Arithmetic approach to *p*-numerical semigroups

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1 Introduction

Let S be a submonoid of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Namely, S satisfies the conditions:

(i) $S \subset \mathbb{N}_0$; (ii) $0 \in S$; (iii) if $a, b \in S$ then $a + b \in S$.

Then S is called a numerical semigroup if and only if $\mathbb{N}_0 \setminus S$ is a finite set, which is equivalent to $1 \in \{x - y | x, y \in S\}$. When S is a numerical semigroup, the maximal element, the cardinality and the sum of the elements of $\mathbb{N}_0 \setminus S$ are called the Frobenius number, the genus or the Sylvester number, and the Sylvester sum, respectively, and denoted by g(S), n(S) and s(S), respectively. Several generalized Frobenius numbers have been introduced, but we study the one focusing on the number of representations (nonnegative integer solutions). Let $A := \{a_1, a_2, \ldots, a_k\}$ be the set of positive integers with $k \geq 2$. The denumerant $d(n) = d(n; a_1, a_2, \ldots, a_k)$ is the number of representations to $n = a_1x_1 + a_2x_2 + \cdots + a_kx_k$ for a given nonnegative integer n. When S is a numerical semigroup and $A \subseteq S$, it is called that S is generated by A and denoted by $S = \langle A \rangle$ if for all $n \in S$, there exist $a_1, a_2, \ldots, a_k \in A$ and $x_1, x_2, \ldots, x_k \in \mathbb{N}_0$ such that $n = \sum_{j=1}^k a_j x_j$. A is called a minimal set of generators of S if $S = \langle A \rangle$ and no proper subset of A has its property. $S = \langle A \rangle$ is called the canonical form description of S.

For an nonnegative integer p, let S_p be the set of integers whose nonnegative integral linear combinations of given positive integers a_1, a_2, \ldots, a_k are expressed in more than pways. By emphasizing the fact that S is generated by the set A, we also write $S_p(A)$ as S_p . We can see that the set $\mathbb{N}_0 \backslash S_p$ is finite if and only if $gcd(a_1, a_2, \ldots, a_k) = 1$. Then there exists the largest integer $g_p(A) := g(S_p)$ in $\mathbb{N}_0 \backslash S_p$, which is called the *p*-Frobenius number. The cardinality of $\mathbb{N}_0 \backslash S_p$ is called the *p*-genus and denoted by $n_p(A) := n(S_p)$. Its sum of elements is called the *p*-Sylvester sum and denoted by $s_p(A) := s(S_p)$.

Several different generalizations have been introduced and studied. However, our generalization is very natural and efficient in terms of the following p-Apéry sets.

2 Preliminaries

Using the elements in the *p*-Apéry set, we can obtain the *p*-Frobenius, the *p*-genus, the *p*-Sylvester sum and so on very efficiently. Without loss of generality, put $a_1 = \min(A)$. For a nonnegative integer *p*, the *p*-Apéry set is given by

$$Ap_p(A) = Ap_p(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\}.$$
 (1)

Here, for each $0 \le i \le a_1 - 1$, the positive integer $m_i^{(p)}$ is determined uniquely as

(i)
$$m_i^{(p)} \equiv i \pmod{a_1}$$
, (ii) $m_i^{(p)} \in S_p(A)$, (iii) $m_i^{(p)} - a_1 \notin S_p(A)$.

So, the set $Ap_p(A)$ is a complete residue system modulo $a_1, \{0, 1, \ldots, a_1 - 1\}$. In general, the following formula for power sum is given ([13]).

Theorem 1. Let k, p and μ be integers with $k \ge 2$, $p \ge 0$ and $\mu \ge 0$. Assume that gcd(A) = 1. We have

$$s_{p}^{(\mu)}(A) := \sum_{n \in \mathbb{N}_{0} \setminus S_{p}(A)} n^{\mu}$$
$$= \frac{1}{\mu + 1} \sum_{\kappa = 0}^{\mu} {\binom{\mu + 1}{\kappa}} B_{\kappa} a_{1}^{\kappa - 1} \sum_{i=0}^{a_{1} - 1} {\binom{m_{i}^{(p)}}{\mu^{i+1-\kappa}}} + \frac{B_{\mu+1}}{\mu + 1} (a_{1}^{\mu+1} - 1) ,$$

where B_n are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \,.$$

When $\mu = 0, 1$ in Theorem 1, together with $g_p(A)$ we have formulas for the *p*-Frobenius number, the *p*-Sylvester number and the *p*-Sylvester sum.

Corollary 1. Assume that gcd(A) = 1. For a nonnegative integer p, we have

$$g_p(A) = \max_{0 \le i \le a_{1}-1} m_i^{(p)} - a_1,$$

$$n_p(A) = \frac{1}{a_1} \sum_{i=0}^{a_1-1} m_i^{(p)} - \frac{a_1 - 1}{2},$$

$$s_p(A) = \frac{1}{2a_1} \sum_{i=0}^{a_1-1} (m_i^{(p)})^2 - \frac{1}{2} \sum_{i=0}^{a_1-1} m_i^{(p)} + \frac{a_1^2 - 1}{12}$$

When p = 0 in Corollary 1, the formulas are reduced to the classically known ones.

$$g(A) = \left(\max_{1 \le i \le a_1 - 1} m_i\right) - a_1, \quad [4]$$

$$n(A) = \frac{1}{a_1} \sum_{i=1}^{a_1 - 1} m_i - \frac{a_1 - 1}{2}, \quad [22]$$

$$s(A) = \frac{1}{2a_1} \sum_{i=1}^{a_1 - 1} m_i^2 - \frac{1}{2} \sum_{i=1}^{a_1 - 1} m_i + \frac{a_1^2 - 1}{12}. \quad [24]$$

Notice that $m_0 = m_0^{(0)} = 0$ is applied when p = 0. Hence, the sum runs from i = 1.

The elements of the *p*-Apéry set are uniquely determined, but it is not easy to obtain them for the general case. Furthermore, it is even more difficult to find any regularity.

