DIFFERENTIAL OPERATORS ON SIEGEL MODULAR FORMS AND AUTOMORPHY FACTORS

TOMOYOSHI IBUKIYAMA
OSAKA UNIVERSITY

The title of my talk at the RIMS workshop was "Differential operators on Siegel modular forms and Laplace transforms". There I talked on the content of my paper [19]. The paper has been already published, so instead of repeating the precise content here, I try to give a survey in an informal style and also give some guidance for references.

1. Automorphy factors

Although our theory applies to various hermitian symmetric domains, here to avoid an unnecessary complication of the formulation, we consider only the case of Siegel upper half space H_n defined by

$$H_n = \{Z = X + iY \in M_n(\mathbb{C}); X = {}^tX, Y = {}^tY \in M_n(\mathbb{R}), Y > 0\},$$

where $Y > 0$ means that Y is positive definite. Then the real symplectic group $Sp(n,\mathbb{R}) \subset M_{2n}(\mathbb{R})$ of real rank n acts on H_n by

$$gZ = (AZ + B)(CZ + D)^{-1} \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}).$$

It is well known that the group $Aut(H_n)$ of biholomorphic automorphism of H_n is $Sp(n,\mathbb{R})/\{\pm 1_{2n}\}$. We consider a finite dimensional vector space V and the space $Hol(H_n, V)$ of V-valued holomorphic functions F on H_n. Let G be a subgroup of $Sp(n,\mathbb{R})$. We consider a $GL(V)$ valued function $J(g,Z)$ on $G \times H_n$ such that

$$J(g_1g_2, Z) = J(g_1, g_2Z)J(g_2, Z) \qquad \text{for all } g_1, g_2 \in G. \tag{1}$$

This condition means that the operation

$$Hol(H_n, V) \ni F(Z) \to F|_J[g] = J(g,Z)^{-1}F(gZ) \in Hol(H_n, V)$$

is the action of the group G on $Hol(H_n, V)$. Such $J(g,Z)$ is called an automorphy factor of G. When $G = Sp(n,\mathbb{R})$, then the maximal compact subgroup K of G is isomorphic to the $n \times n$ unitary group $U(n)$ and its complexification is $GL_n(\mathbb{C})$. For any irreducible rational representation (ρ, V) of $GL_n(\mathbb{C})$, we may define an automorphy factor J_ρ by

$$J_\rho(g,Z) = \rho(CZ + D), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}).$$

(For a general theory, see [25].) In this case, we write $F|_{J_\rho}[g] = F|_\rho[g]$. When $\rho(A) = \det(A)^k$ for $A \in GL_n(\mathbb{C})$, we also write $F|_\rho[g] = F|_k[g]$. For most ρ, the action defined by J_ρ on $Hol(H_n, V)$ is irreducible (holomorphic discrete series representation).

Now for $r \geq 2$, fix a partition $n = n_1 + \cdots + n_r$ of n ($n_i \geq 1$) and put $\boldsymbol{n} = (n_1, \ldots, n_r)$. For $\boldsymbol{n}$, consider the following subdomain of H_n.

$$(2) \quad H_{\boldsymbol{n}} = H_{n_1} \times \cdots \times H_{n_r} \ni (Z_1, \ldots, Z_r) \to \begin{pmatrix} Z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Z_r \end{pmatrix} \in H_n.$$

Then $G_{\boldsymbol{n}} = Sp(n_1, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R})$ acts on $H_{\boldsymbol{n}}$ and $G_{\boldsymbol{n}}$ can be naturally regarded as a subgroup of $Sp(n, \mathbb{R})$ compatible with this embedding of the domain. Here we consider representations of (ρ_i, V_i) of $GL(n_i)$ $(1 \leq i \leq r)$ and put $\rho = \rho_1 \otimes \cdots \otimes \rho_{n_r}$. We put $V_{\boldsymbol{n}} = V_1 \otimes \cdots \otimes V_r$. We define $GL(V_{\boldsymbol{n}})$ valued automorphy factor

$$J_\rho((g_1, \ldots, g_r), (Z_1, \ldots, Z_r)) = \rho_1(C_1 Z_1 + D_1) \otimes \cdots \otimes \rho_r(C_r Z_r + D_r)$$

of $G_{\boldsymbol{n}}$ on $Hol(H_{\boldsymbol{n}}, V_{\boldsymbol{n}})$. When all $\rho_i = \det^k$, we write $\rho = \det^k$.

Now we consider a holomorphic linear differential operator $\mathbb{D}$ with constant coefficients to map $Hol(H_n, \mathbb{C})$ to $Hol(H_n, V)$ such that

$$(3) \qquad \mathrm{Res}_{H_{\boldsymbol{n}}}(\mathbb{D}(F|_k[g])) = (\mathrm{Res}_{H_{\boldsymbol{n}}}(\mathbb{D}F))|_{\det^k \otimes \rho}[g]$$

for any holomorphic function F on H_n and any $g = (g_1, \ldots, g_r) \in G_{\boldsymbol{n}}$. Here Res is the restriction of functions on H_n to $H_{\boldsymbol{n}}$

We have several motivations to consider such differential operators.

(i) This differential operator gives an easy way to construct a new modular forms starting from given modular forms. If F is a modular form of weight k w.r.t. a discrete subgroup $\Gamma \subset Sp(n, \mathbb{R})$, that is, if $F|_k[\gamma] = F$ for all $\gamma \in \Gamma$, then by the differential operators as above, we have

$$\mathrm{Res}_{H_{\boldsymbol{n}}}(\mathbb{D}F) = \big(\mathrm{Res}_{H_{\boldsymbol{n}}}(\mathbb{D}F)\big)|_{J_{det^k \otimes \rho}}[\gamma]$$

for $\gamma \in \Gamma \cap G_{\boldsymbol{n}}$. This means that $\mathrm{Res}_{H_{\boldsymbol{n}}}(\mathbb{D}F)$ is a modular form of $\Gamma \cap G_{\boldsymbol{n}}$ of weight $det^k \otimes \rho$. If we replace the pair $(H_n, H_{\boldsymbol{n}})$ by (H_n^r, H_n) where H_n is embedded diagonally in H_n^r, then the operator is often called a Rankin-Cohen operator. This case is formulated similarly in [10] and used very often (See [1], [9], [12]).

