# On the module category of the triplet W-algebra $\mathcal{W}_{p_+,p_-}$

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We study the structure of the category of modules over the triplet Walgebra  $\mathcal{W}_{p_+,p_-}$  defined by Feigin, Gainutdinov, Semikhatov and Tipunin [1]. Since  $\mathcal{W}_{p_+,p_-}$  satisfies the  $C_2$ -cofinite condition, by Huang, Lepowsky and Zhang [2], every simple module has the projective cover and the module categories have the structure of a braided tensor category. We determine the structure of the projective covers of all simple  $\mathcal{W}_{p_+,p_-}$ -modules, and determine certain non-semisimple fusion rules conjectured by Rasmussen [3] and Gaberdiel, Runkel and Wood [4]. This paper is based on the thesis [5].

## $1 \quad \text{Main results on the triplet $W$-algebra $\mathcal{W}_{p_+,p_-}$}$

Fix two coprime integers  $p_+, p_-$  such that  $p_- > p_+ \ge 2$  and let

$$c_{p_+,p_-} := 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-}$$

be a minimal central charge of Virasoro algebra. Let us briefly rewiew the definitions of the triplet W-algebra  $\mathcal{W}_{p_+,p_-}$  and the simple  $\mathcal{W}_{p_+,p_-}$ -modules in accordance with [6, 7, 8].

For  $\alpha \in \mathbb{C}$ , let  $F_{\alpha}$  be the bosonic Fock module generated from the bosonic field

$$Y(|\alpha\rangle, z) = e^{\alpha \hat{a}} z^{\alpha a_0} e^{\alpha \sum_{n \ge 1} \frac{a_{-n}}{n} z^n} e^{-\alpha \sum_{n \ge 1} \frac{a_n}{n} z^{-n}},$$

where

$$[a_m, a_n] = m\delta_{m+n,0} \text{id}, \qquad \qquad [\hat{a}, a_n] = \delta_{n,0} \text{id}.$$

Let

$$T := \frac{1}{2} \left( a_{-1}^2 - (\alpha_+ + \alpha_-) a_{-2} \right) |0\rangle, \qquad \alpha_+ := \sqrt{\frac{2p_-}{p_+}}, \quad \alpha_- := -\sqrt{\frac{2p_+}{p_-}}$$

be a confomal vector. By T, each Fock module  $F_{\alpha}$  becomes a Virasoro module whose central charge  $c_{p_+,p_-}$ .

For  $r, s, n \in \mathbb{Z}$  we introduce the following symbols

$$\alpha_{r,s;n} := \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- + \frac{\sqrt{2p+p-1}}{2}n.$$

Let  $F_{r,s;n} := F_{\alpha_{r,s;n}}$ .

As detailed in [9], we can define the complex screening operators

$$Q_{+}^{[r]} = \oint_{z=0} dz \int_{[\Delta_{r-1}]} Q_{+}(z)Q_{+}(zy_{1})\cdots Q_{+}(zy_{r-1})dy_{1}\cdots dy_{r-1} \in \operatorname{Hom}_{\mathbb{C}}(F_{r,k;l}, F_{-r,k;l}),$$
$$Q_{-}^{[s]} = \oint_{z=0} dz \int_{[\Delta_{s-1}]} Q_{-}(z)Q_{-}(zy_{1})\cdots Q_{-}(zy_{s-1})dy_{1}\cdots dy_{s-1} \in \operatorname{Hom}_{\mathbb{C}}(F_{k,s;l}, F_{k,-s;l}),$$

where  $Q_{\pm}(z) = Y(|\alpha_{\pm}\rangle, z)$  and  $[\Delta_n]$  is a regularized cycle constructed from the simplex  $\Delta_n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid 1 > y_1 > \cdots y_n > 0\}$ . Let  $Q_+^{[r]}$  and  $Q_-^{[s]}$  be the zero modes of  $Q_+^{[r]}(z)$  and  $Q_-^{[s]}(z)$ . These zero modes commute with every Virasoro mode of Y(T, z) and are called screening operators.

