 5.4])**.** Let $\mathcal{W}(x)$ be a meromorphic function in x satisfying

$$\mathcal{W}(q^{\pm 1/2}x) = (k\xi)^{\mp 1}\mathcal{W}(x). \tag{1.5.8}$$

Then, the function $f(x) = \mathcal{W}(x)u(k^{-2}qx^{-2})$ is a meromorphic solution of the q-difference equation

$$(L^x_{p_1} f)(x) = (\xi + \xi^{-1}) f(x)$$

if and only if $u(z)$ is a meromorphic solution of the q-difference equation

$$(D^z_H(k^2, k^2\xi^2, q\xi^2)u)(z) = 0, \quad z = k^{-2}qx^{-2}.$$

Proof. A direct computation yields that the operator $D^z_H(a, b, c; q)$ in (1.5.6) is proportional to

$$D'(a, b, c; q) := (c/q - abz)T^2_{q,z} - (1 + c/q - (a+b)z)T_{q,z} + (1 - z).$$

If $a/b = q/c$, then $D'(a, ac/q, c; q) = (c/q)(1 - a^2z)T^2_{q,z} - (1 + c/q)(1 - az)T_{q,z} + (1 - z)$. Hence, defining

$$D''(a, c; q) := T^{-1}_{q,z}\frac{1}{1-az}D'_z(a, ac/q, c; q) = cq^{-1}\frac{1 - a^2z/q}{1 - az/q}T_{q,z} + \frac{1 - z/q}{1 - az/q}T^{-1}_{q,z} - (1 + c/q),$$

we have $(D_H^z(a, ac/q, c; q)u)(z) = 0 \iff (D''(a, c; q)u)(z) = 0$. If moreover $z = k^{-2}qx^{-2}$, $a = k^2$ and $c = q\xi^2$, then we have

$$\big(D_H^z(k^2, k^2\xi^2, q\xi^2; q)u\big)(z) = 0 \iff \big(\xi^{-1}D''(k^2, q\xi^2; q)u\big)(z) = 0$$
$$\iff \Big(\frac{1-k^2x^{-2}}{1-x^{-2}}\xi T_{q,z} + \frac{1-k^{-2}x^{-2}}{1-x^{-2}}\xi^{-1}T_{q,z}^{-1} - (\xi+\xi^{-1})\Big)u(z) = 0.$$

On the other hand, by the expression (1.5.2) and the condition (1.5.8), we have

$$\big((L_{p_1}^x - (\xi+\xi^{-1}))f\big)(x) = 0$$
$$\iff \Big(\frac{k-k^{-1}x^{-2}}{1-x^{-2}}k^{-1}\xi^{-1}T_{q,z}^{-1} + \frac{k^{-1}-kx^{-2}}{1-x^{-2}}k\xi T_{q,z} - (\xi+\xi^{-1})\Big)u(z) = 0$$
$$\iff \Big(\frac{1-k^2x^{-2}}{1-x^{-2}}\xi T_{q,z} + \frac{1-k^{-2}x^{-2}}{1-x^{-2}}\xi^{-1}T_{q,z}^{-1} - (\xi+\xi^{-1})\Big)u(z) = 0.$$

Thus we have the desired equivalence. □

Now we give an explicit bispectral solution of (1.5.1).

Proposition 1.5.3 (c.f. [NSh, Theorems 2.1, 2.2, (3.13)], [St14, Cor. 5.5])**.** We denote $\nu \coloneqq q^{1/2}$.

(1) Define the function $f^{A_1}(x,\xi)$ by

$$f^{A_1}(x,\xi) = f^{A_1}(x,\xi;k,q) \coloneqq \mathcal{W}^{A_1}(x,\xi;1/k,q)\,\varphi^{A_1}(x,\xi;k,q),$$
$$\varphi^{A_1}(x,\xi) = \varphi^{A_1}(x,\xi;k,q) \coloneqq \frac{(q\xi^2;q)_\infty}{(k^{-2}q\xi^2;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} k^2,\ k^2\xi^2 \\ q\xi^2 \end{matrix};q,\ \frac{q}{k^2x^2}\right]. \tag{1.5.9}$$

Here we used the function $\mathcal{W}^{A_1}(x,\xi;1/k,q)$ in (1.5.4), and assumed $\left|k^{-2}qx^{-2}\right| < 1$. Then f^{A_1} satisfies the following properties.

(i) It is a solution of the bispectral problem (1.5.1).

(ii) It has the symmetry (the inversion invariance in [St14])

$$f^{A_1}(x,\xi) = f^{A_1}(x^{-1};\xi) = f^{A_1}(x,\xi^{-1}).$$

(iii) It has the self-duality

$$f^{A_1}(x,\xi;k,q) = f^{A_1}(\xi^{-1};x^{-1};k^*,q),$$

using the redundant notation $k^* = k$ for the comparison with the $(C_1^\vee, C_1)$ case.

Recalling the $\mathbb{W}$-action on $\mathbb{K} = \mathcal{M}(T\times T)$ in (1.2.12), we express the subset of $\mathrm{SOL}_{\mathrm{bMR}}(1/k,q)$ satisfying these properties as

$$\mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q) \coloneqq \{f \in \mathrm{SOL}_{\mathrm{bMR}}(1/k,q) \mid \text{ (ii), (iii)}\}.$$

Thus, we can restate the claim as

$$f^{A_1} \in \mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q).$$

(2) Defining $\xi_n \coloneqq k^{-1}\nu^{-n}$ for $n\in\mathbb{N}$, we have

$$f^{A_1}(x,\xi_n) = c_n P_n^{A_1}(x),$$
$$c_n \coloneqq \frac{(-k)^{-n}\nu^{-\binom{n+1}{2}}}{\theta(-k^2\nu^{n+\frac12};\nu)}\frac{(k^{-2}q^{1-n};q)_\infty}{(k^{-4}q^{1-n};q)_\infty}, \quad P_n^{A_1}(x) \coloneqq x^n\,{}_2\phi_1\left[\begin{matrix} k^2,\ q^{-n} \\ k^{-2}q^{1-n} \end{matrix};q,\ \frac{q}{k^2x^2}\right]. \tag{1.5.10}$$

The function $P_n^{A_1}(x)$ satisfies the following three conditions.

(i) It is an eigenfunction of the Macdonald-Ruijsenaars q-difference operator $L_{p_1}^x$ of type A_1.

(ii) It is a Laurent polynomial in x belonging to $x^n\mathbb{C}[x^{-1}]$, and is invariant under the replacement $x\mapsto x^{-1}$.

Moreover, these conditions uniquely determine the function $P_n^{A_1}(x)$ up to constant multiplication, and the eigenvalue in (i) is $p_1(\xi_n^{-1}) = \xi_n^{-1} + \xi_n$.

We will give an almost self-consistent proof, except the following equality (1.5.11).

Fact 1.5.4 ([NSh, (4.11)])**.** The function $\varphi^{A_1}(x,\xi)$ satisfies

$$\varphi^{A_1}(x,\xi) = \frac{(k^2, qx^{-2}\xi^2;q)_\infty}{(k^{-2}qx^{-2}, k^{-2}q\xi^2;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} k^{-2}qx^{-2},\ k^{-2}q\xi^2 \\ qx^{-2}\xi^2 \end{matrix};q,\ k^2\right] \tag{1.5.11}$$

under the condition $|k| < 1$. In particular, we have

$$\varphi^{A_1}(x, \xi) = \varphi^{A_1}(\xi^{-1}; x^{-1}). \tag{1.5.12}$$

The equality (1.5.11) can be shown using Heine's transformation formula for ${}_2\phi_1$ series [GR04, (1.4.1)]. See also [NSh, (4.10)] for the calculation.

Proof of Proposition 1.5.3. For (1), we follow the argument of [St14, Lemma 2.18]. Let us denote $\mathcal{W}^{A_1}(x, \xi) := \mathcal{W}^{A_1}(x, \xi; 1/k, q)$ for simplicity, and recall the quasi-periodicity and the self-duality:

$$\mathcal{W}^{A_1}(\xi^{-1}, x^{-1}) = \mathcal{W}^{A_1}(x, \xi), \quad \mathcal{W}^{A_1}(\nu x, \xi) = (k\xi)^{-1}\mathcal{W}(x, \xi). \tag{1.5.13}$$

The first equality of (1.5.13) and (1.5.12) yield the self-duality (iii). The second equality of (1.5.13) is nothing but the condition (1.5.8), so that Lemma 1.5.2 and (1.5.7) yield

$$L^x_{p_1} f^{A_1}(x, \xi) = (\xi + \xi^{-1}) f^{A_1}(x, \xi) = p_1(\xi^{-1}) f^{A_1}(x, \xi). \tag{1.5.14}$$

On the other hand, (1.5.2) shows $L^{\xi}_{p_1} = L(\xi; k^{-1}, q^{-1}) = IL(\xi; k, q)I$, where I is the operator $g(\xi) \mapsto (Ig)(\xi) := g(\xi^{-1})$ for a function $g(\xi)$. Then, the self-duality (iii) and the eigen-property (1.5.14) imply

$$\begin{aligned} L^{\xi}_{p_1} f^{A_1}(x, \xi) &= \big(IL(\xi; k, q)I\big) f^{A_1}(\xi^{-1}; x^{-1}) = IL(\xi; k, q) f^{A_1}(\xi; x^{-1}) = I\big(p_1(x) f^{A_1}(\xi; x^{-1})\big) \\ &= p_1(x) f^{A_1}(\xi^{-1}; x^{-1}) = (x + x^{-1}) f^{A_1}(x, \xi). \end{aligned}$$

Hence (iii) holds.

Before showing (1) (ii), we show (2). The equality in the statement is a consequence of

$$\mathcal{W}^{A_1}(x, \xi_n; 1/k, q) = (-\nu^{-1/2} k^{-1} x)^n \nu^{-\binom{n}{2}} = x^n c_n,$$

which can be checked using $\theta(x; q) = (q, x, q/x; q)_\infty$. The condition (2) (i) is a consequence of (1.5.14). The condition (2) (ii) can be checked by the formula 1.5.10 (see also Remark 1.5.5 (1)). The uniqueness is well-known in the theory of Macdonald polynomials (see also Remark 1.5.5 (1)).

Now we show the remaining (1) (ii). By (2) (ii), we have $f^{A_1}(x, \xi_n) = f^{A_1}(x^{-1}; \xi_n)$ for any $l \in \mathbb{N}$. Then, applying the identity theorem in complex analysis to the analytic function $g(\xi) := f^{A_1}(x, \xi) - f^{A_1}(x^{-1}; \xi)$, we have $f^{A_1}(x, \xi) = f^{A_1}(x^{-1}; \xi)$ for any ξ in the domain of definition. Combining it with the self-duality (1) (iii), we have $f^{A_1}(x, \xi) = f^{A_1}(x, \xi^{-1})$. Hence we have (1) (ii). □

Remark 1.5.5. Some comments on Proposition 1.5.3 are in order.

(1) Defining $\beta \in \mathbb{C}$ by $k = \nu^\beta$, the Laurent polynomial $P^{A_1}_n$ is equal to

$$P^{A_1}_n(x) = \begin{bmatrix} \beta + n - 1 \\ n \end{bmatrix}_q^{-1} \sum_{i+j=n} \begin{bmatrix} \beta + i - 1 \\ i \end{bmatrix}_q \begin{bmatrix} \beta + j - 1 \\ j \end{bmatrix}_q x^{i-j}, \tag{1.5.15}$$

where we used the q-binomial coefficient (0.2.3). It is nothing but the Macdonald symmetric polynomial of type A_1 [M03, (6.3.7)], and is proportional to the continuous q-ultraspherical polynomial, or the Rogers polynomial. See [M03, §6.3, pp.156–157] for the detail.

(2) In [NSh], Noumi and Shiraishi gave an explicit bispectral solution $f(x_1, \ldots, x_n; s_1, \ldots, s_n)$ of type GL_n. The above solution $f^{A_1}(x, \xi)$ is obtained by specializing $(x_1, x_2) = (x, x^{-1})$ and $(s_1, s_2) = (\xi, \xi^{-1})$ in the solution $f(x_1, x_2; s_1, s_2)$ of type GL_2. See also Stokman [St14, Corollary 5.5] for the uniqueness of $f(x_1, x_2; s_1, s_2)$.

Let us cite another bispectral solution.