3 Explicit expressions

In the case of two variables, namely, $a_1 = a$ and $a_2 = b$, by $\{m_i | 0 \le i \le a - 1\} = \{b(pa + i) | 0 \le i \le a - 1\}$, from Corollary 1 we can get the explicit expressions.

Corollary 2. For a nonnegative integer p, we have

$$g_p(a,b) = (p+1)ab - a - b,$$

$$n_p(a,b) = pab + \frac{(a-1)(b-1)}{2},$$

$$s_p(a,b) = \frac{p^2 a^2 b^2}{2} + \frac{p(ab - a - b)ab}{2} + \frac{(a-1)(b-1)(2ab - a - b - 1)}{12}.$$

However, for $k \geq 3$, even though p = 0, g(A) cannot be given by any set of closed formulas which can be reduced to a finite set of certain polynomials [5]. For k = 3, there are several useful algorithms to obtain the Frobenius number (e.g., [9, 21, 7]). For the concretely given three positive integers, if the conditions are met, the Frobenius number can be calculated by the method of case-dividing in [25]. Only some special cases, <u>explicit closed formulas</u> have been found, including arithmetic, geometric, Fibonacci, Mersenne, repunits and triangular.

When p > 0 and $k \ge 3$, the situation is even harder. Though any explicit formula had not been found even for particular triples, recently we have been finally successful to obtain closed formulas for arithmetic [17], triangular [11], repunit [12], Fibonacci [16] and Jacobsthal [?].

4 Some relations related to *p*-Frobenius numbers

In this section, we show some fundamental relations among $g_p(A)$, $n_p(A)$ and $s_p(A)$.

Lemma 1. For $p \ge 0$, we have

$$n_p(A) \ge \frac{g_p(A) + 1}{2}.$$

Proof. For a non-negative integer s, if $s \in S_p$, then $g_p(A) - s \notin S_p$. Hence, by $n_p(A) \ge \#\{s \in S_p | s < g_p(A)\} = g_p(A) + 1 - n_p(A)$, we get the result.

For emphasis, write $\operatorname{Ap}(S_p, a_1)$ as the *p*-Apéry set $\operatorname{Ap}_p(A)$ from $S_p(A)$ with a_1 as the least element of the set A.

Proposition 1. Assume that $S_p(A)$ is minimally generated by a_1, \ldots, a_k . Set $d = \gcd(a_2, \ldots, a_k)$ and $T_p(A) = \{n \in \mathbb{N}_0 | d(n; a_1, a_2/d, \ldots, a_k/d) > p\}$ Then we have $\operatorname{Ap}(S_p, a_1) = d\operatorname{Ap}(T_p, a_1)$.

Proof. From the definition of the Apéry set, $w \in \operatorname{Ap}(S_p, a_1)$ implies that $w - a_1 \notin S_p$. Since $w \in \langle a_2, \ldots, a_k \rangle$, we have $w/d \in \langle a_2/d, \ldots, a_k/d \rangle$. If $w/d - a_1 \in T_p(A)$, as $w - da_1 \notin S_p$, $w/d - a_1 \notin T_p(A)$. Hence, $w/d \in \operatorname{Ap}(S_p, a_1)$, which implies that $w \in \operatorname{Ap}(S_p, a_1)$.

On the other hand, if $w \in \operatorname{Ap}(T_p, a_1)$, then $w \in \langle a_2/d, \ldots, a_k/d \rangle$, implying that $dw \in \langle a_2, \ldots, a_k \rangle \subseteq S_p$. We shall see that $dw - a_1 \notin S_p(A)$, entailing that $dw \in S_p(A)$. Otherwise, for non-negative integers y_1, \ldots, y_k , $dw - a_1 = a_1y_1 + \cdots + a_ky_k$, implying that $w = a_1(y_1+1)/d + (a_2/d)y_2 + \cdots + (a_k/d)y_k$ and $d|(y_1+1)$. But this is impossible because $w - a_1 \notin T_p(A)$.

By Proposition 1, we can obtain the relations between the *p*-Frobenius numbers $g_p(A)$ and the *p*-Sylvester numbers $n_p(A)$. For simplicity, we write $g_p(A_d) = g(T_p(A))$ and $n_p(A_d) = n(T_p(A))$.

Corollary 3. As the same setting as above, we have

(i)
$$g_p(A) = dg_p(A_d) + (d-1)a_1.$$

(ii) $n_p(A) = dn_p(A_d) + \frac{(d-1)(a_1-1)}{2}.$
(iii) $s_p(A) = d^2s_p(A_d) + \frac{a_1d(d-1)}{2}n_p(A_d) + \frac{(a_1-1)(d-1)(2a_1d-a_1-d-1)}{2}.$

Proof. We shall prove (iii). By Corollary 1,

$$\begin{split} s_p(A) &= \frac{1}{2a_1} \sum_{w \in \operatorname{Ap}_p(S;a_1)} w^2 - \frac{1}{2} \sum_{w \in \operatorname{Ap}_p(S;a_1)} w + \frac{a_1^2 - 1}{12} \\ &= \frac{d^2}{2a_1} \sum_{w \in \operatorname{Ap}_p(T;a_1)} w^2 - \frac{d}{2} \sum_{w \in \operatorname{Ap}_p(T;a_1)} w + \frac{a_1^2 - 1}{12} \\ &= d^2 \left(\frac{1}{2a_1} \sum_{w \in \operatorname{Ap}_p(T;a_1)} w^2 - \frac{1}{2} \sum_{w \in \operatorname{Ap}_p(T;a_1)} w + \frac{a_1^2 - 1}{12} \right) \\ &- \frac{d}{2} \sum_{w \in \operatorname{Ap}_p(T;a_1)} w + \frac{a_1^2 - 1}{12} \\ &= d^2 s_p(A_d) + \frac{a_1 d(d-1)}{2} n_p(A_d) + \frac{(a_1 - 1)(d-1)(2a_1 d - a_1 - d - 1)}{2} \,. \end{split}$$

Example 1. Let $S = \langle 20, 30, 17 \rangle$ and $T = \langle 2, 3, 17 \rangle = \langle 2, 3 \rangle$ with d = 10. Then, for p = 3, by $s_3(A_{10}) = 136$ and $n_3(A_{10}) = 17$, we get $s_3(A) = 10^2 s_3(A_{10}) + 17 \cdot 10 \cdot 9/2n_3(A_{10}) + 16 \cdot 9(2 \cdot 17 \cdot 10 - 17 - 10 - 1)/12 = 30349$.