#### Definition 1.1.

The lattice vertex operator algebra  $\mathcal{V}_{[p_+,p_-]}$  is the tuple

 $(\mathcal{V}_{1,1}^+, |0\rangle, T, Y),$ 

where underlying vector space of  $\mathcal{V}_{[p_+,p_-]}$  is given by

$$\mathcal{V}_{1,1}^+ = \bigoplus_{n \in \mathbb{Z}} F_{1,1;2n} = \bigoplus_{n \in \mathbb{Z}} F_{n\sqrt{2p+p_-}},$$

and  $Y(|\alpha_{1,1;2n}\rangle; z) = V_{\alpha_{1,1;2n}}(z)$  for  $n \in \mathbb{Z}$ .

It is a known fact that simple  $\mathcal{V}_{[p_+,p_-]}$ -modules are given by the following  $2p_+p_-$  direct sum of Fock modules

$$\mathcal{V}_{r,s}^{+} = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n}, \qquad \qquad \mathcal{V}_{r,s}^{-} = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n+1},$$

where  $1 \le r \le p_+, 1 \le s \le p_-$ .

Note that the two screening operators  $Q_+$  and  $Q_-$  act on  $\mathcal{V}_{1,1}^+$ . We define the following vector subspace of  $\mathcal{V}_{1,1}^+$ :

$$\mathcal{K}_{1,1} = \ker Q_+ \cap \ker Q_- \subset \mathcal{V}_{1,1}^+.$$

**Definition 1.2** ([1]). The triplet W-algebra

$$\mathcal{W}_{p_+,p_-} = \left( \mathcal{K}_{1,1}, \left| 0 \right\rangle, T, Y \right)$$

is a sub vertex operator algebra of  $\mathcal{V}_{[p_+,p_-]}$ , where the vacuum vector, conformal vector and vertex operator map are those of  $\mathcal{V}_{[p_+,p_-]}$ .

Let

$$r^{\vee} = p_+ - r, \qquad s^{\vee} = p_- - s$$

For each  $1 \leq r \leq p_+$ ,  $1 \leq s \leq p_-$ , let  $\mathcal{X}_{r,s}^{\pm}$  be the following vector subspace of  $\mathcal{V}_{r,s}^{\pm}$ :

- 1. For  $1 \le r \le p_+ 1$ ,  $1 \le s \le p_- 1$ ,  $\mathcal{X}^+_{r,s} = Q^{[r^{\vee}]}_+(\mathcal{V}^-_{r^{\vee},s}) \cap Q^{[s^{\vee}]}_-(\mathcal{V}^-_{r,s^{\vee}}), \quad \mathcal{X}^-_{r,s} = Q^{[r^{\vee}]}_+(\mathcal{V}^+_{r^{\vee},s}) \cap Q^{[s^{\vee}]}_-(\mathcal{V}^+_{r,s^{\vee}}).$
- 2. For  $1 \le r \le p_+ 1$ ,  $s = p_-$ ,

$$\mathcal{X}_{r,p_{-}}^{+} = Q_{+}^{[r^{\vee}]}(\mathcal{V}_{r^{\vee},p_{-}}^{-}), \qquad \qquad \mathcal{X}_{r,p_{-}}^{-} = Q_{+}^{[r^{\vee}]}(\mathcal{V}_{r^{\vee},p_{-}}^{+})$$

3. For  $r = p_+, 1 \le s \le p_- - 1$ ,

$$\mathcal{X}_{p_{+},s}^{+} = Q_{-}^{[s^{\vee}]}(\mathcal{V}_{p_{+},s^{\vee}}^{-}), \qquad \qquad \mathcal{X}_{p_{+},s}^{-} = Q_{-}^{[s^{\vee}]}(\mathcal{V}_{p_{+},s^{\vee}}^{+}).$$

4.  $r = p_+, s = p_-,$ 

$$\mathcal{X}_{p_+,p_-}^+ = \mathcal{V}_{p_+,p_-}^+, \qquad \qquad \mathcal{X}_{p_+,p_-}^- = \mathcal{V}_{p_+,p_-}^-.$$

