Lower bounds for Maass forms on semisimple groups

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1 Introduction

This is a resume of the paper [4], by Farrell Brumley and myself. Let Y be a closed Riemannian manifold of dimension n and with Laplace operator Δ. Let $\{\psi_i\}$ be an orthonormal basis of Laplace eigenfunctions for $L^2(Y)$, which satisfy $\|\psi_i\|_2 = 1$ and $(\Delta+\lambda_i^2)\psi_i = 0$. We assume that $\{\psi_i\}$ are ordered by eigenvalue, so that $0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots$. It is an important question in harmonic analysis to determine the asymptotic size of ψ_i, i.e. the growth rate of $\|\psi_i\|_\infty$ in terms of λ_i. The basic upper bound for $\|\psi_i\|_\infty$, proved by Avacumović [1] and Levitan [9], is given by

$$\|\psi_i\|_\infty \ll \lambda_i^{(n-1)/2}. \tag{1}$$

This bound is sharp on the round n-sphere. Indeed, the zonal spherical harmonics have peaks of maximal size at the poles of the axis of rotation. More generally, Sogge and Zelditch [19] have shown that the compact Riemannian manifolds saturating (1) necessarily have points which are fixed by an appropriately large number of geodesic returns, in the sense that every geodesic passing through such a point is a loop.

On the other hand, it is reasonable to expect[1] that if Y has strictly negative curvature then the strong bound

$$\|\psi_i\|_\infty \ll_\epsilon \lambda_i^\epsilon \tag{2}$$

holds with density one. This is akin to the Ramanujan conjecture in the theory of automorphic forms [17]: a generic sequence of eigenfunctions is tempered. Any sequence violating (2) will be called *exceptional*.

In the paper [4] we attempt to give sufficient conditions for a negatively curved manifold to support exceptional sequences. Although the question is of interest in this general setting, our techniques are limited to arithmetic locally symmetric spaces. Put succinctly, we show that an arithmetic manifold supports exceptional sequences whenever it has a point with strong Hecke return properties.

[1] The Random Wave Model would predict that almost all sequences satisfy this bound. There are in fact known examples of negatively curved Y admitting sequences violating it. In each of these examples, to be recalled later in the introduction, the offending sequences are in fact of zero density.

1.1 Comparison to quantum ergodicity

Before stating our results we would like to discuss the conjectural generic behavior (2), which to our knowledge has not been previously stated in the literature.

One way of approaching this question is via the link between the asymptotics of sequences of eigenfunctions and the dynamical properties of the geodesic flow. Roughly speaking, microlocal analysis suggests that eigenfunctions on Y should exhibit the same degree of chaotic behaviour as the geodesic flow. On S^n, for instance, the geodesic flow is totally integrable, and this is reflected in the fact that one can both write down an explicit basis of eigenfunctions, and find eigenfunctions with large peaks. Conversely, if Y is negatively curved then its geodesic flow is highly chaotic, and one expects this to be reflected in the asymptotics of the eigenfunctions. For example, the quantum ergodicity (QE) theorem of Schnirelman [18], Colin de Verdière [5], and Zelditch [21] states that the L^2-mass of a generic sequence of Laplacian eigenfunctions on a negatively curved manifold equidistributes to the uniform measure. More precisely, every orthonormal basis of eigenfunctions admits a density one subsequence ψ_i such that the measures $|\psi_i|^2 dV$ tend weakly to dV. It would be interesting to see if existing microlocal techniques such as those used in the QE theorem can be used to prove the expected generic behavior (2) of the sup norm, or weaker versions involving power savings off the local bound (1).

Recall now the conjectural strengthening of the quantum ergodicity theorem by Rudnick and Sarnak [14], known as the Quantum Unique Ergodicity (QUE) conjecture. It states that in negative curvature the L^2-mass of *any* sequence of eigenfunctions equidistributes to the uniform measure. A similar conjecture has been made relative to sup norms for compact hyperbolic surfaces (but unfortunately lacks a catchy name): Iwaniec and Sarnak [7] conjecture that (2) should hold for *all* eigenfunctions of a compact hyperbolic surface. In our terminology, this is saying that hyperbolic surfaces do not support exceptional sequences. This is a very difficult problem, even for arithmetic surfaces (where QUE has actually been proved [10]). In fact, the bound (2) is often referred to as a Lindelöf type bound, as it implies the classical Lindelöf conjecture on the Riemann zeta function in the case of the modular surface. The Iwaniec-Sarnak conjecture is consistent with the Random Wave Model, which itself can be thought of as the eigenfunction analog of the Sato-Tate conjecture in the theory of automorphic forms [17]. Moreover, it is supported by numerical computations as well as a power improvement (in arithmetic settings) over (1) established in [7].