Example 2. When $S = \langle a, b \rangle$, by putting d = b, we get

$$g_p(A_d) = ap - 1$$
, $n_p(A_d) = ap$ and $g_p(A_d) = \frac{a^2p^2 - ap}{2}$.

Therefore, Corollary 2 is reduced again.

4.1 *p*-Hilbert series

For a non-negative integer p, the p-Hilbert series of $S_p(A)$ is defined by

$$H_p(A; x) := H(S_p; x) = \sum_{s \in S_p(A)} x^s$$

When p = 0, the 0-Hilbert series is the original Hilbert series. In addition, the *p*-gaps generating function is defined by

$$\Psi_p(A; x) = \sum_{s \in \mathbb{N}_0 \setminus S_p(A)} x^s \,,$$

satisfying $H_p(A; x) + \Psi_p(A; x) = 1/(1-x)$ (|x| < 1). By using *p*-Apéry set, we see that $S_p(A) = \operatorname{Ap}_p(A; a) + a\mathbb{N}_0$, with $a = \min(A)$. Hence,

$$H_p(A;x) = \frac{1}{1-x^a} \sum_{m \in Ap_p(A;a)} x^m \,.$$
(2)

For three or more variables, it is not easy to obtain an explicit form of the p-Hilbert series. However, the p-Hilbert series may be explicitly given when the structure of the p-Apéry set is known. We give one of the simplest cases, though the expression of the p-Hilbert series often becomes very complicated.

For example, let $A := \{a, a+1, \dots, 2a-1\}$ for an integer with $a \ge 3$. Then its *p*-Apéry set is given as follows.

Lemma 2. Let $a \ge 3$. For p = 0, we have

$$Ap_0(a, a+1, \dots, 2a-1) = \{0, a+1, a+2, \dots, 2a-1\}$$

and $1 \leq p \leq (a-1)/2$, we have

$$Ap_p(a, a+1, \dots, 2a-1) = \{3a, \dots, 3a+2p-1, 2a+2p, \dots, 3a-1\}.$$

Therefore, by (2), when p = 0, we obtain

$$H_0(a, a+1, \dots, 2a-1; x) = \frac{1}{1-x^a} (x^{a+1} + x^{a+2} + \dots + x^{2a-1})$$
$$= \frac{x^{a+1}(1-x^{a-1})}{(1-x)(1-x^a)}.$$

When p > 0, we obtain

$$H_p(a, a+1, \dots, 2a-1; x) = \frac{1}{1-x^a} (x^{2a+2p} + x^{2a+2p+1} + \dots + x^{3a+2p-1})$$
$$= \frac{x^{2a+2p}}{1-x}.$$

This result looks simple, but the expression of the *p*-Hilbert series usually becomes very complicated because its structure of the corresponding *p*-Apéry set is uncertain or complicated. For example, concerning the sequence of consecutive odd integers A := $\{2a + 1, 2a + 3, ..., 4a + 3\}$ $(a \ge 1)$, no exact explicit form of the *p*-Apéry set has been found for general *p*.

5 *p*-symmetric semigroup

By arranging the elements $m_i^{(p)}$ $(0 \le i \le a_1 - 1)$ of the Apéry set in (1) in ascending order, let $\ell_0(p) < \ell_1(p) < \cdots < \ell_{a_1-1}(p)$. That is, the sequence $\ell_0(p), \ell_1(p), \ldots, \ell_{a_1-1}(p)$ is the ascending permutation of $m_0^{(p)}, m_1^{(p)}, \ldots, m_{a_1-1}^{(p)}$.

The *p*-numerical semigroup $S_p = S_p(A)$ is called *p*-symmetric if for all $x \in \mathbb{Z} \setminus S_p$, $\ell_0(p) + g_p(A) - x \in S_p$, where $\ell_0(p)$ is the least element of S_p , that is the *p*-multiplicity of S_p if $p \ge 1$; $\ell_0(p) = 0$ if p = 0. When p = 0, "0-symmetric" is "symmetric". If a *p*-symmetric numerical semigroup S_p further satisfies $\ell_0(p) = g_p(A) + 1 := c_p(A)$, which is called *p*-conductor, then S_p is called *p*-completely-symmetric.

From the definition, the following is obvious.

Proposition 2. For a p-semigroup S_p $(p \ge 0)$, the following conditions are equivalent.

- (i) S_p is p-symmetric.
- (ii) $\#S_p \cap \{\ell_0(p), \dots, g_p(A)\} = \#(\mathbb{N}_0 \setminus S_p) \cap \{\ell_0(p), \dots, g_p(A)\} = \frac{g_p(A) \ell_0(p) + 1}{2}.$
- (iii) If $x + y = \ell_0(p) + g_p(A)$, then exactly one of non-negative integers x and y belongs to S_p and another to $\mathbb{N}_0 \setminus S_p$.

Example

When $A = \{4, 5, 6\}$, we get that

$$S_8 = \{36, 38, 40, 41, \mapsto\},$$
$$\mathbb{N}_0 \setminus S_8 = \{0, 1, \dots, 35, 37, 39\}.$$

Then we know that

$$g_8(A) = 39$$
 and $Ap_8(A) = \{36 = m_0^{(8)}, 38 = m_2^{(8)}, 41 = m_1^{(8)}, 43 = m_3^{(8)}\}.$

Hence, we see that

$$36 + 39 = 38 + 37 = 40 + 35 = 41 + 34 = \dots = 75 + 0 = 76 + (-1) = \dots$$

Therefore, $S_8(A)$, where $A = \{4, 5, 6\}$, is 8-symmetric. In fact, among the elements in $Ap_8(A)$, we can obtain

$$36 + 43 = 38 + 41$$

This fact is explained in the next lemma, which is a generalization of the result by Apéry [1].