We define the interior Kac table  $\mathcal{T}$  as the following quotient set

$$\mathcal{T} = \{ (r, s) | \ 1 \le r < p_+, 1 \le s < p_- \} / \sim$$

where  $(r, s) \sim (r', s')$  if and only if  $r' = p_+ - r, s' = p_- - s$ . Note that  $\#\mathcal{T} = \frac{(p_+-1)(p_--1)}{2}$ . For  $(r, s) \in \mathcal{T}$ , let  $L(h_{r,s})$  be the Virasoro minimal simple module defined by

$$L(h_{r,s}) = \operatorname{Ker}_{F_{r,s;0}} Q_+^{[r]} / \operatorname{Im}_{F_{r^{\vee},s;-1}} Q_+^{[r^{\vee}]}.$$

**Theorem 1.3** ([6, 7, 8]). The  $\frac{(p_+-1)(p_--1)}{2} + 2p_+p_-$  vector spaces

$$L(h_{r,s}), \ (r,s) \in \mathcal{T}, \qquad \qquad \mathcal{X}_{r,s}^{\pm}, \ 1 \le r \le p_+, \ 1 \le s \le p_-$$

become simple  $\mathcal{W}_{p_+,p_-}$ -modules and give all simple  $\mathcal{W}_{p_+,p_-}$ -modules.

We use the following symbols for the projective covers of the simple modules.

**Definition 1.4.** Let  $1 \le r < p_+, 1 \le s < p_-$ .

- 1. Let  $\mathcal{P}^+_{r,s}$  and  $\mathcal{P}^-_{r,s}$  be the projective covers of the simple modules  $\mathcal{X}^+_{r,s}$  and  $\mathcal{X}^-_{r,s}$ , respectively.
- 2. Let  $\mathcal{P}(h_{r,s})$  be the projective cover of the minimal simple module  $L(h_{r,s})$ .
- 3. Let  $\mathcal{Q}(\mathcal{X}_{r,p_{-}}^{\pm})_{r^{\vee},p_{-}}$  be the projective covers of the simple modules  $\mathcal{X}_{r,p_{-}}^{+}$ and  $\mathcal{X}_{r,p_{-}}^{-}$ , respectively.
- 4. Let  $\mathcal{Q}(\mathcal{X}_{p_{+},s}^{\pm})_{p_{+},s^{\vee}}$  be the projective covers of the simple modules  $\mathcal{X}_{p_{+},s}^{+}$ and  $\mathcal{X}_{p_{+},s}^{-}$ , respectively.

**Theorem 1.5** ([5]). The projective modules  $\mathcal{P}_{r,s}^{\pm}$ ,  $\mathcal{Q}(\mathcal{X}_{r,p_{-}}^{\pm})_{r^{\vee},p_{-}}$  and  $\mathcal{Q}(\mathcal{X}_{p_{+},s}^{\pm})_{p_{+},s^{\vee}}$  have the following socle series:

1. For  $\mathcal{P}_{r,s}^+$ , we have

$$S_{1} = \mathcal{X}_{r,s}^{+},$$

$$S_{2}/S_{1} = \mathcal{X}_{r,s^{\vee}}^{-} \oplus \mathcal{X}_{r,s^{\vee}}^{-} \oplus L(h_{r,s}) \oplus \mathcal{X}_{r^{\vee},s}^{-} \oplus \mathcal{X}_{r^{\vee},s}^{-},$$

$$S_{3}/S_{2} = \mathcal{X}_{r,s}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+} \oplus \mathcal{X}_{r,s^{\vee}}^{+},$$

$$S_{4}/S_{3} = \mathcal{X}_{r^{\vee},s}^{-} \oplus \mathcal{X}_{r^{\vee},s}^{-} \oplus L(h_{r,s}) \oplus \mathcal{X}_{r,s^{\vee}}^{-} \oplus \mathcal{X}_{r,s^{\vee}}^{-},$$

$$\mathcal{P}_{r,s}^{+}/S_{4} = \mathcal{X}_{r,s}^{+}.$$

where  $S_i = \operatorname{Soc}_i$ .

