NEWFORM THEORY FOR GL_n

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1. Introduction

Let $\mathcal{M}_k(q,\chi)$ denote the finite-dimensional vector space of holomorphic modular forms of weight k, level q, and nebentypus χ, where χ is a primitive Dirichlet character of conductor $q_\chi \mid q$. The classical theory of newforms due to Atkin and Lehner [AL70] states that for each $q' \mid q$ with $q' \neq q$ and $q' \equiv 0 \pmod{q_\chi}$ and for each $\ell \mid \frac{q}{q'}$, the function $(\iota_\ell f)(z) \coloneqq f(\ell z)$ defines an element of $\mathcal{M}_k(q,\chi)$ whenever $f \in \mathcal{M}_k(q',\chi)$. We call $\iota_\ell f$ an oldform. Moreover, the orthogonal complement with respect to the Petersson inner product of the vector subspace of oldforms has an orthonormal basis consisting of newforms, which are eigenfunctions of the n-th Hecke operator not just for each positive integer n for which $(n,q) = 1$ but for all $n \in \mathbb{N}$.

Casselman [Cas73], building on the seminal work of Jacquet and Langlands [JL70], gave an adèlic reformulation of the Atkin–Lehner theory of newforms. Due to the fact that automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ have a tensor product factorisation in terms of representations of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ for each prime p, this reformulation is purely local and is in terms of distinguished vectors in certain classes of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ determined in terms of congruence subgroups. Such a theory of newforms has been extended to the setting of generic irreducible admissible smooth representations of $\mathrm{GL}_n(F)$, where F is a nonarchimedean local field [JP-SS81]. Below, we discuss some aspects of this theory, its recent development in the archimedean setting by the author, and mention some open questions in this field.

2. Nonarchimedean Newform Theory for GL_n

2.1. **Representations.** Let F be a nonarchimedean local field, so that F is either a finite extension of the p-adic numbers $\mathbb{Q}_p$ for some prime p or F is the field of formal Laurent series $\mathbb{F}_q((t))$. Write $\mathcal{O}$ for the ring of integers of F and $\mathfrak{p}$ for its maximal ideal, and set $q \coloneqq \#\mathcal{O}/\mathfrak{p}$.

Given representations $(\pi_1, V_{\pi_1}), \ldots, (\pi_r, V_{\pi_r})$ of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_r}(F)$, where $n_1 + \cdots + n_r = n$, we form the representation $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ of $\mathrm{M}_\mathrm{P}(F)$, where $\boxtimes$ denotes the outer tensor product and $\mathrm{M}_\mathrm{P}(F)$ denotes the block-diagonal Levi subgroup of the standard (upper) parabolic subgroup $\mathrm{P}(F) = \mathrm{P}_{(n_1,\ldots,n_r)}(F)$ of $\mathrm{GL}_n(F)$. We then extend this representation trivially to a representation of $\mathrm{P}(F)$. By normalised parabolic induction, we obtain an induced representation (π, V_π) of $\mathrm{GL}_n(F)$,

$$\pi \coloneqq \mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{GL}_n(F)} \mathop{\boxtimes}_{j=1}^{r} \pi_j,$$

where V_π denotes the space of smooth functions $f : \mathrm{GL}_n(F) \to V_{\pi_1} \otimes \cdots \otimes V_{\pi_r}$ that satisfy

$$f(umg) = \delta_\mathrm{P}^{1/2}(m)\pi_1(m_1) \otimes \cdots \otimes \pi_r(m_r) \cdot f(g),$$

for any $u \in \mathrm{N}_\mathrm{P}(F)$, $m = \mathrm{blockdiag}(m_1,\ldots,m_r) \in \mathrm{M}_\mathrm{P}(F)$, and $g \in \mathrm{GL}_n(F)$, and the action of π on V_π is by right translation, namely $(\pi(h) \cdot f)(g) \coloneqq f(gh)$. We call π the isobaric sum of $\pi_1, \ldots, \pi_r$, which we denote by

$$\pi = \mathop{\boxplus}_{j=1}^{r} \pi_j.$$

A representation π of $\mathrm{GL}_n(F)$ is said to be an *induced representation of Whittaker type* if it is the isobaric sum of $\pi_1, \ldots, \pi_r$ and each π_j is irreducible and essentially square-integrable. If each