Lemma 3. For a non-negative integer p, $S_p = S_p(A)$ is p-symmetric if and only if $\ell_i(p) + \ell_{a-i-1}(p) = g_p(A) + \ell_0(p) + a$ $(i = 1, 2, ..., \lfloor a/2 \rfloor).$

If one element $m_i^{(p)}$ in $\operatorname{Ap}_p(A; a)$ with $a = \min(A)$ can extend such that $m_i^{(p)} \equiv i \pmod{a}$ for any *i*, Lemma 3 can be restated as follows. For simplicity, put $g = g_p(A)$ and $\ell = \ell_0(p)$.

Lemma 4. For a non-negative integer p, S_p is p-symmetric if and only if $m_{(g+\ell+1)/2+j}(p) + m_{(g+\ell-1)/2+j}(p) = g_p + \ell + a \ (j \in \mathbb{Z}).$

From Lemma 3 or Lemma 4, we have a relation between *p*-Frobenius number $g_p(A)$ and *p*-Sylvester number $n_p(A)$.

Proposition 3. For a non-negative integer p, $S_p = S_p(A)$ is p-symmetric if and only if

$$n_p(A) = \frac{g_p(A) + \ell_0(p) + 1}{2}.$$

Let us consider the two variables' case. For any integer $n \in S_p(A)$ for $A = \{a, b\}$ with gcd(a, b) = 1 and a < b, let x_0 be the largest integer x satisfying n = ax + by $(y \ge 0)$. Then there exists the least non-negative integer y_0 such that $n = ax_0 + by_0$, which is called the *standard form* of the representation of n. Since $S_p(A) \subseteq \mathbb{N}_0 \subseteq \mathbb{Z}$ and \mathbb{Z} is Euclidean domain, the standard form is unique.

Lemma 5. Let $n = ax_0 + by_0$ be the standard form of n. Then

- (i) $0 \le y_0 \le a 1$.
- (ii) For any integer $n \in S(A) = S_0(A)$, $n \in S_p(A)$ if and only if $x_0 \ge pb$.

By Proposition 3 and Lemma 5, together with the formulas in Corollary 2, we can show the p-symmetric property for two variables.

Theorem 2. For any non-negative integer p, $S_p(a, b)$ with gcd(a, b) = 1 is p-symmetric.

Proof. When $A = \{a, b\}$ with gcd(a, b) = 1, by Lemma 5, the least integer whose number of representations in terms of a and b is more than p is pab. That is, the non-negative integral solutions of ax + by = pab are (x, y) = (jb, (p - j)a) (j = 0, 1, ..., p). Since $\ell_0(p) = pab$, by Proposition 3 together with the formulas in Corollary 2, we have

$$\frac{g_p(A) + \ell_0(p) + 1}{2} = \frac{(p+1)ab - a - b + pab + 1}{2}$$
$$= pab + \frac{(a-1)(b-1)}{2} = n_p(a,b).$$

At the end of this section, we consider a *p*-symmetric property in terms of the valuation. Let $R_0 := \mathbb{K}[[t^s | s \in S_p^{(0)}(A)]], \bar{R}_0$ be the integral closure of R_0 , f be the algebraic conductor from $t^{\ell_0(p)}R_0$ to \bar{R}_0 , $c_p = g_p + 1$ (*p*-conductor). Since R_0 is the ring associated to a numerical semigroup $S_p^{(0)}(A)$, it is a discrete valuation ring with the valuation v.

Lemma 6. $f = \{x \in \overline{R}_0 | v(x) \ge c_p + \ell_0(p) \}.$

Proof. For any $x \in f$ and $r \in \overline{R}_0$, we have $rx \in t^{\ell_0(p)}R_0$. So $v(rx) = v(x) + v(r) \in v(t^{\ell_0(p)}R_0)$. For $x = t^{\ell_0(p)}x'$ we get $v(r) + v(t^{\ell_0(p)}) + v(x') \in v(t^{\ell_0(p)}R_0) = v(R_0) + v(t^{\ell_0(p)})$. By the arbitrariness of r and $v(r) \geq 0$, we have $v(x) \geq c_p + \ell_0(p)$, that is, $f \subseteq \{x \in \overline{R}_0 | v(x) \geq c_p + \ell_0(p)\}$.

For any $x \in \overline{R}_0$ and $v(x) \ge c_p + \ell_0(p)$, we have v(x) = v(r) for some $r \in t^{\ell_0(p)}R \subseteq R$. Then for any $r' \in \overline{R}_0$, we have $v(xr') = v(x) + v(r') = v(r) + v(r') \ge c_p + \ell_0(p)$. By the definition of c_p , we have $xr' \in t^{\ell_0(p)}R_0$. So, $f \supseteq \{x \in \overline{R}_0 | v(x) \ge c_p + \ell_0(p)\}$.

For simplicity, let d_1 and d_2 be the lengths of ideal of R_0/f and of R_0 -submodule of \overline{R}_0/f , respectively, and d_3 be the number of elements in $S_p(A) \cap \{1, 2, \ldots c_p + \ell_0(p) - 1\}$.

Theorem 3. $S_p(A)$ is p-symmetric if and only if $d_1 = \frac{d_2}{2}$.

Proof. By Proposition 2 together with the facts that all the elements in $\{1, \ldots i_p - 1\}$ are in $\mathbb{N}_0 \setminus S_p(A)$ and $\{g_p + 1, \ldots \ell_0(p) + g_p - 1\}$ are all in $S_p(A)$, $S_p(A)$ is *p*-symmetric if and only if $d_3 = \frac{\ell_0(p) + g_p - 1}{2}$.

Consider the ideal chain $R_0 \supset R_1 \supset R_2 \cdots \supset R_{d_3} \supset f$, where $R_i = \{r \in R_0 | v(r) \ge v_i\}$ and $v_1 < v_2 < \cdots < v_{d_3}$ are the elements in $S_p(A) \cap \{1, 2, \ldots c_p + \ell_0(p) - 1\}$ arranged in ascending order. This sequence is maximal because if we adjoin an element $r \in R_0$ of value v_{i-1} to R_i , we get all of R_{i-1} . So, $d_1 = d_3 + 1$.