2. For  $\mathcal{P}^-_{r^{\vee},s}$ , we have

$$S_{1} = \mathcal{X}^{-}_{r^{\vee},s},$$

$$S_{2}/S_{1} = \mathcal{X}^{+}_{r^{\vee},s^{\vee}} \oplus \mathcal{X}^{+}_{r^{\vee},s^{\vee}} \oplus \mathcal{X}^{+}_{r,s} \oplus \mathcal{X}^{+}_{r,s},$$

$$S_{3}/S_{2} = \mathcal{X}^{-}_{r^{\vee},s} \oplus \mathcal{X}^{-}_{r^{\vee},s} \oplus \mathcal{X}^{-}_{r,s^{\vee}} \oplus L(h_{r,s}) \oplus L(h_{r,s}) \oplus \mathcal{X}^{-}_{r,s^{\vee}} \oplus \mathcal{X}^{-}_{r,s^{\vee}} \oplus \mathcal{X}^{-}_{r^{\vee},s},$$

$$S_{4}/S_{3} = \mathcal{X}^{+}_{r,s} \oplus \mathcal{X}^{+}_{r,s} \oplus \mathcal{X}^{+}_{r^{\vee},s^{\vee}} \oplus \mathcal{X}^{+}_{r^{\vee},s^{\vee}},$$

$$\mathcal{P}^{-}_{r^{\vee},s}/S_{4} = \mathcal{X}^{-}_{r^{\vee},s}.$$

3. Let  $(a, b, c, d, \epsilon)$  be an element in { $(r, p_{-}, r^{\vee}, +), (r^{\vee}, p_{-}, r, p_{-}, -), (p_{+}, s, p_{+}, s^{\vee}), (p_{+}, s^{\vee}, p_{+}, s)$ }. Then, for the socle series of  $\mathcal{Q}(\mathcal{X}_{a,b}^{\epsilon})_{c,d}$ , we have

$$\begin{aligned} &\operatorname{Soc}_{1} = \mathcal{X}_{a,b}^{\epsilon}, \\ &\operatorname{Soc}_{2}/\operatorname{Soc}_{1} = \mathcal{X}_{c,d}^{-\epsilon} \oplus \mathcal{X}_{c,d}^{-\epsilon} \\ &\mathcal{Q}(\mathcal{X}_{a,b}^{\epsilon})_{c,d}/\operatorname{Soc}_{2} = \mathcal{X}_{a,b}^{\epsilon}. \end{aligned}$$

**Definition 1.6.** By taking quotients of  $\mathcal{P}^+_{r,s}$ ,  $\mathcal{P}^+_{r^{\vee},s^{\vee}}$ ,  $\mathcal{P}^-_{r^{\vee},s}$  and  $\mathcal{P}^-_{r,s^{\vee}}$ , we obtain eight indecomposable modules  $\mathcal{Q}(\mathcal{X}^{\epsilon}_{a,b})_{b,c}$  where

$$\{ (\epsilon, a, b, c, d) \} = \{ (+, r, s, r^{\vee}, s), (+, r, s, r, s^{\vee}), (+, r^{\vee}, s^{\vee}, r^{\vee}, s), (+, r^{\vee}, s^{\vee}, r, s^{\vee}), (-, r^{\vee}, s, r, s), (-, r^{\vee}, s, r^{\vee}, s^{\vee}), (-, r, s^{\vee}, r, s), (-, r, s^{\vee}, r^{\vee}, s^{\vee}) \},$$

and each socle series is given by:

1. For  $\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}$ ,

$$Soc_1 = \mathcal{X}^+_{a,b},$$
  

$$Soc_2/Soc_1 = \mathcal{X}^-_{c,d} \oplus L(h_{a,b}) \oplus \mathcal{X}^-_{c,d},$$
  

$$\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}/Soc_2 = \mathcal{X}^+_{a,b}.$$

2. For  $\mathcal{Q}(\mathcal{X}_{a,b}^{-})_{c,d}$ ,

$$Soc_1 = \mathcal{X}_{a,b}^-,$$
  

$$Soc_2/Soc_1 = \mathcal{X}_{c,d}^+ \oplus \mathcal{X}_{c,d}^+,$$
  

$$\mathcal{Q}(\mathcal{X}_{a,b}^-)_{c,d}/Soc_2 = \mathcal{X}_{a,b}^-.$$

Using the structure of the center of the Zhu algebra  $A(\mathcal{W}_{p_+,p_-})$  [6, 7, 8], we can determine the structure of the projective modules  $\mathcal{P}(h_{r,s})$ .

