Arithmetic-geometric mean norm inequality for positive operators

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1. Introduction. First of all, this note is based on a joint work [10] with R. Nakamoto.

The arithmetic-geometric mean inequality has been developed for positive operators acting on a Hilbert space. We pay our attention to norm inequalities. We first recall the Heinz inequality [12]: Let A and B be positive operators on a Hilbert space. Then

$$\|AQ + QB\| \geq \|A^{1-s}QB^s + A^sQB^{1-s}\|$$

holds for $0 \leq s \leq 1$ and arbitrary operators Q. In particular, as in the case $s = \frac{1}{2}$, we have

$$2\|A^{\frac{1}{2}}QB^{\frac{1}{2}}\| \leq \|AQ + QB\| \tag{HIh}$$

It is understood as an arithmetic-geometric mean inequality by norm. McIntosh [14] gave an alternative as follows:

$$2\|SQT\| \leq \|S^*SQ + QTT^*\| \tag{MI}$$

holds for arbitrary S, T and Q. Moreover we discussed several norm inequalities equivalent to them, e.g. Corach-Porta-Recht inequality [4]

$$\|\mathrm{Re}\ STS^{-1}\| \geq \|T\| \quad \text{for selfadjoint } S, T,$$

and

$$\|\mathrm{Re}\ A^2T\| \geq \|ATA\| \quad \text{for } A \geq 0 \text{ and selfadjoint } T,$$

see [5], [6], [7], [8], [9], [12] and [14].

Now the following inequality is quite standard as an arithmetic-geometric mean inequality by norm.

$$\|AB\| \leq \|\frac{1}{2}(A + B)\|^2 \quad \text{for } A, B \geq 0 \tag{AGM}$$

It is known in [3] that it holds for positive semidefinite matrices. But we cannot find a proof for positive operators. In this note, we give two proofs to it and discuss some related inequalities.

2. A proof of (AGM). In this section, we give a proof to (AGM), which is based on the proof for positive semidefinite matrices due to Bhatia and Kittaneh [3].

For convenience, we cite the following lemma:

Lemma 1. Let $A, B > 0$. Then

$$\|A - B\| \leq \max\{\|A\|, \|B\|\}.$$

Proof. It is proved by $-B \leq A - B \leq A$.

Theorem 2. Let $A, B > 0$. Then

$$4\|AB\| \leq \|(A+B)^2\|.$$

Proof. The frame of our proof is just the same as that of Bhatia-Kittaneh. We put $Y = (A^{\frac{1}{2}}\ B^{\frac{1}{2}})$ and $Q = A^{\frac{1}{2}}B^{\frac{1}{2}}$. Then

$$YY^* = A + B \text{ and } Y^*Y = \begin{pmatrix} A & Q \\ Q^* & B \end{pmatrix},$$

and so

$$(Y^*Y)^2 = \begin{pmatrix} A^2 + QQ^* & AQ + QB \\ Q^*A + BQ^* & B^2 + Q^*Q \end{pmatrix}.$$

Next we take a unitary $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, since

$$U(Y^*Y)^2U = \begin{pmatrix} A^2 + QQ^* & -(AQ + QB) \\ -(Q^*A + BQ^*) & B^2 + Q^*Q \end{pmatrix},$$

we have

$$(Y^*Y)^2 - U(Y^*Y)^2U = 2\begin{pmatrix} 0 & AQ + QB \\ Q^*A + BQ^* & 0 \end{pmatrix},$$

Hence it implies that

$$\|(Y^*Y)^2 - U(Y^*Y)^2U\| = 2\|AQ + QB\|.$$

Finally it follows from (HIh) and Lemma 2 that

$$\begin{aligned} 4\|AB\| &= 4\|A^{\frac{1}{2}}QB^{\frac{1}{2}}\| \\ &\leq 2\|AQ + QB\| \\ &= \|(Y^*Y)^2 - U(Y^*Y)^2U\| \\ &\leq \max\{\|(Y^*Y)^2\|, \|U(Y^*Y)^2U\|\} \\ &= \|(Y^*Y)^2\| \\ &= \|(YY^*)^2\| \\ &= \|(A+B)^2\|, \end{aligned}$$

as desired.