(ii) If we apply our differential operator $\mathbb{D}$ on Siegel Eisenstein series $E_{n_1+n_2}(Z)$ of degree $n_1 + n_2$ and restrict $(\mathbb{D}E_{n_1+n_2})(Z)$ to $H_{n_1} \times H_{n_2} \subset H_n$ by the embedding (2), then this is a linear combination of the tensors of Siegel modular forms of degree n_1 and n_2. The coefficients of this linear combination are given by critical values of the standard L function of Siegel cusp forms (due to P. Garrett, S. Boecherer, N. Kozima and so on). So if $\mathbb{D}$ and the Fourier coefficients of $E_{n_1+n_2}$ are given concretely, then this gives a way to calculate the critical values of the standard L functions explicitly ([8], [13], [14]). This also gives a method to give a congruence between eigenvalues of Siegel modular forms ([14], [2], [3].) This kind of application on congruences was started by Katsurada(e.g. [22]). For this application, we often need a very concrete formulas of differential operators beyond existence theorem.

(iii) But more than the above two reasons, this theory of differential operators give an interesting theory of special functions. Since we assumed that $\mathbb{D}$ has constant coefficients, we have a V-valued polynomial P in partial derivatives ∂_Z of variables of H_n such that $\mathbb{D} = P(\partial_Z)$. More precisely, for $Z = (z_{ij}) \in H_n$, we put a symmetric matrix of partial derivation

$$\partial_Z = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right)_{1 \le i,j \le n}.$$

We consider a V-valued polynomial $P(T)$ in components of $n \times n$ symmetric matrix T and a differential operator $P(\partial_Z)$. Then polynomials P such that $P(\partial_Z)$ satisfies the condition (3) give a very interesting special polynomials, including classical Gegenbauer polynomials. This means that we also have a general theory of special functions of several variables behind this, like a system of differential equations that has our polynomials as its solutions. Then we can also ask non-polynomial solutions similarly as Gegenbauer functions (See [20]).

General characterization of our V-valued polynomial P has been given in [10]. The theory in [10] treat two cases: the case $H_{\boldsymbol{n}} \subset H_n$, and the case $H_n \subset H_n^r$. In this note, we treat only the former case. In this case, the claim of the theory is (under some mild condition on n and k) as follows. A V-valued polynomial P satisfies the condition (3) if and only if the following two conditions (a), (b) are satisfied.

Condition 1.1. *(a) Take an $n_i \times 2k$ matrix X_i of variables $(1 \le i \le r)$. We put*

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix},$$

and write $T = X\,{}^tX$. Then $P(X\,{}^tX)$ is a V-valued polynomial such that it is pluri-harmonic with respect to each X_i with $1 \le i \le r$. Here we say that a polynomial $P(Y)$ in $y_{i\nu}$ for an $n_0 \times 2k$ matrix $Y = (y_{i\nu})$ is pluri-harmonic with respect to Y if $\Delta_{ij}(Y)P(Y) = 0$ for

$$\Delta_{ij}(Y) = \sum_{\nu=1}^{2k} \frac{\partial^2}{\partial y_{i\nu} \partial y_{j\nu}}$$

for any i and j with $1 \le i, j \le n_0$.
(b) Embed any $A = (A_1, \ldots, A_r) \in GL_{n_1}(\mathbb{C}) \times \cdots \times GL_{n_r}(\mathbb{C})$, to

$GL_n(\mathbb{C})$ by

$$A \to \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_r \end{pmatrix}.$$

Then we have $P(AT\,{}^tA) = \rho(A)P(T)$, where $\rho = \rho_1 \otimes \cdots \otimes \rho_r$.

For the proof, see [10] Theorem 1. Even only by this characterization, we can understand a lot of things. For example, since we are taking $T = X\,{}^tX$, our polynomial $\widetilde{P}(X) = P(X\,{}^tX)$ is invariant by the action $X \to Xh$ for $h \in O(2k)$. But we know a complete classification of irreducible representations of $GL(n_i) \times O(2k)$ realized on pluri-harmonic polynomials in X_i by [21]. Our polynomial is a tensor of pluri-harmonic polynomials for each X_i with $X = {}^t(X_1, \ldots, X_r)$, and if we assume that each X_i factor corresponds to a representation (τ_i, h_i) of $GL_{n_i}(\mathbb{C}) \times O(2k)$ for each i, then our $\widetilde{P}(X)$ should belong to the trivial representations in $h_1 \otimes \cdots \otimes h_r$. So the multiplicity of this trivial representation is the dimension of our P. For example, when $r = 2$, P exists only when τ_1 and τ_2 correspond with the same Young diagram, and such P is unique up to constant(See [10]).

In some cases, directly from this result we can give concrete closed formulas of polynomials P as given in [10]. But in general, to give concrete P is not so easy. In most application in the paper we quoted before, we need a very concrete formula of P. There are many tries for this. For example, [11] treated the case that $r = 2$, ρ_1 and ρ_2 are $\det^\ell$. For example, when $\ell = 2$, a concrete closed formula is given in [11] p. 289 for general n, and a constructive method to give P is explained in 4.2.1 of the same paper. This paper contains a theory of associated system of differential equations. The paper [18] gives a kind of generalization of the classical Rodrigues formula as in the case of Legendre polynomial. This gives a one-line formula for the polynomial P for general ρ_i. This is rather a theoretical formula since if we calculate the operator by this formula, then the computer would give you back a mess. The paper [17] explains a certain generic differential operator which can be regarded as a source of all the operators we need. Still we cannot call most of the above results as a closed formula to the extent that we can write down the coefficients of polynomials completely.

