L series and orthogonality in number theory

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Abstract

We consider the importance of orthogonality in number theory, and particularly in the theory of L series, through two distinct contexts: Dirichlet L series and L series attached to Maass cusp forms. The former, more familiar, context will be viewed in a perhaps nonstandard light, so as to help elucidate and illuminate the latter, less familiar context.

Our discussion culminates with some recent results concerning orthogonality of Fourier coefficients of Maass cusp forms on the generalized upper half plane.

1 L series are good things

Our first central observation and main conviction is this:

L series, meaning series of the form

$$L(s,(a_m)) = \sum_{m=1}^{\infty} a_m m^{-s},$$

where s is a complex number and (a_m) is a sequence of complex numbers, are of critical importance in many areas of number theory.

We support this conviction with some examples.

Example 1.1. The Riemann zeta function

$$\zeta(s) = L(s,(1)) = \sum_{m=1}^{\infty} m^{-s}$$

has applications to the distribution of prime numbers, and to practically everything else (in number theory, at least).

Example 1.2. Let χ denote a Dirichlet character mod q. That is, χ is a homomorphism on $(\mathbb{Z}/q\mathbb{Z})^*$, extended to all integers in a natural way (namely, for $m \in \mathbb{Z}$ and $\overline{m}$ the equivalence class of m in $(\mathbb{Z}/q\mathbb{Z})^*$, we define $\chi(m) = \chi(\overline{m})$ if $(m,q) = 1$, and $\chi(m) = 0$ if $(m,q) > 1$). The *Dirichlet L series*

$$L(s,\chi) = \sum_{m=1}^{\infty} \chi(m) m^{-s}$$

is a (the?) key ingredient in the proof of Dirichlet's theorem on primes in arithmetic progression—namely, that there are infinitely many primes in the set $\{nq + a : n \in \mathbb{Z}_{>0}\}$, if $(q,a) = 1$.

Example 1.3. Given an elliptic curve E, the L series

$$L(s, E) = \sum_{m=1}^{\infty} a_E(m) m^{-s},$$

where $a_E(p)$, for p prime, measures the number of $(x, y) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ that lie on the curve mod p (and $a_E(m)$ is defined in terms of $\{a_E(p) | \ p \text{ is a prime dividing } m\}$), is a powerful tool for understanding the curve. Such L series have applications to things like the congruent number problem (which asks: which positive integers arise as areas of right triangles with rational sidelengths?) and Fermat's Last Theorem.

Example 1.4. There are *Artin L series* $L(s, \rho, K/k)$ associated to certain representations ρ of Galois groups of extensions K/k of number fields, with applications to groups of units in number fields etc.

Example 1.5. Given a modular form

$$\phi(z) = \sum_{m=0}^{\infty} a_\phi(m) e^{2\pi i m z}$$

or a cusp Maass form

$$\phi(z) = \sum_{m \in \mathbb{Z}} a_\phi(m) W_\alpha(|m|y) e^{2\pi i m x}$$

of type α on the upper half-plane $\mathfrak{h}^2 = \{z = x + iy : x \in \mathbb{R},\ y > 0\}$ (more on $\mathfrak{h}^2$ Maass cusp forms in Section 2.2.1 below), the L series

$$L(s, \phi) = \sum_{m=1}^{\infty} a_\phi(m) m^{-s}$$

encodes useful, valuable information about ϕ (and vice versa).

Example 1.6. The previous example generalizes to modular forms and Maass cusp forms ϕ on the generalized upper half-plane $\mathfrak{h}^n$. We'll discuss Maass cusp forms on $\mathfrak{h}^n$ in Section 2.2.2 below.

Of course, L series are not the only good things in number theory. The next good thing that we consider is intimately connected to these series.

2 Orthogonality is a good thing

Our second central observation and main conviction is this:

In the study of L series $L(s, (a_m))$, orthogonality of the a_m's is often a good thing to study.

We support this conviction through the consideration of two distinct contexts: (I) Dirichlet characters (cf. Example 1.2 above), and (II) Maass forms on $\mathfrak{h}^n$ (cf. Example 1.6 above), with particular attention paid to the case $n = 2$ (cf. Example 1.5 above).

2.1 Part I: Dirichlet characters

In the application of Dirichlet L series $L(s,\chi)$ to the study of primes in arithmetic progression, the following *orthogonality relation* (denoted (OR_χ), connoting "orthogonality relation for Dirchlet characters χ") is key:

Proposition 2.1. (Orthogonality of Dirichlet characters mod q) *Define* $\varphi(q) = |(\mathbb{Z}/q\mathbb{Z})^*|$. *For given* $\ell, m \in (\mathbb{Z}/q\mathbb{Z})^*$, *we have*

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(\ell)\overline{\chi(m)} = \delta_{\ell,m} = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{otherwise,} \end{cases} \tag{OR_χ}$$

the sum being over all Dirichlet characters χ *mod* q.

It will be instructive to consider a particular (non-standard) kind of proof of (OR_χ), which makes use of indicator functions.

Proof of $(\mathrm{OR}_{\boldsymbol{\chi}})$. Consider the inner product

$$\langle f, g\rangle = \sum_{k\in(\mathbb{Z}/q\mathbb{Z})^*} f(k)\overline{g(k)}$$

of functions f and g on $(\mathbb{Z}/q\mathbb{Z})^*$.

Our proof entails computing $\langle \mathbb{1}_m, \mathbb{1}_\ell\rangle$, where $\mathbb{1}_\ell$ and $\mathbb{1}_m$ are *indicator functions* on $(\mathbb{Z}/q\mathbb{Z})^*$, in two distinct ways. Both ways will require the following lemma.

