On equivariant holomorphic differential operators starting from vector-valued cases

by
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Abstract

The theory of Rankin-Cohen bilinear holomorphic differential operators is well explored for scalar-valued cases, mainly by the work of Ibukiyama. Not so much is known when we start from vector-valued automorphy factors. We will describe some constructions starting from nonholomorphic operators of Maaß-Shimura type. We focus on operators of order one, but by some compatibility with tensor products we can cover more general situations. For the case of symmetric tensor representations we can however give quite complete results by a direct approach. Some parts of the presentation are based on the Mannheim PhD-thesis 2021 by Julia Meister.

1 Introduction

Rankin-Cohen operators are a usefull tool in the theory of holomorphic modular forms on hermitian symmetric domains. Also, by their combinatorial and representation theoretic properties, they are of independent interest. We refer to [5] and many subsequent papers by Ibukiyama in this regard. So far, the focus was always on cases, where one starts from scalar valued functions. In the present note however, we explore some cases, where the starting point are vector-valued functions, equipped with stroke operators attached to higher dimensional representations.
To explain our method, we recall the classical Rankin-Cohen operators for SL_2, changing weights from k and l to $k+l+2$:

$$[f,g] = l \cdot f' \cdot g - k \cdot f \cdot g' \tag{1}$$

for arbitrary (holomorphic) functions f and g on the classical upper half plane. The equivariance property of this bilinear operator is best explained by the identity

$$[f,g] = l \cdot \delta_k(f) \cdot g - k \cdot f \cdot \delta_l(g) \tag{2}$$

where $\delta_k = \frac{k}{2iy} + \frac{\partial}{\partial z}$ is a nonholomorphic operator changing weights from k to $k+2$.
In the three main sections 2-4 of this paper we shall generalize the identities (1), (2) from above from different viewpoints.

In section 1 we give the preliminaries, in particular, we explain the framework of equivariant differential operators following Shimura [7]. In section 2 we show that (1), (2) can be generalized to give Rankin-Cohen brackets which change arbitrary automorphy factors ρ and ρ' to (irreducible sub-) representations of $\rho \otimes \rho' \otimes Sym^2$. This result is obtained using some multiplicity one property and does not provide an explicit formula.
Section 3 then shows that in the framework of symmetric tensor representations, we can get a complete and explicit construction of R-C-brackets changing automorphy factors $\rho = Sym^l$ and $\rho' = Sym^m$ to Sym^{l+m+p} for any m, l, p.
In section 4 we presents the main result of [6], namely an explicit construction of an R-C-bracket changing the automorphy factors $\rho = Sym^1 \otimes \det^k$ and $\rho' = \det^l$ to $\rho \otimes Sym^2 \otimes \det^{k+l}$. This result (although predicted by the abstract considerations of section 2) is obtained independently of section 2 by explicit compution of the R-C bracket in question. A nice feature here is that this explicit construction is compatible with taking arbitrary tensor products of the standard representation $Stand = Sym^1$ with itself. Taking into account that (by H.Weyl) any polynomial irreducible representation occurs in some $Stand^\nu$, we therefore get a construction of R-C-brackets starting from arbitrary (irreducible) ρ and $\rho' = \det^l$; this construction is then explicit provided that the embedding of ρ into some $Stand^{\otimes\nu}$ is made explict.

2 Preliminaries: Maaß-Shimura-differential operators and R-C-brackets.

2.1 Basic notations

As usual, we denote by $\mathbb{H}_n = \{Z = X + iY \in \mathbb{C}^{(n,n)} \mid Z = Z^t, Y > 0\}$ the Siegel upper half space, and by $(g, Z) \mapsto g < Z > := (AZ + B)(CZ + D)^{-1}$ the action of the symplectic group $Sp(n, \mathbb{R})$ on $\mathbb{H}_n$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$. For a polynomial representation $\rho : GL(n, \mathbb{C}) \longrightarrow$