2. Operation on automorphy factors and descending basis

Let's forget for a while that we are considering a differential operator that preserves automorphy under restriction. Let's consider a general differential operator $P(\partial_Z)$ for a scalar valued polynomial P in components of $n \times n$ symmetric matrix T. Let's consider a simple automorphy factor $J_k(g, Z) = \det(CZ+D)^k$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$

and $Z \in H_n$. Now we ask what is $P(\partial_Z)\det(CZ+D)^{-k}$. For example, when $C = 1_n$, $D = 0$, we have $J_k(g, Z) = \det(Z)^k$, and the following Cayley type formula is known (e.g. [6]).

$$\det(\partial_Z)\det(Z)^s = s\left(s+\frac{1}{2}\right)\cdots\left(s+\frac{n-1}{2}\right)\det(Z)^{s-1}.$$

Here s can be any complex number and a branch of $\det(Z)^s$ is defined well in a certain way. In [26], Shimura asked how to generalize this and answered as follows. Consider a representation of $GL_n(\mathbb{C})$ on the ring $\mathbb{C}[T]$ of polynomials $P(T)$ in components of $n \times n$ symmetric matrix T defined by $(\pi(g)P)(T) = P({}^t gTg)$ for $g \in GL_n(\mathbb{C})$. Take an irreducible representation (τ, V_τ) in $(\pi, \mathbb{C}[T])$ and assume that $P(T) \in V_\tau \subset \mathbb{C}[T]$. Then for a certain gamma factor $\beta_n(s)$ we have

$$P(\partial_Z)\det(Z)^s = \beta_n(s)\det(Z)^s P(Z^{-1}).$$

For example, when $P(T) = \det(T)$ then this gives a representation $g \to \det(g)^2$, and $P(Z^{-1}) = \det(Z)^{-1}$, so $\det(Z)^s P(Z^{-1}) = \det(Z)^{s-1}$, and this is nothing but the above Cayley type formula. Here we use $\det(Z)^s$ instead of $\det(CZ+D)^s$, but we will explain now that this does not give much difference. For $Z = (z_{ij}) \in H_n$, we write

$$\partial_{Z,ij} = \frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}}.$$

If we give a general formula of derivatives of $\det(CZ+D)^{-k}$ by any $\partial_{Z,ij}$, then by iteration we can at least have an algorithm to calculate $P(\partial_Z)\det(CZ+D)^{-k}$ for any P. To write this down, we fix $g \in Sp(n, \mathbb{R})$ and (C, D), and for simplicity, we write $\delta = \det(CZ+D)$ and $\Delta = (CZ+D)^{-1}C$. It is well known and easy to see that Δ is a symmetric matrix. For column vectors $x = {}^t(x_i)$, $y = {}^t(y_i) \in \mathbb{C}^n$, put

$$\partial[x, y] = x\partial_Z\, {}^t y = \sum_{1\le i\le j\le n}\frac{x_i y_j + x_j y_i}{2}\frac{\partial}{\partial z_{ij}}.$$

Then as shown in Ibukiyama:保型形式特論（共立出版 2018), we have

$$\begin{aligned}
\partial[x, y]\delta &= \delta(x\Delta\, {}^t y)\\
\partial[x, y]\delta^{-k} &= -k\delta^{-k}(x\Delta\, {}^t y)\\
\partial[x, y](\Delta) &= -\Delta({}^t xy + {}^t yx)\Delta/2.
\end{aligned}$$

This means that

Lemma 2.1. *For any i, j, p, $q \in \{1, \ldots, n\}$ and for any complex number k, we have*

$$\begin{aligned}
\partial_{Z,ij}\delta^{-k} &= -k\delta^{-k}\Delta_{ij}\\
\partial_{Z,ij}\Delta_{pq} &= -\frac{1}{2}(\Delta_{ip}\Delta_{jq} + \Delta_{iq}\Delta_{jp})
\end{aligned}$$

So by iterate use of these formulas, it is clear that for any polynomial $P(T)$, we have some polynomial $Q(T)$ such that

$$P(\partial_Z)\delta^{-k} = \delta^{-k}Q(\Delta).$$

Now the shape of the formula in Lemma 2.1 does not depend on C and D. So as far as Δ_{ij} is generic (for example if Δ_{ij} $(1 \le i \le j \le n)$ are algebraically independent), the polynomial Q is determined independently of the choice of C, D. So Q can be determined as far as we know $P(\partial_Z)\det(Z)^{-k}$. The mapping from $P(T)$ to $Q(T)$ (depending of course on k) is a linear map from the vector space $\mathbb{C}[T]$ to $\mathbb{C}[T]$ over $\mathbb{C}$, and we denote this by ϕ_k as

$$Q = \phi_k(P).$$

There exists a kind of formula to describe $\phi_k(P)$ for any $P(T)$ (see [19] Theorem 1), but we omit it here, since we can give a better formulation and we give a deeper property of ϕ_k later.

It is also well known that we can define the same sort of simple automorphy factor for any tube domain, and Shimura developed a general theory on that. This is a beautiful theory. But his theory does not fit our demand. In most cases, for our differential operators $\mathbb{D} = P(\partial_Z)$, the polynomial P does not belong to any irreducible representation of $GL_n(\mathbb{C})$. Our polynomial should belong to a representation space of the group $GL_{n_1}(\mathbb{C})\times\cdots\times GL_{n_r}(\mathbb{C}) \subset GL_n(\mathbb{C})$, and not representations of $GL_n(\mathbb{C})$. In fact, we have a very nice basis of $\mathbb{C}[T]$ which behaves very well on $\det(Z)^s$. Such a basis is called a descending basis, and has been (essentially) defined in [15] by completely different motivation. We explain this next.

Let $P(T)$ be a polynomial in $\mathbb{C}[T]$. For our differential operators, we had two conditions on P in Condition 1.1. But let's forget a condition on the action of $GL_{n_1}(\mathbb{C}) \times \cdots \times GL_{n_r}(\mathbb{C})$ for a while. In this section, we consider a general polynomial $P(T)$ and ask what is $P(\partial_Z)\det(Z)^s$. Let $X = (x_{i,\nu})_{1\le i\le n, 1\le \nu\le 2k}$ be an $n\times 2k$ matrix of variables. For each i, j with $1 \le i, j \le n$, we put

$$\Delta_{ij}(X) = \sum_{\nu=1}^{2k} \frac{\partial^2}{\partial x_{i\nu}\partial x_{j\nu}}.$$

To see Condition 1.1 (a) on P for T, it is natural to write down the operation of $\Delta_{ij}(X)$ on $P(X\,{}^tX)$ by variables of $T = X\,{}^tX$. We write $T = (t_{ij})$ and put

$$\partial_{ij} = (1+\delta_{ij})\frac{\partial}{\partial t_{ij}}.$$

It has been shown in [15] that if we put

$$D_{ij} = 2k\partial_{ij} + \sum_{k,\ell=1}^{n} t_{k\ell}\partial_{ik}\partial_{j\ell},$$

then we have

$$\Delta_{ij}(P(X\,{}^tX)) = (D_{ij}P)(X\,{}^tX),$$

so we can write the pluri-harmonicity condition by t_{ij}. But D_{ij} and $\Delta_{ij}(X)$ have one difference. To define $\Delta_{ij}(X)$, the number $2k$ should be an integer, but the operator D_{ij} is well-defined for any complex number k. This is a big difference in the theory we explain later.