Lemma. For $\ell \in (\mathbb{Z}/q\mathbb{Z})^*$, *define the indicator function* $\mathbb{1}_\ell$ *on* $(\mathbb{Z}/q\mathbb{Z})^*$ *by*

$$\mathbb{1}_\ell(k) = \delta_{k,\ell}.$$

Then for any function $h : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$, *we have*

$$\langle h, \mathbb{1}_\ell\rangle = h(\ell).$$

Proof of Lemma.

$$\langle h, \mathbb{1}_\ell\rangle = \sum_{k\in(\mathbb{Z}/q\mathbb{Z})^*} h(k)\overline{\mathbb{1}_\ell(k)} = \sum_{k\in(\mathbb{Z}/q\mathbb{Z})^*} h(k)\overline{\delta_{k,\ell}} = h(\ell). \qquad \square$$

Returning to our proof of (OR_χ), we now let $\ell, m \in (\mathbb{Z}/q\mathbb{Z})^*$. On the one hand we have, by the above lemma with $h = \mathbb{1}_m$,

$$\langle \mathbb{1}_m, \mathbb{1}_\ell\rangle = \mathbb{1}_m(\ell) = \delta_{\ell,m}. \tag{$\mathrm{Way1}_\chi$}$$

On the other hand, it's a **fact** that the set

$$\{\varphi(q)^{-1/2}\chi : \chi \text{ is a Dirichlet character mod } q\} \tag{2.2}$$

forms an orthonormal basis for the space of all functions on $(\mathbb{Z}/q\mathbb{Z})^*$, with respect to the above inner product. Because of this we have, for $f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ and $j \in (\mathbb{Z}/q\mathbb{Z})^*$,

$$f(j) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \langle f, \chi\rangle\, \chi(j) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\langle \chi, f\rangle}\, \chi(j). \tag{2.3}$$

Taking the inner product of each side of equation (2.3) with a function g on $(\mathbb{Z}/q\mathbb{Z})^*$ then gives

$$\langle f, g\rangle = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\langle \chi, f\rangle}\, \langle \chi, g\rangle.$$

In particular, for $\ell, m \in (\mathbb{Z}/q\mathbb{Z})^*$, we have

$$\langle \mathbb{1}_m, \mathbb{1}_\ell\rangle = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\langle \chi, \mathbb{1}_m\rangle}\, \langle \chi, \mathbb{1}_\ell\rangle$$

or, by the lemma,

$$\langle \mathbb{1}_m, \mathbb{1}_\ell\rangle = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(m)}\, \chi(\ell). \tag{Way2$_\chi$}$$

Comparing (Way1$_\chi$) with (Way2$_\chi$) gives us exactly (OR$_\chi$). □

Remark 2.4. The above **fact**, that the set (2.2) of Dirichlet characters mod q forms an orthonormal basis for the space of all functions on $(\mathbb{Z}/q\mathbb{Z})^*$, amounts to the **fact** that, given Dirichlet characters χ and ψ mod q, we have

$$\frac{1}{\varphi(q)} \sum_{\ell \in (\mathbb{Z}/q\mathbb{Z})^*} \chi(\ell)\overline{\psi(\ell)} = \delta_{\chi,\psi} := \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

This latter relation (which is a sum over integers ℓ mod q, for fixed characters χ and ψ) may be considered "dual" to (OR$_\chi$) (which is a sum over characters χ mod q, for fixed integers ℓ and m).

Summary:

- In studying primes in arithmetic progressions, Dirichlet L series are good things to consider.
- In studying Dirichlet L series, orthogonality of Dirichlet characters is a good thing to consider.
- The key orthogonality relation (OR$_\chi$) may be deduced by expressing a certain inner product in two different ways.

Remark 2.5. Computing a certain inner product in two different ways and equating results, as we did above, amounts – in appropriate circumstances – to invoking a so-called *trace formula*.

We now turn to a perhaps less familiar, but still in some sense "classical," context for orthogonality.

2.2 Part II: Maass forms on $\mathfrak{h}^n$

2.2.1 The case $n = 2$

Consider the Poincaré upper half-plane

$$\mathfrak{h}^2 = \{z = x + iy : x \in \mathbb{R},\, y > 0\}.$$

Note that the group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 invertible, real matrices acts on $\mathfrak{h}^2$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),\ z \in \mathfrak{h}^2\right).$$

Define

$$\Gamma^2 = \mathrm{SL}(2,\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) : a,b,c,d \in \mathbb{Z}\right\}.$$

We have:

Definition 2.6. A *Maass cusp form on $\mathfrak{h}^2$, of Langlands parameter* $\alpha \in \mathbb{C}$, is a smooth function $\phi : \mathfrak{h}^2 \to \mathbb{C}$ such that:

1. $\phi(\gamma z) = \phi(z) \quad$ for all $\gamma \in \Gamma^2,\ z \in \mathfrak{h}^2$.
2. ϕ is an eigenfunction of the Laplacian differential operator $\Delta = -y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$ on $\mathfrak{h}^2$, with eigenvalue $\frac14 - \alpha^2$. That is, $\Delta\phi = \big(\frac14 - \alpha^2\big)\phi$.
3. $\phi \in L^2(\Gamma^2\backslash\mathfrak{h}^2)$, meaning
$$\int_{\Gamma^2\backslash\mathfrak{h}^2} |\phi(z)|^2 \,\frac{dxdy}{y^2} < \infty,$$
where $\Gamma^2\backslash\mathfrak{h}^2$ is a fundamental domain for the action of Γ^2 on $\mathfrak{h}^2$.

Remark 2.7. In Definition 2.6, one may replace $\Gamma^2 = \mathrm{SL}(2,\mathbb{Z})$ with other discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$. For our purposes, though, it will suffice to consider the case $\Gamma^2 = \mathrm{SL}(2,\mathbb{Z})$. Similarly, the constructs on $\mathfrak{h}^n$, cf. Section 2.2.2 below, have analogs where the group $\Gamma^n = \mathrm{SL}(n,\mathbb{Z})$ is replaced by other discrete groups of $\mathrm{SL}(n,\mathbb{R})$. But we will not consider these more general situations further in this paper.

We wish to define an L series $L(s,\phi)$ associated to ϕ. To do so, of course, we'll need to associate a sequence of $a_\phi(m)$'s to ϕ. This sequence will comprise the *Fourier coefficients of* ϕ, defined as follows.

Definition 2.8. (Fourier-Whittaker expansion and Fourier coefficients of a Maass cusp form on $\mathfrak{h}^2$) Let ϕ be a Maass cusp form on $\mathfrak{h}^2$, as above. We observe that:

(a) Because $\phi(\gamma z) = \phi(z)\ \forall\gamma \in \Gamma^2$ and $z \in \mathfrak{h}^2$, and because $\big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big) \in \Gamma^2$, and because $\big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)z = z+1$, we have $\phi(z+1) = \phi(z)\ \forall z \in \mathbb{Z}$, so ϕ is 1-periodic in x, and therefore has a Fourier series
$$\phi(z) = \sum_{m\in\mathbb{Z}} A_{\phi,m}(y)e^{2\pi imx}.$$
We call $A_{\phi,m}(y)$ the m^{th} *Fourier-Whittaker coefficient of* ϕ.