$GL(X)$ we put

$$C^\infty_\rho(\mathbb{H}_n, X) := \{f : \mathbb{H}_n \longrightarrow X \mid f \text{ is } C^\infty\}$$

and by $Hol_\rho(\mathbb{H}_n, X)$ we denote the subspace of X-valued holomorphic functions. On both spaces there is an action of $Sp(n,\mathbb{R})$, defined by a stroke operator

$$(f \mid_\rho g)(Z) := \rho(CZ+D)^{-1}(f(g < Z >)$$

2.2 Differential operators (following Shimura)

Let ρ be a representation of $GL(n,\mathbb{C})$ and $T := Sym_n(\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} | X = X^t\}$. For finite-dimensional complex vector spaces X, Y, we denote by $S_1(Y,X) = Hom_{\mathbb{C}}(Y,X)$ the vector space of all $\mathbb{C}$-linear maps of Y into X. We now define a representation (see [7, (12.7a) für p=1])

$$\rho \otimes \tau : GL(n,\mathbb{C}) \to GL(S_1(T,X))$$

by

$$(\rho\otimes\tau)(g)(h)(u) = \rho(g)(h(g^t u g))$$

with $g \in GL(n,\mathbb{C})$, $h \in S_1(T,X)$ and $u \in T$. For the special case $X = \mathbb{C}$ and $\rho =$ the trivial representation, we get a represention τ of $GL(n,\mathbb{C})$ on $S_1(T)$ by

$$\tau(g)(h)(u) = h(g^t u g)$$

with $g \in GL(n,\mathbb{C})$, $h \in S_1(T)$ and $u \in T$.

Remark*[7, S.94, (12.19)] We can identify $S_1(T,X)$ with $S_1(T)\otimes X$ by the rule $(h\otimes x)(u) = h(u)x$ for $h \in S_1(T)$, $x \in X$ and $u \in T$, in particular we can identify $S_1(T)$ with Sym^2 (as $GL(n,\mathbb{C})$-representations.*

Now we define a differential operator introduced by Shimura ([7, (12.12 a)]) by

$$D : C^\infty(\mathbb{H}_n, X) \to C^\infty(\mathbb{H}_n, S_1(T,X))$$

$$D(f)(u) = \sum_{1\le i\le j\le n} u_{i,j}\frac{\partial f}{\partial z_{i,j}}$$

with $u \in T = Sym_n(\mathbb{C}^{n\times n})$.

We also need the following

Proposition *[7, (12.18 & 12.10, e=1)] Let* $\rho : GL(n,\mathbb{C}) \to GL(X)$ *and* $f \in C^\infty(\mathbb{H}_n, X)$ *we define* $D_\rho f \in C^\infty_{\rho\otimes\tau}(\mathbb{H}_n, S_1(T,X))$ *by*

$$D_\rho(f)(u) := \rho(Y)^{-1} D\left(\rho(Y) f\right)(u).$$

For $M \in Sp(n,\mathbb{R})$ *the operator satisfies* $D_\rho(f \mid_\rho M) = D_\rho(f) \mid_{\rho\otimes\tau} M$.

To define RC-brackets in general, we start from three polynomial representations ρ, ρ', ρ'' with representationm spaces X, X', X''. A bilinear map

$$[,] : C^\infty_\rho(\mathbb{H}_n, X) \times C^\infty_{\rho'}(\mathbb{H}_n, X) \longrightarrow C^\infty_{\rho''}(\mathbb{H}_n, X'')$$

is called RC-bracket, if it is equivariant, i.e. for all $g \in Sp(n,\mathbb{R})$ we have

$$[F \mid_\rho g, G \mid_{\rho'} g] = [F, G] \mid_{\rho''} g$$

and it should be described by partial holomorphic derivatives of F and G. It is called to be of order ν, if the total of derivatives is ν .

3 RC-brackts of order one: the general case

We want to construct such RC-operators for the case, where f and g are vector-valued, starting from Maass-Shimura operators $D_\rho(f)$ and $D_{\rho'}(g)$ and then using linear combinations of $D_\rho(f) \otimes g$ and $f \otimes D_{\rho'}(g)$.

We start from the observation that $D_\rho(f)$ breaks up naturally into two parts, following [7, 31.28]

$$D_\rho(f)(u) = P(y^{-1}, u) f + D(f)(u).$$

We aim at some uniqueness properties of P.
First we observe that

$$\psi : \begin{cases} T \times X & \longrightarrow \quad S_1(T, X) \sim T \otimes X \\ (v, x) & \mapsto \quad u \mapsto P(v, u)(x) \end{cases}$$

defines a bilinear map and hence an endomorphism $\tilde{\psi}$ of $T \otimes X$.

