Around Golden-Thompson inequality

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1. Introduction

For $n \in \mathbb{N}$, $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ denotes the space of all $n \times n$ complex matrices. Let $A = (a_{ij}) \in \mathbb{M}_n$. The trace of A is the sum of the diagonal entries:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

A norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if

$$|||A||| = |||UAV|||$$

for all $A \in \mathbb{M}_n$ and all unitaries $U, V \in \mathbb{M}_n$. Let A and B be Hermitian matrices in $\mathbb{M}_n$. The partial ordering $A \geq B$ holds if $A - B$ is positive semi-definite, or equivalently

$$x^*Ax \geq x^*Bx$$

for all vectors $x \in \mathbb{C}^n$.

In the commutative case, if A and B are Hermitian matrices, then $e^{A+B} = e^A e^B$. However, in the noncommutative case, it is entirely no relation between e^{A+B} and $e^A e^B$ under the usual order. The celebrated Golden-Thompson trace inequality, independently proved by Golden[5] and Thompson[13], says as follows:

Theorem 1. *If A and B are Hermitian matrices in $\mathbb{M}_n$, then*

$$\mathrm{Tr}\,(e^{A+B}) \leq \mathrm{Tr}\,(e^A e^B). \tag{1}$$

Moreover, Hiai-Petz in [6] showed the following unitarily invariant norm version of Theorem 1:

Theorem 2. *If A and B are Hermitian matrices in $\mathbb{M}_n$, then*

$$\left|\left|\left| e^{A+B} \right|\right|\right| \leq \left|\left|\left| \left(e^{pA/2} e^{pB} e^{pA/2}\right)^{1/p} \right|\right|\right| \qquad \text{for all } p > 0 \tag{2}$$

for every unitarily invariant norm $|||\cdot|||$, and the right hand side of (2) converges to $\left|\left|\left| e^{A+B} \right|\right|\right|$ as $p \downarrow 0$. In particular,

$$\left|\left|\left| e^{A+B} \right|\right|\right| \leq \left|\left|\left| e^{A/2} e^B e^{A/2} \right|\right|\right| \leq \left|\left|\left| e^A e^B \right|\right|\right|. \tag{3}$$

Let A and B be positive definite matrices in $\mathbb{M}_n$ and $\alpha \in [0,1]$. The weight geometric matrix mean $A \,\sharp_\alpha\, B$ is deinied as

$$A \,\sharp_\alpha\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.$$

Ando-Hiai [2] showed the following complemented Golden-Thompson inequalities:

Theorem 3. *If A and B are Hermitian matrices in $\mathbb{M}_n$ and $\alpha \in [0,1]$, then*

$$\left|\!\left|\!\left|(e^{pA} \sharp_\alpha e^{pB})^{1/p}\right|\!\right|\!\right| \le \left|\!\left|\!\left|e^{(1-\alpha)A+\alpha B}\right|\!\right|\!\right| \tag{4}$$

for all $p > 0$ and the left hand side of (4) increases to $\left|\!\left|\!\left|e^{(1-\alpha)A+\alpha B}\right|\!\right|\!\right|$ as $p \downarrow 0$ for any unitarily invariant norm $|\!|\!|\cdot|\!|\!|$. In particular,

$$\operatorname{Tr}\,(e^{pA} \sharp_\alpha e^{pB})^{1/p}) \le \operatorname{Tr}\,(e^{(1-\alpha)A+\alpha B}) \qquad \text{for all } p > 0.$$

Remark 4. *If we put $p = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3 and replecing A, B by $2A, 2B$ respectively, then we have the lower bound of the Golden-Thompson inequality* (3)*:*

$$\left|\!\left|\!\left|e^{2A} \sharp e^{2B}\right|\!\right|\!\right| \le \left|\!\left|\!\left|e^{A+B}\right|\!\right|\!\right| \le \left|\!\left|\!\left|e^{A}e^{B}\right|\!\right|\!\right|$$

for any unitarily invariant norm $|\!|\!|\cdot|\!|\!|$. In particular,

$$\operatorname{Tr}\,(e^{2A} \sharp e^{2B}) \le \operatorname{Tr}\,(e^{A+B}) \ \le \operatorname{Tr}\,(e^{A}e^{B}).$$