(b) Because ϕ is square-integrable on $\Gamma^2\backslash\mathfrak{h}^2$, it turns out that $A_{\phi,0}(y) = 0$.

(c) The fact that ϕ satisfies the differential equation $\Delta\phi = \big(\frac14 - \alpha^2\big)\phi$ implies that
$$A_{\phi,m}(y) = \int_0^1 \phi(z)\, e^{-2\pi imx}\, dx$$
must satisfy its own differential equation in y. FACT: the space of smooth solutions to the latter differential equation is spanned by two functions $W_\alpha^{(2)}(|m|y)$ and $M_\alpha^{(2)}(|m|y)$, where $W_\alpha^{(2)}$ and $M_\alpha^{(2)}$ are certain "Whittaker functions."

(d) The Whittaker function $W_\alpha^{(2)}(y)$ decays rapidly, while the Whittaker function $M_\alpha^{(2)}(y)$ grows rapidly, as $y \to \infty$. The square-integrability of ϕ then implies that $M_\alpha^{(2)}$ *cannot* appear in the Fourier expansion of ϕ. Therefore,

$$A_{\phi,m}(y) = a_\phi(m) W_\alpha^{(2)}(|m|y),$$

for some complex number $a_\phi(m)$, called the m^{th} *Fourier coefficient of* ϕ.

(e) Putting all of the above together, we find that

$$\phi(z) = \sum_{m \neq 0} a_\phi(m) W_\alpha^{(2)}(|m|y) e^{2\pi i m x}. \tag{2.9}$$

We call (2.9) the *Fourier-Whittaker expansion of* ϕ.

We are now ready to define our requisite L series $L(s,\phi)$ associated to ϕ: namely, we define

$$L(s,\phi) = \sum_{m=1}^{\infty} a_\phi(m) m^{-s}. \tag{2.10}$$

(In cases of interest, $a_\phi(m)$ and $a_\phi(-m)$ are related, so one need only consider positive integers m.) It is known that this series will converge for $\operatorname{Re} s$ sufficiently large.

We seek an orthogonality relation for the $a_\phi(m)$'s. Our goal, in analogy with the earlier relation

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(\ell)\overline{\chi(m)} = \delta_{\ell,m}, \tag{OR$_\chi$}$$

is to obtain a relation of the form

$$\frac{1}{p(T)} \sum_{j=1}^{\infty} a_{\phi_j}(\ell)\overline{a_{\phi_j}(m)} h_j(T) = \delta_{\ell,m} + o(1) \tag{OR$_\phi$}$$

as $T \to \infty$ (for fixed integers $\ell, m \in \mathbb{Z}^+$). We explain:

- $p(T)$ is an explicit "normalizing factor." In fact, $p(T)$ turns out to be essentially a power of T.

- Just as Dirichlet characters form a basis for functions on $(\mathbb{Z}/q\mathbb{Z})^*$, so does the space of square-integrable functions on $\Gamma^2\backslash\mathfrak{h}^2$ have a maximal orthonormal set $\{\phi_j : j \in \mathbb{Z}_{>0}\}$ of Maass cusp forms ϕ_j of Langlands parameter α_j. These are the ϕ_j's appearing in (OR$_\phi$). We arrange them in order of increasing Laplace eigenvalue $\lambda_j = \frac{1}{4} - \alpha_j^2$. (Recall that $\Delta\phi_j = \lambda_j\phi_j$ for each j. It is known that the λ_j's are real, positive – in fact, $\lambda_j \geq \frac{1}{4}$ – and unbounded above.)

- The sum on j of the quantities $a_{\phi_j}(\ell)\overline{a_{\phi_j}(m)}$ in (OR$_\phi$) does not actually converge, so we need to multiply each summand by a "cutoff function" $h_j(T)$. Think of $h_j(T)$ as being more or less equal to one for $\lambda_j \leq T$, and 0 otherwise. (It's not this, exactly, but it behaves similarly.) If we could actually take the limit as $T \to \infty$, then (OR$_\phi$) would yield a true orthogonality relation. But we can't, for convergence reasons, so we really have an "asymptotic orthogonality relation." Still, we can take T as large as we want.

- The "$+o(1)$" on the right side of (OR$_\phi$) indicates that the sum on the left differs from $\delta_{\ell m}$ by something that goes to zero as T gets large.

We proceed with the proof of (OR_ϕ). As in the case of (OR_χ), our proof will entail the expression of a certain inner product in two ways. This time, we use the inner product

$$\langle f, g\rangle = \int_{\Gamma^2\backslash\mathfrak{h}^2} f(z)\,\overline{g(z)}\,\frac{dxdy}{y^2}$$

on $L^2(\Gamma^2\backslash\mathfrak{h}^2)$, and instead of indicator functions $\mathbb{1}_\ell$ and $\mathbb{1}_m$, we use "Poincaré series," defined as follows.

Definition 2.11. Given a function g on $(0,\infty)$, define the "Whittaker transform" $\widehat{g}$ of g by

$$\widehat{g}(\alpha) = \int_0^\infty \overline{g(y)} W_\alpha^{(2)}(y)\frac{dy}{y^2}, \tag{2.12}$$

where $W_\alpha^{(2)}$ is the Whittaker function of Definition 2.8.

For $z = x + iy \in \mathfrak{h}^2$, let $\psi(z) = e^{2\pi i x}$, and let $g_T(y)$ be a function chosen specifically so that

$$\left|\widehat{g_T}(\alpha_j)\right|^2 = h_j(T)/p(T), \tag{2.13}$$

where $p(T)$ is the normalizing factor, and $h_j(T)$ the cutoff function, appearing in (OR_ϕ).

Let $\ell \in \mathbb{Z}^+$. Also let $\Gamma^2_\infty \subset \Gamma^2$ be the set of matrices of the form $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ $(k \in \mathbb{Z})$. Then the *Poincaré series associated with g_T* is the infinite sum

$$P_\ell(z) = \ell^{-1}\sum_{\gamma\in\Gamma^2_\infty\backslash\Gamma^2}\psi(\ell\gamma z)\,g_T(\ell\gamma z).$$

Remark 2.14. (a) In the above definition, we use the standard practice of summing a function on $\mathfrak{h}^2$ over translates by elements $\gamma \in \Gamma^2$, to get a function on $\Gamma^2\backslash\mathfrak{h}^2$ (that is, a function on $\mathfrak{h}^2$ invariant under $z \to \gamma z$, for $\gamma \in \Gamma^2$). Actually, summing over all of Γ^2 yields a series that diverges (because of infinite redundancy in summands), so we instead sum over $\Gamma^2_\infty\backslash\Gamma^2$. It's readily checked that, by definition of ψ and g_T, this sum is independent of the choice of coset representatives for Γ^2_∞ in Γ^2, so that P_ℓ is well-defined.