3. A simple proof of (AGM). In this section, we present a simple proof of (AGM) by the use of (HIh) and (MI). For convenience, we cite the latter again:

$$2\|SQT\| \leq \|S^*SQ + QTT^*\| \tag{MI}$$

Proof of (AGM). It follows that

$$\begin{aligned}
4\|AB\| &= 4\|A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{\frac{1}{2}})B^{\frac{1}{2}}\| \\
&\leq 2\|AA^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}B\| \\
&= 2\|A^{\frac{1}{2}}(A+B)B^{\frac{1}{2}}\| \\
&= 2\|A^{\frac{1}{2}}(A+B)^{\frac{1}{2}}(A+B)^{\frac{1}{2}}B^{\frac{1}{2}}\| \\
&\leq \|(A+B)^{\frac{1}{2}}A(A+B)^{\frac{1}{2}} + (A+B)^{\frac{1}{2}}B(A+B)^{\frac{1}{2}} \\
&= \|(A+B)^2\|,
\end{aligned}$$

as desired.

Remark. We may expect a generalization of Theorem 2 as follows:

$$4\|AXB\| \leq \|(A+B)X(A+B)\|.$$

Unfortunately, we have a counterexample for this: We take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = B.$$

Then $\|AXB\| = \|AB\| = \sqrt{2}$ and

$$\|(A+B)X(A+B)\| = \left\| \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \right\| = 5 < 4\sqrt{2} = 4\|AQB\|,$$

as required.

4. Weak (AGM) inequalities. It is well-known that $\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \|AB\|^{\frac{1}{2}}$ for $A, B \geq 0$. So we remark that (AGM) implies that

$$2\|AB\| \leq \|A^2 + B^2\|$$

for $A, B \geq 0$. Moreover it is equivalent to a simple version of (MI), i.e.,

$$2\|ST\| \leq \|S^*S + TT^*\|$$

via polar decompositions of S, T^*. We here mention that the statement (i) in below is shown by Tao [15] in the case of matrices.

Summing up, we have the following norm inequalities which are mutually equivalent.

Theorem 3. The following inequalities hold and mutually equivalent:
(1) $K = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0 \;\Rightarrow\; \|K\| \geq 2\|X\|$ for $A, B \geq 0$ and arbitrary X.
(2) $\|A - B\| \leq \max\{\|A\|, \|B\|$ for $A, B \geq 0$.
(3) $2\|XY^*\| \leq \|X^*X + Y^*Y\|$ and arbitrary S, T.
(4) $2\|AB\| \leq \|A^2 + B^2\|$ for $A, B \geq 0$.

Proof. (1) $\Rightarrow$ (2): Put $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$. Since

$$K_1 = \frac{1}{2}\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} = U\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}U^* \geq 0,$$

it follows from (1) that

$$\|A - B\| \leq \|K_1\| = \|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\| = \max\{\|A\|, \|B\|\}.$$

(2) $\Rightarrow$ (3): It is the same argument as a proof for (AGM) in Section 2. Put $T = \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}$ and $U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.Then it follows from (2) that

$$\begin{aligned} 2\|XY^*\| &= 2\|\begin{pmatrix} 0 & XY^* \\ YX^* & 0 \end{pmatrix} \\ &= \|TT^* - UTT^*U\| \\ &\leq \max\{\|TT^*\|, \|UTT^*U\|\} \\ &= \|TT^*\| = \|T^*T\| \\ &= \|X^*X + Y^*Y\|. \end{aligned}$$

(3) $\Leftrightarrow$ (4): It is obvious.

(3) $\Rightarrow$ (1): Suppose that $K = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$. We may assume $A > 0$ by replacing A to $A + \epsilon$ for some $\epsilon > 0$. Put $R = A^{\frac{1}{2}}$ and $S = A^{-\frac{1}{2}}X$. Then

$$0 \leq \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & RX \\ X^*R & B \end{pmatrix}.$$

We here note that $\begin{pmatrix} I & Z \\ Z^* & B \end{pmatrix} \geq 0$ if and only if $B \geq Z^*Z$. As a matter of fact, if $\begin{pmatrix} I & Z \\ Z^* & B \end{pmatrix} \geq 0$, then it follows that

$$0 \leq \langle\begin{pmatrix} I & Z \\ Z^* & B \end{pmatrix}\begin{pmatrix} Zx \\ -x \end{pmatrix}, \begin{pmatrix} Zx \\ -x \end{pmatrix}\rangle = \langle Bx, x\rangle - \|Zx\|^2.$$

Anyway we have $B \geq (RX)^*(RX) = X^*A^{-1}X$, so that

$$\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \| \geq \| \begin{pmatrix} A & X \\ X^* & X^*A^{-1}X \end{pmatrix} \| = \| \begin{pmatrix} R^*R & R^*S \\ S^*R & S^*S \end{pmatrix} \| = \|RR^* + SS^*\|.$$

Applying (3), we have

$$\|RR^* + SS^*\| \geq 2\|R^*S\| = 2\|X\|,$$

which completes the proof.