Apparently there seems no relation between what is $P(\partial_Z)\det(Z)^{-k}$ and P being pluri-harmonic. But soon we will see that there exists a mysterious relation between these, and this is explained by our descending basis. To explain this, we consider the following set of indices

$$\mathcal{N} = \{\boldsymbol{\nu} = (\nu_{ij}) = {}^t\boldsymbol{\nu} \in M_n(\mathbb{Z}); \nu_{ij} \geq 0, \nu_{ii} \equiv 0 \bmod 2 \text{ for } 1 \leq i \leq n\}.$$

For $\boldsymbol{\nu} \in \mathcal{N}$ and $T = (t_{ij}) = {}^tT$, we write

$$T^{\boldsymbol{\nu}} = \prod_{1\leq i,j\leq n} t_{ij}^{\nu_{ij}/2} = \prod_{i=1}^{n} t_{ii}^{\nu_{ii}/2} \prod_{1\leq i<j\leq n} t_{ij}^{\nu_{ij}}. \tag{4}$$

Since these elements give all the monomials in $\mathbb{C}[T]$, it is natural that elements of a basis of $\mathbb{C}[T]$ is indexed by elements of $\mathcal{N}$ in some way. If we consider a monomial $T^{\boldsymbol{\nu}}$ and the degree a_i of $(X\,{}^tX)^{\boldsymbol{\nu}}$ with respect to $x_{i\nu}$ for a fixed i, then obviously we have

$$a_i = \sum_{j=1}^{n} \nu_{ij}.$$

We call $\mathbf{a} = (a_1, \ldots, a_n)$ a multi-degree of $T^{\boldsymbol{\nu}}$. Of course $\mathbf{a}$ does not determine $\boldsymbol{\nu}$ at all.

In [15], we considered subspace $\mathcal{H}$ of polynomials $P(T)$ in $\mathbb{C}[T]$ such that $D_{ii}P = 0$ for all i. In Condition 1.1 (a), this is the case where $r = n$ and $n_1 = \cdots = n_r = 1$. In [15], we considered two canonical bases of $\mathcal{H}$. One is called a monomial basis $P^M_{\boldsymbol{\nu}}(T)$. This is explained as follows. We put $\mathcal{N}_0 = \{\boldsymbol{\nu} = (\nu_{ij}) \in \mathcal{N}; \nu_{ii} = 0 \text{ for all } i\}$. We can show (under certain mild condition on k and n) that for each $\boldsymbol{\nu} \in \mathcal{N}_0$, there exists the unique polynomial $P^M_{\boldsymbol{\nu}}(T)$ such that $D_{ii}P^M_{\boldsymbol{\nu}}(T) = 0$ for all $i = 1, \ldots, n$ and that $P^M_{\boldsymbol{\nu}}(T) = T^{\boldsymbol{\nu}} \bmod (t_{11}, \ldots, t_{nn})$. The last condition means that one of the monomials in $P^M_{\boldsymbol{\nu}}(T)$ is $T^{\boldsymbol{\nu}}$ with coefficient 1 and all the other monomials contain t_{ii} for some i.

The other canonical basis is called descending basis described in the following theorem. In [15], this basis was given only for $\mathcal{H}$, but here we state it for the whole space $\mathbb{C}[T]$, since the proof is the same.

Notation: Let $\mathcal{I}_n$ be the set of integers α such that $\alpha < n$. We denote by $\widehat{\mathbb{C}_n}$ the set of complex numbers that do not belong to $\mathcal{I}_n$. We denote by e_{ij} the $n \times n$ matrix element whose (i,j) component is 1 and all the other components are 0. We put $\mathbf{e}_{ij} = e_{ij} + e_{ji}$.

Theorem 2.2. *For the index* $\mathbf{0} = (0) \in \mathcal{N}$*, we define* $P_{\mathbf{0}}^D(T) = 1$*. Assume that* $2k \in \widehat{\mathbb{C}_n}$*. Then there exists a basis of* $\mathbb{C}[T]$ *consisting of unique polynomials* $P_{\boldsymbol{\nu}}^D(T)$ *(*$\boldsymbol{\nu} \in \mathcal{N}$*) such that*

$$D_{ij}P_{\boldsymbol{\nu}}^D(T) = P_{\boldsymbol{\nu}-\mathbf{e}_{ij}}^D(T).$$

Here, if any component of $\boldsymbol{\nu} - \mathbf{e}_{ij}$ *is negative, we regard* $P_{\boldsymbol{\nu}-\mathbf{e}_{ij}}^D(T) = 0$.

We see an example. Put $n = 3$ and put

$$\boldsymbol{\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $P = P_{\boldsymbol{\nu}}^D(T)$ is characterized by the following conditions.

$$\begin{aligned} &D_{11}P = D_{22}P = D_{33}P = 0, \\ &D_{12}P = P_{\mathbf{0}} = 1, \\ &D_{13}P = D_{23}P = 0. \end{aligned}$$

It is easy to see that such a polynomial is uniquely given by $t_{12}/2k$.

A proof of Theorem 2.2 is not so easy. If we consider only indices $\boldsymbol{\nu} \in \mathcal{N}_0$, then polynomials $P_{\boldsymbol{\nu}}^D(T)$ for $\boldsymbol{\nu} \in \mathcal{N}_0$ give a basis of $\mathcal{H}$, and this basis is the dual basis of the monomial basis with respect to a certain natural inner product of $\mathcal{H}$. The two bases of $\mathcal{H}$ we described above were introduced independently by two authors of [15], the monomial basis by Zagier and the descending basis by Ibukiyama. Since it turns out that they are dual, we were convinced that we can call them canonical bases.