(b) Of course, P_ℓ depends on g_T – and therefore on T – as well, but it will be convenient to suppress this dependence from the notation.

Next, we evaluate $\langle P_m, P_\ell\rangle$ in two different ways. The first way will require the following deep fact.

Theorem 2.15. (The Spectral Decomposition of $L^2(\Gamma^2\backslash\mathfrak{h}^2)$) *For* $\mathrm{Re}\, s > 1$, *define the* $\mathrm{SL}(2,\mathbb{Z})$ Eisenstein series $E_s\colon \mathfrak{h}^2 \to \mathbb{C}$ *by*

$$E_s(z) = \sum_{\gamma\in\Gamma^2_\infty\backslash\Gamma^2}(\mathrm{Im}\,\gamma z)^s.$$

(Note that the construction of E_s is similar to that of P_ℓ, cf. Definition 2.11 above.) Then, as a function of s, E_s has meromorphic continuation to $\mathbb{C}$, for any $z \in \mathfrak{h}^2$. Further, there exists a maximal orthonormal set $\{\phi_j\}_{j=1,2,\ldots}$ of Maass cusp forms on $\mathfrak{h}^2$ such that, for any $f \in L^2(\Gamma^2\backslash\mathfrak{h}^2)$, we have

$$f(z) = \sum_{j=1}^{\infty}\langle f, \phi_j\rangle\,\phi_j(z) + \int_{\mathrm{Re}\, s=1/2}\langle f, E_s\rangle\,E_s(z)\,ds.$$

For proofs of Theorem 2.15, see, for example, [Art79], [Lan76], and [MW95].

We apply the above theorem to P_m, to get

$$P_m(z) = \sum_{j=1}^{\infty} \langle P_m, \phi_j \rangle \, \phi_j(z) + \int_{\operatorname{Re} s = 1/2} \langle P_m, E_s \rangle \, E_s(z) \, ds.$$

Taking the inner product of either side with P_ℓ then gives us

$$\langle P_m, P_\ell \rangle = \sum_{j=1}^{\infty} \langle P_m, \phi_j \rangle \, \langle \phi_j, P_\ell \rangle + \int_{\operatorname{Re} s = 1/2} \langle P_m, E_s \rangle \, \langle E_s, P_\ell \rangle \, ds.$$

Now the integral on the right hand side can be estimated quite explicitly, and it turns out that this integral tends to zero as $T \to \infty$. So we get

$$\langle P_m, P_\ell \rangle = \sum_{j=1}^{\infty} \langle P_m, \phi_j \rangle \, \langle \phi_j, P_\ell \rangle + o(1) \quad \text{as } T \to \infty. \tag{2.16}$$

We need to examine the sum on j, in (2.16). Regarding the second inner product in this sum, we have, by standard techniques,

$$\begin{aligned}
\langle \phi_j, P_\ell \rangle &= \int_{\Gamma^2 \backslash \mathfrak{h}^2} \phi_j(z) \overline{P_\ell(z)} \, \frac{dxdy}{y^2} = \int_{\Gamma^2 \backslash \mathfrak{h}^2} \phi_j(z) \, \ell^{-1} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \overline{\psi(\ell \gamma z) \, g_T(\ell \gamma z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[now interchange the integral with the sum]} \\
&= \ell^{-1} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \int_{\Gamma^2 \backslash \mathfrak{h}^2} \phi_j(z) \overline{\psi(\ell \gamma z) \, g_T(\ell \gamma z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[now substitute } z \to \gamma^{-1} z;\ \phi_j \text{ and } dxdy/y^2 \text{ are invariant under this]} \\
&= \ell^{-1} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \int_{\gamma(\Gamma^2 \backslash \mathfrak{h}^2)} \phi_j(z) \overline{\psi(\ell z) \, g_T(\ell z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[the sum of the integrals is the integral of the union]} \\
&= \ell^{-1} \int_{\Gamma^2_\infty \backslash \mathfrak{h}^2} \phi_j(z) \overline{\psi(\ell z) \, g_T(\ell z)} \, \frac{dxdy}{y^2}.
\end{aligned}$$

Now a fundamental domain for the action of Γ^2_∞ on $\mathfrak{h}^2$ is just $\{x + iy : x \in (0,1],\ y \in (0, \infty)\}$. So, recalling that $\psi(z) = e^{2\pi i x}$, the above gives

$$\langle \phi_j, P_\ell \rangle = \ell^{-1} \int_0^\infty \int_0^1 \phi_j(z) \, e^{-2\pi i \ell x} \, \overline{g_T(\ell y)} \, \frac{dxdy}{y^2} = \ell^{-1} \int_0^\infty \overline{g_T(\ell y)} \left[\int_0^1 \phi_j(z) \, e^{-2\pi i \ell x} \, dx \right] \frac{dy}{y^2}.$$

The quantity in square-brackets is just the ℓ^{th} Fourier-Whittaker coefficient $A_{\phi_j,\ell}(y)$ of ϕ_j, cf. Definition 2.8(a). As we've seen before (cf. Definition 2.8(d)),

$$A_{\phi_j,\ell}(y) = a_{\phi_j}(\ell) W^{(2)}_{\alpha_j}(\ell y),$$

where, again, $a_{\phi_j}(\ell)$ is the ℓ^{th} Fourier coefficient of ϕ_j, and $W^{(2)}_{\alpha_j}(\ell y)$ is a Whittaker function. So

$$\langle \phi_j, P_\ell \rangle = \ell^{-1} a_{\phi_j}(\ell) \int_0^\infty \overline{g_T(\ell y)} W^{(2)}_{\alpha_j}(\ell y) \, \frac{dy}{y^2},$$

or, substituting $y \to y/\ell$,

$$\langle \phi_j, P_\ell \rangle = a_{\phi_j}(\ell) \int_0^\infty \overline{g_T(y)} W^{(2)}_{\alpha_j}(y) \, \frac{dy}{y^2} = a_{\phi_j}(\ell) \, \widehat{g_T}(\alpha_j),$$

the last step by the definition (2.12) of the Whittaker transform $\widehat{g}$.