**Theorem 1.7** ([5]). Each projective module  $\mathcal{P}(h_{r,s})$  has the following length five socle series:

$$\begin{aligned} \operatorname{Soc}_{1}(\mathcal{P}(h_{r,s})) &= L(h_{r,s}), \\ \operatorname{Soc}_{2}(\mathcal{P}(h_{r,s})) / \operatorname{Soc}_{1}(\mathcal{P}(h_{r,s})) &= \mathcal{X}_{r,s}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+}, \\ \operatorname{Soc}_{3}(\mathcal{P}(h_{r,s})) / \operatorname{Soc}_{2}(\mathcal{P}(h_{r,s})) &= 2\mathcal{X}_{r^{\vee},s}^{-} \oplus L(h_{r,s}) \oplus 2\mathcal{X}_{r,s^{\vee}}^{-}, \\ \operatorname{Soc}_{4}(\mathcal{P}(h_{r,s})) / \operatorname{Soc}_{3}(\mathcal{P}(h_{r,s})) &= \mathcal{X}_{r,s}^{+} \oplus \mathcal{X}_{r^{\vee},s^{\vee}}^{+}, \\ \mathcal{P}(h_{r,s}) / \operatorname{Soc}_{4}(\mathcal{P}(h_{r,s})) &= L(h_{r,s}). \end{aligned}$$

In the following, we introduce the structure of certain fusion rules of  $\mathcal{W}_{p_+,p_-}$ . Let us define the following indecomposable modules.

#### Definition 1.8.

1. For 
$$1 \le r \le p_+ - 1$$
,  $1 \le s \le p_- - 1$ ,  
 $\mathcal{K}_{r,s} := \mathcal{W}_{p_+,p_-}$ .  $|\alpha_{r,s}\rangle$ 

2. For  $1 \le r \le p_+, \ 1 \le s \le p_-,$  $\mathcal{K}_{r,p_-} := \mathcal{X}^+_{r,p_-}, \qquad \qquad \mathcal{K}_{p_+,s} := \mathcal{X}^+_{p_+,s}.$ 

Let  $\mathcal{C}_{p_+,p_-}$  be the category of  $\mathcal{W}_{p_+,p_-}$ -modules and let  $(\mathcal{C}_{p_+,p_-},\boxtimes,\mathcal{K}_{1,1})$  be the braided tensor category on  $\mathcal{C}_{p_+,p_-}$ , where  $\mathcal{K}_{1,1}$  is the unit object.

Similar to the arguments in [10, 11, 12], we can show the following theorem.

**Theorem 1.9.** The indecomposable modules  $\mathcal{K}_{1,2}$  and  $\mathcal{K}_{2,1}$  are rigid and selfdual.

Using the self-duality of  $\mathcal{K}_{1,2}$  and  $\mathcal{K}_{2,1}$ , we obtain the following theorems.

**Theorem 1.10** ([5]). All indecomposable modules of types  $\mathcal{K}_{r,s}$ ,  $\mathcal{Q}(\mathcal{X}_{r,s}^{\pm})_{\bullet,\bullet}$ and  $\mathcal{P}_{r,s}^{\pm}$  are rigid and self-dual in  $(\mathcal{C}_{p_+,p_-},\boxtimes,\mathcal{K}_{1,1})$ .

**Theorem 1.11** ([5]). *1.* For  $1 \le r \le p_+$ ,  $1 \le s \le p_-$ ,

 $\mathcal{K}_{1,1}^* \boxtimes \mathcal{K}_{r,s} = \mathcal{K}_{r,s}^*,$ 

where  $\mathcal{K}_{r,s}^*$  is the contragredient of  $\mathcal{K}_{r,s}$ .