π_j is additionally of the form $\sigma_j \otimes |\det|^{t_j}$, where σ_j is irreducible, unitary, and square-integrable, and $\Re(t_1) \geq \cdots \geq \Re(t_r)$, then π is said to be an *induced representation of Langlands type.* Every irreducible admissible smooth representation π of $\mathrm{GL}_n(F)$ is isomorphic to the unique irreducible quotient of some induced representation of Langlands type. If π is also generic, so that it has a Whittaker model $\mathcal{W}(\pi, \psi)$, then it is isomorphic to some (necessarily irreducible) induced representation of Langlands type [CS98].

2.2. **Newform Theory.** Let $K = K_n := \mathrm{GL}_n(\mathcal{O})$ be the maximal compact subgroup of $\mathrm{GL}_n(F)$, which is unique up to conjugation. For a nonnegative integer m, we define the following finite index subgroup of K:

$$K_1(\mathfrak{p}^m) := \{k \in K : k_{n,1}, \ldots, k_{n,n-1}, k_{n,n} - 1 \in \mathfrak{p}^m\}.$$

Given an induced representation of Langlands type (π, V_π) of $\mathrm{GL}_n(F)$, we define the vector subspace $V_\pi^{K_1(\mathfrak{p}^m)}$ of V_π consisting of $K_1(\mathfrak{p}^m)$-fixed vectors:

$$V_\pi^{K_1(\mathfrak{p}^m)} := \{v \in V_\pi : \pi(k) \cdot v = v \text{ for all } k \in K_1(\mathfrak{p}^m)\}.$$

The following theorem is due to Casselman [Cas73, Theorem 1] for $n = 2$ and Jacquet, Piatetski-Shapiro, and Shalika for arbitrary n.

Theorem 2.1 (Jacquet–Piatetski-Shapiro–Shalika [JP-SS81, Théorème (5)])**.** *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. There exists a minimal nonnegative integer m for which $V_\pi^{K_1(\mathfrak{p}^m)}$ is nontrivial. For this minimal value of m, $V_\pi^{K_1(\mathfrak{p}^m)}$ is one-dimensional.*

Definition 2.2. We define the conductor exponent of π to be this minimal nonnegative integer m and denote it by $c(\pi)$; we then call the ideal $\mathfrak{p}^{c(\pi)}$ the conductor of π. The newform of π is defined to be the nonzero vector $v^\circ \in V_\pi^{K_1(\mathfrak{p}^{c(\pi)})}$, unique up to scalar multiplication.

The uniqueness of the newform may be thought of as being a multiplicity-one theorem for newforms. The reason for naming this distinguished vector a newform is due to its relation to the classical theory of modular forms: as shown by Casselman [Cas73, Section 3], an automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ whose associated Whittaker function is a pure tensor composed of newforms in the Whittaker model is the adèlic lift of a classical newform in the sense of Atkin and Lehner [AL70].

If $c(\pi) = 0$, so that $K_1(\mathfrak{p}^{c(\pi)}) = K$, then π must be a spherical representation and we say that π is unramified. If $c(\pi) > 0$, then π is said to be ramified. In this sense, the conductor exponent is a measure of the extent of ramification of π: it quantifies how ramified π may be. Moreover, the conductor exponent is *additive* with respect to isobaric sums and appears in the epsilon factor associated to π.