A basis suited for our purpose here is the descending basis, so we do not talk on monomial basis from now on. There are many good surprising properties of the descending basis, and we will explain some of them.

For any index $\boldsymbol{\nu} = (\nu_{ij}) \in \mathcal{N}$, we write

$$\deg(\boldsymbol{\nu}) = \frac{1}{2}\sum_{1\le i,j\le n} \nu_{ij} = \frac{1}{2}\sum_{i=1}^{n} \nu_{ii} + \sum_{1\le i<j\le n} \nu_{ij}.$$

If we put $a_i = \sum_{j=1}^{n} \nu_{ij}$, then this means that

$$\deg(\boldsymbol{\nu}) = \frac{1}{2}\sum_{i=1}^{n} a_i$$

For the matrix Δ and an index $\boldsymbol{\nu} \in \mathcal{N}$, we define $\Delta^{\boldsymbol{\nu}}$ as in (4). We also put

$$\boldsymbol{\nu}! = \prod_{i=1}^{n} \nu_{ii}!! \prod_{1\le i<j\le n} \nu_{ij}!,$$

where we put

$$\nu_{ii}!! = \nu_{ii}(\nu_{ii}-2)\cdots 1 = 2^{\nu_{ii}/2}(\nu_{ii}/2)!.$$

Theorem 2.3. *We assume that* $2k \in \widehat{\mathbb{C}_n}$.
(i) For an index $\boldsymbol{\nu} \in \mathcal{N}$ *and the descending basis* $P^D_{\boldsymbol{\nu}}(T)$ *of index* $\boldsymbol{\nu}$, *we have*

$$P^D_{\boldsymbol{\nu}}(\partial_Z)(\delta^{-k}) = \delta^{-k} \times \frac{(-1)^{\deg(\boldsymbol{\nu})}}{2^{\deg(\boldsymbol{\nu})}\boldsymbol{\nu}!}\Delta^{\boldsymbol{\nu}}.$$

(ii) The linear map ϕ_k *from* $\mathbb{C}[T]$ *to* $\mathbb{C}[T]$ *defined by* $P(\partial_Z)\delta^{-k} = \delta^{-k}\phi_k(P)(\Delta)$ *is an isomorphism and commutes with the action* π *of* $GL_n(\mathbb{C})$ *on* $\mathbb{C}[T]$ *defined by* $(\pi(A)P)(T) = P({}^tATA)$. *That is, we have*

$$\phi_k(\pi(A)P) = \pi(A)(\phi_k(P)), \qquad P(T) \in \mathbb{C}[T].$$

The first equality means that $P^D_{\boldsymbol{\nu}}(\partial_Z)$ on δ^{-k} gives essentially a multiplication of monomial $\Delta^{\boldsymbol{\nu}}$ of Δ_{ij}, and this is very striking. We explain the meaning of (ii) more. As seen in (i), the images of $P^D_{\boldsymbol{\nu}}$ by ϕ_k are monomials. So among the images of descending basis, the action of $GL_n(\mathbb{C})$ on RHS of (ii) means the action on monomials. Since ϕ_k is an isomorphism, the action on monomials reflects on the action of $GL_n(\mathbb{C})$ on polynomials $P^D_{\boldsymbol{\nu}}(T)$. For example, for the sake of simplicity, we first consider the case $n = 2$. For a matrix

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix},$$

consider the case $Q(T) = t_{12}^{\ell}$. Then by Theorem 2.3 (i), the polynomial P such that $\phi_k(P) = Q$ is given up to constant by

$$P^D_{\boldsymbol{\nu}}(T)$$

where

$$\boldsymbol{\nu} = \begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix}.$$

Then by Theorem 2.3 (ii), for $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, we have

$$\phi_k(\pi(A)P^D_{\boldsymbol{\nu}}) = \phi_k(P^D_{\boldsymbol{\nu}}({}^tATA)) = \pi(A)(t_{12}^{\ell}) = a_1^{\ell}a_2^{\ell}t_{12}^{\ell} = a_1^{\ell}a_2^{\ell}\phi_k(P^D_{\boldsymbol{\nu}}(T)).$$

Since ϕ_k is a linear isomorphism by Theorem 2.3 (i), we have

$$\pi(A)P^D_{\boldsymbol{\nu}}(T) = P^D_{\boldsymbol{\nu}}({}^tATA) = a_1^{\ell}a_2^{\ell}P^D_{\boldsymbol{\nu}}(T).$$

This is nothing but Condition 1.1 (b) on $GL_1(\mathbb{C}) \times GL_1(\mathbb{C})$. Besides, by the definition of the descending basis, it is clear that $D_{11}P_{\boldsymbol{\nu}}(T) = D_{22}P_{\boldsymbol{\nu}}(T) = 0$. This means that $P_{\boldsymbol{\nu}}$ satisfies the necessary pluriharmonicity condition. So for any holomorphic function $F(Z)$ on $Z \in H_2$, any $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(\mathbb{R})$ $(i = 1, 2)$, and

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in Sp(2, \mathbb{R}),$$

we have

$$\text{(5)}\quad \mathrm{Res}_{H_1\times H_1} P^D_{\boldsymbol{\nu}}(\partial_Z)\bigl(F(gZ)\det(CZ+D)^{-k}\bigr)$$
$$= (P^D_{\boldsymbol{\nu}}(\partial_Z)F)\begin{pmatrix} g_1z_1 & 0 \\ 0 & g_2z_2\end{pmatrix}(c_1z_1+d_1)^{-k-\ell}(c_2z_2+d_2)^{-k-\ell}$$

for z_1, $z_2 \in H_1$. So by our theory, we can describe the property of $P^D_{\boldsymbol{\nu}}(\partial_Z)$ without knowing ϕ_k or $P^D_{\boldsymbol{\nu}}$ precisely. Any differential operator satisfying (5) is a constant times $P^D_{\boldsymbol{\nu}}(\partial_Z)$. The actual formula for $P^D_{\boldsymbol{\nu}}(T)$ in this case is given by the *homogenized* Gegenbauer polynomials that will be explained below. For the sake of simplicity, we write $P^D_\ell(T) = P^D_{\boldsymbol{\nu}}(T)$ for $\boldsymbol{\nu}$ defined above for each ℓ. To describe a neat generating function of P^D_ℓ for all $\ell \geq 0$, we must change P^D_ℓ by constant times for each ℓ. This is a bit tricky point. The differential operator is determined only up to constant for each ℓ, but we have various ℓ, so we can multiply various different constants to P^D_ℓ depending on ℓ. So as a whole, we have infinitely many different normalization. But to give a neat generating function, we should define a certain nice normalization, and there is no definite theory for such choices. We may say if the result is beautiful, then it is a good normalization. Anyway, we have a well known formula in this case. We consider the following series.