Following [7, prop.13.15(4)] we have for all $h \in GL(n,\mathbb{C})$

$$P(hvh^t, h^{-t} u h^{-1})(\rho(x)) = \rho(h) P(v, u)(x)$$

We rephrase this for the map $\hat{\psi}$:

$$(\rho \otimes \tau)(h) \circ \tilde{\psi} = (\tilde{\psi} \circ \rho \otimes \tau)(h)$$

We do the same for $D_{\rho'}(g)(u) = Q(y^{-1}, u) + D(g)(u)$ and obtain an endomorphism $\tilde{\phi}$ of $T \otimes X'$ with equivariance for $\rho' \otimes \tau$.
We extend both $\tilde{\psi}$ and $\tilde{\phi}$ to endomorphsims $\hat{\psi}$ and $\hat{\phi}$ of $X \otimes X' \otimes T$ which are now $\rho \otimes \rho' \otimes \tau$-equivariant.
We now consider the restriction of $\hat{\psi}$ and $\hat{\phi}$ to an irreducible subspace $\mathfrak{X}$ of $X \otimes X' \otimes T$; then on $\mathfrak{X}$ the endomorphisms $\hat{\psi}$ and $\hat{\phi}$ are proportional. We may therefore choose a linear combination of $D_\rho(F) \otimes g$ and $f \otimes D_{\rho'}(g)$ such that (at least on $\mathfrak{X}$) the nonholomorphic parts $P(y^{-1}, u)$ and $Q(y^{-1}, u)$ get cancelled: We obtain

Theorem *Let ρ, ρ' be polynomial representions of $GL(n, \mathbb{C})$ and let $\mathfrak{X}$ be an irreducible subspace of $X \otimes X' \otimes T$. Then there is a nonzero linear combination*

$$\alpha D_\rho(f) \otimes g + \beta f \otimes D_{\rho'}(g) \tag{3}$$

which after restriction to $\mathfrak{X}$ defines a nonzero RC-bracket

$$[,] : Hol_\rho(\mathbb{H}_n, X) \times Hol_{\rho'}(\mathbb{H}_n, X') \longrightarrow Hol(\mathbb{H}_n, \mathfrak{X})$$

Remark *The explict calculations in section 4 will show that the coefficients α, β in (3) will indeed not be the same on all the irreducible subspaces $\mathfrak{X} \subset X \otimes X' \otimes T$.*

Remark *We do not claim here that this is the only possible $\mathfrak{X}$-valued RC-bracket of order one. We also do not claim that both α and β are nonzero. Already for $n = 1$ one can see that for weights k and l being zero, there are two linearly independent RC-brackets.*

4 RC-brackets for symmetric tensor representations

Here we study the case of syymetric tensor representations. We use the explicit direct approach from [1] for the nonholomorphic differential operators. Let $(x_1, \dots, x_n)$ be a row vector consisting of n inderminates. We put $V = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. we identify the $l-$ symmetric tensor product $V^{(l)}$ of V

with $\mathbb{C}[x_1, \ldots, x_n]_l$, the space of polynomials homogeneous of degree l. Then $GL(n, \mathbb{C})$ acts on $V^{(l)}$ by

$$(gv)(x) = \det(g)^k \cdot v(xg)$$

for $g \in Gl(n, \mathbb{C})$ and $v \in V^{(l)}$.
For a $V^{(l)}$-valued function $f \in C^\infty(\mathbb{H}_n, V^{(l)})$ we define elements of $C^\infty(\mathbb{H}_n, V^{(l+2)})$ by

$$\mathcal{D}(f) := \frac{\partial}{\partial Z} f[x]$$

$$Nf := \frac{1}{2i} Y^{-1} f[x] = \frac{1}{2i} Y^{-1}[x] \cdot f$$

We put

$$\delta_k f := k \cdot Nf + Df$$

Then

$$\delta_{k+l}(f \mid_{k,l} M) = \delta_{k+l} \mid_{k,l+2} M$$

holds for any $M \in Sp(n, \mathbb{R})$.
Furthermore, we have the commutation rule

$$DN^\nu = -\nu N^{\nu+1} + N^\nu D \qquad (\nu \geq 0) \tag{4}$$

Using the $r-$fold iteration

$$\delta^r_{k+l} := \delta_{k+l+2r-2} \circ \cdots \circ \delta_{k+l}$$

we can now define a nonholomorphic RC-bracket by

$$C^\infty_{Sym^l \otimes \det^k}(\mathbb{H}_n, V^l) \times C^\infty_{Sym^m \otimes \det^{k'}}(\mathbb{H}_n, V^m) \longmapsto C^\infty_{Sym^{l+m+q} \otimes \det^{k+k'}}(\mathbb{H}_n, V^{l+m})$$