Unlike the setting of the QUE conjecture, there do in fact exist compact manifolds Y of negative curvature which support exceptional sequences, in the sense of violating (2). The first such example was given by Rudnick and Sarnak [14]. They showed the existence of an arithmetic hyperbolic 3-manifold Y and a sequence of L^2-normalised eigenfunctions ψ_i on Y for which $\|\psi_i\|_\infty \gg_\epsilon \lambda_i^{1/2-\epsilon}$. Straining somewhat, one can view this result as being parallel to the early discovery by Piatetski-Shapiro of counter examples to the Ramanujan conjecture for non-generic cusp forms on the symplectic group.

1.2 Statement of results

Our main theorem is modelled on a result of Milićević [13], which, building on [14], provides a structural framework for the class of arithmetic hyperbolic 3-manifolds supporting exceptional sequences.

First recall that an arithmetic hyperbolic 3-manifold arises from the following general construction. Let E be a number field having exactly one complex embedding, up to equivalence, and let F be its maximal totally real subfield. For a division quaternion algebra B over E, ramified at all real places of E, denote by G the restriction of scalars of B^1 from E to F. Then any arithmetic hyperbolic 3-manifold is commensurable with a congruence manifold associated with G.

Following [13], an arithmetic hyperbolic 3-manifold as above is said to be of Maclachlan-Reid type if E is quadratic over F and there exists a quaternion division algebra A over F satisfying $B = A \otimes_F E$. The main result of *loc. cit.* is that Maclachlan-Reid type manifolds support exceptional sequences (in fact, satisfying the same lower bounds as the examples of Rudnick-Sarnak).

Notice that when $B = A \otimes_F E$, the following properties hold. Let v_0 be the unique archimedean place of F which ramifies in E. By [11, Theorem 9.5.5] we may assume that A is ramified at v_0. Then

1. $G_{v_0} = \mathrm{SL}_2(\mathbb{C})$ is noncompact, and non-split (as an $\mathbb{R}$-group);
2. $G_v = \mathbf{H}^1$ (the norm-one Hamiltonian quaternions) is compact for all real $v \neq v_0$;
3. the global involution $\theta : g \mapsto \sigma(g)$ of G, where σ is the unique non-trivial element in the Galois group of E over F, induces a Cartan involution on G_{v_0}. Indeed, $G^\theta = A^1$, so that $G^\theta(F_{v_0}) = \mathbf{H}^1$ is the maximal compact SU(2) inside $G_{v_0} = \mathrm{SL}_2(\mathbb{C})$.

Our main result is an extension of this to a wide range of compact congruence manifolds.

Theorem 1.1. *Let F be a totally real number field, and let v_0 be a real place of F. Let G/F be a connected anisotropic semisimple F-group. We make the following additional assumptions on G.*

1. *G_{v_0} is noncompact, not split, and $\mathbb{R}$-almost simple.*
2. *G_v is compact for all real $v \neq v_0$.*
3. *There is an involution θ of G defined over F that induces a Cartan involution of G_{v_0}.*

Let Y be a congruence manifold associated to G. Then there exists $\delta > 0$ and a sequence of linearly independent Laplacian eigenfunctions ψ_i on Y that satisfy

$$\|\psi_i\|_2 = 1, \quad (\Delta + \lambda_i^2)\psi_i = 0, \quad \textit{and} \quad \|\psi_i\|_\infty \gg \lambda_i^\delta.$$

1.3 Remarks on main theorem

A well-known theorem of Borel [3] addresses the question of whether one can find many groups satisfying the rationality hypothesis (3). One consequence of his theorem is that for any connected, simply-connected, semisimple algebraic $\mathbb{R}$-group G satisfying condition (1), Theorem 1.1 produces a manifold Y of the form $\Gamma\backslash G/K$ with an exceptional sequence of eigenfunctions. See Section 1.5 for more details and a concrete example.