$$\frac{1}{(1-2tz+z^2)^\lambda} = \sum_{a=0}^{\infty} C^\lambda_a(t)z^a.$$

Here z and t are variables. Then $C^\lambda_a(t)$ is a polynomial in t of degree a and called a Gegenbauer polynomial of degree a. More explicitly this is given by

$$C^\lambda_a(t) = \sum_{0\leq s\leq a/2}(-1)^s\binom{a-s+\lambda-1}{a-s}\binom{a-s}{s}(2t)^{a-2s}.$$

If we put

$$P_\ell(T) = (t_{11}t_{22})^{\ell/2}C^{k-1}_\ell\left(\frac{t_{12}}{\sqrt{t_{11}t_{22}}}\right),$$

then we see that $P_\ell(T)$ is a constant times $P^D_\ell(T)$. (The constant can be calculated but omitted here.) The generating function of $P_\ell(T)$ is given by

$$\sum_{\ell=0}^{\infty} P_\ell(T)z^\ell = \frac{1}{(1-2t_{12}z+t_{11}t_{22}z^2)^{k-1}}.$$

Here $\phi_k(P^D_\ell) = t^\ell_{12}$ is simple, but we see $P^D_\ell(T)$ itself is not so simple.

In the same way, put

$$T = \begin{pmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22}\end{pmatrix}$$

for a $2m\times 2m$ symmetric matrix T and $m\times m$ symmetric matrices T_{11} and T_{22} of variables. For A_1, $A_2 \in GL_m(\mathbb{C})$. we have $\det({}^tA_1T_{12}A_2)^\ell =$

$\det(A_1)^\ell \det(A_2)^\ell \det(T_{12})^\ell$. Consider the polynomial $P(T) \in \mathbb{C}[T]$ such that $\phi_k(P(T)) = \det(T_{12})^\ell$. Then by Theorem 2.3 (ii), we have

$$P({}^tATA) = \det(A_1)^\ell \det(A_2)^\ell P(T) \quad \text{for } A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Of course $P(T)$ is a certain linear combination of the descendind basis corresponding to indices of monomials in the expansion of $\det(T_{12})^\ell$. The indices $\boldsymbol{\nu} = (\nu_{ij}) \in \mathcal{N}$ appearing here satisfy

$$\nu_{ij} = 0 \text{ for all } 1 \le i, j \le m, \text{ and } m+1 \le i, j \le 2m.$$

This means that we have

$$D_{ij}P = 0 \text{ fo } 1 \le i, j \le m,\ m+1 \le i, j \le 2m.$$

These conditions mean that $P(T)$ satisfies Condition 1.1 for $\rho = \det^\ell \otimes \det^\ell$. So for any function $F(Z)$ on $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ {}^tZ_{12} & Z_{22} \end{pmatrix} \in H_{2m}$, any elements $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in Sp(m, \mathbb{R})$ $(i = 1, 2)$, and

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \in Sp(2m, \mathbb{R}),$$

we have

$$\text{(6)} \quad \mathrm{Res}_{H_m \times H_m} P(\partial_Z)(F(gZ)\det(CZ+D)^{-k})$$
$$= (P(\partial_Z)F)\begin{pmatrix} g_1Z_{11} & 0 \\ 0 & g_2Z_{22} \end{pmatrix} \det(C_1Z_{11}+D_1)^{-k-\ell} \det(C_2Z_{22}+D_2)^{-k-\ell}.$$

The explicit closed formula for P is not known except for the case $n = 2$, $n = 4$ for general ℓ, or for the case $\ell = 1, 2$ for general $n = 2m$, but there are several algorithm to obtain $P(T)$ explicitly ([11],[19]).

In the above, we explained the case $r = 2$ and $\rho = \det^\ell \otimes \det^\ell$, but the general cases are essentially the same. We continue a little more on the case $r = 2$ and $n_1 = n_2 = m$. To adjust notation for a later use, we change the notation from T to an $2m \times 2m$ symmetric matrix $S = (S_{ij})$ with $m \times m$ symmetric matrices S_{ii} and consider polynomials $P(S) \in \mathbb{C}[S]$ in components s_{ij} of S. Then the pluriharmonicity conditions on a vector P in Condition 1.1 suggest that the components P_i of P (i.e. coefficients of a vector P with respect to some basis) should satisfy that the polynomial $\phi_k(P_i)$ is a linear span of monomials $S^{\boldsymbol{\nu}}$ with $\boldsymbol{\nu} = (\nu_{ij})$ such that $\nu_{ij} = 0$ for $1 \le i, j \le m$ and $m+1 \le i, j \le n$. Now consider the ring $\mathbb{C}[S_{12}]$ of polynomials generated by s_{ij} with $1 \le i \le m$ and $m+1 \le j \le 2m$, i.e. components of S_{12}. The ring $\mathbb{C}[S_{12}]$ can be regarded as a representation space of $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$ by $S_{12} \to {}^tA_1S_{12}A_2$ for $A_1, A_2 \in GL_m(\mathbb{C})$. It is well known that this representation on $\mathbb{C}[S_{12}]$ is decomposed into the sum of irreducible representations $\rho_{m,\lambda}\otimes\rho_{m,\lambda}$ of $GL_m(\mathbb{C})\times GL_m(\mathbb{C})$ corresponding to a Young diagram $\lambda=(\lambda_1,\ldots,\lambda_m)$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0$. We will describe the representation space V_λ in $\mathbb{C}[S_{12}]$ of $\rho_{m,\lambda}\otimes\rho_{m,\lambda}$ much later, but here we prepare notation. We write $W=S_{12}$. For any integer i with $1\leq i\leq m$, denote by W_i the $i\times i$ principal minor of W (i.e. the determinant of the first i rows and columns of W). We put

$$W_\lambda = W_1^{\lambda_1-\lambda_2}W_2^{\lambda_2-\lambda_3}\cdots W_m^{\lambda_m}. \tag{7}$$