Theorem 2.3 (Jacquet–Piateski-Shapiro–Shalika [JP-SS83, Theorem (3.1), Section 5])**.** *For an induced representation of Langlands type $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ of $\mathrm{GL}_n(F)$, we have that*

$$c(\pi) = \sum_{j=1}^{r} c(\pi_j).$$

Moreover, the epsilon factor $\varepsilon(s, \pi, \psi)$ satisfies

$$\varepsilon(s, \pi, \psi) = \varepsilon\left(\frac{1}{2}, \pi, \psi\right) q^{-c(\pi)\left(s - \frac{1}{2}\right)}.$$

2.3. **Oldform Theory.** While Jacquet, Piatetski-Shapiro, and Shalika merely show that $V_\pi^{K_1(\mathfrak{p}^{c(\pi)})}$ is one-dimensional, one can also calculate the dimension of $V_\pi^{K_1(\mathfrak{p}^m)}$ for all $m \geq c(\pi)$ in terms of a binomial coefficient; for $n = 2$, this is due to Casselman [Cas73, Corollary to the Proof], while Reeder has proven this result for arbitrary n.

Theorem 2.4 (Reeder [Ree91, Theorem 1]). *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$ with $n \geq 2$. We have that*

$$\dim V_\pi^{K_1(\mathfrak{p}^m)} = \begin{cases} \dbinom{m - c(\pi) + n - 1}{n-1} & \text{if } m \geq c(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

Casselman and Reeder also give a basis for each of these spaces in terms of the action of certain Hecke operators on the newform. For $m > c(\pi)$, we call $V_\pi^{K_1(\mathfrak{p}^m)}$ the space of oldforms of exponent m. Once again, the reason for naming these distinguished vectors oldforms is due to their relation to the classical theory of modular forms: an automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ whose associated Whittaker function is a pure tensor composed of Whittaker newforms at all but finitely many places and of Whittaker oldforms at the remaining places corresponds to an oldform in the sense of Atkin and Lehner [AL70].

2.4. **K-Types.** Since π is admissible, $\mathrm{Hom}_K(\tau, \pi|_K)$ is finite-dimensional for each irreducible smooth representation τ of K. We say that such a representation τ is a *K-type* of π if $\mathrm{Hom}_K(\tau, \pi|_K)$ is nontrivial, and we call $\dim \mathrm{Hom}_K(\tau, \pi|_K)$ the *multiplicity* of τ in π. The complexity of an irreducible smooth representation τ of K can be measured by its *level* m, which is the least nonnegative integer m for which τ factors through the finite group $\mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^m)$. In [Hum22], the author proved the existence of a distinguished K-type of π that occurs with multiplicity one and is closely associated to the newform and the conductor exponent.

Theorem 2.5 ([Hum22, Theorem 4.11]). *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. Among the K-types of π whose restriction to*

$$K_{n-1,1} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in K_n : a \in K_{n-1},\ b \in \mathrm{Mat}_{(n-1)\times 1}(\mathcal{O}) \right\}$$

contains the trivial representation, there exists a unique K-type τ° of minimal level. Furthermore, τ° occurs with multiplicity one in π, the level of τ° is equal to the conductor exponent $c(\pi)$, and the subspace of V_π of τ°-isotypic $K_{n-1,1}$-invariant vectors is equal to the one-dimensional subspace $V_\pi^{K_1(\mathfrak{p}^{c(\pi)})}$ spanned by the newform v°.

Definition 2.6. We call the distinguished K-type τ° the *newform K-type*.

The author additionally showed that spaces of oldforms can be described in terms of distinguished K-types.

Theorem 2.7 ([Hum22, Theorem 4.11]). *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. For each $m \geq c(\pi)$, there exists a unique K-type τ_m of π of level m whose restriction to $K_{n-1,1}$ contains the trivial representation. Furthermore, this K-type occurs with multiplicity*

$$\binom{m - c(\pi) + n - 2}{n-2},$$

and the direct sum indexed by nonnegative integers $\ell \in \{c(\pi), \ldots, m\}$ of the subspaces of V_π of τ_ℓ-isotypic $K_{n-1,1}$-invariant vectors is equal to $V_\pi^{K_1(\mathfrak{p}^m)}$, the space of oldforms of exponent m.

