Summary on the Potential Function of the Colored Jones Polynomial with Arbitrary Colors

Shun Sawabe
Waseda University

1 Introduction

Kashaev [8] observed that a certain limit of the Kashaev invariant for some hyperbolic knots is equal to the hyperbolic volume of their complements. Murakami-Murakami [10] proved that the Kashaev invariant for a link L coincides with the colored Jones polynomial $J_N(L;q)$ evaluated at the root of unity $\xi_N = e^{\frac{2\pi\sqrt{-1}}{N}}$, and reformulated Kashaev's conjecture as the volume conjecture.

Conjecture 1.1 (the volume conjecture [10])**.** *For any knot K,*

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K;q=\xi_N)|}{N} = v_3||K||,$$

where v_3 is the volume of the ideal regular tetrahedron in the three-dimensional hyperbolic space and $||\cdot||$ is the simplicial volume for the complement of K.

Many variations and generalizations of the volume conjecture have been proposed. One of them is the Chen-Yang conjecture.

Conjecture 1.2 (the Chen-Yang conjecture [3])**.** *For any 3-manifold M with a complete hyperbolic structure of the finite volume,*

$$2\pi \lim_{r\to\infty} \frac{\log TV_r(M, u=\xi_r)}{r} = \mathrm{Vol}(M),$$

where r runs over all odd integers, $TV(M)$ is a Turaev-Viro invariant of M and $\mathrm{Vol}(M)$ is a hyperbolic volume of M.

Detcherry-Kalfragianni-Yang [4] proved that for an odd integer $r \geq 3$ and r-th root of unity u, the Turaev-Viro invariant $TV(S^3\setminus L, u)$ for the complement of a link L can be written as a summation of $|J_{\boldsymbol{i}}(L;u^2)|^2$ with respect to $\boldsymbol{i}$, and proved Conjecture 1.2 for the complements of the figure-eight knot and the Borromean rings.

These conjectures are still open, but we can provide theoretical evidence. One idea of proof of the volume conjecture is to use the saddle point method. For a sufficiently large integer N, if a certain quantity Q_N depending on N can be asymptotically written of the form

$$Q_N \sim \int \cdots \int_\Omega P_N e^{\frac{N}{2\pi\sqrt{-1}}\Phi(z_1,\ldots,z_\nu)} dz_1 \cdots dz_\nu,$$

where P_N grows at most polynomially and Ω is a region in $\mathbb{C}^\nu$, then, a saddle point of the function $\Phi(z_1, \ldots, z_\nu)$ contributes to the limit of Q_N.

Definition 1.3. *We call the function $\Phi(z_1, \ldots, z_\nu)$ a potential function of Q_N.*

Yokota [15] established the relationship between a saddle point equation of the potential function of the Kashaev invariant and the triangulation of a hyperbolic knot complement. Cho-Murakami [2] considered a potential function of the colored Jones polynomial evaluated at the root of unity. The upshot of their works is as follows: The saddle point equation of the potential function coincides with the 'gluing equation' of the triangulation. In this study, we will consider the potential function of $J_{\boldsymbol{i}}(L; q = \xi_N)$ and explore geometric meanings. Specifically, we obtain the following theorem:

Theorem 1.4 ([11])**.** *Let D be a diagram of a hyperbolic link with n components, and let $\mathbf{1}$ be $(1, \ldots, 1) \in \mathbb{Z}^n$. The point $(\mathbf{1}, \sigma_1(\mathbf{1}), \ldots, \sigma_\nu(\mathbf{1}))$ is a saddle point of the function $\Phi_D(a_1, \ldots, a_n, w_1, \ldots, w_\nu)$ and gives a complete hyperbolic structure to the link complement.*

Besides, a parametrized potential function is conjectured to lead the A-polynomial of a knot [7, 16]. On the other hand, the AJ conjecture [5] also states the relationship between the colored Jones polynomial and the A-polynomial. In this context, we will view the AJ conjecture from the perspective of the potential function.