Similarly consider the maximal R_0 -submodule chain of \overline{R}_0/f : $\overline{R}_0 = b_0 \supset b_1 \supset b_2 \cdots \supset b_{\ell_0(p)+g_p+1} = f$ where $b_i = \{r \in \overline{R}_0 | v(r) \ge i\}$. So we have $d_2 = \ell_0(p) + g_p + 1$. Hence, $S_p(A)$ is p-symmetric if and only if $d_1 - 1 = \frac{d_2 - 2}{2}$.

5.1 *p*-pseudo-symmetric semigroup

For a non-negative integer p, let $S_p(A)$ be a p-numerical semigroup. $x \in \mathbb{Z}$ is called a ppseudo-Frobenius number if $x \notin S_p(A)$ and $x+s-\ell_0(p) \in S_p(A)$ for all $s \in S_p(A) \setminus \{\ell_0(p)\}$, where $\ell_0(p)$ is the least element of $S_p(A)$, so is of $\operatorname{Ap}_p(A; a)$ with $a = \min(A)$. The set of p-pseudo-Frobenius numbers is denoted by $\operatorname{PF}_p(A)$. The p-type is denoted by $t_p(A) :=$ $\#(\operatorname{PF}_p(A))$. Notice that the p-Frobenius number is given by $g_p(A) = \max(\operatorname{PF}_p(A))$.

For $p \ge 0$, the *p*-numerical semigroup $S_p = S_p(A)$ is called *p*-pseudo-symmetric if for all $x \in \mathbb{Z} \setminus S_p$ with $x \ne (\ell_0(p) + g_p(A))/2 \in \mathbb{Z}$, $\ell_0(p) + g_p(A) - x \in S_p$, where $\ell_0(p)$ is the least element of S_p . When p = 0, "0-pseudo-symmetry" is "pseudo-symmetry".

For simplicity, put the *p*-Frobenius number as $g := g_p(A)$ and the *p*-multiplicity as $\ell := \ell_0(p) \ (p \ge 1)$ with $\ell_0(0) = 0$. Denote the *p*-Apéry set by $\operatorname{Ap}_p(A; a)$ with $a = \min(A)$.

Theorem 4. For a non-negative integer p, the following conditions are equivalent:

(i) $S_p = S_p(A)$ is p-pseudo-symmetric

$$m_{(g+\ell)/2+j}^{(p)} + m_{(g+\ell)/2-j}^{(p)} = g + \ell + \begin{cases} 2a & \text{if } j = 0 \text{ and } (g+\ell)/2 \in \mathbb{N}_0 \setminus S_p(A); \\ 0 & \text{if } j = 0 \text{ and } (g+\ell)/2 \in S_p(A); \\ a & \text{if } j > 0. \end{cases}$$

(iii)
$$n_p(A) = \frac{g+\ell}{2} + \begin{cases} 1 & \text{if } (g+\ell)/2 \in \mathbb{N}_0 \setminus S_p(A); \\ 0 & \text{if } (g+\ell)/2 \in S_p(A). \end{cases}$$

Corollary 4. Let $S_p(A)$ be a *p*-numerical semigroup. The following conditions are equivalent.

- (i) S_p is p-symmetric.
- (ii) $\operatorname{PF}_p(A) = \{g_p(A)\}$ with $g_p(A) \not\equiv \ell_0(p) \pmod{2}$.
- (iii) $t_p(A) = 1$ with $g_p(A) \not\equiv \ell_0(p) \pmod{2}$.

Corollary 5. Let $S_p(A)$ be a *p*-numerical semigroup. The following conditions are equivalent.

(i) S_p is p-pseudo-symmetric.

(ii)
$$\operatorname{PF}_{p}(A) = \begin{cases} \{g_{p}(A), (g_{p}(A) + \ell_{0}(p))/2\} & \text{if } (g_{p}(A) + \ell_{0}(p))/2 \in \mathbb{N}_{0} \setminus S_{p}(A); \\ \{g_{p}(A)\} & \text{if } (g_{p}(A) + \ell_{0}(p))/2 \in S_{p}(A). \end{cases}$$

(iii) $t_{p}(A) = \begin{cases} 2 & \text{if } (g_{p}(A) + \ell_{0}(p))/2 \in \mathbb{N}_{0} \setminus S_{p}(A); \\ 1 & \text{if } (g_{p}(A) + \ell_{0}(p))/2 \in S_{p}(A). \end{cases}$

For $a, b \in \mathbb{Z}$, define a partial order relation $a \leq_{S_p} b$ (or $a \leq_S b$ for short) as $b - a \in S_p$. The set of *p*-pseudo-Frobenius numbers $PF_p(A)$ can be determined with this order relation in terms of the *p*-maximal gaps.

Proposition 4. For a p-numerical semigroup $S_p = S_p(A)$, we have

$$\operatorname{PF}_p(A) = \operatorname{Maximals}_{\leq_S}(\mathbb{N}_0 \setminus S_p).$$

The set of *p*-pseudo-Frobenius numbers $PF_p(A)$ can be also determined in terms of the *p*-Apéry set.

Proposition 5. Let $S_p = S_p(A)$ be a *p*-numerical semigroup with $a = \min(A)$. Then for $n \in S_p$ we have

$$\operatorname{PF}_p(A) = \{ w - a | w \in \operatorname{Maximals}_{\leq s} \operatorname{Ap}_p(A; a) \}.$$

At the end of this subsection, we mention a partially corresponding result to Theorem 3.

Theorem 5. If $S_p(A)$ is p-pseudo-symmetric, then $2d_1 + 1 = d_2$.

Proof. If $S_p(A)$ is p-pseudo-symmetric, then we have $2d_3 = \ell_0(p) + g_p - 2$. Again, consider the maximal ideal chain $R_0 \supset R_1 \supset R_2 \cdots \supset R_{d_3} \supset f$ as in the proof of Theorem 3. Thus, we get $d_1 = d_3 + 1$. And consider the R_0 -submodule chain of \overline{R}_0/f : $\overline{R}_0 = b_0 \supset b_1 \supset b_2 \cdots \supset b_{\ell_0(p)+g_p+1} = f$. We have $d_2 = \ell_0(p) + g_p + 1$. Hence, if $S_p(A)$ is p-pseudo-symmetric, then $2(d_1 - 1) = d_2 - 3$.