These K-types have a particular structure. If τ is an irreducible smooth representation of K whose restriction to $K_{n-1,1}$ contains the trivial representation, then τ has as a model a *space of spherical harmonics*, namely a distinguished subspace of the space $C^\infty(S^{n-1})$ of complex-valued locally constant functions on the $\mathfrak{p}$-adic n-sphere

$$S^{n-1} := \{(x_1, \ldots, x_n) \in F^n : \max\{|x_1|, \ldots, |x_n|\} = 1\} \cong K_{n-1,1} \backslash K.$$

The decomposition of $C^\infty(S^{n-1})$ into irreducible K-modules was analysed by the author [Hum22, Theorem 2.16] by extending earlier work of Petrov [Pet82].

2.5. **Nongeneric Nonarchimedean Newform Theory.** Recently, Atobe, Kondo, and Yasuda developed some aspects of newform theory for *nongeneric* representations of $\mathrm{GL}_n(F)$.

Theorem 2.8 (Atobe–Kondo–Yasuda [AKY22, Theorem 1.1])**.** *Let (π, V_π) be a* nongeneric *irreducible admissible smooth representation of $\mathrm{GL}_n(F)$. Then π has a local newform, which is invariant under the action of some subgroup of the form*

$$K_{n,\lambda} := \left\{k \in K : k_{i,j} \equiv \delta_{i,j} \pmod{\mathfrak{p}^{\lambda_i}}\right\}$$

for some minimal $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. Moreover, the conductor exponent of π is $c(\pi) = \lambda_1 + \cdots + \lambda_n$.

Some natural questions remain open in this nongeneric setting.

Question 2.9. *Let F be a nonarchimedean local field.*

(1) *Is there a theory of oldforms for nongeneric representations of $\mathrm{GL}_n(F)$?*
(2) *Can one describe the newform of a nongeneric representation in terms of a K-type?*

We expect that a resolution of the latter question must involve branching not just from K_n to $K_{n-1,1}$, as in the generic setting, but instead branching in stages to smaller subgroups of K_n. This branching in stages should involve both branching from

$$K'_{n-m+1} := \left\{\begin{pmatrix} a & 0 \\ 0 & 1_{m-1} \end{pmatrix} \in K_n : a \in K_{n-m+1}\right\}$$

to

$$K_{n-m,m} := \left\{\begin{pmatrix} a & b \\ 0 & 1_m \end{pmatrix} \in K_n : a \in K_{n-m},\ b \in \mathrm{Mat}_{(n-m)\times m}(\mathcal{O})\right\}$$

and from $K_{n-m,m}$ to K_{n-m}, where $m \in \{1, \ldots, n-1\}$.

3. Archimedean Newform Theory for GL_n

3.1. **Representations.** We now turn our attention to the archimedean setting, so that F is an archimedean local field, namely either $F = \mathbb{R}$ or $F = \mathbb{C}$. We define induced representations of Langlands type of $\mathrm{GL}_n(F)$ in the same fashion as in the nonarchimedean setting. As well as being admissible and smooth, such a representation is additionally a Fréchet representation of moderate growth and of finite length.

3.2. **K-Types.** One cannot define the newform of an induced representation of Langlands type of $\mathrm{GL}_n(F)$ with F archimedean in terms of congruence subgroups, since $\mathrm{GL}_n(F)$ lacks compact open subgroups akin to $K_1(\mathfrak{p}^m)$. Instead, we proceed directly via K-types, where K denotes the maximal compact subgroup of $\mathrm{GL}_n(F)$, which is unique up to conjugacy, namely

$$K = K_n = \begin{cases} \mathrm{O}(n) & \text{if } F = \mathbb{R}, \\ \mathrm{U}(n) & \text{if } F = \mathbb{C}. \end{cases}$$