We then have

$$\langle P_m, \phi_j \rangle = \overline{\langle \phi_j, P_m \rangle} = \overline{a_{\phi_j}(m) \widehat{g_T}(\alpha_j)},$$

so that, by (2.16),

$$\begin{aligned}
\langle P_m, P_\ell \rangle &= \sum_{j=1}^{\infty} \langle P_m, \phi_j \rangle \, \langle \phi_j, P_\ell \rangle + o(1) \\
&= \sum_{j=1}^{\infty} a_{\phi_j}(\ell) \overline{a_{\phi_j}(m)} \, \left| \widehat{g_T}(\alpha_j) \right|^2 + o(1) \quad \text{as } T \to \infty,
\end{aligned}$$

or, finally, since we have chosen $\widehat{g_T}(\alpha_j)$ so that $\left| \widehat{g_T}(\alpha_j) \right|^2 = h_j(T)/p(T)$,

$$\langle P_m, P_\ell \rangle = \frac{1}{p(T)} \sum_{j=1}^{\infty} a_{\phi_j}(\ell) \overline{a_{\phi_j}(m)} h_j(T) + o(1) \quad \text{as } T \to \infty. \tag{Way1$_\phi$}$$

The formula (Way1$_\phi$) constitutes our first way of expressing $\langle P_m, P_\ell \rangle$.

We now derive an *a priori* different expression, which we shall denote by (Way2$_\phi$), for $\langle P_m, P_\ell \rangle$. This time, we evaluate this inner product not by way of a spectral decomposition, but instead by simply expanding out one of the Poincaré series, and approximating the other. Like this:

$$\begin{aligned}
\langle P_m, P_\ell \rangle &= \int_{\Gamma^2 \backslash \mathfrak{h}^2} P_m(z) \, \overline{P_\ell(z)} \, \frac{dxdy}{y^2} = \ell^{-1} \int_{\Gamma^2 \backslash \mathfrak{h}^2} P_m(z) \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \overline{\psi(\ell \gamma z) \, g_T(\ell \gamma z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[now interchange the integral with the sum]} \\
&= \ell^{-1} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \int_{\Gamma^2 \backslash \mathfrak{h}^2} P_m(z) \overline{\psi(\ell \gamma z) \, g_T(\ell \gamma z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[now substitute } z \to \gamma^{-1} z;\ P_m \text{ and } dxdy/y^2 \text{ are invariant under this]} \\
&= \ell^{-1} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2} \int_{\gamma(\Gamma^2 \backslash \mathfrak{h}^2)} P_m(z) \overline{\psi(\ell z) \, g_T(\ell z)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[the sum of the integrals is the integral of the union]} \\
&= \ell^{-1} \int_{\Gamma^2_\infty \backslash \mathfrak{h}^2} P_m(z) e^{-2\pi i \ell x} \, \overline{g_T(\ell y)} \, \frac{dxdy}{y^2} \\
&\qquad \text{[use the same fundamental domain for } \Gamma^2_\infty \backslash \mathfrak{h}^2 \text{ as before]} \\
&= \ell^{-1} \int_0^\infty \overline{g_T(\ell y)} \left[\int_0^1 P_m(z) e^{-2\pi i \ell x} dx \right] \frac{dy}{y^2}.
\end{aligned}$$

Now P_m is, again, a sum over $\gamma = \Gamma^2_\infty \backslash \Gamma^2_\infty$ of terms of the form $m^{-1} \psi(m\gamma z) \, g_T(m\gamma z)$. But it turns out that, in the present context, the $\gamma =$ Identity term of P_m is the *dominant* term; all of the other terms contribute $o(1)$ as $T \to \infty$. So the above calculations yield

$$\langle P_m, P_\ell \rangle = (\ell m)^{-1} \int_0^\infty \overline{g_T(\ell y)} \left[\int_0^1 \psi(mz)\, g_T(mz)\, e^{-2\pi i \ell x} dx \right] \frac{dy}{y^2} + o(1)$$
$$= (\ell m)^{-1} \int_0^\infty \overline{g_T(\ell y)} g_T(my) \left[\int_0^1 e^{2\pi i m x}\, e^{-2\pi i \ell x} dx \right] \frac{dy}{y^2} + o(1)$$
[invoke orthogonality of complex exponentials]
$$= \delta_{\ell m} \cdot (\ell m)^{-1} \int_0^\infty \overline{g_T(\ell y)} g_T(my)\, \frac{dy}{y^2} + o(1) \quad \text{as } T \to \infty.$$

For carefully chosen g_T, we can show that

$$(\ell m)^{-1} \int_0^\infty \overline{g_T(\ell y)} g_T(my)\, \frac{dy}{y^2} = 1 + o(1) \quad \text{as } T \to \infty, \tag{2.17}$$

so we conclude that

$$\langle P_m, P_\ell \rangle = \delta_{\ell m} + o(1) \quad \text{as } T \to \infty. \tag{Way2$_\phi$}$$

The left hand sides of (Way1$_\phi$) and (Way2$_\phi$) are equal, whence so are the right hand sides. That is,

$$\frac{1}{p(T)} \sum_{j=1}^{\infty} a_{\phi_j}(\ell) \overline{a_{\phi_j}(m)} h_j(T) = \delta_{\ell,m} + o(1),$$

which is our desired (asymptotic) orthogonality relation (OR$_\phi$). □

Remark 2.18. (a) Strictly speaking the Poincaré series P_ℓ, as we've defined it, does not converge. However, this obstruction is easily treated as follows: we multiply each summand in the series defining P_ℓ by an appropriate factor $I_\nu(\gamma z)$ such that the new series converges for ν sufficiently large, and such that $I_0(\gamma z) = 1$. We take all inner products and do all calculations assuming, initially, that ν is large, and then let $\nu \to 0$ afterwards.

(b) Regarding the functions g_T, $\widehat{g_T}$, and $h_j(T)$ figuring in the orthogonality relation (OR$_\phi$) and its proof, we make a few observations. First: as stipulated in (2.13), and as utilized in the above proof, we have the requirement

$$\left| \widehat{g_T}(\alpha_j) \right|^2 = h_j(T)/p(T).$$

Second: our above proof also requires the asymptotic condition (2.17) on g_T.