Fact 1.5.6 ([St14, Theorem 4.6, (5.18)])**.** Define a meromorphic function $\mathcal{E}^{A_1}_+(x, \xi) = \mathcal{E}^{A_1}_+(x, \xi; k, q) \in \mathbb{K} = \mathcal{M}(x, \xi)$ by

$$\begin{aligned} \mathcal{E}^{A_1}_+(x, \xi; k, q) &:= \frac{\theta(-\nu^{1/2}k; \nu)}{\theta(-\nu^{1/2}\xi; \nu)} \frac{(k^2\xi^{-2}, k^2; q)_\infty}{(\xi^{-2}, k^4; q)_\infty} \widehat{\mathcal{W}}^{A_1}(x, \xi; 1/k, q)\, {}_2\phi_1\left[\begin{matrix} k^2, \ k^2\xi^2 \\ q\xi^2 \end{matrix}; q, \ \frac{q}{k^2x^2}\right] + (\xi \mapsto \xi^{-1}) \\ &= \frac{\theta(-\nu^{1/2}k, -\nu^{1/2}kx\xi; \nu)}{\theta(-\nu^{1/2}\xi, -\nu^{1/2}x; \nu)} \frac{(k^2\xi^{-2}, k^2; q)_\infty}{(\xi^{-2}, k^4; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} k^2, \ k^2\xi^2 \\ q\xi^2 \end{matrix}; q, \ \frac{q}{k^2x^2}\right] + (\xi \mapsto \xi^{-1}), \end{aligned} \tag{1.5.16}$$

where the second term is obtained by replacing ξ in the first term with ξ^{-1}. Then the function $\mathcal{E}^{A_1}_+$ enjoys the following properties (i)–(iii).

(i) It is a solution of the bispectral problem (1.5.1).

(ii) It has the symmetry (the inversion invariance in [St14])
$$\mathcal{E}_+^{A_1}(x,\xi) = \mathcal{E}_+^{A_1}(x^{-1};\xi) = \mathcal{E}_+^{A_1}(x,\xi^{-1}).$$
(iii) It has the self-duality
$$\mathcal{E}_+^{A_1}(x,\xi;k,q) = \mathcal{E}_+^{A_1}(\xi^{-1};x^{-1};k^*,q),$$
using the redundant notation $k^* = k$ for the comparison with the $(C_1^\vee, C_1)$ case.

Recalling the $\mathbb{W}$-action on $\mathbb{K} = \mathcal{M}(x,\xi)$ in (1.2.12), we express the subset of $\mathrm{SOL}_{\mathrm{bMR}}(1/k,q)$ satisfying these properties as
$$\mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q) := \{f \in \mathrm{SOL}_{\mathrm{bMR}}(1/k,q) \mid \text{(ii), (iii)}\}.$$
Thus, we can restate the claim as
$$\mathcal{E}_+^{A_1} \in \mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q).$$
Following [St14], we call it *the basic hypergeometric function of type A_1*.

Remark 1.5.7. Some comments on the function $\mathcal{E}_+^{A_1}$ are in order.

(1) As explained right after [St14, Definition 2.19], we have the basic hypergeometric function of arbitrary type. The reduced case, including the above $\mathcal{E}_+^{A_1}(x,\xi;k,q)$, was introduced by Cherednik [C97, C09] under the name of global spherical function. The non-reduced case (type $(C_1^\vee, C_1)$) was introduced by Stokman [St02], and the uniform approach was discussed in [St14]. The GL_2 type is written down in [St14, (5.18)], from which we can recover the A_1 case.

(2) Although we take (1.5.16) as the definition of the basic hypergeometric function $\mathcal{E}_+^{A_1}$, the actual statement of [St14, Theorem 4.6] is that $\mathcal{E}_+$ (of arbitrary type) has *the c-function expansion* with respect to the self-dual basic Harish-Chandra series Φ (see Fact 1.4.6 for type A_1), and defined for generic $\eta \in T$. The c-function expansion is given in the form
$$\mathcal{E}_+(t,\gamma;k,q) = \sum_{w \in W_0} \mathfrak{c}(t, w\gamma; k,q)\Phi(t,w\gamma;k,q).$$

2. Type $(C_1^\vee, C_1)$

We discuss the type $(C_1^\vee, C_1)$, or the non-reduced type. See also [St14, §3, §5.2].

2.1. **Extended affine Hecke algebra.** First, we recall the affine root system of type $(C_1^\vee, C_1)$ and the extended affine Weyl group, following [M03, §1, §2, §6.4].

We consider the one-dimensional real Euclidean space $(V, \langle\cdot,\cdot\rangle)$ with
$$V = \mathbb{R}\epsilon, \quad \langle \epsilon, \epsilon \rangle = 1.$$
Similarly as in §1.1.1, we denote by F the space of affine real functions on V, and identify it with $V \oplus \mathbb{R}c$. Using the gradient map $D\colon F \to V$, we extend $\langle\cdot,\cdot\rangle$ to F.

Let $S(C_1^\vee, C_1) := \{m(\pm\epsilon + \frac{1}{2}n) \mid m \in \{1,2\},\, n \in \mathbb{Z}\}$ be the affine root system $S(C_1^\vee, C_1)$ in the sense of Macdonald [M03]. A basis is given by $\{a_0 := \frac{1}{2}c - \epsilon, a_1 := \epsilon\}$, and the corresponding simple reflections $s_i\colon V \to V$ for $i = 0,1$ are given by the formula (1.1.2) with $a_i^\vee := 2a_i/\langle a_i, a_i\rangle = 2a_i \in F$. Explicitly, we have
$$s_1(r\epsilon) = -r\epsilon, \quad s_0(r\epsilon) = (1-r)\epsilon \quad (r \in \mathbb{R}). \tag{2.1.1}$$
We denote $W_0 := \langle s_1 \rangle \subset O(V, \langle\cdot,\cdot\rangle)$, which is isomorphic to $\mathfrak{S}_2$. The W_0-action (2.1.1) on V preserves
$$\Lambda := \mathbb{Z}\epsilon \subset V,$$
the coroot lattice of the root system $R(C_1) = \{\pm 2\epsilon\}$ generated by $(2\epsilon)^\vee = \epsilon$. We also denote by $\mathrm{t}(\Lambda) = \{\mathrm{t}(\lambda) \mid \lambda \in \Lambda\}$ is the abelian group with relations $\mathrm{t}(\lambda)\,\mathrm{t}(\mu) = \mathrm{t}(\lambda+\mu)$ for $\lambda, \mu \in \Lambda$. The group $\mathrm{t}(\Lambda)$ acts on V by translation (1.1.4). Then, the extended affine Weyl group W of $S(C_1^\vee, C_1)$ is defined to be the subgroup of the isometries on $(V, \langle\cdot,\cdot\rangle)$ generated by W_0 and $\mathrm{t}(\Lambda)$.
$$W := W_0 \ltimes \mathrm{t}(\Lambda). \tag{2.1.2}$$
In particular, we have the relation
$$s_1\, \mathrm{t}(\lambda) s_1 = \mathrm{t}(s_1(\lambda)) \quad (\lambda \in \Lambda) \tag{2.1.3}$$
with $s_1(\lambda)$ given by (2.1.1).

As an abstract group, W is generated by s_0 and s_1 with fundamental relations
$$s_0^2 = s_1^2 = e. \tag{2.1.4}$$

The following relations hold in W.

$$\mathrm{t}(\epsilon) = s_0 s_1, \quad \mathrm{t}(-\epsilon) = s_1 s_0. \tag{2.1.5}$$

Compare the first relation with (1.1.9): denoting $s_i^{A_1}$ $(i = 0, 1)$ for the generators of the extended Weyl group W^{A_1} of $S(A_1)$, we have $\mathrm{t}(\alpha) = s_0^{A_1} s_1^{A_1}$.

Next, we recall the extended affine Hecke algebra H associated to the affine root system $S(C_1^\vee, C_1)$. For the detail, see [M03, §4, §6.4]. Hereafter we fix nonzero complex numbers k_1, k_0, l_1, l_0 and denote

$$\underline{k} := (k_1, k_0), \quad \underline{l} := (l_1, l_0). \tag{2.1.6}$$

The symbols k_1 and k_0 are borrowed from [NSt04].

Remark 2.1.1. Our parameters (k_1, k_0, l_1, l_0) correspond to $(t_1^{1/2}, t_0^{1/2}, l_1^{1/2}, l_0^{1/2})$ in [N95] and [T10].

Definition 2.1.2. The extended affine Hecke algebra $H(\underline{k})$ is the $\mathbb{C}$-algebra generated by T_1 and T_0 with fundamental relations

$$(T_i - k_i)(T_i + k_i^{-1}) = 0 \quad (i = 1, 0). \tag{2.1.7}$$

In this § 2, we denote $H := H(\underline{k})$ for simplicity.

As in § 1.1, we denote by $\ell(w)$ the length of $w \in W$. If we have a reduced expression $w = s_{i_1} \cdots s_{i_l}$, $i_j \in \{0, 1\}$, then $\ell(w) = l$. For such $w \in W$, we set

$$T_w := T_{i_1} \cdots T_{i_l} \in H.$$

Then T_w is independent of the choice of reduced expression. We also define $Y^{\pm 1} \in H$ by

$$Y := T_0 T_1, \quad Y^{-1} := T_1^{-1} T_0^{-1}, \tag{2.1.8}$$

which can be regarded as deformations of $\mathrm{t}(\epsilon) \in W$ given in (2.1.5). As in the case of type A_1 (§ 1.1), the monomials in $\mathbb{C}[Y^{\pm 1}] \subset H$ are denoted as $Y^\lambda := Y^l$ for $\lambda = l\epsilon \in \Lambda$, $l \in \mathbb{Z}$. We also have a $\mathbb{C}$-linear isomorphism $H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}]$, where

$$H_0 := \mathbb{C} + \mathbb{C}T_1$$

is the subalgebra of H generated by T_1.

Remark 2.1.3. Our choice (2.1.8) of the Dunkl operator Y follows [M03, §6.4], which is the opposite of [N95, T10, St14]. The choice (2.1.8) is compatible with the choice for type A_1 (see (1.1.12)).

Next, we review Noumi's [N95] basic representation $\rho_{\underline{k},\underline{l},q}$ of $H = H(\underline{k})$. Choose and fix a parameter $q^{1/2} \in \mathbb{C}^\times$. The extended affine Weyl group W acts on the Laurent polynomial ring $\mathbb{C}[x^{\pm 1}]$ by

$$(s_{1,q} f)(x) = f(x^{-1}), \quad (s_{0,q} f)(x) = f(qx^{-1}), \quad (\mathrm{t}(\epsilon)_q f)(x) = f(qx) = (T_{q,x} f)(x), \tag{2.1.9}$$

where $T_{q,x}$ denotes the q-shift operator on the variable x. Then, we have an algebra embedding

$$\rho_{\underline{k},\underline{l},q} \colon H(\underline{k}) \longrightarrow \mathrm{End}(\mathbb{C}[x^\pm]), \quad \rho(T_i) := c(x_i; k_i, l_i) s_{i,q} + b(x_i; k_i, l_i) \quad (i = 1, 0) \tag{2.1.10}$$

with $x_1 := x^2$, $x_0 := qx^{-2}$ and

$$\begin{aligned} c(z; k, l) &:= k^{-1} \frac{(1 - klz^{1/2})(1 + kl^{-1} z^{1/2})}{1 - z}, \\ b(z; k, l) &:= k - c(z; k, l) = \frac{(k - k^{-1}) + (l - l^{-1}) z^{1/2}}{1 - z}. \end{aligned} \tag{2.1.11}$$

Here we understand $x_1^{1/2} = x$ and $x_0^{1/2} = q^{1/2} x^{-1}$. We call $\rho_{\underline{k},\underline{l},q}$ *the basic representation of* $H(\underline{k})$.

Definition 2.1.4. *The double affine Hecke algebra (DAHA) of type* $(C_1^\vee, C_1)$, denoted as

$$\mathbb{H} = \mathbb{H}(\underline{k}, \underline{l}, q) = \mathbb{H}^{(C_1^\vee, C_1)}(\underline{k}, \underline{l}, q),$$

is defined to be the $\mathbb{C}$-subalgebra of $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ generated by the multiplication operators by $x^{\pm 1}$ and the image $\rho_{\underline{k},\underline{l},q}(H(\underline{k}))$.