When is a *p*-numerical semigroup *p*-symmetric, and when *p*-pseudo-symmetric? Let $A = \{6, 7, 17, 28\}$. S_0 is pseudo-symmetric. In addition,

		C /		,	~	-		v					
S_1	=	{	24	30	31	34	35	36	37	38	40	\mapsto	}
G_1	=	{	39	33	32	29	28	27	26	25	23	\mapsto	}
S_2	=	{	41	42			47	48	49	51	\mapsto	}	
G_2	=	{	50		46	45	44	43		40	\mapsto	}	
S_3	=	{	48				54	55		58	\mapsto	}	
G_3	=	{	57	56	53	52	51	50	49	47	\mapsto	}	
S_4	=	{	65	66	68	\mapsto	}						
G_4	=	{	67		64	\mapsto	}						
G_4	=	{	67		64	\mapsto	}						

Hence, S_1 is symmetric, S_2 and S_3 are not symmetric, and S_4 is pseudo-symmetric. By continuing, S_p is *p*-symmetric for $p = 1, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, \underline{11}, \underline{12}, \underline{13}, 15, \underline{17}, \underline{18}, \underline{21}, \underline{22}, \underline{24}, \ldots$ (For underlined *p*'s, they are completely-symmetric.) S_p is *p*-pseudo-symmetric for $p = 0, 4, 5, 19, 20, 23, 25, \ldots$ S_p is neither *p*-symmetric nor *p*-pseudo-symmetric for $p = 2, 3, 14, 16, \ldots$

Conjecture 1. If $gcd(a_i, a_j) = 1$ $(i \neq j)$ for $A := \{a_1, a_2, \ldots, a_k\}$, then S_p is p-completely-symmetric for enough large p.

5.2 *p*-irreducible numerical semigroup

A numerical semigroup S is irreducible if it cannot be expressed as the intersection of two proper oversemigroups. A p-numerical semigroup $S_p = S_p(A)$ is called *p-irreducible* if it is either *p*-symmetric or *p*-pseudo-symmetric. It is known that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups. By Theorem 2, we have the *p*-irreducible property for two variables.

Corollary 6. For any non-negative integer p, $S_p(a, b)$ with gcd(a, b) = 1 is p-irreducible.

Every p-numerical semigroup can be also expressed as a finite intersection of irreducible numerical semigroups ([2]).

Proposition 6. For a non-negative integer p, let S_p be a p-numerical semigroup. Then, there exist finitely many irreducible numerical semigroups S_1, \ldots, S_r such that $S_p = S_1 \cup \cdots \cup S_r$.

Remark. It has not been known that for any fixed non-negative integer p, a p-numerical semigroup can be expressed as an intersection of all p-irreducible numerical semigroups. **Example.** For $A = \{5, 9, 16\}$, we see that $S_2(A) = \{41, 45, 46, 48, 50, \mapsto\}$, which is neither (2-)symmetric nor (2-)pseudo-symmetric. But it can be expressed as an intersection of two (0-)numerical semigroups: $S_2(A) = S(A_1) \cup S(A_2)$ with $A_1 = \{41, 43, 45, 46, 48, 50, \mapsto\}$ and $A_2 = \{41, 45, 46, 47, 48, 50, \mapsto\}$. Here both $S(A_1)$ and $S(A_2)$ are (0-)pseudo-

symmetric. In addition, these 0-numerical semigroups are given by canonical forms:

$$S(A_1) = \langle 41, 43, 45, 46, 48, \underbrace{50, \dots, 81}_{83, 85} \rangle,$$

$$S(A_2) = \langle 41, 45, 46, 47, 48, \underbrace{50, \dots, 81}_{83, 84, 85} \rangle,$$

6 Lipman semigroup and dual

For simplicity, set $S_p^{\circ} = S_p \cup \{0\}$. Then the *p*-dual of $S_p(A)$ is defined to be

$$B(S_p) := (S_p^{\circ} - S_p) = (S_p - S_p)$$

Note that for ideals I and J, $I + J = \{i + j | i \in I, j \in J\}$, and $lI = \underbrace{I + \cdots + I}_{I}$.

The *p*-Lipman semigroup is defined to be $L_p(S) = L(S_p^\circ) := \bigcup_{h \ge 1} (hS_p - hS_p)$. Then two kinds of chains of semigroups are obtained by duals and blow-ups, respectively:

$$S_p \coloneqq B_0(S_p) \subseteq B(B_0(S_p)) \coloneqq B_1(S_p) \subseteq \cdots \subseteq B(B_h(S_p)) \equiv B_{h+1}(S_p) \subseteq \cdots$$
$$S_p \coloneqq L_0(S_p) \subseteq L(L_0(S_p)) \equiv L_1(S_p) \subseteq \cdots \subseteq L(L_h(S_p)) \equiv L_{h+1}(S_p) \subseteq \cdots$$

If these sequences coincide, the semigroup S is called the *p*-Arf numerical semigroup. $\beta_p(S) = \beta(S_p)$ and $\lambda_p(S) = \lambda(S_p)$ denote the least integers such that $B_{\beta_p(S)} = L_{\lambda_p(S)} = \mathbb{N}_0$. Two chains play a role to characterize classes of certain local Noetherian domains.

The following is a generalization of the result in [3].

Proposition 7. For a nonnegative integer p, let $S_p(A)$ be a p-numerical semigroup with canonical form $\langle A \rangle$. Then

(i) $g_p(B_p(A)) = g_p(S_p(A)) - \ell_0(p)$, where $\ell_0(p)$ is the least non-zero element of $S_p(A)$.

(ii)
$$L_p(A) = \langle \ell_1(p) - \ell_0(p), \ell_2(p) - \ell_0(p), \dots, \ell_{a_1-1}(p) - \ell_0(p) \rangle$$

Example. For $A = \{5, 9, 16\}$, we see that

 $S_0(A) = \{0, 5, 9, 10, 14, 15, 16, 18, 19, 20, 21, 23, \mapsto\},$ $\mathbb{N}_0 \setminus S_0(A) = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 17, 22\}.$

So, we see that $B_0(A) = (M - M) = \{0, 5, 9, 10, 11, 14, 15, 16, 18, \mapsto\}$. Hence, $g_0(S_0(A)) - \ell_0(0) = 22 - 5 = 17 = g_0(B_0(A))$.