Since π is admissible, $\mathrm{Hom}_K(\tau, \pi|_K)$ is finite-dimensional for each irreducible representation τ of K. We say that such a representation τ is a *K-type* of π if $\mathrm{Hom}_K(\tau, \pi|_K)$ is nontrivial, and we call $\dim \mathrm{Hom}_K(\tau, \pi|_K)$ the *multiplicity* of τ in π. The complexity of an irreducible smooth representation τ of K can be measured by its *Howe degree* $m = \deg \tau$, which is defined by $\deg \tau := \sum_{j=1}^n |\mu_j|$ with $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ the highest weight of τ. In [Hum20], the author proved the existence of a distinguished K-type of π that occurs with multiplicity one and *defined* the newform and the conductor exponent via this K-type.

Theorem 3.1 ([Hum20, Theorem 4.7])**.** *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. Among the K-types of π whose restriction to*

$$K'_{n-1} := \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in K : a \in K_{n-1}\right\}$$

contains the trivial representation, there exists a unique K-type τ° of minimal Howe degree. Furthermore, τ° occurs with multiplicity one in π and the subspace of V_π of τ°-isotypic K'_{n-1}-invariant vectors is one-dimensional and spanned by a vector v° that is unique up to scalar multiplication.

Definition 3.2. We call the distinguished vector v° the *newform*, the distinguished K-type τ° the *newform K-type*, and the distinguished nonnegative integer $c(\pi) := \deg \tau^\circ$ the *conductor exponent*.

Just as in the nonarchimedean setting, the conductor exponent is additive with respect to isobaric sums and appears in the epsilon factor associated to π.

Theorem 3.3 ([Hum20, Theorem 4.15])**.** *For an induced representation of Langlands type $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ of $\mathrm{GL}_n(F)$, we have that*

$$c(\pi) = \sum_{j=1}^{r} c(\pi_j).$$

Moreover, the epsilon factor $\varepsilon(s, \pi, \psi)$ satisfies

$$\varepsilon(s, \pi, \psi) = i^{-c(\pi)}.$$

The author additionally showed that spaces of oldforms can be described in terms of distinguished K-types.

Theorem 3.4 ([Hum20, Theorem 4.12])**.** *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. For each $m \geq c(\pi)$ for which $m \equiv c(\pi) \pmod 2$, there exists a unique K-type τ_m of π of Howe degree m whose restriction to K'_{n-1} contains the trivial representation. Furthermore, this K-type occurs with multiplicity*

$$\binom{\frac{m-c(\pi)}{2} + n - 2}{n-2}.$$

Definition 3.5. For each $m > c(\pi)$ for which $m \equiv c(\pi) \pmod 2$, we call the subspace of V_π of τ_m-isotypic K'_{n-1}-invariant vectors the space of *oldforms* of exponent m.

Note that if τ is an irreducible representation of K whose restriction to K'_{n-1} contains the trivial representation, then τ has as a model a *space of spherical harmonics*, namely a distinguished subspace of the space $C^\infty(S^{n-1})$ of complex-valued locally constant functions on the n-sphere

$$\begin{aligned} S^{n-1} &:= \begin{cases} \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\} & \text{if } F = \mathbb{R}, \\ \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1\overline{x_1} + \cdots + x_n\overline{x_n} = 1\} & \text{if } F = \mathbb{C}, \end{cases} \\ &\cong K'_{n-1}\backslash K. \end{aligned}$$

The decomposition of $C^\infty(S^{n-1})$ into irreducible K-modules is simply the classical theory of spherical harmonics.

3.3. **Nongeneric Archimedean Newform Theory.** As yet, there is no newform theory of *nongeneric* representations of $\mathrm{GL}_n(F)$ when F is archimedean. One can raise some natural questions in this nongeneric setting.