5. Positivity of operator matrices. The following result on the positivity of operator matrices is used implicitly in this note. For convenience, we cite it with brief proof.

Theorem 4. The following statements are mutually equivalent for $A, B \geq 0$ and arbitrary X:

(1) $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$.

(2) $\langle Ax, x\rangle\langle By, y\rangle \geq |\langle Xy, x\rangle|^2$ for any vectors x, y.

(3) $\langle Ax, x\rangle + \langle By, y\rangle \geq 2|\langle Xy, x\rangle|$ for any vectors x, y.

(4) $B \geq X^*(A + \epsilon)^{-1}X$ for all $\epsilon > 0$.

(5) $X = A^{\frac{1}{2}}RB^{\frac{1}{2}}$ for some contraction R.

To prove this, we prepare a lemma for the basic case.

Lemma 5. (1) Suppose $B \geq 0$. Then $\begin{pmatrix} I & X \\ X^* & B \end{pmatrix} \geq 0$ if and only if $B \geq X^*X$.

(2) Suppose $A \geq 0$. Then $\begin{pmatrix} A & X \\ X^* & I \end{pmatrix} \geq 0$ if and only if $A \geq XX^*$.

(3) If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, then $\|B\|A \geq XX^*$ and so ran $A^{\frac{1}{2}}$ contains ran X.

Proof. (1) If $B \geq X^*X$, then $\begin{pmatrix} I & X \\ X^* & B \end{pmatrix} \geq \begin{pmatrix} I & X \\ X^* & X^*X \end{pmatrix} \geq 0$.

Conversely, if $\begin{pmatrix} I & X \\ X^* & B \end{pmatrix} \geq 0$, then we have

$$0 \leq \langle \begin{pmatrix} I & X \\ X^* & B \end{pmatrix} \begin{pmatrix} Xx \\ -x \end{pmatrix}, \begin{pmatrix} Xx \\ -x \end{pmatrix} \rangle = \langle Bx, x\rangle - \|Xx\|^2.$$

(2) Since $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} B & X^* \\ X & A \end{pmatrix}$, (2) follows from (1).

(3) We first note that $\begin{pmatrix} A & X \\ X^* & \|B\| \end{pmatrix} \geq \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$. Thus it follows from (2) and Douglas' majorization theorem. □

Proof of Theorem 4. (1) $\Leftrightarrow$ (3): (1) is equivalent to $\langle \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = \langle Ax, x\rangle + \langle By, y\rangle + 2\mathrm{Re}\langle Xy, x\rangle| \geq 0$ for any $x, y \in H$, which implies (3). The reverse (3) $\Rightarrow$ (1) is shown by similar argument.

Next (3) $\Rightarrow$ (2) is shown by replacing x to tx for arbitrary real numbers t, and (2) $\Rightarrow$ (3) is implied by AG mean inequality.

(1) $\Leftrightarrow$ (4): We assume $A > 0$ for simplicity. Multiplying $\begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix}$ on both side, it follows from the fact that $\begin{pmatrix} I & Z \\ Z^* & B \end{pmatrix} \geq 0$ if and only if $B \geq Z^*Z$.

(5) $\Rightarrow$ (1): We note that if we put $X = A^{\frac{1}{2}}RB^{\frac{1}{2}}$, then

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & R \\ R^* & I \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix}$$

(1) $\Rightarrow$ (5) It is almost same as Ando's proof in [1]:

Fix $y \in H$. It follows from Lemma 5 (3) that $Xy \in \mathrm{ran}A^{1/2}$, or $Xy = A^{1/2}z$ for a unique $z \in \ker A^{\perp}$. Hence we have

$$\langle Ax, x\rangle\langle By, y\rangle \geq |\langle Xy, x\rangle|^2 = |\langle A^{1/2}z, x\rangle|^2 = |\langle z, A^{1/2}x\rangle|^2,$$

so that

$$\|B^{1/2}y\| \geq |\langle z, w\rangle| \quad \text{for } w \in \mathrm{ran}A^{1/2} \text{ with } \|w\| = 1.$$

Since $z \in \ker A^{\perp}$, we have $\|B^{1/2}y\| \geq \|z\|$. Hence there exists a contraction W such that $WB^{1/2}y = z$, and so $A^{1/2}WB^{1/2}y = A^{1/2}z = Xy$, which completes the proof.