Both of these requirements may be met by taking $\widehat{g_T}(\alpha)$ to be a certain function of exponential decay in α^2, times a certain polynomial in α, times some Gamma functions depending on α. (Defining $\widehat{g_T}$ also defines g_T, through an inversion formula for the Whittaker transform $g \mapsto \widehat{g}$.)

Proving that such a function $\widehat{g_T}$ has the desired properties, and that the requisite approximations described in the above proof follow, both require explicit information concerning the Whittaker function $W_\alpha^{(2)}$. Here, one especially needs formulas for the *Mellin transform*

$$T_\alpha^{(2)}(s) = \int_0^\infty W_\alpha^{(2)}(y)\, y^s\, \frac{dy}{y^2}$$

of $W_\alpha^{(2)}$.

It's well-known that, for appropriately normalized $W_\alpha^{(2)}$, we have

$$T_\alpha^{(2)}(s) = \Gamma\Big(\frac{s+\alpha}{2}\Big)\Big(\frac{s-\alpha}{2}\Big). \tag{2.19}$$

Properties of the gamma function (growth estimates, analytic continuation, functional equation, etc.) are well-understood, and this makes the necessary estimates and calculations possible.

(c) The role played by the Poincaré series P_ℓ in the proof of (OR_ϕ) is, in many ways, analogous to the role played by the indicator function $\mathbb{1}_\ell$ in the proof of (OR_χ).

(d) The first results along the lines of (OR_ϕ) are due to R. Bruggeman [Bru78]. Related results have been obtained by P. Sarnak [Sar87]; B. Conrey, W. Duke, and D. Farmer [CDF97]; and J.P. Serre [Ser97].

2.2.2 The general case

The above ideas concerning L series and Maass cusp forms on $\mathfrak{h}^2$ generalize to the context of the "generalized upper half-plane" $\mathfrak{h}^n$. The starting point for this generalization is the observation that the standard upper half-plane $\mathfrak{h}^2$ can be realized as the quotient space

$$\mathfrak{h}^2 \cong \mathrm{GL}(2,\mathbb{R})/(\mathrm{O}(2,\mathbb{R}) \times \mathbb{R}^*).$$

It then makes sense to define $\mathfrak{h}^n$ by

$$\mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R}) \times \mathbb{R}^*).$$

All of the above constructs on $\mathfrak{h}^2$ then have analogs on $\mathfrak{h}^n$.

For $n > 2$, various orthogonality relations of the form (OR_ϕ) are known. In the case $n = 3$, such relations were first obtained by D. Goldfeld and A. Kontorovich [GK12], and independently by V. Blomer [Blo13]. Related results for $\mathfrak{h}^3$ have been obtained by V. Blomer, J. Buttcane, and N. Raulf [BBR14]; and by J. Guerreiro [Gue15].

Recently, in work with D. Goldfeld and M. Woodbury (see [GSW21] and [GSW22]), we have obtained results of the form (OR_ϕ) on $\mathfrak{h}^n$, for all $n \geq 2$. Our results are unconditional in the case $n \leq 5$; for larger n, our proof relies on two conjectures, to be described below (See Conjectures 2.23 and 2.26).

Our proof makes fundamental use of certain *Poincaré series on* $\mathfrak{h}^n$, which are defined analogously to the above Poincaré series P_ℓ on $\mathfrak{h}^2$ (cf. Definition 2.11). Further, our proof on $\mathfrak{h}^n$, like the above (sketch of a) proof on $\mathfrak{h}^2$, entails the evaluation of an inner product of two such Poincaré series in two different ways—this method amounts to application of the *Kuznetsov trace formula* (see [Kuz81], [CPS90], and [Gol06, Section 11.6]).

To describe our results, we need to say a few words about harmonic analysis on $\mathfrak{h}^n$. (See, for example, [Gol06] for details and proofs concerning the following discussion.)

The orthonormal set $\{\phi_j\}_{j=1,2,\ldots}$ of Maass cusp forms appearing in Theorem 2.15 has an analog in the spectral decomposition of $L^2(\Gamma^n\backslash\mathfrak{h}^n)$, where

$$\Gamma^n = \mathrm{SL}(n,\mathbb{Z}).$$

As in the case $n = 2$, the ϕ_j's on $\mathfrak{h}^n$ are eigenfunctions of the Laplacian differential operator there. We assume that these ϕ_j's are arranged in order of increasing eigenvalue under this Laplacian.

In the general case, the ϕ_j's are, in fact, eigenfunctions of *all* $\mathrm{GL}(n,\mathbb{R})$ invariant differential operators on $\mathfrak{h}^n$. The eigenvalues of ϕ_j under these operators may be indexed by a *Langlands parameter* $\alpha^{(j)}$, which is an n-tuple

$$\alpha^{(j)} = \left(\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_n^{(j)}\right) \in \mathbb{C}^n$$

satisfying

$$\alpha_1^{(j)} + \alpha_2^{(j)} + \ldots + \alpha_n^{(j)} = 0.$$

Remark 2.20. Earlier (cf. Definition 2.6), we defined the Langlands parameter of a Maass cusp form ϕ on $\mathfrak{h}^2$ to be a *complex number* α such that ϕ has eigenvalue $\frac{1}{4} - \alpha^2$ under the Laplacian, rather than a *complex pair* (α_1, α_2). This is really an abuse of notation: to be more consistent with the language of the previous paragraph, we should really say that the Langlands parameter of such a ϕ, in the case $n = 2$, is the complex pair $(\alpha, -\alpha)$. (Of course, $\frac{1}{4} - \alpha^2 = \frac{1}{4} - (-\alpha)^2$.)

(In the case of general n, as in the case $n = 2$, the eigenvalues of a Maass cusp form ϕ under the invariant differential operators on $\mathfrak{h}^n$ are expressible in terms of the Langlands parameter α of ϕ, but we will not need these precise expressions here.)

We say that a Maass cusp form ϕ on $\mathfrak{h}^n$ is *tempered* if all coordinates α_i of the Langlands parameter for ϕ are purely imaginary. It is known that all Maass cusp forms on $\mathfrak{h}^n$ are tempered, and it is conjectured that the analogous result holds for any n.