As an abstract algebra, the DAHA $\mathbb{H}$ of type $(C_1^\vee, C_1)$ is presented with generators $T_1, T_0, T_1^\vee, T_0^\vee$ and relations

$$\begin{aligned} &(T_i - k_i)(T_i + k_i^{-1}) = 0 \quad (T_i^\vee - l_i)(T_i^\vee + l_i^{-1}) = 0 \quad (i = 1, 0), \\ &T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}. \end{aligned} \tag{2.1.12}$$

See [Sa99], [NSt04], [M03, §4.7] and [C05] for the detail. The symbols $T_i^\vee$ are borrowed from [NSt04]. To recover Definition 2.1.4, we put

$$T_1^\vee = X^{-1}T_1^{-1}, \quad T_0^\vee = q^{-1/2}T_0^{-1}X, \tag{2.1.13}$$

by which we can extend the map $\rho_{\underline{k},\underline{l},q}$ of (2.1.12) to the embedding $\rho_{\underline{k},\underline{l},q} \colon \mathbb{H} \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$.

Similarly as the type A_1, we have the Poincaré-Birkhoff-Witt decomposition of $\mathbb{H}$:

$$\mathbb{H} \cong \mathbb{C}[X^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}], \tag{2.1.14}$$

and *the duality anti-involution*

$$* \colon \mathbb{H}(\underline{k},\underline{l},q) \longrightarrow \mathbb{H}(\underline{k}^*,\underline{l}^*,q), \quad h \longmapsto h^*, \tag{2.1.15}$$

which is a unique $\mathbb{C}$-algebra anti-involution determined by

$$T_1^* := T_1, \quad (Y^\lambda)^* := x^{-\lambda}, \quad (x^\lambda)^* := Y^{-\lambda}$$

for $\lambda \in \Lambda$ and

$$(\underline{k}^*,\underline{l}^*) = (k_1^*,k_0^*,l_1^*,l_0^*) := (k_1,l_1,k_0,l_0). \tag{2.1.16}$$

We also denote by

$$H(\underline{k},\underline{l})^* \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}]) \tag{2.1.17}$$

the image of $H(\underline{k},\underline{l}) \subset \mathbb{H}(\underline{k},\underline{l},q)$ under the duality anti-involution $*$.

2.2. **Bispectral quantum Knizhnik-Zamolodchikov equation.** Let us explain the bispectral qKZ equation of the affine root system $S(C_1^\vee,C_1)$, mainly following [T10, §4.1, §4.2]. Hereafter we choose and fix $k_1,k_0,l_1,l_0,q^{1/2} \in \mathbb{C}^\times$, and consider the affine Hecke algebra $H = H(\underline{k})$, the basic representation $\rho_{\underline{k},\underline{l},q} \colon H(\underline{k}) \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ and the DAHA $\mathbb{H} = \mathbb{H}(\underline{k},\underline{l},q)$.

2.2.1. *The affine intertwiners.* Following [C05, §1.3] and [T10, §4.2], we introduce the affine intertwines of type $(C_1^\vee,C_1)$. We set $x_1 := x^2$, $x_0 := qx^{-2}$, and define $\widetilde{S}_1,\widetilde{S}_0 \in \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ by

$$\widetilde{S}_i := d_i(x)s_i, \quad d_i(x) = d_i(x;\underline{k},\underline{l},q) := k_i^{-1}(1-k_il_ix_i^{1/2})(1+k_il_i^{-1}x_i^{1/2}) \quad (i=0,1). \tag{2.2.1}$$

The elements $\widetilde{S}_1$ and $\widetilde{S}_0$ belong to the subalgebra $\mathbb{H} \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ since

$$\widetilde{S}_i = (1-x_i)\rho_{\underline{k},\underline{l},q}(T_i) - (k_i-k_i^{-1}) - (l_i-l_i^{-1})x_i^{1/2}. \tag{2.2.2}$$

More generally, for each $w \in W$, taking a reduced expression $w = s_{j_1}\cdots s_{j_r}$ with $j_1,\dots,j_r \in \{0,1\}$, we define the element $\widetilde{S}_w \in \mathbb{H}$ by

$$\widetilde{S}_w := d_{j_1}(x)\cdot(s_{j_1}d_{j_2})(x)\cdot\cdots\cdot(s_{j_1}\cdots s_{j_{r-1}}d_{j_r})(x)\cdot w, \tag{2.2.3}$$

The element $\widetilde{S}_w \in \mathbb{H}$ is independent of the choice of reduced expression $w = s_{j_1}\cdots s_{j_r}$ by the same argument as the type A_1 case, using

$$d_w(x) := d_{j_1}(x)\cdot(s_{j_1}d_{j_2})(x)\cdot\cdots\cdot(s_{j_1}\cdots s_{j_{r-1}}d_{j_r})(x) \tag{2.2.4}$$

Also, by [T10, §4.1], we have

$$\widetilde{S}_w = \widetilde{S}_{j_1}\cdots\widetilde{S}_{j_r}. \tag{2.2.5}$$

We call the elements $\widetilde{S}_w$ in (2.2.3) *the affine intertwiners of type* $(C_1^\vee,C_1)$.

2.2.2. *The double extended affine Weyl group.* As in the case of type A_1 (§1.2.2), let us consider the ring

$$\mathbb{L} := \mathbb{C}[x^{\pm 1},\xi^{\pm 1}] \cong \mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[\xi^{\pm 1}].$$

We can regard $\mathbb{H}$ as an $\mathbb{L}$-module by

$$(f \otimes g)h := f(x)\,h\,g(Y) \tag{2.2.6}$$

for $f = f(x) \in \mathbb{C}[x^{\pm 1}]$, $g = g(\xi) \in \mathbb{C}[\xi^{\pm 1}]$ and $h \in \mathbb{H}$, where x is understood as the multiplication operator by x itself, and Y is the Dunkl operator. By the PBW type decomposition (2.1.14), we have an $\mathbb{L}$-module isomorphism

$$\mathbb{H} \cong H_0^{\mathbb{L}} := \mathbb{L} \otimes H_0. \tag{2.2.7}$$

As in the case of type A_1, we regard $f(x,\xi) \in H_0^{\mathbb{L}}$ as a function of x,ξ valued in H_0.

The double extended Weyl group $\mathbb{W}$ is introduced in the same way (1.2.10) as the type A_1 case. Let ι denote the nontrivial element of the group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, and define $\mathbb{W}$ to be the semi-direct product group

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W),$$

with $\iota \in \mathbb{Z}_2$ acting on $W \times W$ by $\iota(w, w') = (w', w)\iota$ for $(w, w') \in W \times W$.

The group W acts on $\mathbb{L}$ in the same way as the type A_1 (see § 1.2.2). Define the involution $\diamond\colon W \to W$ by (1.2.11), i.e., $w^\diamond := w$ for $w \in W_0$ and $\mathrm{t}(\lambda)^\diamond := \mathrm{t}(-\lambda)$ for $\lambda \in \Lambda$. Then the $\mathbb{W}$-action on $\mathbb{L}$ is given by

$$(wf)(x) := (w_q f)(x), \quad (w'g)(\xi) := ((w'^\diamond)_q g)(\xi), \quad (\iota F)(x, \xi) = F(\xi^{-1}, x^{-1}) \tag{2.2.8}$$

for $w \in W = W \times \{e\} \subset \mathbb{W}$, $w' \in W = \{e\} \times W \subset \mathbb{W}$ and $f = f(x), g = g(\xi), F = F(x, \xi) \in \mathbb{L}$. Here w_q denotes the W-action in (2.1.9).

We also define $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_\iota \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})$ by

$$\widetilde{\sigma}_{(w,w')}(h) := \widetilde{S}_w h \widetilde{S}^*_{w'}, \quad \widetilde{\sigma}_\iota(h) := h^*$$

for $h \in \mathbb{H}$, where $*$ is the duality anti-involution (2.1.15). Then, as in Fact 1.2.4, we have

$$\widetilde{\sigma}_{(w,w')}(fh) = ((w, w')f)\widetilde{\sigma}_{(w,w')}(h), \quad \widetilde{\sigma}_\iota(fh) = (\iota f)\widetilde{\sigma}_\iota(h) \tag{2.2.9}$$

for $h \in \mathbb{H}$, $f \in \mathbb{L}$ and $w, w' \in W$. The proof is essentially the same as Fact 1.2.4 ([vM11, Lemma 3.5]).

2.2.3. *The cocycle.* As in the case of type A_1 (see (1.2.15)), we denote by

$$\mathbb{K} := \mathcal{M}(x, \xi)$$

the meromorphic functions of variables x, ξ, and define

$$H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0 \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H},$$

We can express an element $f \in H_0^{\mathbb{K}}$ as (1.2.16): $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$, $f_w \in \mathbb{K}$. The $\mathbb{W}$-action (2.2.8) on $\mathbb{L}$ naturally extends to that on $\mathbb{K}$, and we have a $\mathbb{W}$-action on $H_0^{\mathbb{K}}$ by the formula (1.2.17):

$$\mathbf{w} f := \sum_{w \in W_0} (\mathbf{w} f_w) T_w \tag{2.2.10}$$

for $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

By the argument right before Fact 1.2.5, we have $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_\iota \in \mathrm{End}_{\mathbb{C}}(H_0^{\mathbb{K}})$ such that the formulas (2.2.9) are valid for $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. Then, similarly as Fact 1.2.5, we have:

Fact 2.2.1 ([T10, §4.2])**.** There is a unique group homomorphism $\tau\colon \mathbb{W} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$ satisfying

$$\tau(w, w')(f) = d_w(x)^{-1} d^*_{w'}(\xi^{-1})^{-1} \cdot \widetilde{\sigma}_{(w,w')}(f), \quad \tau(\iota)(f) = \widetilde{\sigma}_\iota(f)$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. Here we denoted by $d^*_{w'}$ the image of $d_{w'}$ under the duality anti-involution $*$ in (2.1.17), and $\cdot$ denotes the $\mathbb{L}$-action (2.2.6).

By the $\mathbb{W}$-action (2.2.10) on $H_0^{\mathbb{K}}$, we can regard $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as a $\mathbb{W}$-group via the corresponding conjugation action:

$$(\mathbf{w}, A) \longmapsto \mathbf{w} A \mathbf{w}^{-1} \quad (\mathbf{w} \in \mathbb{W},\ A \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})).$$

Then, we have the following analogue of Fact 1.2.7.

Fact 2.2.2 ([T10, §4.2])**.** The map

$$\mathbf{w} \longmapsto C_{\mathbf{w}} := \tau(\mathbf{w})\mathbf{w}^{-1} \tag{2.2.11}$$

is a cocycle of $\mathbb{W}$ with values in the $\mathbb{W}$-group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K} \otimes \mathrm{GL}_{\mathbb{C}}(H_0)$.

We denote $C_{\mathbf{w}}(x, \xi)$ to stress that the cocycle can be regarded as a meromorphic function of x, ξ valued in $\mathrm{GL}_{\mathbb{C}}(H_0)$

Definition 2.2.3. Denote $C_{l,m} := C_{(\mathrm{t}(l\epsilon), \mathrm{t}(m\epsilon))}$ for $l, m \in \mathbb{Z}$. The system of q-difference equations

$$C_{l,m}(x, \xi) f(q^{-l}x, q^m \xi) = f(x, \xi) \quad (l, m \in \mathbb{Z})$$

for $f = f(x, \xi) \in H_0^{\mathbb{K}}$ is called *the bispectral quantum KZ equations* (*the bqKZ equations* for short) *of type* $(C_1^\vee, C_1)$. We also denote

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)} = \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)}(\underline{k}, \underline{l}, q) := \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } (C_1^\vee, C_1)\}.$$

In this § 2, we abbreviate $\mathrm{SOL}_{\mathrm{bqKZ}} := \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)}$.

Similarly as Lemma 1.2.12, we can compute the action of $C_{1,0}$ and $C_{0,1}$ on $H_0^{\mathbb{K}}$. We define an algebra homomorphisms $\eta_L\colon H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_L(A)\Big(\sum_{w\in W_0} f_w T_w\Big) \coloneqq \sum_{w\in W_0} f_w(AT_w), \tag{2.2.12}$$

for $A \in H$ and $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$. Similarly, using the subspace $H^* \subset \mathbb{H}$ in (2.1.17), we define an algebra anti-homomorphism $\eta_R\colon H^* \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_R(A)\Big(\sum_{w\in W_0} f_w T_w\Big) \coloneqq \sum_{w\in W_0} f_w(T_w A) \tag{2.2.13}$$

for $A \in H^*$ and $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$.