To show the reverse of Theorem 3, we need some preliminaries. We present an important constant due to Specht [12], who estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \ldots, x_n \in [m, M]$

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n} \le S(h)\sqrt[n]{x_1 x_2 \cdots x_n} \tag{5}$$

where $h = \frac{M}{m}$ and the Specht ratio is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}\ (h \ne 1) \quad \text{and} \quad S(1) = 1. \tag{6}$$

We note that the Specht theorem (5) means a ratio type reverse inequality of the arithmetic-geometric mean inequality.

Now, in [4], we showed a noncommutative version of the Specht theorem (5):

Theorem 5. *Let A be a positive definite matrix in $\mathbb{M}_n$ such that $0 < m \le A \le M$ for some scalars $0 < m < M$ and put $h = \frac{M}{m}$. Then*

$$e^{\langle \log A\, x, x\rangle} \le \langle Ax, x\rangle \le S(h)e^{\langle \log A\, x, x\rangle} \tag{7}$$

holds for every unit vector $x \in \mathbb{C}^n$.

We mention some basic properties of the Specht ratio $S(h)$ in [3, Theorem 2.16, Theorem 2.17]:

Lemma 6. *Let $h > 0$ and $\alpha \in \mathbb{R}$.*

(i) $S(1) = \lim_{h\to 1} S(h) = 1$.
(ii) $S(h) = S(h^{-1})$.
(iii) *A function $S(h)$ is strictly decreasing for $0 < h < 1$ and strictly increasing for $h > 1$.*
(iv) $\lim_{\alpha\to 0} S(h^\alpha)^{1/\alpha} = 1$.
(v) $\lim_{\alpha\to\infty} S(h^\alpha)^{1/\alpha} = h$ *for $h > 1$ and* $\lim_{\alpha\to\infty} S(h^\alpha)^{1/\alpha} = h^{-1}$ *for $0 < h < 1$.*
(vi) $\lim_{r\to 0} K(h^r, \frac{\alpha}{r}) = S(h^\alpha)$.

We showed reverses of the complemented Golden-Thompson inequality (4) due to Ando-Hiai in terms of the Specht ratio in [11]:

Theorem 7. *Let A and B be Hermitian matrices such that $m \le A, B \le M$ for some scalars $m < M$, and let $\alpha \in [0,1]$. Then*

$$(8) \qquad \left(\left|\!\left|\!\left| (e^{pA} \sharp_\alpha e^{pB})^{\frac{1}{p}} \right|\!\right|\!\right| \le \right) \ |||e^{(1-\alpha)A+\alpha B}||| \le S(e^{p(M-m)})^{\frac{1}{p}} \left|\!\left|\!\left| (e^{pA} \sharp_\alpha e^{pB})^{\frac{1}{p}} \right|\!\right|\!\right|$$

for all $p > 0$ and every unitarily invariant norm $|||\cdot|||$, and the right-hand side of (8) converges to the middle hand side as $p \downarrow 0$. In particular,

$$(|||e^{2A} \sharp e^{2B}||| \le) \ |||e^{A+B}||| \le S(e^{2(M-m)}) \, |||e^{2A} \sharp e^{2B}|||$$

and

$$(\mathrm{Tr}\,(e^{2A} \sharp e^{2B}) \le) \ \mathrm{Tr}\,(e^{A+B}) \le S(e^{2(M-m)}) \mathrm{Tr}\,(e^{2A} \sharp e^{2B}).$$

The obvious generalization of the Golden-Thompson trace inequality (1), namely,

$$\mathrm{Tr}(e^{A+B+C}) \le \mathrm{Tr}(e^A e^B e^C)$$

is not true in general. We would like to consider a $k(\ge 3)$-variable version of the Golden-Thompson trace inequality and its complements.