by

$$[f, g]^p_{k,l;k',m} := \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \frac{\Gamma(k+l+p)\Gamma(m+k'+p)}{\Gamma(k+l+p-\nu)\Gamma(m+k'+\nu)} \delta^{p-\nu}_{k+l} f \cdot \delta^\nu_{m+k'} g \tag{5}$$

We claim that this expression is actually holomorphic:
Using (4) one can rewrite (5) as a polynomial in N (multiplied by holomorphc derivatives of f and g as coefficients (on the right). Our claim is then equivalent to the statement that these coefficients vanish identically for nontrivial

powers of N. Fortunately, the coefficients of (4) and (5) are both independent of the degree n and depend only on $k+l$ and $k'+m$; the claim follows from the same statement for degree $n=1$, which is given in Shimura's book [7].
Looking only at the constant term in this polynomial in N we get (as in the degree one case of Shimura):

Theorem RC-brackets of order p for symmetric tensor representations of degree n

$$\mathcal{RC}^p_{k,l;k',m}(f,g) = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \frac{\Gamma(k+l+p)\Gamma(m+k'+p)}{\Gamma(k+l+p-\nu)\Gamma(m+k'+\nu)} D^{p-\nu}(f) \cdot D^\nu(g)$$

defines a bilinear RC-bracket

$$Hol_{Sym^l \otimes \det^k}(\mathbb{H}_n, V^l) \times Hol_{Sym^m \otimes \det^{k'}}(\mathbb{H}_n, V^m) \longrightarrow Hol_{Sym^{l+m} \otimes \det^{k+k'}}(\mathbb{H}_n, V^{l+m})$$

5 Explicit calculation for $\rho = Sym^1 \otimes \det^k$ and $\rho' = \det^\ell$

This section contains the main results from [6]. Throughout, we look at the special situation

$$\rho = Sym^1 \otimes \det^k, \qquad \rho' = \det^\ell, \qquad n \geq 2.$$

Using the notation from section 2, we have

$$D_\rho(f)(u) = P(Y^{-1}, u)(f) + D(f)(u)$$

$$D_{\det^\ell}(g)(u) = Q(Y^{-1}, u) + D(g)(u)$$

Both $D_\rho(f) \cdot g$ and $f \otimes D_{\det_\ell}(g)$ are now $\mathfrak{X} = Sym^1 \otimes Sym^2 \otimes \det^{k+l}$-valued; here we tacity identify the representations with their representation spaces. One knows from representation theory ("Pieri's rule", see e.g.[4, Corollary 9.2.4] that $Sym^1 \otimes Sym^2$ decomposes into two irreducible components, which we call Pieri-component $\mathcal{P}$ and Co-Pieri component $\mathcal{C}P$, i.e.

$$\mathfrak{X} = \mathfrak{X}^{\mathcal{P}} \oplus \mathfrak{X}^{\mathcal{C}P}. \tag{6}$$

Here the assumption $n \geq 2$ comes in.
Accordingly, we may decompose

$$P(Y^{-1},u)(f)\otimes g = \big(P(Y^{-1},u)(f)\otimes g\big)^{\mathcal{P}} + \big(P(Y^{-1},u)(f)\otimes g\big)^{\mathcal{C}P}$$

$$f\otimes Q(Y^{-1},u)g = \big(f\otimes Q(Y^{-1},u)g\big)^{\mathcal{P}} + \big(f\otimes Q(Y^{-1},u)g\big)^{\mathcal{C}P}$$

We put

$$\begin{aligned}
\alpha^{\mathcal{P}} &= \ell \\
\beta^{\mathcal{P}} &= 1+k \\
\alpha^{\mathcal{C}P} &= \ell \\
\beta^{\mathcal{C}P} &= -\frac{1}{2}+k
\end{aligned}$$