Since $2M = \{10, 14, 15, 18, 19, 20, 21, 23, \mapsto\}$, we get $(2M - 2M) = \{0, 5, 9, 10, 11, 13, \mapsto\}$. Since $3M = \{15, 19, 20, 23, \mapsto\}$, we get $(3M - 3M) = \{0, 4, 5, 8, \mapsto\}$, which is also equal to (4M - 4M) because $4M = \{20, 24, 25, 28, \mapsto\}$. Thus, $L(S) = \langle 5, 9 - 5, 16 - 5 \rangle = \langle 4, 5, 11 \rangle = \{0, 4, 5, 8, \mapsto\} = \cup_{h=1}^{3} (hM - hM)$.

Since

$$G_1(A) = \{0, \dots, 24, 26, 27, 28, 29, 31, 33, 38\},\$$

$$S_1(A) = \{25, 30, 32, 34, 35, 36, 37, 39, \mapsto\},\$$

we see that $B_1(A) = (S_1(A) - S_1(A)) = \{0, 5, 7, 9, 10, 11, 12, 14, \mapsto\}$. Hence, $g_1(S_1(A)) - \ell_0(1) = 38 - 25 = 13 = g_1(B_1(A))$. In addition, $B_1^{(2)} = \{0, 5, 7, 9, \mapsto\}$, $B_1^{(3)} = \{0, 2, 4, \mapsto\}$, $B_1^{(4)} = \{0, 2, \mapsto\}$ and $B_1^{(5)} = \mathbb{N}$. Since $2S_1(A) = \{50, 55, 57, 59, 60, 61, 62, 64, \mapsto\}$, we get $(2S_1(A) - 2S_1(A)) = \{0, 5, 7, 9, 10, 11, 12, 14, \mapsto\}$, which is the same for $(hS_1(A) - S_1(A))$ when $h \ge 3$. Hence, $L_1(S) = B_1(S) = \{30 - 25, 32 - 25, 34 - 25, 35 - 25, 36 - 25, 37 - 25, 39 - 25, \mapsto\}$. Similarly, $L_1^{(2)}(S) = \{0, 2, 4, 5, 7, 9, \mapsto\}$, $L_1^{(3)}(S) = \{0, 2, \mapsto\}$ and $L_1^{(4)}(S) = \mathbb{N}_0$.

Since $S_2(A) = \{41, 45, 46, 48, 50, \mapsto\}$, we see that $B_2(A) = \{0, 5, 7, 9, \mapsto\}$, $B_2^{(2)}(A) = \{0, 2, 4, \mapsto\}$, $B_2(A)^{(3)} = \{0, 2, \mapsto\}$ and $B_2^{(4)}(A) = \mathbb{N}_0$. We have $g_2(S_2(A)) - \ell_0(2) = 49 - 41 = 8 = g_1(B_2(A))$. In addition, $L_2(S) = \{0, 4, 5, 7, \mapsto\} = \langle 45 - 41, 46 - 41, 48 - 41, 50 - 41 \rangle$ and $L_2^{(2)}(S) = \mathbb{N}_0$.

7 *p*-Arf numerical semigroup

A numerical semigroup S is called an Arf numerical semigroup if for every $x, y, z \in S$ such that $x \ge y \ge z$, then $x + y - z \in S$. Arf semigroups help to characterize Arf rings, an important class of rings in commutative algebra and algebraic geometry. **Proposition 8.** If S(A) for $A = \{a, b\}$ with gcd(a, b) = 1 is an Arf numerical semigroup, then $S_p(A)$ is also an Arf numerical semigroup.

Proof. Assume that for every $x, y, x \in S_p$ such that $x \ge y \ge z$. We write x, y and z in the standard form as $x = ak_1 + bh_1$, $y = ak_2 + bh_2$ and $z = ak_3 + bh_3$. Then by Lemma 5, $k_i \ge pb$ (i = 1, 2, 3). Put $x' = x - pb = a(k_1 - pb) + bh_1$, $y' = y - pb = a(k_y - pb) + bh_y$ and $z' = z - pb = a(k_3 - pb) + bh_3$. Since $k_1 - pb \ge 0$ and $h_i \ge 0$ (i = 1, 2, 3), we get $x', y', z' \in S$ with $x' \ge y' \ge z$. As S is an Arf, we have $x' + y' - z' \in S$. Hence, x' + y' - z' has the standard form $x' + y' - z' = ak_0 + bh_0$ with $k_0, h_0 \ge 0$. Then by $x + y - z = x' + y' - z' + pab = a(pb + k_0) + bh_0$ and Lemma 5, we have $x + y - z \in S_p$, so S_p is also an Arf.

Proposition 9. Let S = S(A) be an Arf numerical semigroup with $a = \min(A)$. For a nonnegative integer p, let p-conductor be c_p , that is, $c_p = g_p(A) + 1$. $\overline{c_p}$ denotes the residue modulo a, that is $c_p \equiv \overline{c_p} \pmod{a}$ with $0 \leq \overline{c_p} < a$. Then, we have

(i) $m_1^{(p)} = \begin{cases} c_p + 1 & \text{if } c_p \equiv 0 \pmod{a} \\ c_p - \overline{c_p} + a + 1 & \text{otherwise.} \end{cases}$

(ii)
$$m_{a-1}^{(p)} = c_p - \overline{c_p} + a - 1.$$

Proof. As $a \nmid g_p(A)$, we see that $c_p \not\equiv 1 \pmod{a}$. Let $c_p \equiv 0 \pmod{a}$. Since $ah + 1 \notin S_p$ and $ah + a - 1 \notin S_p$ for $h < c_p/a$, we have $m_1^{(p)} = a(c_p/a) + 1 = c_p + 1$ and $m_{a-1}^{(p)} = a(c_p/a) + a - 1 = c_p + a - 1$.