Theorem 1.1 goes some distance toward answering the basic question of determining the precise conditions under which one should expect a Lindelöf type bound on a compact congruence negatively curved manifold. The three numbered conditions on the group G are a particularly convenient way of asking that a large enough compact subgroup of G_∞ admits a rational structure, which is a key ingredient in our proof. Although the condition that G_{v_0} is not split should be necessary, we expect that the other conditions can be relaxed somewhat. For example, throughout most of the paper, the condition that G_{v_0} is $\mathbb{R}$-almost simple could be weakened to G being F-almost simple. The stronger form of this condition is only used in one lemma, to simplify the application of a theorem of Blomer and Pohl [2, Theorem 2] and Matz-Templier [12, Proposition 7.2].

Besides the results of Rudnick-Sarnak and Milićević that we already mentioned, both in the context of arithmetic hyperbolic 3-manifolds, there are other results in the literature which provide examples of arithmetic manifolds supporting exceptional sequences. For instance, the techniques of Rudnick-Sarnak were generalised to n-dimensional hyperbolic manifolds for $n \geqslant 5$ by Donnelly [6]. And Lapid and Offen in [8] discovered a series of arithmetic quotients of $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$ admitting large eigenforms through the link with automorphic L-functions (conditionally on standard conjectures on the size of automorphic L-functions at the edge of the critical strip). Note that Theorem 1.1 includes the examples of Rudnick-Sarnak, Donnelly, and Milićević, although without explicit exponents. It is unable to reproduce the examples of Lapid-Offen due to the compactness requirement, but – as was indicated above – it can produce compact quotients of $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$ with an exceptional sequence of eigenfunctions. In fact, non-compact quotients should also be amenable to our techniques, via an application of simple trace formulae, but we have not pursued this here.

Finally, while our approach was largely inspired by that of Milićević, we have made an effort to emphasize the common features it shares with the techniques of Rudnick-Sarnak and Lapid-Offen. A synthesis of the subject, as well as a general conjecture restricting the possible limiting exponents for exceptional sequences, can be found in the influential letter [16].

1.4 A hybrid result in the level-eigenvalue aspect

We in fact prove a stronger result than that described in Theorem 1.1, establishing a lower bound in the level and eigenvalue aspects simultaneously. We present this separately, as it requires more care to state; indeed, any notion of non-trivial lower bound must overcome the lower bound one may prove when the eigenspaces have large dimension. More precisely, if M is a compact Riemannian manifold and V is the space of $\psi \in L^2(M)$ with a given Laplace eigenvalue, one may show that there is $\psi \in V$ satisfying $\|\psi\|_\infty \geqslant$

$\sqrt{\dim V}\|\psi\|_2$.

If we consider a tower of congruence covers Y_N of Y, then the Laplace eigenspaces will have growing dimension because of multiplicities in the corresponding representations at places dividing N. Computationally, one observes that this (and its stronger form involving Arthur packets) is the only source of dimension growth. Although we believe that the dimensions of the joint eigenspaces we consider should be small (partly as a result of our choice of "large" congruence subgroup), we do not know how to prove this in general. As a result, we shall be satisfied if we can beat the bound $\sqrt{\dim V}$, where V is now a space of Hecke-Maass forms with the same Laplace and Hecke eigenvalues. This motivates the following definitions.

Let G be as in Theorem 1.1. Let H be the identity component of the group of fixed points of θ. We let D be a positive integer such that G and H are unramified at places away from D and ∞. Let K and K_H be compact open subgroups of $G(\mathbb{A}_f)$ and $H(\mathbb{A}_f)$ that are hyperspecial away from D. If N is a positive integer prime to D, we let $K(N)$ be the corresponding principal congruence subgroup of K, and define $Y_N = G(F)\backslash G(\mathbb{A})/K(N)K_H K_\infty$. We give each Y_N the probability volume measure.