Up to now, we assumed that $r=2$ and also $n_1=n_2=m$. Now we treat the case that n_1 might be different from n_2. For a Young diagram $\lambda=(\lambda_1,\lambda_2,\ldots,)$, we put $m=\max\{d:\lambda_d\neq 0\}$ and call this a depth of λ. Then there exists a vector $P(T)$ for an $n\times n$ symmetric matrix T with $n=n_1+n_2$ satisfying Condition 1.1 only when $\rho_1=\rho_{n_1,\lambda}$, $\rho_2=\rho_{n_2,\lambda}$ with $m\leq\min(n_1,n_2)$. We write an $n\times n$ matrix T as

$$T=\begin{pmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22}\end{pmatrix}$$

for an $n_1\times n_1$ matrix T_{11} and an $n_2\times n_2$ symmetric matrix T_{22}.

We prepare an $m\times n_1$ matrix U and an $m\times n_2$ matrix V and put

$$\mathbb{U}=\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}.$$

Then we have $2m\times 2m$ matrix

$$\mathbb{U}T\,{}^t\mathbb{U}=\begin{pmatrix} UT_{11}\,{}^tU & UT_{12}\,{}^tV \\ V\,{}^tT_{12}\,{}^tU & VT_{22}\,{}^tV\end{pmatrix}$$

As before, we consider an $2m\times 2m$ matrix S and

$$S=\begin{pmatrix} S_{11} & S_{12} \\ {}^tS_{12} & S_{22}\end{pmatrix} \qquad S_{11},\ S_{22} \text{ are } m\times m \text{ matrices.}$$

Now, for a Young diagram λ with depth m, we consider a polynomial $P_{0,\lambda}(S)\in\mathbb{C}[S]$ such that $\phi_k(P_{0,\lambda})=(S_{12})_\lambda$, where $(S_{12})_\lambda$ is defined as (7). Then put

$$P_\lambda(U,V,T)=P_0\begin{pmatrix} UT_{11}\,{}^tU & UT_{12}\,{}^tV \\ V\,{}^tT_{12}\,{}^tU & VT_{22}\,{}^tV\end{pmatrix}$$

Then this is a realization of the representation $\lambda_{n_1,\lambda}\otimes\lambda_{n_2,\lambda}$ of $GL_{n_1}(\mathbb{C})\times GL_{n_2}(\mathbb{C})$ with respect to a basis of the tensors of bideterminants of U and V. Here bideterminants realization is explained as follows. For any subset of $I\subset\{1,\ldots,m\}$ and $J\subset\{1,\ldots,n_1\}$ with $|I|=|J|$, we denote by U_{IJ} the minor of U taking rows whose numbers are in I and columns whose numbers are in J. For any i with $1\leq i\leq m$, we put

$$[i]=\{1,2,\ldots,i\}.$$

For λ, we consider a vector space V of polynomials $p(U,V)$ in components of U and V spanned by

$$p(U,V)=\prod_{q=1}^{m}\prod_{\ell=1}^{\lambda_q-\lambda_{q+1}} U_{[q],I_l^{(q)}}V_{[q],J_\ell^{[q]}},$$

where $I_\ell^{(q)}$ and $J_\ell^{(q)}$ run over $\lambda_q-\lambda_{q+1}$ subsets of $[n_1]$ and $[n_2]$ such that $|I_\ell^{(q)}|=|J_\ell^{(q)}|=q$ (here put $\lambda_{m+1}=0$). These are called bideterminants. Then $p(U,V)\to p(UA_1,VA_2)$ for $(A_1,A_2)\in GL_{n_1}(\mathbb{C})\times GL_{n_2}(\mathbb{C})$ on $\langle p\rangle$ gives a realization of $\rho_{n_1,\lambda}\otimes\rho_{n_2,\lambda}$. For example, if $\lambda_1=\cdots=\lambda_m=\ell$ and $n_1=n_2=m$, then $V=\mathbb{C}\det(UV)^\ell$. In general we have $\phi_k(P(U,V,T))=(UT_{12}^tV)_\lambda$. Here $P_\lambda(UA_1,VA_2,T)=P_\lambda(U,V,AT\,{}^tA)$ for $A=diag(A_1,A_2)$. This is a linear combination of bideterminants with polynomial coefficients in $\mathbb{C}[T]$. The bideterminants themselves are not linearly independent, so if we want to write a vector w.r.t. a basis, we should choose some basis, but we omit details.

Now notation being as above, the differential operator $P_\lambda(\partial_Z)$ gives a differential operator from weight det^k to $det^k\rho_{n_1,\lambda}\otimes det^k\rho_{n_2,\lambda}$.

Next we consider the case that $r>2$. In this case, if we write

$$T=(T_{ij})_{1\le i,j\le r}\quad\text{where } T_{ij} \text{ is an } n_i\times n_j \text{ matrix}$$

then for the polynomial $P(T)$ satisfying Condition 1.1 for the partition $n_1,\ \dots,\ n_r$, the polynomial $\phi_k(P)$ should be a polynomial in components of T_{ij} for $i\neq j$. The group $GL_{n_1}(\mathbb{C})\times\cdots\times GL_{n_r}(\mathbb{C})$ acts on the space $\mathbb{C}[T_{ij};i\neq j]$, and the irreducible decomposition of this representation is given in principle by using the Littelwood-Richardson rule but it seems not so simple(see [17] Theorem 3.4).

As a conclusion, in any case, Theorem 2.3 characterizes the differential operators that satisfies Condition 1.1.