Lemma 2.2.4. The cocycles $C_{1,0}, C_{0,1} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K}\otimes \mathrm{GL}(H_0)$, regarded as functions of x and ξ are expressed as

$$C_{1,0} = R_0^L(x_0)R_1^L(x_1'), \quad C_{0,1} = R_0^R(\xi_0')R_1^R(\xi_1'), \tag{2.2.14}$$

where we denoted $x_0 \coloneqq qx^{-2}$, $x_1' \coloneqq q^2x^{-2}$, $\xi_0' \coloneqq q\xi^2$, $\xi_1' \coloneqq q^2\xi^2$ and

$$\begin{aligned} R_i^L(z) &\coloneqq c_i(z)^{-1}\big(\eta_L(T_i) - b_i(z)\big) \\ &= \frac{k_i}{(1-k_il_iz^{1/2})(1+k_il_i^{-1}z^{1/2})}\big((1-z)\eta_L(T_i) - (k_i - k_i^{-1}) - (l_i - l_i^{-1})z^{1/2}\big), \\ R_i^R(z) &\coloneqq c_i^*(z)^{-1}\big(\eta_R(T_i^*) - b_i^*(z)\big) \\ &= \frac{k_i^*}{(1-k_i^*l_i^*z^{1/2})(1+k_i^*(l_i^*)^{-1}z^{1/2})}\big((1-z)\eta_R(T_i^*) - (k_i^* - (k_i^*)^{-1}) - (l_i^* - (l_i^*)^{-1})z^{1/2}\big) \end{aligned}$$

for $i = 0, 1$, using the duality anti-involution $*$ in (2.1.15).

Proof. We denote by s_i^x and s_i^ξ for $i = 0, 1$ the action (2.2.8) of s_i in terms of variables x and ξ of $\mathbb{K} = \mathcal{M}(x, \xi)$. Explicitly, for $f(x, \xi) \in \mathbb{K}$, we have

$$\begin{aligned} (s_1^x f)(x,\xi) &= f(x^{-1},\xi), \quad (s_0^x f)(x,\xi) = f(qx^{-1},\xi), \\ (s_1^\xi f)(x,\xi) &= f(x,\xi^{-1}), \quad (s_0^\xi f)(x,\xi) = f(x,q^{-1}\xi^{-1}). \end{aligned}$$

By a similar calculation as Lemma 1.2.12, the cocycle values for (s_1, e) and (s_0, e) are given by $C_{(s_1,e)} = R_1^L(x_1)$ with $x_1 \coloneqq x^2$ and $C_{(s_0,e)} = R_0^L(x_0)$, respectively. Then the cocycle condition gives

$$C_{1,0} = C_{(s_0s_1,e)} = C_{(s_0,e)}(C_{(s_1,e)})^{(s_0,e)} = R_0^L(x)\big(s_0^x R_1^L(x_1)\big) = R_0^L(x_0)R_1^L(x_1'),$$

where s_0^x means the (s_1, e)-action given in (2.2.8).

Next, using the duality anti-involution $*$ and the $\mathbb{K}$-action (2.2.6), the cocycle values for (e, s_1) and (e, s_0) are given by $C_{(e,s_1)} = R_1^L(x_1)^* = R_1^R(\xi^{-2})$ and $C_{(e,s_0)} = R_0^L(x_0)^* = R_0^R(\xi_0')$ with $\xi_0' = (x_0)^* = q\xi^2$. Thus, we have

$$C_{0,1} = C_{(e,s_0s_1)} = C_{(e,s_0)}(C_{(e,s_1)})^{(e,s_0)} = R_0^R(\xi_0')\big(s_0^\xi R_1^R(\xi^{-2})\big) = R_0^R(\xi_0')R_1^R(\xi_1').$$

□

Remark 2.2.5. Some comments on Lemma 2.2.4 are in order.

(1) Explicitly, we have

$$C_{1,0} = J_0(x)J_1(x), \quad C_{1,0} = K_0(\xi)K_1(\xi) \tag{2.2.15}$$

with

$$\begin{aligned} J_0(x) &\coloneqq \frac{k_0}{(1-k_0l_0q^{1/2}x^{-1})(1+k_0l_0^{-1}q^{1/2}x^{-1})} \cdot \\ &\quad\cdot \big((1-qx^{-2})\eta_L(T_0) - (k_0 - k_0^{-1}) - (l_0 - l_0^{-1})q^{1/2}x^{-1}\big), \\ J_1(x) &\coloneqq \frac{k_1}{(1-k_1l_1qx^{-1})(1+k_1l_1^{-1}qx^{-1})}\big((1-q^2x^{-2})\eta_L(T_1) - (k_1 - k_1^{-1}) - (l_1 - l_1^{-1})qx^{-1}\big), \\ K_0(\xi) &\coloneqq \frac{l_1}{(1-l_1l_0q^{1/2}\xi)(1+l_1l_0^{-1}q^{1/2}\xi)}\big((1-q\xi^2)\eta_R(T_0^*) - (l_1 - l_1^{-1}) - (l_0 - l_0^{-1})q^{1/2}\xi\big), \\ K_1(\xi) &\coloneqq \frac{k_1}{(1-k_1k_0q\xi)(1+k_1k_0^{-1}q\xi)}\big((1-q^2\xi^2)\eta_R(T_1) - (k_1 - k_1^{-1}) - (k_0 - k_0^{-1})q\xi\big). \end{aligned}$$

(2) As in Remark 1.2.13, we have
$$C_{(e,w)}(x,\xi) = C_\iota C_{(w,e)}(\xi^{-1}, x^{-1}) C_\iota \tag{2.2.16}$$
for any $w \in W$. The formulas (2.2.14) are compatible with 2.2.16.
(3) The formulas (2.2.14) are also consistent with the computation of $C_{0,1}$ in the final paragraph of [T10, §4.2]. Note that we are working on the different choice (2.1.8) of Y from loc. cit.

For later use, we give a $(C_1^\vee, C_1)$-analogue of Fact 1.2.14.

Lemma 2.2.6. Let $\mathcal{A} := \mathbb{C}[x^{-1}] \subset \mathbb{L} = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]$, and $\mathcal{Q}_0(\mathcal{A})$ be the subring of the quotient field $\mathcal{Q}(\mathcal{A}) = \mathbb{C}(x)$ consisting of rational functions which are regular at $x^{-1} = 0$. Considering $\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm 1}]$ as subring of $\mathbb{C}(x,\xi)$, we have
$$C_{1,0} \in (\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm 1}]) \otimes \operatorname{End} H_0. \tag{2.2.17}$$
Moreover, setting $C_{1,0}^{(0)} := C_{1,0}|_{x^{-1}=0} \in \mathbb{C}[\xi^{\pm 1}] \otimes \operatorname{End} H_0$, we have
$$C_{1,0}^{(0)} = k_1 k_0 \eta_L(T_1 Y^{-1} T_1^{-1}). \tag{2.2.18}$$
Similarly, defining $\mathcal{B} := \mathbb{C}[\xi] \subset \mathbb{L}$ and $\mathcal{Q}_0(\mathcal{B}) \subset \mathcal{Q}(\mathcal{B}) = \mathbb{C}(\xi)$ to be the subring consisting of rational functions which are regular at the point $\xi = 0$, we have
$$C_{0,1} \in (\mathbb{C}[x^{\pm 1}] \otimes \mathcal{Q}_0(\mathcal{B})) \otimes \operatorname{End} H_0.$$
Moreover, setting $C_{0,1}^{(0)} := C_{0,1}|_{\xi=0} \in \mathbb{C}[x^{\pm 1}] \otimes \operatorname{End} H_0$, we have
$$C_{0,1}^{(0)} = k_1 l_1 \eta_R(T_1 Y^{-1} T_1^{-1}). \tag{2.2.19}$$

Proof. We only show the statements for $C_{1,0}$. By the expression (2.2.14) of $C_{1,0}$, we have $C_{1,0} \in (\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm 1}]) \otimes \operatorname{End} H_0$. To get (2.2.18), we compute
$$\begin{aligned} \lim_{x\to\infty} C_{1,0} &= \big(\lim_{x\to\infty} J_1(x)\big)\big(\lim_{x\to\infty} J_0(x)\big) = k_0(\eta_L(T_0) - k_0 + k_0^{-1}) k_1(\eta_L(T_1) - k_1 + k_1^{-1}) \\ &= k_1 k_0 \eta_L(T_0^{-1}) \eta_L(T_1^{-1}) = k_1 k_0 \eta_L(T_1 Y T_1^{-1}). \end{aligned}$$
Here we used $T_i^{-1} = T_i - k_i + k_i^{-1}$ from (2.1.7) and $Y = T_1 T_0$ from (2.1.8). □

Let us also record the $(C_1^\vee, C_1)$-version of Fact 1.2.15.

Fact 2.2.7 (c.f. [vM11, Lemma 4.2])**.** For $w \in W_0$, we set
$$\tau_w := \eta_L(\widetilde{S}^*_{w^{-1}}) T_e \in \mathbb{C}[\xi^{\pm 1}] \otimes H_0 \subset H_0^{\mathbb{K}}.$$
Then the following statements hold.
(1) $\{\tau_w \mid w \in W_0\}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of eigenfunctions for the η_L-action of $\mathbb{C}[Y^{\pm 1}] \subset \mathbb{H}$ on $H_0^{\mathbb{K}}$.
(2) For $p \in \mathbb{C}[\xi^{\pm 1}]$ and $w \in W_0$, we have $\eta_L(p(Y))\tau_w(\xi) = (w^{-1}p)(\xi)\tau_w(\xi)$ as H_0-valued regular functions in ξ.

The proof for the reduced type in [vM11] also works for the non-reduced type $(C_1^\vee, C_1)$, so we omit it.

2.3. **Bispectral Askey-Wilson q-difference equation.** As in § 1.3, we consider the crossed product algebra
$$\mathbb{D}_q^{\mathbb{W}} := \mathbb{W} \ltimes \mathbb{C}(x,\xi)$$
where $\mathbb{W}$ acts on $\mathbb{C}(x,\xi)$ by (1.2.12), and also consider the subalgebra
$$\mathbb{D}_q := \big(\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)\big) \ltimes \mathbb{C}(x,\xi) \subset \mathbb{D}_q^{\mathbb{W}},$$
which is identified with the algebra of q-difference operators on $\mathbb{C}(x,\xi)$. We can expand $D \in \mathbb{D}_q^{\mathbb{W}}$ as
$$D = \sum_{\mathbf{w} \in \mathbb{W}} f_{\mathbf{w}} \mathbf{w} = \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}} \mathbf{s}, \tag{2.3.1}$$
where $f_{\mathbf{w}} \in \mathbb{C}(T \times T)$ and $D_{\mathbf{s}} = \sum_{\mathbf{t} \in \mathrm{t}(\Lambda)\times \mathrm{t}(\Lambda)} g_{\mathbf{ts}} \mathbf{t} \in \mathbb{D}_q$. We also use $\operatorname{Res}\colon \mathbb{D}_q^{\mathbb{W}} \to \mathbb{D}_q$ given by
$$\operatorname{Res}(D) := \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}}. \tag{2.3.2}$$

Next, following (1.3.3) and (1.3.4), we introduce two realizations of the basic representation of type $(C_1^\vee, C_1)$. Let us denote
$$(1/\underline{k}, 1/\underline{l}) := (1/k_1, 1/k_0, 1/l_1, 1/l_0).$$
Then, the first is given by the algebra homomorphism
$$\rho^x_{1/\underline{k},1/\underline{l},q} \colon H(1/\underline{k}) \longrightarrow \mathbb{C}(x)[W \times \{e\}] \subset \mathbb{D}^{\mathbb{W}}_q \tag{2.3.3}$$
given by the map $\rho_{1/\underline{k},1/\underline{l},q}$ in (2.1.10). The second is
$$\rho^\xi_{\underline{k}^*,\underline{l}^*,1/q} \colon H(\underline{k}^*) \longrightarrow \mathbb{C}(\xi)[\{e\} \times W] \subset \mathbb{D}^{\mathbb{W}}_q. \tag{2.3.4}$$
Then, recalling Definitions 1.3.1 and 1.3.3, let us introduce:

Definition 2.3.1. For $h \in H(1/\underline{k})$ and $h' \in H(\underline{k}^*)$, we define $D^x_h, D^\xi_{h'} \in \mathbb{D}^{\mathbb{W}}_q$ by
$$D^x_h := \rho^x_{1/\underline{k},1/\underline{l},q}(h), \quad D^\xi_{h'} := \rho^\xi_{\underline{k}^*,\underline{l}^*,1/q}(h').$$
Also, for an invariant polynomial $p = p(z) \in \mathbb{C}[z^{\pm 1}]^{W_0} = \mathbb{C}[z + z^{-1}]$, we define $L^x_p, L^\xi_p \in \mathbb{D}_q$ by
$$L^x_p = L^x_p(\underline{k}, \underline{l}, q) := \mathrm{Res}(D^x_{p(Y)}), \quad L^\xi_p = L^\xi_p(\underline{k}, \underline{l}, q) := \mathrm{Res}(D^\xi_{p(Y)}), \tag{2.3.5}$$
where we regarded $p(Y) \in H(1/\underline{k})$ for L^x_p, and $p(Y) \in H(\underline{k}^*)$ for L^ξ_p, and used the map Res in (2.3.2).