2. For any simple modules  $\mathcal{X}_{r,s}^{\pm}$  and  $\mathcal{X}_{r',s'}^{\pm}$ , we have

$$\mathcal{X}_{r,s}^{\pm} \boxtimes \mathcal{X}_{r',s'}^{\pm} = (\mathcal{K}_{r,s} \boxtimes \mathcal{K}_{r',s'}) \boxtimes \mathcal{K}_{1,1}^{*}.$$

Let us introduce the free abelian group  $P^0(\mathcal{C}_{p_+,p_-})$  of rank  $8p_+p_- - 4p_+ - 4p_- + 2$ 

$$P^{0}(\mathcal{C}_{p_{+},p_{-}}) = \bigoplus_{r=1}^{p_{+}} \bigoplus_{s=1}^{p_{-}} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{X}_{r,s}^{\epsilon}]_{P} \oplus \bigoplus_{r=1}^{p_{+}-1} \bigoplus_{s=1}^{p_{-}-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{P}_{r,s}^{\epsilon}]_{P}$$
$$\oplus \bigoplus_{r=1}^{p_{+}-1} \bigoplus_{s=1}^{p_{-}-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}_{r,s}^{\epsilon})_{r^{\vee},s}]_{P} \oplus \bigoplus_{r=1}^{p_{+}-1} \bigoplus_{s=1}^{p_{-}-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}_{r,s}^{\epsilon})_{r,s^{\vee}}]_{P}$$
$$\oplus \bigoplus_{r=1}^{p_{+}-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}_{r,p_{-}}^{\epsilon})_{r^{\vee},p_{-}}]_{P} \oplus \bigoplus_{s=1}^{p_{-}-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}_{p_{+},s}^{\epsilon})_{p_{+},s^{\vee}}]_{P}.$$

For any  $M \in \mathcal{C}_{p_+,p_-}$  which have minimal simple modules in the Socle, let  $\pi_0(M)$  be the quotient module of M quotiented by all the minimal simple modules in the Socle. We define a  $\pi \in \operatorname{Hom}(\mathcal{C}_{p_+,p_-})$  such that for any M in  $\mathcal{C}_{p_+,p_-}$ 

$$\pi(M) = \begin{cases} \pi_0(M) & M \text{ contains minimal simple modules in Soc}(M) \\ M & \text{otherwise} \end{cases}$$

$$[\bullet]_P \cdot [\bullet]_P = [\pi(\bullet \boxtimes \bullet)]_P.$$

The three operators

$$X = \pi(\mathcal{X}_{1,2}^+ \boxtimes -), \qquad Y = \pi(\mathcal{X}_{2,1}^+ \boxtimes -), \qquad Z = \pi(\mathcal{X}_{1,1}^- \boxtimes -)$$

define  $\mathbb{Z}$ -linear endomorphism of  $P^0(\mathcal{C}_{p_+,p_-})$ . Thus  $P^0(\mathcal{C}_{p_+,p_-})$  is a module over  $\mathbb{Z}[X,Y,Z]$ . We define the following  $\mathbb{Z}[X,Y,Z]$ -module map

$$\psi: \mathbb{Z}[X, Y, Z] \to P^0(\mathcal{C}_{p_+, p_-}),$$
  
$$f(X, Y, Z) \mapsto f(X, Y, Z) \cdot [\mathcal{X}_{1,1}^+]_P.$$

**Theorem 1.13** ([5]). The  $\mathbb{Z}[X, Y, Z]$ -module map  $\psi$  is surjective, and, through  $\psi$ , we have the isomorphism of rings

$$P^{0}(\mathcal{C}_{p_{+},p_{-}}) \simeq \frac{\mathbb{Z}[X,Y] \oplus \mathbb{Z}[X,Y]Z}{\langle Z^{2} - 1, U_{2p_{-}-1}(X) - 2ZU_{p_{-}-1}(X), U_{2p_{+}-1}(Y) - 2ZU_{p_{+}-1}(Y) \rangle},$$

where  $U_n(A)$  is the Chebyshev polynomials defined recursively

$$U_0(A) = 1,$$
  $U_1(A) = A,$   
 $U_{n+1}(A) = AU_n(A) - U_{n-1}(A).$ 

**Remark 1.14.** By using this theorem, we can obtain the non-semisimple fusion rules conjectured by [3] and [4].

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