3. A generating series of descending basis

We have a sort of universal generating series of descending basis. Since basis is indexed by indices in $\mathcal{N}$, we use a dummy $n\times n$ symmetric matrix $X=(x_{ij})$ of variables and we define $X^{\boldsymbol{\nu}}$ for $\boldsymbol{\nu}\in\mathcal{N}$ as before by (4). We consider a formal power series

$$G_n(T,X)=\sum_{\boldsymbol{\nu}\in\mathcal{N}}P_{\boldsymbol{\nu}}(T)X^{\boldsymbol{\nu}}$$

in variables x_{ij} such that $P_{\boldsymbol{\nu}}(T)$ is a constant multiple of $P_{\boldsymbol{\nu}}^D(T)$. In order to obtain a neat generating series, we must put

$$P_{\boldsymbol{\nu}}(T)=2^\nu(k)_\nu(2k-2)_\nu P_{\boldsymbol{\nu}}^D(T),$$

where $\nu=\deg(\boldsymbol{\nu})=(1/2)\sum_{i,j=1}^n\nu_{ij}$ and $(x)_\nu=x(x+1)\cdots(x+\nu-1)$. Then there exists a formula to describe the series $G_n(T,X)$. Since this is explained in other places several times (e.g. [15], [17]), we do not repeat the details, but we state the essence very shortly.

For each $i = 0, \dots, n$, we define a polynomial $\sigma_i(T, X)$ in components of T and X by

$$\det(x1_n - TX) = \sum_{i=0}^{n} (-1)^i \sigma_i(T, X) x^{n-i}.$$

So we have $\sigma_0(T, X) = 1$ and $\sigma_n(T, X) = \det(T)\det(X)$. We assume $2k \in \widehat{\mathbb{C}}$. Then $G_n(T, X)$ has the following property.
(1) G_n is a formal power series $\mathcal{G}_n(\sigma_1, \dots, \sigma_n)$ in σ_1 to σ_n. In particular, we have

$$\mathcal{G}_1(\sigma_1) = \left(1 - \frac{\sigma_1}{2}\right)^{2-2k}.$$

and

$$\mathcal{G}_2(\sigma_1, \sigma_2) = \left(\left(1 - \frac{\sigma_1}{2}\right)^2 - \sigma_2\right)^{1-k}.$$

This $\mathcal{G}_2$ can be regarded as the generating function of Gegenbauer polynomials. We also have a nice closed formula for $\mathcal{G}_3(\sigma_1, \sigma_2, \sigma_3)$, but we omit it here (See [15]).

(2) We may regard $\sigma_1, \dots, \sigma_n$ as algebraically independent variables below. Then we have

$$\mathcal{G}_{n-1}(\sigma_1, \dots, \sigma_{n-1}) = \mathcal{G}_n(\sigma_1, \dots, \sigma_{n-1}, 0).$$

Here the original meaning of σ_i depends on n, but we are ignoring the difference. In other words, the above equality means as a formal equality, as well as the equality between series in σ_i defined for $n-1$.
(3) We define the partial derivative $\partial_a = \frac{\partial}{\partial \sigma_a}$ for each a. For each n, we have a certain explicitly written second order differential operator $\mathcal{M}_n$ of ∂_a ($1 \leq a \leq n-1$) whose coefficients are constants or constant times σ_b for some $1 \leq b \leq n-1$, such that for some explicit constants c_i we have

$$\mathcal{G}_n(\sigma_1, \dots, \sigma_n) = \sum_{i=0}^{\infty} c_i \sigma_n^i \mathcal{M}_n^i \mathcal{G}_{n-1}(\sigma_1, \dots, \sigma_{n-1}).$$

Since $\mathcal{G}_n$ is obtained by iteratedly differntiating generating series of smaller degrees in a unified way, our generating function would be called a *universal* generating series. If we consider $G_n(\partial_Z, X)$, then this is a *generic* differential operator since it is a source of any differential operators satisfying Condition 1.1 under some representation theoretical mapping (See [17] Theorem 3.1).

4. Pullback formulas

Finally we explain pullback formulas that is one of the strong motivations to our theory. For any even integer k with $k > n+1$ and any partition $n = n_1 + n_2$ ($n_i \in \mathbb{Z}_{\geq 1}$), we denote by $E_k^n(Z)$ the Siegel Eisenstein series of degree n of weight k of $\Gamma_n = Sp(n, \mathbb{Z})$ defined by

$$E_k^n(Z) = \sum_{(C,D)} \det(CZ + D)^{-k},$$

where (C, D) runs over representatives of coprime symmetric pairs by the multiplication by $GL_n(\mathbb{Z})$ from the left.

Theorem 4.1 ([7],[4],[27],[23],[24],[19])**.** *Let P be a polynomial satisfying Condition 1.1 for (n_1, n_2) for a Young diagram λ of depth $m \geq 1$. Then we have*

$$(P(\partial_Z)E_k^n(Z)) \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \sum_{t=m}^{\min(n_1,n_2)} c_t \sum_{j=1}^{e_r} D(f_{t,j})[f_{t,j}]_t^{n_1}(Z_1) \otimes [f_{t,j}]_t^{n_2}(Z_2)$$

for some **explicitly given constants** *c_t for $P = P_\lambda^D$ independent of $f_{t,j}$.*

Here, for each t, we fix an orthonormal basis $\{f_{t,j}\}_{1 \leq j \leq e_r}$ of $S_{det^k \rho_{t,\lambda}}(\Gamma_t)$. Of course this depends on a choice of the Petersson inner product of $S_{\det^k \rho_{t,\lambda}}(\Gamma_t)$, but we have some standard choice, and we can give c_t explicitly for that choice. We denote by $[f_{t,j}]_t^{n_i}$ the Klingen lift of $f_{t,j}$ to $A_{\det^k \rho_{n_i,\lambda}}(\Gamma_{n_i})$. We put

$$D(f_{t,j}) = \zeta(k)^{-1} \prod_{i=1}^{t} \zeta(2k - 2i)^{-1} L(k - t, f_{t,j}, St)$$

where $L(s, f, St)$ is the standard L function of a Siegel modular form f. For the details, see [19].

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Department of Mathematics, Graduate School of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka,Osaka, 560-0043 Japan

E-mail address: `ibukiyam@math.sci.osaka-u.ac.jp`