Question 3.6. *Let F be an archimedean local field.*

(1) *Is there a theory of newforms and oldforms for nongeneric representations of* $\mathrm{GL}_n(F)$*?*
(2) *Can one describe the newform of a nongeneric representation in terms of a K-type?*

Once more, we expect that a resolution of the latter question must involve branching not just from K_n to K'_{n-1}, as in the generic setting, but instead branching in stages from

$$K'_{n-m+1} \coloneqq \left\{\begin{pmatrix} a & 0 \\ 0 & 1_{m-1}\end{pmatrix} \in K_n : a \in K_{n-m+1}\right\}$$

to K'_{n-m}, where $m \in \{1, \ldots, n-1\}$.

4. Rankin–Selberg Integrals

4.1. **The Test Vector Problem.** There is a close connection between newform theory and the theory of test vectors for Rankin–Selberg integrals. We recall that given induced representations of Whittaker type π of $\mathrm{GL}_n(F)$ and π' of $\mathrm{GL}_m(F)$ with $m \leq n$, and given Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \overline{\psi})$, the local $\mathrm{GL}_n \times \mathrm{GL}_m$ Rankin–Selberg integral is defined by

$$\Psi(s, W, W') \coloneqq \int\limits_{\mathrm{N}_m(F)\backslash \mathrm{GL}_m(F)} W\begin{pmatrix} g & 0 \\ 0 & 1_{n-m}\end{pmatrix} W'(g) \left|\det g\right|^{s-\frac{n-m}{2}} \, dg \quad \text{for } m < n,$$

$$\Psi(s, W, W', \Phi) \coloneqq \int\limits_{\mathrm{N}_n(F)\backslash \mathrm{GL}_n(F)} W(g) W'(g) \Phi(e_n g) \left|\det g\right|^{s} \, dg \quad \text{for } m = n,$$

where $\Phi \in \mathscr{S}(\mathrm{Mat}_{1\times n}(F))$ is a Schwartz–Bruhat function and $e_n \coloneqq (0, \ldots, 0, 1) \in \mathrm{Mat}_{1\times n}(F) = F^n$. These integrals converge absolutely for $\Re(s)$ sufficiently large and extend meromorphically to the entire complex plane.

The Rankin–Selberg integral is always a holomorphic multiple of the Rankin–Selberg L-function. In particular, for nonarchimedean F, the Rankin–Selberg L-function $L(s, \pi \times \pi')$ is the generator of the $\mathbb{C}[q^s, q^{-s}]$-fractional ideal of $\mathbb{C}(q^{-s})$ generated by the family of Rankin–Selberg integrals $\Psi(s, W, W')$ (or $\Psi(s, W, W', \Phi)$ if $m = n$) with $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \overline{\psi})$ (and $\Phi \in \mathscr{S}(\mathrm{Mat}_{1\times n}(F))$ if $m = n$). For archimedean F, the quotient $\Psi(s, W, W')/L(s, \pi \times \pi')$ (or $\Psi(s, W, W', \Phi)/L(s, \pi \times \pi')$ if $m = n$) is entire and of finite order in vertical strips.

While this quotient is always entire regardless of the choice of Whittaker functions W and W' (and Schwartz–Bruhat function Φ if $m = n$), for many applications, one requires something stronger, namely that for particular choices of W, W', and Φ', this quotient be nicely behaved — in particular, nonvanishing apart from a prescribed collection of values of $s \in \mathbb{C}$. When the representations π and π' are both unramified, one can take W and W' to be the spherical Whittaker functions, and additionally explicitly choose the Schwartz–Bruhat function Φ if $m = n$, such that this quotient is *exactly* equal to 1. This motivates the following problem.

Test Vector Problem. *Given induced representations of Langlands type π of $\mathrm{GL}_n(F)$ and π' of $\mathrm{GL}_m(F)$, determine the existence of Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \overline{\psi})$, and additionally a Schwartz–Bruhat function $\Phi \in \mathscr{S}(\mathrm{Mat}_{1\times n}(F))$ if $m = n$, for which*

$$L(s, \pi \times \pi') = \begin{cases} \Psi(s, W, W') & \text{for } m < n, \\ \Psi(s, W, W', \Phi) & \text{for } m = n. \end{cases}$$

We call such a tuple (W, W'), or a triple (W, W', Φ) if $m = n$, a *test vector* for the Rankin–Selberg integral.