One is to consider the Hadamard product instead of the usual product. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{M}_n$, the Hadamard product is defined to be the entrywise product

$$A \circ B = (a_{ij} b_{ij}).$$

The following resilt due to Ando is already shown in [1]:

Theorem 8. *Let $A_1, \ldots, A_k$ be Hermitian matrices, and $\circ$ the Hadamard product. Then*

$$|||e^{A_1+\cdots+A_k}||| \le |||e^{U^*A_1U} \circ \cdots \circ e^{U^*A_kU}|||$$

for some unitary U and every unitarily invariant norm $|||\cdot|||$.

In the commutative case, we have

$$e^{A+B+C} = e^A e^B e^C = \left(e^{3A} e^{3B} e^{3C}\right)^{1/3},$$

that is, the right hand side is regarded as the geometric mean of e^{3A}, e^{3B}, e^{3C}. Thus, the other is to consider a k-variable geometric mean version instead of the matrix geometric mean in Theorem 7.

In the next section, we will proceed with a discussion in this direction.

2. k-variable version

First of all, we recall the k-variable version of the matrix geometric mean: We start with the Karcher mean of positive definite matrices in $\mathbb{M}_n$: In 2012, Lim and Pálfia [10] established the formulation of the geometric mean for k (≥ 3) positive definite matrices which is a nice extension of the matrix geometric mean in the Kubo-Ando theory [8]. They showed that there exists the unique positive definite solution of the Karcher equation

$$(9) \qquad \sum_{i=1}^{k} \omega_i \log X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} = 0$$

for given k positive definite matrices $A_1, \ldots, A_k$, where $\omega = (\omega_1, \ldots, \omega_k)$ is a weight vector, i.e., $\omega_1, \ldots, \omega_k \ge 0$ and $\sum_{i=1}^{k} \omega_i = 1$. We say the solution X of (9) the Karcher mean for n positive definite matrices $A_1, \ldots, A_k$ and denote it by $G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)$. In the case of $k = 2$, the Karcher mean $G_{\mathrm{K}}((1-\alpha, \alpha); A, B)$ coincides with the weighted matrix geometric mean

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2} \qquad \text{for } \alpha \in [0,1].$$

We list some properties of the Karcher mean which we need later, also see [9]:

(P1) Consistency with scalars: $G_{\mathrm{K}}(\omega; A_1, \ldots, A_k) = A_1^{\omega_1} \cdots A_k^{\omega_k}$ if the A_i's commute;
(P2) Joint homogeneity: $G_{\mathrm{K}}(\omega; a_1A_1, \ldots, a_kA_k) = a_1^{\omega_1} \cdots a_k^{\omega_k} G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)$;
(P3) Permutation invariance: $G_{\mathrm{K}}(\omega_\sigma; A_{\sigma(1)}, \ldots, A_{\sigma(k)}) = G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)$ where $\omega_\sigma = (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(k)})$ and σ is any permutation;
(P4) Transformer inequality: $T^*G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)T \le G_{\mathrm{K}}(\omega; T^*A_1T, \ldots, T^*A_kT)$ for every operator T;
(P5) Self-duality: $G_{\mathrm{K}}(\omega; A_1^{-1}, \ldots, A_k^{-1})^{-1} = G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)$;
(P6) Information monotonicity: $\Phi(G_{\mathrm{K}}(\omega; A_1, \ldots, A_k)) \le G_{\mathrm{K}}(\omega; \Phi(A_1), \ldots, \Phi(A_k))$ for any unital positive linear map Φ;
(P7) AGH weighted mean inequality:

$$\left(\sum_{i=1}^k \omega_i A_i^{-1}\right)^{-1} \le G_{\mathrm{K}}(\omega; A_1, \ldots, A_k) \le \sum_{i=1}^k \omega_i A_i.$$