We note, finally, that any Maass cusp form ϕ on $\mathfrak{h}^n$ has a *Fourier-Whittaker expansion* involving *Fourier coefficients* $a_\phi(M)$ and a *Whittaker function* $W_\alpha^{(n)}$. The Fourier coefficients $a_\phi(M)$ are indexed by integer $(n-1)$-tuples $M = (m_1, m_2, \ldots, m_{n-1})$, and are analogous to the Fourier coefficients $a_\phi(m)$ of Maass cusp forms on $\mathfrak{h}^2$, cf. Definition 2.8(d). Similarly, the Whittaker function $W_\alpha^{(n)}$ is analogous to the function $W_\alpha^{(2)}$ of Definition 2.8(c)(d).

We may now state the following result (cf. [GSW21] for the case $n = 4$, and [GSW22] for the general case).

Theorem 2.21. (Asymptotic orthogonality on $\mathfrak{h}^n$) *Fix $n \geq 2$. Let $\{\phi_j\}_{j=1,2,\ldots}$ denote the maximal orthonormal set of Maass cusp forms on $\mathfrak{h}^n$ arising in the spectral decomposition of $L^2(\Gamma^n\backslash\mathfrak{h}^n)$. Denote the Langlands parameter of ϕ_j by*

$$\alpha^{(j)} = \left(\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_n^{(j)}\right).$$

Assume that each ϕ_j is tempered.

For a positive integer m, denote the Fourier coefficient $a_{\phi_j}((m,1,1,\ldots 1))$ by $\lambda_j(m)$.

Let $\ell, m \in \mathbb{Z}_{>0}$, and assume $T, R \in \mathbb{R}_{>0}$ are sufficiently large. Assume that Conjectures 2.23 and 2.26 below are true (which is known to be the case for $n \leq 5$). We then have, for an appropriate cutoff function $h_{T,R}^{(n)}$, and for any $\varepsilon > 0$,

$$\frac{1}{\mathfrak{c}_1 T^{R\cdot\left(\binom{2n}{n}-2^n\right)+n-1}} \sum_{j=1}^{\infty} \lambda_j(\ell)\, \overline{\lambda_j(m)}\, h_{T,R}^{(n)}\left(\alpha^{(j)}\right)$$
$$= \delta_{\ell,m}\left(1 + \sum_{i=2}^{n-1} \mathfrak{c}_i \cdot T^{1-i}\right) + \mathcal{O}_{\varepsilon,R,n}\left((\ell m)^{\frac{n^2+13}{4}} \cdot T^{1-n+\varepsilon}\right)$$

as $T \to \infty$. Here $\delta_{\ell,m}$ is the Kronecker symbol, and $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} > 0$ are constants that depend at most on R and n.

Remark 2.22. The positive number R figuring in the above theorem is required in our proof for technical reasons that make certain estimates possible.

To describe the conjectures upon which Theorem 2.21 rests, we need to discuss a few additional notions concerning analysis on $\mathfrak{h}^n$.

We begin with the background necessary to describe the first conjecture. This background regards the so-called *Langlands Eisenstein series* for $\mathrm{SL}(n, \mathbb{Z})$. (See [Gol06, Chapter 10] for details and proofs concerning these series.)

The Eisenstein series $E_s(z)$ appearing in the spectral decomposition of $L^2(\Gamma^2 \backslash \mathfrak{h}^2)$ (cf. Theorem 2.15) must, in the analogous theory for $\mathfrak{h}^n$, be replaced by a (generally infinite) set of functions called Langlands Eisenstein series, denoted $E_{\mathcal{P},\Phi}(z, s)$. Here $z \in \mathfrak{h}^n$, s is a vector of complex numbers, $\mathcal{P}$ is a *parabolic subgroup* of $\mathrm{GL}(n, \mathbb{R})$, and Φ is a *Maass cusp form associated to* $\mathcal{P}$.

More specifically: there is a parabolic subgroup $\mathcal{P}_{n_1,n_2,\ldots,n_r}$ associated to each partition $n = n_1 + n_2 + \cdots + n_r$ of n. Then a Maass cusp form Φ associated to $\mathcal{P}_{n_1,n_2,\ldots,n_r}$ amounts essentially to an r-tuple $(\phi_1, \phi_2, \ldots, \phi_r)$, where ϕ_j is the constant function 1 whenever $n_j = 1$, and ϕ_j Maass cusp form on $\mathfrak{h}^{n_j}$ whenever $n_j > 1$.

Let $\mathcal{P} = \mathcal{P}_{n_1,n_2,\ldots,n_r}$, let Φ be a Maass cusp form associated to $\mathcal{P}$, and let

$$s = (s_1, s_2, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}.$$

The Langlands Eisenstein series $E_{\mathcal{P},\Phi}(z, s)$ is defined as a sum of translates, over $(\mathcal{P} \cap \Gamma^n) \backslash \Gamma^n$, of certain "power functions" in z and s times a certain product of the ϕ_j's. Such a sum actually converges only for the real parts of the s_i's sufficiently large, but $E_{\mathcal{P},\Phi}(z, s)$ is known to have meromorphic continuation to $s \in \mathbb{C}^{r-1}$.

Each Langlands Eisenstein series $E_{\mathcal{P},\Phi}(z, s)$ has a so-called Fourier-Whittaker expansion, analogous to (but considerably more complicated than) that of Definition 2.8. The typical Fourier coefficient in such an expansion will entail *Rankin-Selberg L functions* $L(s, \phi_i \times \phi_j)$, for all $1 \le i < j \le r$ such that $n_i, n_j > 1$. (See [GSW23].) Because our proof of Theorem 2.21 entails the spectral decomposition on $L^2(\Gamma^n \backslash \mathfrak{h}^n)$ (as was the case for $n = 2$, cf. Section 2.2.1 above), and because this spectral expansion involves the Langlands Eisenstein series $E_{\mathcal{P},\Phi}(z, s)$, our proof of Theorem 2.21 requires certain estimates of $L(s, \phi_i \times \phi_j)$.

We may now state the first conjecture required to validate Theorem 2.21 for arbitrary $n \ge 2$.