Let $A \subset G_\infty$ be a maximal $\mathbb{R}$-split torus with real Lie algebra $\mathfrak{a}$ and Weyl group W. We let $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes \mathbb{C}$. Let G_∞^0 be the connected component of G_∞ in the real topology. Any unramified irreducible unitary representation of G_∞^0 gives rise to an element $\xi \in \mathfrak{a}_\mathbb{C}^*/W$ via the Harish-Chandra isomorphism, which we have normalised so that the tempered spectrum corresponds to $\mathfrak{a}^*/W$. We let $\|\cdot\|$ be the norm on $\mathfrak{a}$ and $\mathfrak{a}^*$ coming from the Killing form and extend it naturally to their complexifications. If $\mu, \lambda \in \mathfrak{a}_\mathbb{C}^*/W$ we will sometimes abuse notation and write $\|\mu - \lambda\|$ to mean the minimum of this norm over representatives for the W-orbits.

By a Hecke-Maass form we mean a joint eigenfunction $\psi \in L^2(Y_N)$ for the Hecke algebra (away from N and D) and the ring of invariant differential operators $\mathcal{D}$ on Y_N. We may view the associated eigenvalues as elements in the unramified unitary dual of G_v at finite places v (via the Satake isomorphism), while at infinity they determine an element $\xi \in \mathfrak{a}_\mathbb{C}^*/W$. We define a spectral datum c for (G, N) to be a choice of element $\xi(c) \in \mathfrak{a}_\mathbb{C}^*/W$ and an element $\pi_v(c)$ in the unramified unitary dual of G_v for all $v \nmid ND\infty$. Given a spectral datum c for (G, N), we define $V(N, c)$ to be the space of Hecke-Maass forms on Y_N whose $\mathcal{D}$-eigenvalues are given by $\xi(c)$ (the spectral parameter) and whose Hecke eigenvalues at $v \nmid ND\infty$ are given by $\pi_v(c)$.

Theorem 1.2. *With the notation and hypotheses of Theorem 1.1, there is $\delta > 0$ and $Q > 1$ with the following property. For any positive integer N with $(N, D) = 1$ and spectral parameter $\xi \in \mathfrak{a}^*$ such that $N(1 + \|\xi\|)$ is sufficiently large, there is a spectral datum c for (G, N) with $\|\xi(c) - \xi\| \leqslant Q$ and a Hecke-Maass form $\psi \in V(N, c)$ such that*

$$\|\psi\|_\infty \gg N^\delta (1 + \|\xi\|)^\delta \sqrt{\dim V(N, c)}\|\psi\|_2.$$

Note that a Hecke-Maass form as in Theorem 1.1 has Laplacian eigenvalue of size roughly $(1 + \|\xi\|)^2$. Theorem 1.2 therefore implies Theorem 1.1. Moreover, taking ξ at distance at least Q from the root hyperplanes ensures that the eigenfunction ψ produced by the theorem is tempered at infinity.

The only previous results giving lower bounds in the level aspect are for GL_2 over a number field, due to Saha [15] and Templier [20]. They use the fact that local Whittaker functions of highly ramified p-adic representations are large high in the cusp, and in particular rely on the noncompactness of the manifold.

1.5 Borel's theorem and a concrete example

A classical theorem of Borel on the existence of rationally defined Cartan involutions on real semisimple Lie algebras can be used to provide examples of groups satisfying the hypotheses of Theorem 1.1.[2] We state this as the following result.

Proposition 1.3. *Let $G'/\mathbb{R}$ be connected, simply connected, and $\mathbb{R}$-almost simple. Let F be a totally real number field, and let v_0 be a real place of F. There is a connected semisimple group G/F with $G_{v_0} \simeq G'$ that satisfies conditions (2) and (3) of Theorem 1.1.*

We now give a concrete example of a family of manifolds to which our theorem can be applied, and which to our knowledge does not already appear in the literature.

Let F be a totally real number field, and let E be a CM extension of F. Let the rings of integers of these fields be $\mathcal{O}_F$ and $\mathcal{O}_E$ respectively. Let v_0 be a distinguished real place of F, and let w_0 be the place of E over v_0. Let V be a vector space of dimension $n+1$ over E with a Hermitian form $\langle \cdot, \cdot \rangle$ with respect to E/F. Assume that $\langle \cdot, \cdot \rangle$ has signature $(n, 1)$ at w_0 and is definite at all other infinite places of E. Let G be the F-algebraic group $SU(V, \langle \cdot, \cdot \rangle)$, so that $G_{v_0} \simeq SU(n, 1)$.