As in Definition 1.3.4, we denote by $p_1(z) := z+z^{-1}$, which is the generator of the invariant polynomial ring $\mathbb{C}[z^{\pm1}]^{W_0}$. Then, similarly as in Proposition 1.3.5, we can compute $L^x_{p_1}$ and $L^\xi_{p_1}$ using the function $c(z; t, l)$ in (2.1.11). Let us denote the action of $w \in W$ on the functions of x given in (2.1.9) as w^x. It is compatible with $\rho^x_{1/k,q}$ in (2.3.3), and explicitly,
$$s^x_0(x) := qx^{-1}, \quad s^x_1(x) = x^{-1}, \quad \mathrm{t}(\varpi)^x(x) = q^{1/2}x. \tag{2.3.6}$$
We also denote by w^ξ the action on functions of ξ. It is compatible with $\rho^\xi_{k,1/q}$ in (2.3.4), and explicitly,
$$s^\xi_0(\xi) := q^{-1}\xi^{-1}, \quad s^\xi_1(\xi) = \xi^{-1}, \quad \mathrm{t}(\varpi)^\xi(\xi) = q^{-1/2}\xi. \tag{2.3.7}$$

Proposition 2.3.2. We have
$$L^x_{p_1} = k_1k_0 + (k_1k_0)^{-1} + (k_1k_0)^{-2}D^x_{\mathrm{AW}}, \qquad D^x_{\mathrm{AW}} := A(x)(T_{q,x} - 1) + A(x^{-1})(T^{-1}_{q,x} - 1), \tag{2.3.8}$$
$$L^\xi_{p_1} = k_1l_1 + (k_1l_1)^{-1} + (k_1l_1)^2D^\xi_{\mathrm{AW}}, \qquad D^\xi_{\mathrm{AW}} := A^*(\xi^{-1})(T_{q,\xi} - 1) + A^*(\xi)(T^{-1}_{q,\xi} - 1) \tag{2.3.9}$$
with
$$A(z) := \frac{(1 - k_1l_1z)(1 + k_1l_1^{-1}z)(1 - k_0l_0q^{-1/2}z)(1 + k_0l_0^{-1}q^{-1/2}z)}{(1 - z^2)(1 - q^{-1}z^2)},$$
$$A^*(z) := \frac{(1 - k_1k_0z)(1 + k_1l_1^{-1}z)(1 - l_1l_0q^{-1/2}z)(1 + l_1l_0^{-1}q^{-1/2}z)}{(1 - z^2)(1 - q^{-1}z^2)}.$$

Proof. Let us compute $L^x_{p_1} = \mathrm{Res}(D^x_{Y+Y^{-1}})$. Since $Y = T_0T_1$ and $s_0 = \mathrm{t}(\epsilon)s_1$, using (2.1.7), (2.3.6) and (2.1.10), we have
$$\begin{aligned}
D^x_{Y+Y^{-1}} &= \rho^x_{1/\underline{k},1/\underline{l},q}(T_0T_1 + T_1^{-1}T_0^{-1}) \\
&= \big(k_0^{-1} + c_0(\mathrm{t}(\epsilon)^x s^x_1 - 1)\big)\big(k_1^{-1} + c_1(s^x_1 - 1)\big) + \big(k_1 + c_1(s^x_1 - 1)\big)\big(k_0 + c_0(\mathrm{t}(\epsilon)^x s^x_1 - 1)\big) \\
&= k_1^{-1}k_0^{-1} + k_1^{-1}c_0(\mathrm{t}(\epsilon)^x s^x_1 - 1) + k_0^{-1}c_1(s^x_1 - 1) + c_0(c'_1\,\mathrm{t}(\epsilon)^x s^x_1 - c_1)(s^x_1 - 1) \\
&\quad + k_1k_0 + k_1c_0(\mathrm{t}(\epsilon)^x s^x_1 - 1) + k_0c_1(s^x_1 - 1) + c_1(c'_0s^x_1 - c_0)(\mathrm{t}(\epsilon)^x s^x_1 - 1),
\end{aligned}$$
where w^x is given by (2.3.6) and, using the function c in (2.1.11), we denoted
$$c_1 := c(x^2; k_1^{-1}, l_1^{-1}), \qquad c'_1 := \mathrm{t}(\epsilon)^x s^x_1(c_1),$$
$$c_0 := c(qx^{-2}; k_0^{-1}, l_0^{-1}), \qquad c'_0 := s^x_1(c_0) = c(qx^2; k_0^{-1}, l_0^{-1}).$$
Then, using $(c'_0s^x_1 - c_0)(\mathrm{t}(\epsilon)^x s^x_1 - 1) = c'_0\,\mathrm{t}(-\epsilon)^x - c'_0s^x_1 - c_0\,\mathrm{t}(\epsilon)^x s^x_1 + c_0$ and
$$\mathrm{Res}(\mathrm{t}(\epsilon)^x s^x_1 - 1) = \mathrm{t}(\epsilon)^x - 1, \quad \mathrm{Res}(s^x_1 - 1) = 0,$$
we have
$$\begin{aligned}
\mathrm{Res}(D^x_{Y+Y^{-1}}) &= k_1^{-1}k_0^{-1} + k_1^{-1}c_0(\mathrm{t}(\epsilon)^x - 1) \\
&\quad + k_1k_0 + k_1c_0(\mathrm{t}(\epsilon)^x - 1) + c_1(c'_0\,\mathrm{t}(-\epsilon)^x - c'_0 - c_0\,\mathrm{t}(\epsilon)^x + c_0)
\end{aligned}$$

$$= k_1k_0 + k_1^{-1}k_0^{-1} + c_0(k_1 + k_1^{-1} - c_1)(\mathrm{t}(\epsilon)^x - 1) + c_1c_0'(\mathrm{t}(-\epsilon)^x - 1).$$

Now, using the identity

$$k_1 + k_1^{-1} - c_1 = k_1^{-1}\frac{(1-k_1l_1x)(1+k_1l_1^{-1}x)}{1-x^2} = c(x^2; k_1, l_1) = c(x^{-2}; k_1^{-1}, l_1^{-1}) = s_1^x(c_1),$$

we have $c_0(k_1 + k_1^{-1} - c_1) = c_0 \cdot s_1^x(c_1) = s_1^x(c_0'c_1)$. Then, by $\mathrm{t}(\epsilon)^x = T_{q,x}$, we have

$$L_{p_1}^x = \mathrm{Res}(D_{Y+Y^{-1}}^x) = k_1k_0 + k_1^{-1}k_0^{-1} + \big(s_1^x(c_0'c_1)\big)(T_{q,x} - 1) + c_0'c_1(T_{q,x}^{-1} - 1).$$

Denoting $A(x) := s_1^x(c_0'c_1)$, we obtain (2.3.8). The formula (2.3.9) of $L_{p_1}^\xi$ is obtained from $L_{p_1}^x$ by replacing $(x, k_0, k_1, l_0, l_1, q)$ with $(\xi, l_1^{-1}, k_1^{-1}, l_0^{-1}, k_0^{-1}, q^{-1})$. □

Remark 2.3.3 (c.f. [N95, pp.54–55])**.** The operators D_{AW}^x and D_{AW}^ξ are equivalent to *the Askey-Wilson second order q-difference operator* [AW85, (5.7)]:

$$D_{\mathrm{AW}}(z; a, b, c, d, q) := A^+(z; a, b, c, d, q)(T_{q,z} - 1) + A^+(z^{-1}; a, b, c, d, q)(T_{q,z}^{-1} - 1),$$

$$A^+(z; a, b, c, d, q) := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

The precise relation with $A(x), A^*(\xi)$ in (2.3.8), (2.3.9) is given by

$$A(x) = A^+(x; a, b, c', d', q), \quad A^*(\xi) = A^+(\xi; a^*, b^*, c'^*, d'^*, q)$$

with the parameters

$$\{a, b, c', d'\} := \{k_1l_1, -k_1l_1^{-1}, q^{-1/2}k_0l_0, -q^{-1/2}k_0l_0^{-1}\},$$

$$\{a^*, b^*, c'^*, d'^*\} := \{k_1^{-1}k_0^{-1}, -k_1^{-1}k_0, q^{-1/2}l_1^{-1}l_0^{-1}, -q^{-1/2}l_1^{-1}l_0\}.$$

The reciprocal parameter q^{-1} appearing above originates from our choice (2.1.8) of the Dunkl operator Y. As mentioned in Remark 2.1.3, the choice in [N95, T10, St14] is the opposite, and for that choice, the above construction of the q-difference operator on x which is equal to the original Askey-Wilson operator $D_{\mathrm{AW}}^x(x; a, b, c, d, q)$.

The ordinary parameters and *the dual parameters* of Askey-Wilson polynomials are given as

$$\{a, b, c, d\} := \{k_1l_1, -k_1l_1^{-1}, q^{1/2}k_0l_0, -q^{1/2}k_0l_0^{-1}\},$$

$$\{a^*, b^*, c^*, d^*\} := \{k_1k_0, -k_1k_0^{-1}, q^{1/2}l_1l_0, -q^{1/2}l_1l_0^{-1}\}.$$

There are related by the duality anti-involution $*$ (see (2.1.15)) as

$$a^* = \sqrt{abcd/q}, \quad b^* = ab/a^*, \quad c^* = ac/a^*, \quad d^* = ad/a^*.$$

By Remark 2.3.3, it is natural to name the bispectral problem as:

Definition 2.3.4. The following system of eigen-equations for $f = f(x, \xi) \in \mathbb{K}$ is called *the bispectral Askey-Wilson q-difference equation of type* $(C_1^\vee, C_1)$, and *the bAW equation* for short.

$$\begin{cases} (L_{p_1}^x f)(x, \xi) &= p_1(\xi^{-1})f(x, \xi), \\ (L_{p_1}^\xi f)(x, \xi) &= p_1(x)f(x, \xi). \end{cases} \tag{2.3.10}$$

The solution space is denoted as

$$\mathrm{SOL}_{\mathrm{bAW}}(\underline{k}, \underline{l}, q) := \{f \in \mathbb{K} \mid f \text{ satisfies } (2.3.4)\}.$$

2.4. **Bispectral qKZ/AW correspondence.** Here we give a $(C_1^\vee, C_1)$-analogue of § 1.4, using the reciprocal parameters

$$(1/\underline{k}, 1/\underline{l}) := (1/k_1, 1/k_0.1/l_1, 1/l_0).$$

Similarly as in Definition 1.4.1, we define a $\mathbb{K}$-linear function $\chi_+ \colon H_0(1/\underline{k}) \to \mathbb{C}$ by

$$\chi_+(T_w) := k_1^{-\ell(w)} \tag{2.4.1}$$

for the basis element $T_w \in H_0(1/\underline{k})$ $(w \in W_0)$. It is extended to $H_0(1/\underline{k})^{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{L}} H_0(1/\underline{k})$ as

$$\chi_+ \colon H_0(1/\underline{k})^{\mathbb{K}} \longrightarrow \mathbb{K}, \quad \sum_{w \in W_0} f_w T_w \longmapsto \sum_{w \in W_0} f_w \chi_+(T_w). \tag{2.4.2}$$

Below is a $(C_1^\vee, C_1)$-analogue of Fact 1.4.3.

Theorem 2.4.1 (c.f. [St14, §3])**.** Assume $0 < q < 1$. Then the map χ_+ restricts to an injective $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(1/\underline{k}, 1/\underline{l}, q) \longrightarrow \mathrm{SOL}_{\mathrm{bAW}}(\underline{k}, \underline{l}, q),$$

where $\mathbb{W}_0$ is the subgroup of $\mathbb{W}$ defined by

$$\mathbb{W}_0 := \mathbb{Z}_2 \ltimes (W_0 \times W_0) \subset \mathbb{W},$$

and $\mathbb{F}$ is the subspace of $\mathbb{K} = \mathcal{M}(T \times T)$ defined by

$$\mathbb{F} := \left\{ f(t,\gamma) \in \mathbb{K} \mid \big((\mathrm{t}(\lambda), \mathrm{t}(\mu)) f\big)(t,\gamma) = f(t,\gamma), \ \forall (\lambda,\mu) \in \Lambda \times \Lambda \right\}.$$

The strategy of proof is the same as the type A_1 (§ 1.4). Denoting $\mathrm{SOL}_{\mathrm{bqKZ}} := \mathrm{SOL}_{\mathrm{bqKZ}}(1/\underline{k}, 1/\underline{l}, q)$ and $\mathrm{SOL}_{\mathrm{bAW}} := \mathrm{SOL}_{\mathrm{bAW}}(1/\underline{k}, 1/\underline{l}, q)$, we can divide the proof into three parts.