Let $c_p \not\equiv 0 \pmod{a}$. Since $ah + 1 \not\in S_p$ and $ah + a - 1 \not\in S_p$ for $h < (c_p - \overline{c_p})/a$, we have $m_{a-1}^{(p)} = a((c_p - \overline{c_p})/a + 1) + 1 = c_p - \overline{c_p} + a + 1$ and $m_{a-1}^{(p)} = a((c_p - \overline{c_p})/a) + a - 1 = c_p - \overline{c_p} + a - 1$.

For a nonnegative integer p and every $i \in \{0, 1, ...\}$, there is a positive integer $k_i^{(p)}$ such that $m_i^{(p)} = k_i^{(p)}a + i$. Then $(k_0^{(p)}, k_1^{(p)}, ..., k_{a-1}^{(p)})$ are called *p*-Kunz coordinates of S_p .

Proposition 10. Let $S_p(A)$ be an Arf numerical semigroup with $a = \min(A)$, p-conductor c_p and p-Kunz coordinates $(k_0^{(p)}, k_1^{(p)}, \ldots, k_{a-1}^{(p)})$. Then,

$$k_1^{(p)} = \left\lceil \frac{c_p}{a} \right\rceil \quad and \quad k_{a-1}^{(p)} = \left\lfloor \frac{c_p}{a} \right\rfloor \,.$$

Proof. When $c_p \equiv 0 \pmod{a}$, by Lemma 9, we have $m_1^{(p)} = k_1^{(p)}a + 1 = c_p + 1$ and $m_{a-1}^{(p)} = k_{a-1}^{(p)}a + a - 1 = c_p + a - 1$. Hence, $k_1^{(p)} = k_{a-1}^{(p)} = c_p/a$. When $c_p \not\equiv 0 \pmod{a}$, by Lemma 9, we have $m_1^{(p)} = k_1^{(p)}a + 1 = c_p - \overline{c_p} + a + 1$ and $m_{a-1}^{(p)} = k_{a-1}^{(p)}a + a - 1 = c_p - \overline{c_p} + a - 1$. Hence, $k_1^{(p)} = (c_p - \overline{c_p})/a + 1$ and $k_{a-1}^{(p)} = (c_p - \overline{c_p})/a$.

Acknowledgement

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

- R. Apéry, Sur les branches superlinéaires des courbes algébriques, C. R. Acad. Sci. Paris 222 (1946), 1198–1200.
- [2] A. Assi, M. D'Anna and P. A. Garcia-Sanchez, Numerical semigroups and applications, Second edition, ME Springer Series, 3. Springer, Cham, 2020.
- [3] V. Barucci, D. E. Dobbs and M. Fontana, Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, Mem. Am. Math. Soc. 598 78 p, (1997).
- [4] A. Brauer and B. M. Shockley, On a problem of Frobenius, J. Reine. Angew. Math. 211 (1962), 215–220.
- [5] F. Curtis, On formulas for the Frobenius number of a numerical semigroup, Math. Scand. 67 (1990), 190–192.
- [6] M. Estrada and A. Lopez, A note on symmetric semigroups and almost arithmetic sequences, Commun. Algebra 22 (1994), No. 10 3903–3905.
- [7] L. G. Fel, Frobenius problem for semigroups $S(d_1, d_2, d_3)$, Funct. Anal. Other Math. 1 (2006), no.2, 119–157.
- [8] I. García-Marco, J. L. Ramírez Alfonsin and Ø. J. Rødseth, Numerical semigroups. II: Pseudo-symmetric AA-semigroups, J. Algebra 470 (2017), 484–498.
- [9] S. M. Johnson, A linear diophantine problem, Canadian J. Math. 12 (1960), 390–398.
- [10] T. Komatsu, Sylvester power and weighted sums on the Frobenius set in arithmetic progression, Discrete Appl. Math. 315 (2022), 110–126.
- [11] T. Komatsu, The Frobenius number for sequences of triangular numbers associated with number of solutions, Ann. Comb. 26 (2022) 757–779.
- [12] T. Komatsu, The Frobenius number associated with the number of representations for sequences of repunits, C. R. Math., Acad. Sci. Paris 361 (2023), 73–89.
- [13] T. Komatsu, On p-Frobenius and related numbers due to p-Apéry set, arXiv:2111.11021v3 (2022).

- [14] T. Komatsu, S. Laishram and P. Punyani, *p*-numerical semigroups of generalized Fibonacci triples, Symmetry 15 (2023), no.4, Article 852, 13 p.
- [15] T. Komatsu and C. Pita-Ruiz, The Frobenius number for Jacobsthal triples associated with number of solutions, Axioms 12 (2023), no.2, Article 98, 18 p.
- [16] T. Komatsu and H. Ying, The p-Frobenius and p-Sylvester numbers for Fibonacci and Lucas triplets, Math. Biosci. Eng. 20 (2023), No.2, 3455–3481.
- [17] T. Komatsu and H. Ying, The p-numerical semigroup of the triple of arithmetic progressions, Symmetry 15 (2023), No.7, Article 1328, 17 p. https://doi.org/10.3390/sym15071328
- [18] T. Komatsu and H. Ying, p-numerical semigroups with p-symmetric properties, J. Algebra Appl. (online ready). https://doi.org/10.1142/S0219498824502165
- [19] G. L. Matthews, On numerical semigroups generated by generalized arithmetic sequences, Commun. Algebra 32 (2004), No. 9, 3459–3469.
- [20] P. Punyani and A. Tripathi, On changes in the Frobenius and Sylvester numbers, Integers 18B (2018), #A8, 12 p.
- [21] Ø. J. Rødseth, On a linear Diophantine problem of Frobenius, J. Reine Angew. Math. 301 (1978), 171–178.
- [22] E. S. Selmer, On the linear diophantine problem of Frobenius, J. Reine Angew. Math. 293/294 (1977), 1–17.
- [23] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, available at oeis.org. (2023).
- [24] A. Tripathi, On sums of positive integers that are not of the form ax + by, Amer. Math. Monthly 115 (2008), 363–364.
- [25] A. Tripathi, Formulae for the Frobenius number in three variables, J. Number Theory 170 (2017), 368–389.

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