(P8) Determinant identity:

$$\det(G_K(\omega : A_1, \ldots, A_k)) = \prod_{i=1}^k \det(A_i)^{\omega_i}.$$

Moreover, Yamazaki in [14] showed the following Ando-Hiai inequality for the Karcher mean:

Theorem 9. *Let $A_1, \ldots, A_k$ be positive definite matrices and $\omega = (\omega_1, \ldots, \omega_k)$ a weight vector. Then*

$$G_K(\omega : A_1, \ldots, A_k) \le I \quad \textit{implies} \quad G_K(\omega : A_1^p, \ldots, A_k^p) \le I \quad \textit{for all } p \ge 1.$$

By Theorem 9, we show a k-variable version of Theorem 3. Put $\|G\|_\infty = \|G_K(\omega : A_1, \ldots, A_k)\|_\infty$, where $\|\cdot\|_\infty$ is matrix norm. Since

$$G_K(\omega : A_1, \ldots, A_k) \le \|G_K(\omega : A_1, \ldots, A_k)\|_\infty,$$

it follows from (P2) that

$$G_K(\omega : \frac{A_1}{\|G\|_\infty}, \ldots, \frac{A_k}{\|G\|_\infty}) \le I.$$

By Theorem 9, we have

$$G_K(\omega : \left(\frac{A_1}{\|G\|_\infty}\right)^p, \ldots, \left(\frac{A_k}{\|G\|_\infty}\right)^p) \le I \qquad \text{for all } p \ge 1$$

and hence

$$G_K(\omega : A_1^p, \ldots, A_k^p) \le \|G_K(\omega : A_1, \ldots, A_k)^p\|_\infty.$$

Therefore we have

$$\|G_K(\omega : A_1^p, \ldots, A_k^p)\|_\infty \le \|G_K(\omega : A_1, \ldots, A_k)^p\|_\infty.$$

For $0 < q < p$, since $p/q \geq 1$, the fact above implies

$$\left\|G_K(\omega : A_1^{p/q}, \ldots, A_k^{p/q})\right\|_\infty \leq \left\|G_K(\omega : A_1, \ldots, A_k)^{p/q}\right\|_\infty.$$

Replacing A_i by A_i^q, we have

$$\left\|G_K(\omega : A_1^p, \ldots, A_k^p)^{1/p}\right\|_\infty \leq \left\|G_K(\omega : A_1^q, \ldots, A_k^q)^{1/q}\right\|_\infty \qquad \text{for all } 0 < q < p.$$

Since Hiai-Petz in [7] showed the Lie-Trotter formula for the Karcher mean:

$$\lim_{q \to 0} G_K(\omega : A_1^q, \ldots, A_k^q)^{1/q} = e^{\omega_1 \log A_1 + \cdots + \omega_k \log A_k},$$

as $q \to 0$ we have

$$\left\|G_K(\omega : A_1^p, \ldots, A_k^p)^{1/p}\right\|_\infty \leq \left\|e^{\omega_1 \log A_1 + \cdots + \omega_k \log A_k}\right\|_\infty.$$

By antisymmetric tensor technique and (P8), we have

$$\left|\left|\left|G_K(\omega : A_1^p, \ldots, A_k^p)^{1/p}\right|\right|\right| \leq \left|\left|\left|e^{\omega_1 \log A_1 + \cdots + \omega_k \log A_k}\right|\right|\right|$$

for every unitarily invariant norm $|||\cdot|||$. See [2] for antisymmetric tensor technique. Hence we have the following Golden-Thompson inequality for the Karcher mean due to Hiai-Petz in [7]:

Theorem 10 (Hiai-Petz [7])**.** *Let $A_1, \ldots, A_k$ be positive definite matrices and $\omega = (\omega_1, \ldots, \omega_k)$ a weight vector. Then*

$$\operatorname{Tr}[G_K(\omega : e^{pA_1}, \ldots, e^{pA_k})^{1/p}] \leq \operatorname{Tr}[e^{\omega_1 A_1 + \cdots + \omega_k A_k}] \qquad \text{for all } p > 0 \tag{10}$$

and the left hand side of (10) *converges to* $\operatorname{Tr}[e^{\omega_1 A_1 + \cdots + \omega_k A_k}]$ *as $p \downarrow 0$. In particular,*

$$\operatorname{Tr}[G_K(\tilde{\omega} : e^{kA_1}, \ldots, e^{kA_k})] \leq \operatorname{Tr}[e^{A_1 + \cdots + A_k}],$$

where a weight vector $\tilde{\omega} = (1/k, \ldots .1/k)$.

Remark 11. *Theorem 10 is just a k-variable version of Theorem 3, that is, if we put $k = 2$ in Theorem 10, then we have Theorem 3.*

Next, we show a k-variable version of Theorem 7. For this, we need the following Lemma:

Lemma 12. *Let $A_1, \ldots, A_k$ be positive definite matrices such that $m \leq A_i \leq M$ for some scalars $0 < m \leq M$ and $\omega = (\omega_1, \ldots, \omega_k)$ a weight vector. Put $h = \frac{M}{m}$. Then*

$$\sum_{i=1}^{k} \omega_i A_i \leq S(h) e^{\sum_{i=1}^{k} \omega_i \log A_i} \tag{11}$$

where the Specht ratio $S(h)$ is defined by (6).

Proof. Put $\mathbb{A} = \operatorname{diag}(A_1, \ldots, A_k), y = (\sqrt{\omega_1}x, \ldots, \sqrt{\omega_k}x)^T$ for every unit vector $x \in \mathbb{C}^n$. By Theorem 5, since $m \leq \mathbb{A} \leq M$, we have

$$\langle \mathbb{A}y, y \rangle \leq S(h)\ e^{\langle \log \mathbb{A}y, y \rangle}.$$

Hence it follows from the Jensen inequality that

$$\begin{aligned}\langle(\sum_{i=1}^{k}\omega_i A_i)x,x\rangle &= \langle \mathbb{A}y,y\rangle \\ &\le S(h)\ e^{\langle \log \mathbb{A}y,y\rangle} \\ &= S(h)\ e^{\langle \sum_{i=1}^{k}\omega_i \log A_i x,x\rangle} \\ &\le S(h)\ \langle e^{\sum_{i=1}^{k}\omega_i \log A_i}x,x\rangle \qquad \text{by (7)}\end{aligned}$$

for every unit vector $x \in \mathbb{C}^n$ and we get (11):

$$\sum_{i=1}^{k}\omega_i A_i \le S(h)\ e^{\sum_{i=1}^{k}\omega_i \log A_i}.$$

□

Theorem 13. *Let $A_1,\dots,A_k$ be positive definite matrices such that $m \le A_i \le M$ for some scalars $0 < m \le M$ and $\omega = (\omega_1,\dots,\omega_k)$ a weight vector. Put $h = \frac{M}{m}$. Then*

$$\left|\!\left|\!\left| e^{\sum_{i=1}^{k}\omega_i A_i} \right|\!\right|\!\right| \le S(e^{p(M-m)})^{1/p} \left|\!\left|\!\left| G_K(\omega : e^{pA_1},\dots,e^{pA_k})^{1/p} \right|\!\right|\!\right| \tag{12}$$

for all $p > 0$ and every unitarily invariant norm $|||\cdot|||$, and the right-hand side of (12) converges to the left hand side as $p \downarrow 0$. In particular,

$$|||e^{A_1+\cdots+A_k}||| \le S(e^{(M-m)})\ |||G_K(\tilde{\omega} : e^{kA_1},\dots,e^{kA_k})|||$$

where a weight vector $\tilde{\omega} = (1/k,\dots.1/k)$, and

$$\mathrm{Tr}[e^{A_1+\cdots+A_k}] \le S(e^{(M-m)})\mathrm{Tr}[G_K(\tilde{\omega} : e^{kA_1},\dots,e^{kA_k})].$$