4.2. **Newforms as Test Vectors.** When $m \in \{n, n-1\}$ and the representation π' of $\mathrm{GL}_m(F)$ is spherical, *newforms* are test vectors. The case $m = n - 1$ is as follows.

Theorem 4.1. *Let π be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. The newform $W^\circ \in \mathcal{W}(\pi, \psi)$ is the unique Whittaker function that is both right K'_{n-1}-invariant, so that*

$$W^\circ\left(g \begin{pmatrix} k & 0 \\ 0 & 1\end{pmatrix}\right) = W^\circ(g)$$

for all $k \in K_{n-1}$*, and is such that for any spherical representation of Langlands type* π' *of* $\mathrm{GL}_{n-1}(F)$ *with spherical Whittaker function* $W'^{\circ} \in \mathcal{W}(\pi', \overline{\psi})$*,*

$$\Psi(s, W, W'^{\circ}) = L(s, \pi \times \pi')$$

for $\Re(s)$ *sufficiently large.*

For F nonarchimedean, this is due to Jacquet, Piatetski-Shapiro, and Shalika [JP-SS81, Théorème (4)] (though the proof was incomplete and was independently corrected by Jacquet [Jac12] and Matringe [Mat13, Corollary 3.3]). For F archimedean, this is [Hum20, Theorem 4.17]. An analogous result holds for the case $m = n$.

Theorem 4.2. *Let* π *be an induced representation of Langlands type of* $\mathrm{GL}_n(F)$*. Then there exists a choice of bi-*K*-finite Schwartz–Bruhat function* $\Phi^{\circ} \in \mathscr{S}(\mathrm{Mat}_{1\times n}(F))$ *such that for any spherical representation of Langlands type* π' *of* $\mathrm{GL}_n(F)$ *with spherical Whittaker function* $W'^{\circ} \in \mathcal{W}(\pi', \overline{\psi})$*, the newform* $W^{\circ} \in \mathcal{W}(\pi, \psi)$ *of* π *satisfies*

$$\Psi(s, W^{\circ}, W'^{\circ}, \Phi^{\circ}) = L(s, \pi \times \pi')$$

for $\Re(s)$ *sufficiently large.*

For F nonarchimedean, this is due to Kim [Kim10, Theorem 2.1.1]; for F archimedean, this is [Hum20, Theorem 4.18].

What about the case $m < n - 1$? For F nonarchimedean, the newform is again a test vector for the Rankin–Selberg integral. For F archimedean, on the other hand, it is widely believed that no test vector exists.

4.3. **Test Vectors for Ramified Representations.** When both π and π' are ramified, newforms are no longer test vectors for Rankin–Selberg integrals, and instead one must look elsewhere to construct test vectors. For example, for F archimedean, there are certain representations for which Whittaker functions lying in the minimal K-type are test vectors [IM22]. In general, however, we do not yet have a way of systematically determining test vectors for Rankin–Selberg integrals.

Question 4.3. *Can one systematically determine* K*-finite test vectors for Rankin–Selberg integrals?*

Let us briefly discuss how one might go about this in the case $m = n - 1$. By the Iwasawa decomposition, we may write

$$\Psi(s, W, W') = \int_{\mathrm{A}_{n-1}(F)} \left|\det a'\right|^{s-\frac{1}{2}} \delta_{n-1}^{-1}(a') \int_{K_{n-1}} W\left(\begin{pmatrix} a' & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} k' & 0 \\ 0 & 1\end{pmatrix}\right) W'(a'k')\, dk'\, d^{\times}a'.$$