Let $L \subset V$ be an $\mathcal{O}_E$ lattice on which the form $\langle \cdot, \cdot \rangle$ is integral. Let L^* be the dual lattice $L^* = \{x \in V : \langle x, y \rangle \in \mathcal{O}_E \text{ for all } y \in L\}$. Let Γ be the group of isometries of V that have determinant 1, preserve L, and act trivially on L^*/L. If $F \neq \mathbb{Q}$, completion at w_0 allows us to consider Γ as a discrete, cocompact subgroup of $SU(n, 1)$, which will be torsion free if L is chosen sufficiently small.

One may associate a complex hyperbolic manifold to Γ in the following way. Let D denote the space of lines in V_{w_0} on which the Hermitian form is negative definite. $SU(n, 1)$ acts on D, and D carries a natural $SU(n, 1)$-invariant metric under which it becomes a model for complex hyperbolic n-space. The quotient $Y = \Gamma \backslash D$ is then a compact complex hyperbolic n-manifold, and is an example of a congruence manifold associated to G as in Theorem 1.1.

If $n \geqslant 2$, G satisfies conditions (1) and (2) of Theorem 1.1. We now show that (3) is satisfied. Let $W \subset V$ be a codimension 1 subspace defined over E such that the Hermitian form is positive definite on W_{w_0}. Let θ be the isometry of reflection in W. Then $g \mapsto \theta g \theta^{-1}$ gives an F-involution of G that is a Cartan involution on G_{v_0}, as required. Theorem 1.1 then implies that there is a sequence of Laplace eigenfunctions $\{\psi_i\}$ on Y satisfying $\|\psi_i\|_\infty \gg \lambda_i^\delta \|\psi_i\|_2$.

[2] It is an interesting question whether condition (3) on the existence of a rational Cartan involution is automatic or not. We believe that it is not when G is almost simple of type A_n, D_n, or E_6, but are unsure otherwise.

1.6 The method of proof

The proof of power growth for arithmetic hyperbolic 3-manifolds of Maclachlan-Reid type by Milićević [13] compares an amplified trace and pre-trace formula. Our proof works by extending this method to general groups. The bulk of the work lies in proving asymptotics for the trace formula.

More precisely, our proof of Theorem 1.2 proceeds by comparing a trace formula on G with a relative trace formula for G with respect to H. If we choose a test function $k \in C_c^\infty(G(\mathbb{A}))$, the main geometric terms of these trace formulae are $k(1)$ and $\Pi_H k(1)$ respectively, where $\Pi_H : L^1(G(\mathbb{A})) \to L^1(X(\mathbb{A}))$ is given by integration over H. In fact, we need to choose test functions of the form $k = \omega * \omega^*$ so that their action on $L^2(G(F)\backslash G(\mathbb{A}))$ is positive semidefinite. After controlling the other geometric terms, we wish to find ω that makes $\Pi_H k(1)$ large relative to $k(1)$. At finite places, we shall take for ω_f an appropriately large sum of L^2-normalized basic Hecke operators $\tau(\nu, v)$, supported on $K_v \nu(\varpi_v) K_v$, where ν is a cocharacter of G. The condition that $\Pi_H k(1)$ be large then boils down to

$$(\Pi_H \tau(\nu, v))(1) = \frac{\mathrm{vol}(H_v \cap K_v \nu(\varpi_v) K_v)}{\mathrm{vol}(K_v \nu(\varpi_v) K_v)^{1/2}}$$

being large, for enough places v. Note that this corresponds to our informal description in terms of Hecke returns at the beginning of the introduction: if the projection of the H-period onto the given locally symmetric Y is simply a point p (rather than a finite collection of such), the right-hand side above is roughly the multiplicity with which p appears in its image by the Hecke correspondence $\tau(\nu, v)$. In any case, we bound this quantity from below, in terms of the H-relative size of ν. The latter is a certain cocharacter inequality, which we show is verified under the conditions of our main theorem.

While writing this paper, Erez Lapid pointed out to us that there was another approach to proving Theorems 1.1 and 1.2 based on a theorem of Sakellaridis on the unramified C^∞ spectrum of symmetric varieties. We have included a discussion of this in the paper. We have also included an explanation of why the condition of G being nonsplit at v_0 is natural, and motivated our choice of test functions in the trace formula, based on a related conjecture of Sakellaridis and Venkatesh on the L^2 spectrum.

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