(i) χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$.
(ii) The image $\chi_+(\mathrm{SOL}_{\mathrm{bqKZ}})$ is contained in $\mathrm{SOL}_{\mathrm{bAW}}$.
(iii) $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathrm{SOL}_{\mathrm{bAW}}$ is injective

We write down the arguments of part (i) and the first half of part (ii). The rest arguments are similar as the type A_1, and we omit them.

Part (i) *of the proof of Theorem 2.4.1.* Similarly as Lemma 1.4.5, we have

$$\chi_+(C_{\mathbf{w}} F) = \chi_+(F) \tag{2.4.3}$$

for each $\mathbf{w} \in \mathbb{W}_0$ and $F \in H_0(1/\underline{k})^{\mathbb{K}}$. The proof is quite similar as Lemma 1.4.5, once we use $C_{(e,s_1)} = C_\iota C_{(s_1,e)} C_\iota$ and replace the expression (1.4.4) of $C_{(s_1,e)} h$ for $h \in H_0$ by

$$C_{(s_1,e)} h = d(x^2; 1/k_1, 1/l_1)^{-1} \big((1-x^2)\eta_L(T_1) - (k_1^{-1} - k_1) - (l_1^{-1} - l_1)x\big) h.$$

Then, in the same way as § 1.4, we can show that χ_+ is $\mathbb{W}_0$-equivariant using (2.2.11), (2.4.3) and (2.2.8), and that χ_+ restricts to an $\mathbb{F}$-linear map $\mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$ using Definition 2.2.3, (2.4.1) and (2.4.2). □

Similarly as the type A_1, the part (ii) of the proof consists of two steps.

- Describe of $\mathrm{SOL}_{\mathrm{bqKZ}}$ in terms of the basic asymptotically free solution Φ.
- Analyze the map χ_+ using Φ.

The second step is quite the same as the type A_1, and we omit the detail. The first step requires the following Proposition 2.4.2, which is a $(C_1^\vee, C_1)$-analogue of Fact 1.4.6, and a simple modification of Fact 1.4.8.

Proposition 2.4.2. Denote $w_0 := s_1 \in W_0$. Let

$$\mathcal{W}(x,\xi) = \mathcal{W}(x,\xi; \underline{k}, \underline{l}, q) \in \mathbb{K} = \mathcal{M}(x,\xi)$$

be a meromorphic function satisfying the q-difference equations

$$\mathcal{W}(q^l x, \xi) = (k_1 k_0 \xi)^{-l} \mathcal{W}(x,\xi) \qquad (l \in \mathbb{Z}) \tag{2.4.4}$$

and the self-duality

$$\mathcal{W}(\xi^{-1}, x^{-1}; \underline{k}^*, \underline{l}^*, q) = \mathcal{W}(x,\xi; \underline{k}, \underline{l}, q). \tag{2.4.5}$$

Then, there is a unique element $\Psi \in H_0(1/\underline{k})^{\mathbb{K}}$ satisfying the following conditions.

(i) We have

$$\Phi := \mathcal{W}\Psi \in \mathrm{SOL}_{\mathrm{bqKZ}}.$$

(ii) We have a series expansion

$$\Psi(x,\xi) = \sum_{m,n \in \mathbb{N}} K_{m,n} x^{-m} \xi^{n\alpha} \quad (K_{\alpha,\beta} \in H_0)$$

for $(x,\xi) \in B_\varepsilon^{-1} \times B_\varepsilon$ with B_ε being some open ball of radius $\varepsilon > 0$, which is normally convergent on compact subsets of $B_\varepsilon^{-1} \times B_\varepsilon$.

(iii) $K_{0,0} = T_{w_0}$.

We call the solution Φ *the basic asymptotically free solution of the bqKZ equation of type* $(C_1^\vee, C_1)$.

Let us give some preliminaries for the proof of Proposition 2.4.2. Given a function $\mathcal{W} \in \mathbb{K}$ satisfying (2.4.4) and (2.4.5), we write

$$D_{1,0}(x,\xi) := \mathcal{W}(x,\xi)^{-1} C_{1,0}(x,\xi) \mathcal{W}(q^{-\epsilon}x,\xi),$$
$$D_{0,1}(x,\xi) := \mathcal{W}(x,\xi)^{-1} C_{0,1}(x,\xi) \mathcal{W}(x,q^{\epsilon}\xi),$$

which are regarded as $\mathrm{End}\big(H_0(1/\underline{k})\big)$-valued meromorphic functions in x,ξ. We have $f \in H_0(1/\underline{k})^{\mathbb{K}}$ if and only if $g := \mathcal{W}(x,\xi)^{-1} f$ satisfies the holonomic system of q-difference equations

$$\begin{cases} D_{1,0}(x,\xi) g(q^{-\epsilon}x,\xi) = g(x,\xi) \\ D_{0,1}(x,\xi) g(x,q^{\epsilon}\xi) = g(x,\xi) \end{cases}$$

as $\mathrm{End}\big(H_0(1/\underline{k})\big)$-valued rational functions in x,ξ. Now recall from Lemma 2.2.6

$$\mathcal{A} := \mathbb{C}[x^{-1}] \subset \mathbb{C}[x^{\pm 1}], \quad \mathcal{B} := \mathbb{C}[\xi] \subset \mathbb{C}[\xi^{\pm 1}]$$

and

$$Q_0(\mathcal{A}) := \big\{ f(x^{-1})/g(x^{-1}) \in Q(\mathcal{A}) \mid g(0) \neq 0 \big\} \subset Q(\mathcal{A}) = \mathbb{C}(x),$$
$$Q_0(\mathcal{B}) := \{ f(\xi)/g(\xi) \in Q(\mathcal{B}) \mid g(0) \neq 0 \} \subset Q(\mathcal{B}) = \mathbb{C}(\xi).$$

Lemma 2.4.3 (c.f. [vMS09, Lemma 5.2])**.** The operators $D_{1,0}$ and $D_{0,1}$ satisfy the following properties.

(1) $D_{1,0} \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}\big(H_0(1/\underline{k})\big)$ and $D_{0,1} \in (\mathcal{A} \otimes Q_0(\mathcal{B})) \otimes \mathrm{End}\big(H_0(1/\underline{k})\big)$

(2) Define $D^{(0)}_{1,0}, D^{(0)}_{0,1} \in \mathrm{End}\big(H_0(1/\underline{k})\big)$ by

$$D^{(0)}_{1,0} := D_{1,0}|_{x^{-1}=0}, \quad D^{(0)}_{0,1} := D_{0,1}|_{\xi=0}.$$

Then, denoting $w_0 := s_1$, we have

$$D^{(0)}_{1,0}(T_{w_0}T_w) = \begin{cases} T_1 & (w=e) \\ 0 & (w=s_1) \end{cases}, \quad D^{(0)}_{0,1}(T_{w_0}T_w) = \begin{cases} T_1 & (w=e) \\ 0 & (w=s_1) \end{cases}. \tag{2.4.6}$$

Proof. For the first half of (1), note that the q-difference equation (2.4.4) with $\lambda = -\epsilon$ yields

$$D_{1,0}(x,\xi) = \mathcal{W}(x,\xi)^{-1} C_{1,0}(x,\xi) \mathcal{W}(q^{-1}x,\xi) = k_1 k_0 \xi C_{1,0}(x,\xi), \tag{2.4.7}$$

By the explicit expression of $C_{1,0}$ (Lemma 2.2.4), we have $D_{1,0} \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}(H_0)$.

For the second half, using (2.4.4) and (2.4.5), we have

$$D_{0,1}(x,\xi) = \mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)^{-1} C_{0,1}(x,\xi) \mathcal{W}^{(C_1^\vee,C_1)}(x,q\xi) = (k_1 u_1 x)^{-1} C_{0,1}(x,\xi).$$

By the explicit expression of $C_{0,1}$ (Lemma 2.2.4), we have $D_{0,1} \in (\mathcal{A} \otimes Q_0(\mathcal{B})) \otimes \mathrm{End}(H_0)$.

Next, we will show the first half of (2). By the above computation (2.4.7) and Lemma 2.2.6, we have

$$D^{(0)}_{1,0} = D_{1,0}|_{x^{-1}=0} = k_1 k_0 \xi C^{(0)}_{1,0}. \tag{2.4.8}$$

Let us compute $D^{(0)}_{1,0}(T_1)$. Since $\eta_L(T_1 Y^{-1} T_1^{-1})(T_1) = \xi^{-1} T_1$, we have

$$D^{(0)}_{1,0}(T_1) = k_1 k_0 \xi C^{(0)}_{1,0}(T_1) = \xi \eta_L(T_1 Y^{-1} T_1^{-1})(T_1) = T_1,$$

using (2.2.18) with reciprocal parameters $1/\underline{k}$ in the second equality. Hence we obtain $D^{(0)}_{1,0}(T_1) = T_1$. For $D^{(0)}_{1,0}(T_e)$, note that $\tau_w := \eta_L(\widetilde{S}^*_{w^{-1}})T_e$ ($w \in W_0$) form a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ (Fact 2.2.7) and $\eta(T_{w_0})\tau_w \in \mathcal{B} \otimes \mathrm{End}(H_0)$. By Fact 2.2.7 and (2.2.18), we obtain

$$D^{(0)}_{1,0}(\eta(T_1)\tau_{s_1}) = k_1 k_0 \xi C^{(0)}_{1,0}(\eta(T_1)\tau_{s_1}) = \xi \eta_L(T_1 Y^{-1} T_1^{-1})(\eta(T_1)\tau_{s_1}) = \xi^2 \eta(T_1)\tau_{s_1}.$$

as identities in $\mathcal{B} \otimes \mathrm{End}(H_0)$. Specializing at $\xi = 0$, we obtain $D^{(0)}_{1,0}(T_e) = 0$.

The second half of (2) can be shown similarly using (2.2.19). We omit the detail. □

Proof of Proposition 2.4.2. Lemma 2.4.3 implies that the operators $D^{(0)}_{1,0}$ and $D^{(0)}_{0,1}$ on $H_0(1/\underline{k}, 1/\underline{l})$ commute with each other. We denote the simultaneous eigenspace decomposition of $H_0(1/\underline{k}, 1/\underline{l})$ as

$$H_0(1/\underline{k}, 1/\underline{l}) = \bigoplus_{(a,b) \in \mathbb{C}^2} H_0[a,b], \quad H_0[a,b] := \Big\{ v \in H_0 \mid D^{(0)}_{1,0}(v) = av, \ D^{(0)}_{0,1}(v) = bv \Big\}$$

Since $H_0(1/\underline{k}, 1/\underline{l})$ is finite dimensional, the subset $S \subset \mathbb{C}^2$ for which $H_0[a,b] \neq 0$ is finite. We also have $(1,1) \in S$ and $H_0[1,1] = \mathbb{C}T_1$ by Lemma 2.4.3. Furthermore, $a, b \in q^{\mathbb{N}}$ for all $(a,b) \in S$. Under these conditions, the holonomic system of q-difference equations 2.4.6 admits a unique solution Ψ satisfying the desired properties by the general theory developed in [vMS09, Theorem A.6]. $\square$

Example 2.4.4. We give an example of the function $\mathcal{W}$ in Proposition 2.4.2. As in the case of type A_1 (Example 1.4.12 (1)), using the Jacobi theta function $\theta(z;q) := (q, z, q/z; q)_\infty$, we define

$$\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi) = \mathcal{W}^{(C_1^\vee,C_1)}(x,\xi;\underline{k},\underline{l}) := \frac{\theta(-q^{1/2}x\xi;q)}{\theta(-q^{1/2}(k_1k_0)^{-1}x, -q^{1/2}k_1l_1\xi;q)}. \tag{2.4.9}$$

It satisfies the q-difference equation (2.4.4) in the form

$$\mathcal{W}^{(C_1^\vee,C_1)}(q^{\pm1}x,\xi) = (k_1k_0\xi)^{\mp1}\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi),$$

and the self-duality (2.4.5) in the form

$$\mathcal{W}^{(C_1^\vee,C_1)}(\gamma^{-1},t^{-1};\underline{k}^*,\underline{l}^*) = \mathcal{W}^{(C_1^\vee,C_1)}(t,\gamma;\underline{k},\underline{l}). \tag{2.4.10}$$

Here we used the duality anti-involution $*$ in (2.1.15).