Conjecture 2.23. (Lower bounds for Rankin-Selberg L-functions) *For a Maass cusp form ϕ on $\mathfrak{h}^n$ with Langlands parameter $\alpha = (\alpha_1, \ldots, \alpha_n)$, let*

$$c(\phi) = (1 + |\alpha_1|)(1 + |\alpha_2|) \cdots (1 + |\alpha_n|)$$

denote the analytic conductor of ϕ, as defined by Iwaniec and Sarnak [IS00]. Let $\varepsilon > 0$ be fixed, and let $n_i, n_j > 1$. Then, for ϕ_i and ϕ_j Maass cusp forms on $\mathrm{SL}(n_i, \mathbb{Z})$ *and* $\mathrm{SL}(n_j, \mathbb{Z})$ *respectively, we have the lower bound*

$$|L(1 + it, \phi_i \times \phi_j)| \gg_\varepsilon \big(c(\phi_i) \cdot c(\phi_j)\big)^{-\varepsilon} \big(|t| + 2\big)^{-\varepsilon}.$$

Remark 2.24. Langlands' conjecture that $\phi_i \times \phi_j$ is automorphic for $\mathrm{SL}(n_i n_j,\ \mathbb{Z})$ (for $n_i, n_j > 1$) implies Conjecture 2.23. This implication can be proved via the method of de la Valée Poussin, cf. [Sar04].

In the case $n_i = n_j = 2$, it was proved by Ramakrishnan [Ram00] that $\phi_i \times \phi_j$ is automorphic for $\mathrm{SL}(4, \mathbb{Z})$, thus proving Conjecture 2.23 for $n \le 4$. Further, for $n_i = 2$ and $n_j = 3$, it was proved by Kim and Shahidi [KS02] that $\phi_i \times \phi_j$ is automorphic for $\mathrm{SL}(6, \mathbb{Z})$, thus proving Conjecture 2.23 for $n \le 5$.

The second conjecture required for our proof of Theorem 2.21 concerns the Whittaker function $W_\alpha^{(n)}$ that arises in the Fourier-Whittaker expansion of a Maass cusp form on $\mathfrak{h}^n$ (as discussed just before the statement of that theorem). This Whittaker function is a function of $n-1$ positive real variables $y_1, y_2, \ldots, y_{n-1}$, and has a (multiple) Mellin transform $T_\alpha^{(n)}$, which is a function of a variable $s = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$.

Remark 2.25. We've seen, in Remark 2.18(b) above, that $T_\alpha^{(2)}$ may be written as a product of two Gamma functions (cf. equation (2.19)). It was shown by D. Bump [Bum84] that $T_\alpha^{(3)}$ is expressible as a product of six Gamma functions, divided by a single Gamma function. See equation (2.28) below. For $n \ge 4$, though, there is no formula for $T_\alpha^{(n)}$ as a simple ratio of Gamma functions; rather, this Mellin transform may be written as a multiple *integral* involving Gamma functions. See [Sta01] and [IS07].

At any rate, for general n, as in the case $n = 2$ (see, again, Remark 2.18(b)), the Mellin transform $T_\alpha^{(n)}$ plays a crucial role in the estimation of various quantities that arise in our proof.

To perform such estimation, we need to assume the following conjecture.

Conjecture 2.26. *Let $m, n \in \mathbb{Z}$ with $1 \le m \le n-1$; let $\delta \in \mathbb{Z}_{\ge 0}$. Let*

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1).$$

Then there exists a positive integer r and, for each i with $1 \le i \le r$, a polynomial $P_i(s, \alpha)$ and an $(n-1)$-tuple $\Sigma_i \in (\mathbb{Z}_{\ge 0})^{n-1}$, such that

$$T_\alpha^{(n)}(s) = \left[\prod_{1 \le j_1 < j_2 < \ldots < j_m \le n} (s_m + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m})_\delta \right]^{-1} \sum_{i=1}^{r} P_i(s, \alpha) T_\alpha^{(n)}(s + \Sigma_i), \tag{2.27}$$

where the m^{th} coordinate of each Σ_i is at least δ. Moreover, for each i, we have

$$\deg(P_i(s, \alpha)) + 2\left|\Sigma_i\right| = \delta \binom{n}{m}.$$

We note that the case $\delta = 0$ of Conjecture 2.26 is trivial. Moreover, for a given m and n with $1 \le m \le n-1$, it's enough to prove the conjecture for $\delta = 1$. The case $\delta > 1$ then follows by applying the case $\delta = 1$ to itself iteratively.

For $\delta = 1$ and $n = 2$, the conjecture follows immediately from equation (2.19) above, together with the functional equation $\Gamma(s+1) = s\Gamma(s)$. . For $\delta = 1$ and $n = 3$, it follows from Bump's formula (see [Bum84])

$$T_\alpha^{(3)}(s) = \frac{\Gamma\left(\frac{s_1+\alpha_1}{2}\right)\Gamma\left(\frac{s_1+\alpha_2}{2}\right)\Gamma\left(\frac{s_1+\alpha_3}{2}\right)\Gamma\left(\frac{s_2-\alpha_1}{2}\right)\Gamma\left(\frac{s_2-\alpha_2}{2}\right)\Gamma\left(\frac{s_2-\alpha_3}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)} \tag{2.28}$$

(for appropriately normalized $W_\alpha^{(n)}$) and the same functional equation. The case $\delta = 1$ and $n = 4$ of Conjecture 2.26 is a consequence of [ST21, equations (21), (29), and (31)].

The case $\delta = 1$ and $n = 5$ has been proved by Taku Ishii (personal correspondence). Ishii's approach entails explicit formulae for $T_\alpha^{(5)}$ (cf. [IS07]), and for the $\mathrm{GL}(n,\mathbb{R})$ invariant differential operators on $\mathfrak{h}^n$ (cf. [IO14]). The essential idea here is that, by definition of the Mellin transform, differential operators on $\mathfrak{h}^5$, acting $W_\alpha^{(5)}$, become *shift* operators in their action on $T_\alpha^{(5)}$. Applying these operators judiciously yields the "shift equation" (2.27), in the case $n = 5$ (and $\delta = 1$). (See [GSW22] for some details on Ishii's approach, which can also be applied when $n \le 4$.)

As a final remark, we note that similar asymptotic orthogonality relations on $\mathrm{GL}(n,\mathbb{R})$ (and on other reductive groups) have been obtained by several other authors, e.g. J. Matz and N Templier [MT15]; P. Sarnak, S. W. Shin, and N. Templier [SST16], T. Finis and J. Matz [FM19], and S. Jana [Jan20], using quite different methods. A unique (to our knowledge) feature of our approach, though, is the presence of the "higher order asymptotics"

$$\mathfrak{c}_i \cdot T^{1-n} \qquad (2 \le i \le n-1).$$

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