Now we point out that (AGM) is equivalent to

$$\begin{pmatrix} \|A+B\|^2 & 4AB \\ 4BA & \|A+B\|^2 \end{pmatrix} \geq 0$$

by virtue of (1) $\Leftrightarrow$ (2) in Theorem 4. So we have a question: Is $H = \begin{pmatrix} (A+B)^2 & 4AB \\ 4BA & (A+B)^2 \end{pmatrix}$ positive? We give it a negative answer: We take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $H = \begin{pmatrix} 5 & 3 & 4 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 4 & 5 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$. Since $H_1 = \begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix} \ngeq 0$, we have $H \ngeq 0$.

Another candidate of H is as follows: $K = \begin{pmatrix} (A+B)^2 & 4|AB| \\ 4|AB| & (A+B)^2 \end{pmatrix}$.

Unfortunately, it is "Negative", too: We take the same matrices as above, i.e.,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $K = \begin{pmatrix} 5 & 3 & 4\sqrt{2} & 0 \\ 3 & 2 & 0 & 0 \\ 4\sqrt{2} & 0 & 5 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$. Since $K_1 = \begin{pmatrix} 5 & 4\sqrt{2} \\ 4\sqrt{2} & 5 \end{pmatrix} \not\geq 0$, we have $K \not\geq 0$.

The Heinz-Kato inequality says that if an operator T satisfies $T^*T \leq A^2,\ TT^* \leq B^2$ for some $A, B \geq 0$, then

$$|\langle Tx, y\rangle \leq \|A^p x\| \|B^q y\|$$

holds for $x, y \in H$ and $p, q \in [0,1]$ with $p + q = 1$.

It is expressed as follows: If an operator T satisfies $T^*T \leq A^2,\ TT^* \leq B^2$ for some $A, B \geq 0$, then $\begin{pmatrix} B^{2q} & T \\ T^* & A^{2p} \end{pmatrix} \geq 0$ for $x, y \in H$ and $p, q \in [0,1]$ with $p + q = 1$.

The Heinz-Kato inequality is generalized by Furuta [11] as follows: If an operator $T = U|T|$ satisfies $T^*T \leq A^2,\ TT^* \leq B^2$ for some $A, B \geq 0$, then

$$|\langle T|T|^{p+q-1}x, y\rangle \leq \|A^p x\| \|B^q y\|$$

holds for $x, y \in H$ and $p, q \in [0,1]$ with $p + q \geq 1$.

The inequality in the conclusion of it is expressed as $\begin{pmatrix} B^{2q} & T|T|^{p+q-1} \\ (T|T|^{p+q-1})^* & A^{2p} \end{pmatrix} \geq 0$.

6. An application of a reverse Heinz inequality. First of all, we mention a reverse Heinz inequality [12], [14]:

$$\|A^{1-s}QB^s - A^sQB^{1-s}\| \leq |2s-1| \|AQ - QB\| \tag{RHI}$$

holds for $s \in [0,1]$.

For reader's convenience, we sketch a proof of (RHI):

Proof of (RHI). It suffices to show the case $s \in [\frac{1}{2}, 1]$. Put

$$J = \{s \in [\frac{1}{2}, 1]; \text{(RHI) holds for } s.\}.$$

Since $\frac{1}{2}, 1 \in J$ clearly, the convexity of J should be shown. So we suppose $r, s \in J$ with $r < s$, $A, B \geq 0$ and Q is an arbitrary operator. Put $t = \frac{1}{2}(r + s)$. Then $s - t = t - r$, that is, $s = 2t - r$. Putting $R = A^r QB^{1-s} - A^{1-s}QB^r$, it follows from (MI) that