Remark 2.4.5. As in the case of type A_1 case (Remark 1.4.13), the function $\mathcal{W}^{(C_1^\vee,C_1)}$ is nothing but the function G of Remark 1.4.13 (2) introduced by [vM11]:

$$G(t,\gamma) := \frac{\vartheta(\mathrm{t}(w_0\gamma)^{-1})}{\vartheta(\gamma_0 t)\,\vartheta((\gamma_0^*)^{-1}\gamma)}$$

whose lattice theta function $\vartheta(t) = \vartheta^{A_1}(t)$ is replaced by

$$\vartheta(t) := \sum_{\lambda\in\Lambda} q^{\langle\lambda,\lambda\rangle/2}t^\lambda, \quad \Lambda = \mathbb{Z}\epsilon,$$

and the parameters γ_0, γ_0^* are replaced by

$$\gamma_0 := (k_1k_0)^{-\epsilon},\ \gamma_0^* := (k_1l_1)^{-\epsilon} \in T. \tag{2.4.11}$$

2.5. **Bispectral Askey-Wilson function.** In this subsection, we cite from [St02, St14] an example of explicit solution of the bispectral Askey-Wilson q-difference equation. As in the previous Theorem 2.4.1, we assume $0 < q < 1$.

Let us write again the bispectral Askey-Wilson q-difference equation (2.3.10) for $f(x,\xi) \in \mathbb{L} = \mathbb{C}[x^{\pm1},\xi^{\pm1}]$ for the reciprocal parameters $\mathrm{SOL}_{\mathrm{bAW}}(1/\underline{k}, 1/\underline{l})$:

$$\begin{cases} (L^x_{p_1}f)(x,\xi) &= (\xi+\xi^{-1})f(x,\xi) \\ (L^\xi_{p_1}f)(x,\xi) &= (x+x^{-1})f(x,\xi) \end{cases}. \tag{2.5.1}$$

By Proposition 2.3.2 and Remark 2.3.3, the operators are given by

$$L^x_{p_1} = k_1k_0 + (k_1k_0)^{-1} + (k_1k_0)^{-1}D^x_{\mathrm{AW}}, \quad L^\xi_{p_1} = k_1l_1 + (k_1l_1)^{-1} + (k_1l_1)D^\xi_{\mathrm{AW}}, \tag{2.5.2}$$

$$D^x_{\mathrm{AW}} := D_{\mathrm{AW}}(x;a,b,c,d,q), \quad D^\xi_{\mathrm{AW}} := D_{\mathrm{AW}}(\xi;(a^*)^{-1},(b^*)^{-1},(c^*)^{-1},(d^*)^{-1},q^{-1}),$$

$$\{a,b,c,d\} := \{k_1l_1, -k_1l_1^{-1}, q^{1/2}k_0l_0, -q^{1/2}k_0l_0^{-1}\}, \tag{2.5.3}$$

$$\{a^*,b^*,c^*,d^*\} := \{k_1k_0, -k_1k_0^{-1}, q^{1/2}l_1l_0, -q^{1/2}l_1l_0^{-1}\} \tag{2.5.4}$$

with

$$D_{\mathrm{AW}}(x;q,a,b,c,d) := A(x)(T_{q,x}-1) + A(x^{-1})(T_{q,x}^{-1}-1),$$

$$A(x) := \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}. \tag{2.5.5}$$

As mentioned in Remark 2.3.3, the q-difference operator D^x_{AW} was introduced by Askey and Wilson [AW85]. Using the symbol $(x_1,\ldots,x_r;q)_n$ in (0.2.1), they showed that the basic hypergeometric polynomial

$$P_n(x;a,b,c,d;q) := \frac{(ab,ac,ad;q)_n}{a^n}{}_4\phi_3\left[\begin{matrix} q^{-n},\ abcdq^{n-1},\ ax,\ a/x \\ ab,\ ac,\ ad \end{matrix};q,\ q\right] \quad (n\in\mathbb{N}) \tag{2.5.6}$$

is an eigenfunction of D^x_{AW}, and the eigenvalue is $-(1-q^{-n})(1-q^{n-1}abcd)$. This claim is restated as

$$L^x_{p_1}P_n(x;a,b,c,d;q) = (q^na^* + q^{-n}(a^*)^{-1})P_n(x;a,b,c,d;q)$$

under the parameter correspondence (2.5.3) and (2.5.4) (c.f. [N95, p.55]). The Laurent polynomial $P_n(x;a,b,c,d;q)$ is called *the Askey-Wilson polynomial.*

In order to treat the bispectral problem (2.5.1), we need to consider non-polynomial eigenfunctions of the Askey-Wilson second order q-difference operator D_{AW}. In literature, such an eigenfunction is given in terms of a very-well-poised ${}_8\phi_7$ series under the name of *the Askey-Wilson function*. Here we give a brief review, and refer to [St02, §3] for more information.

Following Gasper and Rahman [GR04, (2.1.11)], we denote

$$ {}_8W_7(a_1;a_4,a_5,a_6,a_7,a_8;q,z) := {}_8\phi_7\left[\begin{matrix} a_1,\ qa_1^{1/2},\ -qa_1^{1/2},\ a_4,\ a_5,\ a_6,\ a_7,\ a_8 \\ a_1^{1/2},\ -a_1^{1/2},\ \frac{qa_1}{a_4},\ \frac{qa_1}{a_5},\ \frac{qa_1}{a_6},\ \frac{qa_1}{a_7},\ \frac{qa_1}{a_8} \end{matrix};q,\ z\right], $$

which is a very-well-poised basic hypergeometric series in the sense of [GR04, the line after (2.1.9)]. Then, the Askey-Wilson function $\phi_\xi(x) = \phi_\xi(x;a,b,c,d;q)$ is defined by [St02, (3.1)]

$$ \phi_\xi(x) := \frac{(qax\xi/d^*, qa\xi/d^*x, qabc/d;q)_\infty}{(a^*b^*c^*\xi, q\xi/d^*, qx/d, q/dx, bc, qb/d, qc/d;q)_\infty}\, {}_8W_7(a^*b^*c^*\xi/q; ax, a/x, a^*\xi, b^*\xi, c^*\xi; q, q/d^*\xi). $$

It satisfies the eigen-equation

$$ (L^x_{p_1}\phi_\xi)(x) = (\xi+\xi^{-1})\phi_\xi(x), \tag{2.5.7} $$

the self-duality

$$ \phi_\xi(x;a,b,c,d;q) = \phi_x(\xi;a^*,b^*,c^*,d^*;q), \tag{2.5.8} $$

and the symmetry (the inversion invariance in [St14])

$$ \phi_\xi(x) = \phi_\xi(x^{-1}) = \phi_{\xi^{-1}}(x). \tag{2.5.9} $$

The properties (2.5.8) and (2.5.9) are the consequences of the equality [St14, (3.2)]:

$$ \begin{aligned} \phi_\xi(x) &= \frac{(qabc/d;q)_\infty}{(bc, qa/d, qb/d, qc/d, q/ad;q)_\infty}\, {}_4\phi_3\left[\begin{matrix} ax,\ a/x,\ a^*\xi,\ a^*/\xi \\ ab,\ ac,\ ad \end{matrix};q,\ q\right] \\ &+ \frac{(ax, a/x, a^*\xi, a^*/\xi, qabc/d;q)_\infty}{(qx/d, q/dx, q\xi/d^*, q/d^*\xi, ab, ac, bc, qa/d, ad/q;q)_\infty}\, {}_4\phi_3\left[\begin{matrix} qx/d,\ q/dx,\ q\xi/d^*,\ q/d^*\xi \\ qb/d,\ qc/d,\ q^2/ad \end{matrix};q,\ q\right], \end{aligned} $$

which can be shown by a form [GR04, (2.10.10)] of Bailey's transformation formulas. The above equality also yields

$$ \phi_{\xi_n}(x) = \frac{(qabc/d;q)_\infty}{(bc, qa/d, qb/d, qc/d, q/ad;q)_\infty}\, {}_4\phi_3\left[\begin{matrix} q^{-n},\ abcdq^{n-1},\ ax,\ a/x \\ ab,\ ac,\ ad \end{matrix};q,\ q\right], \quad \xi_n := (a^*)^{-1}q^{-n}, $$

which is proportional to the Askey-Wilson polynomial $P_n(x)$ (2.5.6).

Let us consider the asymptotic form of the Askey-Wilson q-difference equation $\big(L^x_{p_1} - (\xi+\xi^{-1})\big)f(x) = 0$ in the region $|x| \gg 1$. Since the functions $A(x)$ and $A(x^{-1})$ in (2.5.5) behave as $A(x) \approx (a^*)^2$ and $A(x^{-1}) \approx 1$, we have the asymptotic form

$$ L^x_{p_1} \approx a^* T_{q,x} + (a^*)^{-1}T^{-1}_{q,x}. $$

Now, recall the function $\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)$ given in (2.4.9):

$$ \mathcal{W}^{(C_1^\vee,C_1)}(x,\xi) = \frac{\theta(-\nu x\xi;q)}{\theta(-\nu x/a^*, -\nu\xi a;q)}, $$

where $\nu := q^{1/2}$. By $\theta(qx;q) = -x^{-1}\theta(x;q)$, we have $T^{\pm1}_{q,x}\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi) = (a^*\xi)^{\mp1}\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)$, which implies that the set $\{\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi^{\pm1})\}$ is a basis of solutions of the asymptotic q-difference equation

$$ \big(a^*T_{q,x} + (a^*)^{-1}T^{-1}_{q,x} - (\xi+\xi^{-1})\big)f(x) = 0. $$

Similarly, the ξ-side asymptotic q-difference equation in the region $|\xi| \ll 1$ is given by

$$ L^\xi_{p_1} \approx aT^{-1}_{q,\xi} + a^{-1}T_{q,\xi}, $$

and since $T^{\pm1}_{q,\xi}\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi) = (a/x)^{\pm1}\mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)$, the set $\{\mathcal{W}^{(C_1^\vee,C_1)}(x^{\pm1},\xi)\}$ is a basis of solutions of the asymptotic equation

$$ \big(a^{-1}T_{q,\xi} + aT^{-1}_{q,\xi} - (x+x^{-1})\big)g(\xi) = 0, $$

By the argument in §2.4, we have a unique element $\widehat{\Phi} := \chi_+(\Phi) \in \mathrm{SOL}_{\mathrm{bAW}}$ of the form $\widehat{\Phi} = \mathcal{W}^{(C_1^\vee,C_1)}g$, where $g = g(x)$ has a convergent series expansion around $|x| = \infty$ with constant coefficient being 1. By [St14, Proposition 5.2, (5.8)], $\widehat{\Phi}$ is written down as

$$ \widehat{\Phi}(x,\xi) = \mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)\cdot\frac{(qa\xi/a^*x, qb\xi/a^*x, qc\xi/a^*x, qa^*\xi/dx, d/x;q)_\infty}{(q/ax, q/bx, q/dx, q^2\gamma^2/dx;q)_\infty} $$

$$\cdot\, {}_8W_7(q\xi^2/dx; q\xi/a^*, q\xi/d^*, b^*\xi, c^*\xi, q/dx; q, d/x).$$

Remark 2.5.1. Our solution $\widehat{\Phi}(x,\xi)$ is equivalent to the solution $\widehat{\Phi}_\eta(t,\gamma)$ in [St14, (5.8)] up to quasi-constant multiplication.

Now we cite a $(C_1^\vee, C_1)$-analogue of Fact 1.5.6.

Fact 2.5.2 (c.f. [St14, Proposition 5.2])**.** The function $\mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi) = \mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi;\underline{k},\underline{l},q)$ given by

$$\mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi) := \frac{(qax\xi/d^*, qa\xi/d^*x, qa/d, q/ad; q)_\infty}{(a^*b^*c^*\xi, q\xi/d^*, qx/d, q/dx; q)_\infty}\, {}_8W_7(a^*b^*c^*\xi/q; ax, a/x, a^*\xi, b^*\xi, c^*\xi; q, q/d^*x).$$

enjoys the following properties.

(i) It is a solution of the bispectral problem (2.5.1).