Proof. By Lemma 12 and (P7), we have

$$G_K(\omega : A_1,\dots,A_k) \le \sum_{i=1}^{k}\omega_i A_i \le S(h)e^{\sum_{i=1}^{k}\omega_i \log A_i}.$$

Replacing A_i by e^{-pA_i} for $i = 1,\dots,k$ and $p > 0$, since $e^{-pM} \le e^{-pA_i} \le e^{-pm}$, it follows that

$$G_K(\omega : e^{-pA_1},\dots,e^{-pA_k}) \le S(e^{p(M-m)})e^{\sum_{i=1}^{k} -\omega_i pA_i}.$$

Taking the inverse of both sides, we have

$$G_K(\omega : e^{-pA_1},\dots,e^{-pA_k})^{-1} \ge S(e^{p(M-m)})^{-1}e^{\sum_{i=1}^{k}\omega_i pA_i}$$

and this and (P5) imply

$$e^{\sum_{i=1}^{k}\omega_i pA_i} \le S(e^{p(M-m)})G_K(\omega : e^{pA_1},\dots,e^{pA_k})$$

for all $p > 0$ and there exists a unitary matrix U such that

$$\left(e^{\sum_{i=1}^{k}\omega_i pA_i}\right)^{1/p} \le S(e^{p(M-m)})^{1/p}U^*G_K(\omega : e^{pA_1},\dots,e_{pA_k})^{1/p}U.$$

Hence we have

$$\left|\!\left|\!\left| e^{\sum_{i=1}^{k}\omega_i A_i} \right|\!\right|\!\right| \le S(e^{p(M-m)})^{1/p} \left|\!\left|\!\left| G_K(\omega : e^{pA_1},\dots,e^{pA_k})^{1/p} \right|\!\right|\!\right|$$

for all $p > 0$ and every unitarily invariant norm $|||\cdot|||$. □

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References

[1] T. Ando, *Hadamard products and Golden-Thompson inequalies*, Linear Algebra Appl. **241/243** (1996), 105-112.

[2] T. Ando and F. Hiai, Log-majorization and complementary Golden-Thompson type inequalities, Linear Algebra Appl., **197,198** (1994), 113–131.

[3] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

[4] J. I. Fujii and Y. Seo, *Determinant for positive operators*, Sci. Math. **1** (1998), 153-156.

[5] S. Golden, *Lower bounds for Helmholtz function*, Phys. Rev., **137** (1965), B1127–B1128.

[6] F. Hiai and D. Petz, The Golden-Thompson trace inequality is complemented, Linear Algebra Appl., **181** (1993), 153–185.

[7] F. Hiai and D. Petz, *Riemannian metrics on positive definite matrices related to means II*, Linear Algebra Appl. **436** (2012), 2117-2136.

[8] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.

[9] J. Lawson and Y. Lim, *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc. Series B **1** (2014), 1–22.

[10] Y. Lim and M. Pálfia, *Matrix power means and the Karcher mean*, J. Funct. Anal. **262** (2012), 1498–1514.

[11] Y. Seo, *Reverses of the Golden-Thopson type inequalities due to Ando-Hiai-Petz*, Banach J. Math. Anal. **2** (2008), 140-149.

[12] W. Specht, *Zur Theorie der elementaren Mittel*, Math. Z. **74** (1960), 91–98.

[13] C.J. Thompson, Inequality with applications in statistical mechanics, J. Math. Phys., **6** (1965), 469–480.

[14] T. Yamazaki, *Riemannian mean and matrix inequalities related to the Ando-Hiai inequality and chaotic order*, Oper. Matrices **6** (2012), 577-588.