The main point is now (in accordance with the considerations of section 2, but obtained independently) the following
Proposition

$$\alpha^{\mathcal{P}}\cdot\big(P(Y^{-1},u)(f)\otimes g\big)^{\mathcal{P}} = \beta^{\mathcal{P}}\cdot\big(f\otimes Q(Y^{-1},u)g\big)^{\mathcal{P}}$$

and

$$\alpha^{\mathcal{C}P}\cdot\big(P(Y^{-1},u)(f)\otimes g\big)^{\mathcal{C}P} = \beta^{\mathcal{C}P}\cdot\big(f\otimes Q(Y^{-1},u)g\big)^{\mathcal{C}P}$$

To prove this proposition, one has not only to compute $P(Y^{-1},u)(f)$ and $Q(Y^{-1},u)g$ explicitly, but also one has to determine explcitly a decomposition into Pieri - and Co-Pieri-components. This is the crucial technical point of this section. Then one gets immediately

Theorem: *For $\rho = Sym^1\otimes\det^k$ and $\rho' = \det^\ell$ we get explicit holomorphic RC-brackets with values in in the Pieri- and Co-Pieri-components of $Sym^1\otimes Sym^2\otimes\det^{k+\ell}$ by*

$$(f,g)\longmapsto \alpha^{\mathcal{P}}\cdot(D_\rho(f)\cdot g)^{\mathcal{P}} - \beta^{\mathcal{P}}\cdot(f\otimes D_{\rho'}(g))^{\mathcal{P}}$$

and

$$(f,g)\longmapsto \alpha^{\mathcal{C}P}\cdot(D_\rho(f)\cdot g)^{\mathcal{C}P} - \beta^{\mathcal{C}P}\cdot(f\otimes D_{\rho'}(g))^{\mathcal{C}P}$$

We may rephrase the statements above without using the decomposition (6):

Theorem': *For* $\rho = Sym^1 \otimes \det^k$ *and* $\rho' = \det^\ell$ *there is a* $GL(n,\mathbb{C})$*- equivariant endomorphism of* $Sym^1 \otimes Sym^2$ *such that*

$$\ell \cdot D_\rho(f) - \mathcal{L} \circ (f \otimes D_{\det^\ell}(g))$$

defines a holomorphic RC-bracket

Using [7, 13.17], one can see that the property above is compatible with taking tensor products, i.e. if the property of theorem' holds for ρ_1 and ρ_2 (instead of ρ), then it also holds for $\rho_1 \otimes \rho_2$. In particular, it therefore holds for $Stand^{\otimes\nu}$ for arbitrary $\nu \geq 1$. Now we take into account that any polynomial representation ρ appears as subrepresentation in some $Stand^{\otimes\nu}$ (see [3, Lecture 6]. In principle, we then get for any ρ a (nonzero) holomorphic RC-bracket mapping X_ρ-valued functions f and scalar-valued functions g (with automorphy factor det^ℓ to $X \otimes Sym^2$-valued functions. This construction is then explicit provided that we have an explicit embedding of ρ into $Stand^{\otimes\nu}$. Hence, in principle, we can cover in this way the case of RC -brackets of order one with ρ arbitrary, but ρ' scalar-valued.

References

[1] S. Böcherer, T. Satoh, T. Yamazaki: On the pullback of a Differential Operator and Its Application to Vector Valued Eisenstein Series. Commentarii Mathematici Universitatis Sancti Pauli, Vol. 41, No. 1 (1992)

[2] S. Böcherer: Holomorphic differential operators with iteration. Proceedings of the 7th Autumn Workshop on Number Theory. "Differential Operators on Modular Forms and Application ". (Hakuba, Japan. October 2004) Editor: T. Ibukiyama. Printed by Ryushi-do, March 2005.

[3] W. Fulton, J. Harris: Representation Theory - A first course. Springer Verlag (1991)

[4] R. Goodman, N.R. Wallach: Symmetry, Representations and Invariants. Springer Verlag (2009)

[5] T. Ibukiyama: On differential operators on automophic forms and pluriharmonic polynomials. Comm.Math.Univ.St.Pauli 48, 103-118 (1999)

[6] Meister,J.: Äquivariane holomorphe Differentialoperatoren mit vektorwertigen Automorphiefaktoren. PhD thesis Mannheim 2021

[7] G. Shimura: Arithmeticity in the Theory of Automorphic Forms. Mathematical Surveys and Monographs Volume 82. American Mathematical Society (2000)

[8] G. Shimura: Elementary Dirichlet Series and Modular Forms. Springer Monographs in Mathematics (2000)

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