$$\begin{aligned}
\|A^tQB^{1-t} - A^{1-t}QB^t\| &= \|A^{t-r}RB^{t-r}\| \\
&\leq \frac{1}{2}\|A^{2(t-r)}R + RB^{2(t-r)}\| \\
&= \frac{1}{2}\|A^sQB^{1-s} - A^{1-r}QB^r + A^rQB^{1-r} - A^{1-s}QB^s\| \\
&\leq \frac{1}{2}\{\|A^sQB^{1-s} - A^{1-s}QB^s\| + \|A^rQB^{1-r} - A^{1-r}QB^r\|\} \\
&\leq \frac{1}{2}\{(2s-1)\|AQ - QB\| + (2r-1)\|AQ - QB\|\} \\
&= (2t-1)\|AQ - QB\|,
\end{aligned}$$

as desired.

Now, Bhatia [2] gave the following estimation on the power of positive operators:

Theorem B. If $A, B \geq aI$ for some $a > 0$, then

$$\|A^r - B^r\| \leq ra^{r-1}\|A - B\|$$

holds for all $r \in [0, 1]$.

We propose an extension of Theorem B by the use of (RHI) as follows:

Theorem 6. If $A, B \geq aI$ for some $a > 0$, then

$$\|A^rQ - QB^r\| \leq ra^{r-1}\|AQ - QB\|$$

holds for all $r \in [0, 1]$ and arbitrary operators Q.

Proof. The point of the proof is (RHI):

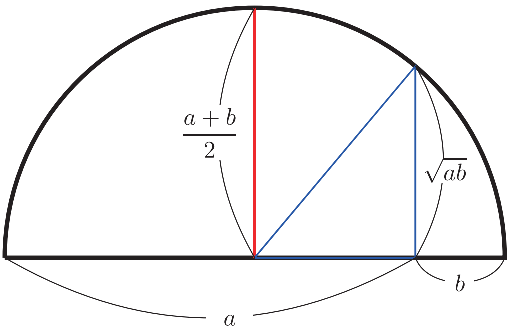
$$\|A^{1-s}QB^s - A^sQB^{1-s}\| \leq |2s-1|\|AQ - QB\|$$

holds for $s \in [0, 1]$. Put $s = \frac{1+r}{2}$ for a given $r \in [0, 1]$. Since $2s - 1 = r$ and $1 - s = \frac{1-r}{2}$, we have

$$\begin{aligned}
\|A^rQ - QB^r\| &= \|A^{s-1}(A^sQB^{1-s} - A^{1-s}QB^s)B^{s-1}\| \\
&\leq \|A^{s-1}\|\|B^{s-1}\|\|A^sQB^{1-s} - A^{1-s}QB^s\| \\
&\leq a^{r-1}|2s-1|\|AQ - QB\| \\
&= ra^{r-1}\|AQ - QB\|
\end{aligned}$$

as desired.

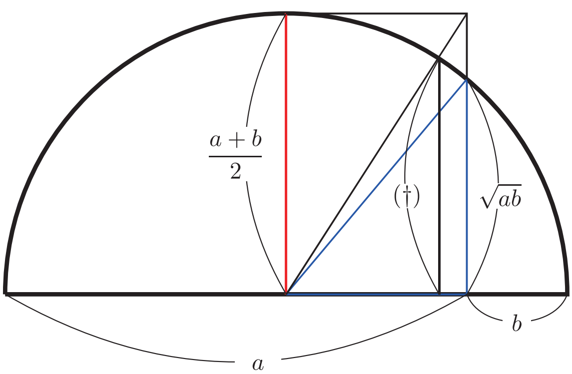
Appendix. In this talk, we posed the following figuare as a graphic proof of the classical arithmetic-geometric mean inequality:



After the talk, Ohwada, one of the organizers, informed an interesting improvement of it in [13], precisely

$$\sqrt{ab} \leq \frac{(a+b)^2}{2\sqrt{2(a^2+b^2)}} \leq \frac{a+b}{2}.$$

The middle term (†) in above appears in the figure as follows:



It is easily generalized in the following way:

Proposition 7. If $k \geq \dfrac{2\sqrt{ab}}{a-b}$ for given $a > b > 0$, then

$$\sqrt{ab} \leq \frac{k(a+b)}{2\sqrt{k^2+1}} \leq \frac{a+b}{2}.$$

In particlar, the previous inequality is obtained by taking $k = \dfrac{a+b}{a-b}$.

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