Here $\mathrm{A}_{n-1}(F)$ denotes the subgroup of $\mathrm{GL}_{n-1}(F)$ of diagonal matrices and $\delta_{n-1}(a')$ denotes the modulus character. Suppose that $W \in \mathcal{W}(\pi, \psi)$ is right K-finite Whittaker function, so that the action of $\pi(k)$ on W for $k \in K_n$ generates a finite-dimensional representation τ of K; similarly, suppose that the action of $\pi'(k')$ on W' for $k' \in K_{n-1}$ generates a finite-dimensional representation τ' of K_{n-1}. By Schur's lemma, the inner integral vanishes unless $\tau \otimes \tau'|_{K_{n-1}}$ contains the trivial representation of K_{n-1}.

With this in mind, we expect that there exist test vectors when τ, τ' are such that the representation $\tau \otimes \tau'$ of $K_n \times K_{n-1}$ is *minimal* among all $K_n \times K_{n-1}$ types of $\pi \otimes \pi'$ in some explicit sense. For F nonarchimedean, this should be in the sense of the *level* of this representation, while for F archimedean, this should be in the sense of the *Howe degree* of this representation. Finally, the level or Howe degree should be associated to the conductor exponent $c(\pi \times \pi')$ of $\pi \times \pi'$, which appears in the epsilon factor as

$$\varepsilon(s, \pi \times \pi', \psi) = \begin{cases} \varepsilon\left(\frac{1}{2}, \pi \times \pi', \psi\right) q^{-c(\pi\times\pi')\left(s-\frac{1}{2}\right)} & \text{if } F \text{ is nonarchimedean,} \\ i^{-c(\pi\times\pi')} & \text{if } F \text{ is archimedean.}\end{cases}$$

4.4. **Nongeneric Representations.** Rankin–Selberg integrals involve Whittaker functions and hence are only defined for representations that admit a Whittaker model. Nonetheless, there ought to be a theory of test vectors for other period integrals for nongeneric representations. In [AKY22], Atobe, Kondo, and Yasuda investigate Rankin–Selberg integrals in the Zelevinsky models and in the Shalika models, which are well-defined even for nongeneric representations. They prove that if F is nonarchimedean, the newform is a test vector for these period integrals when the second representation is unramified. Once more, there are natural open questions in this regard.

Question 4.4.

(1) *Can one extend the test vector result of* [AKY22] *to the setting of F archimedean?*
(2) *Can one systematically determine K-finite test vectors for Rankin–Selberg integrals in the Zelevinsky models and in the Shalika models?*

5. Other Groups

We end by discussing the problem of generalising the notion of newforms and of test vectors to groups other than GL_n.

For nonarchimedean F, there has been a great deal of recent progress in defining newforms via congruence subgroups for groups G other than GL_n. See, for example, [RS07] for $G = \mathrm{PGSp}_4$, [Oka19] for $G = \mathrm{GSp}_4$, [Che22a, Tsa16] for $G = \mathrm{SO}_{n+1,n}$, [AOY22, Che22b] for $G = \mathrm{U}_{n+1,n}$, and [Ato23] for $G = \mathrm{U}_{n,n}$. In several cases, it has additionally been shown that newforms for these groups are test vectors for certain period integrals. For example, in the case of $G = \mathrm{GSp}_4$, the period integral of interest is the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ Rankin–Selberg integral: the newform is a test vector for this integral when the representation of GL_2 is spherical.

From the previous discussion for GL_n, many open questions naturally arise. We state several below as motivation for researchers in the field.

Question 5.1.

(1) *Can one characterise newforms for groups other than $G = \mathrm{GL}_n$ via K-types?*
(2) *Can one determine a theory of newforms for groups other than $G = \mathrm{GL}_n$ when F is archimedean?*
(3) *Can one determine a theory of newforms for groups other than $G = \mathrm{GL}_n$ when the representation is nongeneric?*
(4) *Can one systematically determine K-finite test vectors for period integrals for groups other than $G = \mathrm{GL}_n$?*

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