(ii) It has the symmetry

$$\mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi) = \mathcal{E}_+^{(C_1^\vee,C_1)}(x^{-1};\xi) = \mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi^{-1}).$$

(iii) It has the self-duality

$$\mathcal{E}_+^{(C_1^\vee,C_1)}(x,\xi;\underline{k},\underline{l},q) = \mathcal{E}_+^{(C_1^\vee,C_1)}(\xi^{-1};x^{-1},\underline{k}^*,\underline{l}^*,q). \tag{2.5.10}$$

Thus, defining $\mathrm{SOL}_{\mathrm{bAW}}^{\mathbb{W}^*} := \{f \in \mathrm{SOL}_{\mathrm{bAW}} \mid \text{(ii), (iii)}\}$, we have

$$\mathcal{E}_+^{(C_1^\vee,C_1)} \in \mathrm{SOL}_{\mathrm{bAW}}^{\mathbb{W}^*}.$$

The function $\mathcal{E}_+^{(C_1^\vee,C_1)}$ is called *the basic hypergeometric series of type* $(C_1^\vee, C_1)$.

3. Specialization

In [YY22, §2.6], we introduced four embeddings of affine root systems of type A_1 into type $(C_1^\vee, C_1)$. They are given by certain specializations of the parameters $(\underline{k},\underline{l})$, and are characterized to preserve the Macdonald inner product under which the Macdonald-Koornwinder polynomials are orthogonal. Among the four specializations, the one given by

$$(\underline{k},\underline{l}) = (k,1,1,1) \tag{3.0.1}$$

has the special property that it is also compatible with the duality anti-involution (2.1.15). In this section, we show that this specialization yields the commutative diagram mentioned in § 0:

$$\begin{array}{ccc} \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)} & \xrightarrow{\ \chi_+^{(C_1^\vee,C_1)}\ } & \mathrm{SOL}_{\mathrm{bAW}} \\ \big\downarrow{\scriptstyle \mathrm{sp}} & & \big\downarrow{\scriptstyle \mathrm{sp}} \\ \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1} & \xrightarrow[\ \chi_+^{A_1}\ ]{} & \mathrm{SOL}_{\mathrm{bMR}} \end{array}$$

3.1. The bispectral qKZ equations.

Recall the subalgebras $H_0^{A_1}(k) \subset \mathbb{H}^{A_1}(k,q)$ and $H_0^{(C_1^\vee,C_1)}(\underline{k}) \subset \mathbb{H}^{(C_1^\vee,C_1)}(\underline{k},\underline{l},q)$, both of which have the basis $\{T_e = 1, T_{s_1} = T_1\}$. Let us identify these linear spaces, and denote it by H_0. As in the previous sections, let us use the notation $\mathbb{K} = \mathcal{M}(x,\xi)$ and $H_0^{\mathbb{K}} = \mathbb{K}\otimes H_0$.

Then, the solution spaces of bispectral qKZ equations of type A_1 and of type $(C_1^\vee, C_1)$ (Definition 1.2.8 and Definition 2.2.3) can be expressed as

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{A_1}(k,q) = \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } A_1\},$$
$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)}(\underline{k},\underline{l},q) = \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } (C_1^\vee,C_1)\}.$$

Then we can show:

Proposition 3.1.1. For the specialized parameters $(\underline{k},\underline{l}) = (k,1,1,1)$, we have the relation

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)}(k,1,1,1,q) \subset \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1}(k,q).$$

Proof. Denoting by $c^{A_1}(z;k,q) := c(z;k,q)$ the function in (1.1.17), and by $c^{(C_1^\vee,C_1)}(z;k,l,q) := c(z;k,l,q)$ the function in (2.1.11), we have

$$c^{(C_1^\vee,C_1)}(z;k,1,q) = c^{A_1}(z;k,q).$$

Then, comparing Lemma 1.2.16 and Lemma 2.2.4, we have

$$C^{(C_1^\vee,C_1)}_{1,0}(k,1,1,1,q) = C^{A_1}_{2,0}(k,q), \quad C^{(C_1^\vee,C_1)}_{0,1}(k,1,1,1,q) = C^{A_1}_{0,2}(k,q), \tag{3.1.1}$$

from which we have the claim. □

Theorem 3.1.2. The specialization (3.0.1) yields the commutative diagram

$$\begin{array}{ccc} \mathrm{SOL}^{(C_1^\vee,C_1)}_{\mathrm{bqKZ}}(k,1,1,1,q) & \xrightarrow{\chi_+^{(C_1^\vee,C_1)}} & \mathrm{SOL}_{\mathrm{bAW}}(k,1,1,1,q) \\ \mathrm{sp}\downarrow & & \downarrow\mathrm{sp} \\ \mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k,q) & \xrightarrow[\chi_+^{A_1}]{} & \mathrm{SOL}_{\mathrm{bMR}}(k,q) \end{array} \tag{3.1.2}$$

Proof. We saw the left vertical embedding in Proposition 3.1.1. Thus, it is enough to check that the specialization maps the bispectral Askey-Wilson equation (2.3.10) to the bispectral Macdonald-Ruijsenaars equation (1.3.9). Since $(k_1,k_0,l_1,l_0) = (k,1,1,1)$ yields the Askey-Wilson parameters $\{a,b,c,d\} = \{k,-k,q^{1/2},-q^{1/2}\}$, the specialization of the x-side equation is computed as

$$\begin{aligned} L^x_{(C_1^\vee,C_1)}(k,1,1,1,q) &= k+k^{-1} + \frac{k-k^{-1}x^{-2}}{1-x^{-2}}(T_{q,x}-1) + \frac{k^{-1}-kx^{-2}}{1-x^{-2}}(T_{q,x}^{-1}-1) \\ &= \frac{k-k^{-1}x^{-2}}{1-x^{-2}}T_{q,x} + \frac{k^{-1}-kx^{-2}}{1-x^{-2}}T_{q,x}^{-1} = L^x_{A_1}(k,q^2). \end{aligned}$$

Note that the parameter q^2 in type A_1 is compatible with the relation (3.1.1). The ξ-side is similarly checked directly, or by the compatibility of the duality anti-involution and the specialization. □

So far we give a computational argument to show the commutative diagram (3.1.2). Let us give another, more conceptual argument.

Lemma 3.1.3. There is an isomorphism of algebras

$$\mathbb{H}^{(C_1^\vee,C_1)}(k,1,1,1,q) \xrightarrow{\sim} \mathbb{H}^{A_1}(k,q).$$

Proof. Recall the presentations (1.1.20) of $\mathbb{H}^{A_1}$ and (2.1.12) of $\mathbb{H}^{(C_1^\vee,C_1)}$. The former gives $\mathbb{H}^{A_1}(k,q)$ as the quotient of the free algebra $\mathbb{C}\langle T,U,X\rangle$ by the relations

$$(T-k)(T+k^{-1}) = 0, \quad U^2 = 1, \quad TXT = X^{-1}, \quad UXU = q^{1/2}X^{-1}.$$

Under the specialization $(\underline{k},\underline{l}) = (k,1,1,1)$, the latter gives $\mathbb{H}^{(C_1^\vee,C_1)}(k,1,1,1,q)$ as the quotient of $\mathbb{C}\langle T_1,T_0,T_1^\vee,T_0^\vee\rangle$ by the relations

$$(T_1-k)(T_1+k^{-1}) = 0, \quad (T_0)^2 = (T_1^\vee)^2 = (T_0^\vee)^2 = 1, \quad T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}. \tag{3.1.3}$$

Now, recalling (2.1.13), we find that the correspondence $T_1 = T$, $T_0 = U$ and $T_0^\vee = q^{-1/2}UX$ gives the desired isomorphism □

Since the bispectral correspondence $\chi_+^{A_1}$ is defined in terms of the DAHA $\mathbb{H}^{A_1}(k,q)$, the restriction to the subalgebra $\mathbb{H}^{(C_1^\vee,C_1)}(k,1,1,1,1,q)$ will give the correspondence $\chi_+^{(C_1^\vee,C_1)}$. Thus we have the commutative diagram (3.1.2).

Remark 3.1.4. We leave it for a future study to give an explicit element in $\mathrm{SOL}_{\mathrm{bAW}}(k,1,1,1,q)$ which is mapped to $\mathrm{SOL}_{\mathrm{bMR}}(k,q)$ under the right vertical embedding sp in (3.1.2). Here we only give a clue to find such an element. If the spectral variable ξ is specialized to $\xi_1 = k^{-1}q^{-1/2}$ (see Proposition 1.5.3 (2)), we have

$$P_n^{A_1}(x;k^2,q) := x^n {}_2\phi_1\left[\begin{matrix} k^2,\ q^{-n} \\ q^{1-n}/k^2 \end{matrix}; q,\ \frac{q}{k^2x^2}\right] = \frac{1}{(q^nk^2;q)_n}P_n(x;k,1,1,1;q) = P_n^{(C_1^\vee,C_1)}(x;k,1,1,1;q).$$

We expect that there is an element $f(x,\xi) \in \mathrm{SOL}_{\mathrm{bAW}}(k,1,1,1,q)$ such that the specialized $f(x,\xi_n)$ is equal to $P_n^{(C_1^\vee,C_1)}(x;k,1,1,1;q)$ and the image $\mathrm{sp}(f(x,\xi_n))$ is equal to $P_n^{A_1}(x;t,q)$.

References

[AW85] R. Askey. J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc., **54** (1985), no. 319.

[C92a] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Commun. Math. Phys., **150** (1992), 109–136.

[C92b] I. Cherednik, *Double Affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, Int. Math. Res. Not., **9** (1992), 171–179.

[C94] I. Cherednik, *Induced representations of double affine Hecke algebras and applications*, Math. Res. Lett., **1** (1994), 319–337.

[C95] I. Cherednik, *Macdonald's evaluation conjectures and difference Fourier transform*, Inv. Math., **122** (1995), 119–145.

[C97] I. Cherednik, *Difference Macdonald-Mehta conjecture*, Internat. Math. Res. Notices **1997** (1997), no. 10, 449–467.

[C05] I. Cherednik, *Double Affine Hecke Algebras*, London Math. Soc. Lect. Note Ser., **319**. Cambridge Univ. Press, Cambridge (2005).

[C09] I. Cherednik, *Whittaker limits of difference spherical functions*, Internat. Math. Res. Notices **2009** (2009), no. 20, 3793–3842.

[GR04] G. Gasper, M. Rahman, *Basic hypergeometric series*, 2nd. ed., Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press (2004).

[L76] E. Looijenga, *Root systems and Elliptic curves*, Inv. Math., **38** (1976), 17–32.

[M87] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, preprint, 1987; typed and published in the Sḿinaire Lotharingien de Combinatoire, **45** (2000), B45a; available from arXiv:math/0011046.

[M95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, 1995.

[M03] I. G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics **157**, Cambridge Univ. Press, 2003.

[vM11] M. van Meer, *Bispectral quantum Knizhnik-Zamolodchikov equations for arbitrary root systems*, Sel. Math. New Ser., **17** (2011), 183–221.

[vMS09] M. van Meer, J. V. Stokman, *Double Affine Hecke Algebras and Bispectral Quantum Knizhnik-Zamolodchikov Equations*, Int. Math. Res. Not., **6** (2010), 969–1040.

[N95] M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings* (in Japanese), in *Various Aspects of Hypergeometric Functions* (Kyoto, 1994), RIMS Kôkyûroku **919**, Res. Inst. Math. Sci., Kyoto Univ., Kyoto (1995), 44–55.

[NSh] M. Noumi, J. Shiraishi, *A direct approach to the bispectral problem for the Ruijsenaars-Macdonald q-difference operators*, preprint (2012), arXiv:1206.5364.

[NSt04] M. Noumi, J. V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebra approach*, in *Laredo Lectures on Orthogonal Polynomials and Special Functions*, pp. 111–144, Nova Science Publishers, 2004; arXiv:math/0001033.

[R87] S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys., **110** (1987), 191–213.

[Sa99] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. Math., **150** (1999) 267–282.

[St02] J. V. Stokman, *An expansion formula for the Askey-Wilson function*, J. Approx. Theory, **114** (2002), 308–342.

[St02] J. V. Stokman, *Difference Fourier transforms for nonreduced root systems*, Selecta Math. **9** (2003), 409–494.

[St14] J. V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, Ann. Math., **179** (2014), 253–299.

[T10] Y. Takeyama, *Differential equations compatible with boundary rational qKZ equation*, in *New trends in quantum integrable systems*, pp. 421–450 (2010); arXiv:0908.2288.

[YY22] K. Yamaguchi, S. Yanagida, *Specializing Koornwinder polynomials to Macdonald polynomials of type B, C, D and BC*, J. Algebraic Combin. (2022), online published: https://doi.org/10.1007/s10801-022-01165-8

Graduate School of Mathematics, Nagoya University. Furocho, Chikusaku, Nagoya, Japan, 464-8602.

Email address: d20003j@math.nagoya-u.ac.jp, yanagida@math.nagoya-u.ac.jp