2 Potential function

2.1 Colored Jones polynomial

Let N, N' be integers, and let m, m' be half-integers satisfying $N = 2m + 1$ and $N' = 2m' + 1$. Moreover, let V and V' be an N-dimensional vector space and an N'-dimensional vector space whose bases are $\{e, \ldots, e_m\}$ and $\{e'_{-m'}, \ldots, e'_{m'}\}$ respectively. The colored Jones polynomial is obtained by the following R-matrix $R : V \otimes V' \to V' \otimes V$

$$\begin{aligned} &R_{VV'}(e_i \otimes e'_j) \\ &= \sum_{k=0}^{\min\{m+i,\ m'-j\}} (-1)^{k+k(m+m')+2ij} q^{-ij-\frac{k(i-j)}{2}+\frac{k(k+1)}{4}} \\ &\qquad \times \frac{\{m-i+k\}!\{m'+j+k\}!}{\{k\}!\{m-i\}!\{m'+j\}!} e'_{j+k} \otimes e_{i-k}, \end{aligned}$$

and its inverse [9, 11]. Here, for an integer k,

$$\{k\} = q^{\frac{k}{2}} - q^{-\frac{k}{2}},\ \{k\}! = \{k\}\{k-1\}\cdots\{1\},\ \{0\}! = 1.$$

Remark 2.1. The integer N is a dimension of a $\mathcal{U}_q(sl_2)$-representation assigned to a link component. We call the integer a color.

We put

$$R_{VV'}(e_i \otimes e'_j) = \sum_{k,l} (R^+)_{ij}^{kl} e'_k \otimes e_l,$$
$$R_{VV'}^{-1}(e'_i \otimes e_j) = \sum_{k,l} (R^-)_{ij}^{kl} e_k \otimes e'_l.$$

We assign these coefficients to each crossing of a diagram of a link L as shown in Figure 1. We also assign $(-1)^{N-1}q^{\pm i}$ to the maximum points of the diagram depending on the

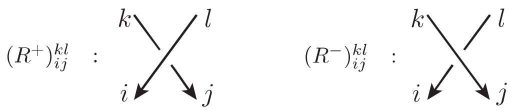


Figure 1: A crossing and the coefficient of the R-matrix.

orientation of the string as shown in Figure 2. Note that indices are labeled to the edges

Figure 2: A maximum point and $(-1)^{N-1}q^{\pm i}$.

of the diagram. For arguments of the potential function later, we change the indices i, j, k, and l to the ones labeled to four regions around the crossing. See Figure 3. Under

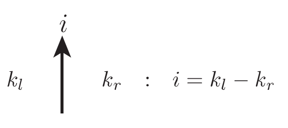


Figure 3: An index i labeled to an edge and indices k_l, k_r labeled to regions.

the change of indices, we obtain the R-matrix $R^{\pm}(m, m', k_{j_1}, k_{j_2}, k_{j_3}, k_{j_4})$, where k_{j_1}, k_{j_2}, k_{j_3}, and k_{j_4} are indeces as shown in Figure 4. Here, the indices i, j, k, l and $k_{j_1}, \ldots, k_{j_4}$ satisfy

$$i = k_{j_2} - k_{j_1},$$
$$j = k_{j_3} - k_{j_2},$$
$$k = k_{j_2} + k_{j_4} - k_{j_1} - k_{j_3}.$$

The colored Jones polynomial $J_{\boldsymbol{i}}(L; q)$ for n-component link L, where $\boldsymbol{i} = (i_1, \ldots, i_n)$ is an n-tuple of colors, is the multiplication of all these factors with modification for the Reidemeister move I. We normalize the colored Jones polynomial so that the polynomial for a trivial link with any colors $\boldsymbol{i}$ is equal to 1.

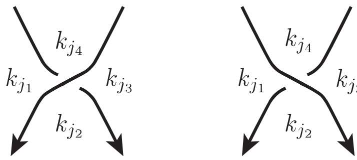


Figure 4: Indices around a crossing.

2.2 Potential function

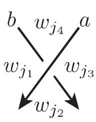
Let L be an n-component hyperbolic link. We fix a diagram D of L. A procedure to obtain a potential function of $J_{\boldsymbol{i}}(L;\xi_N^p)$, where $p = 1$ or 2, is as follows: First, we obtain a local potential function $\Phi_{c,p}^{\pm}$ assigned to each crossing c of D by approximating the R-matrix with continuous functions. At this step, each quantum factorial becomes the dilogarithm function defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-x)}{x}dx.$$

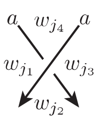
Let a_N and b_N be colors. We put

$$a = \lim_{N\to\infty}\frac{a_N}{N}, \text{ and } b = \lim_{N\to\infty}\frac{b_N}{N}.$$

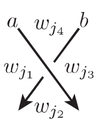
Functions $\Phi_{c,p}^{\pm}$ are as follows:



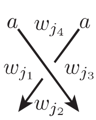
: $\Phi_{c,p}^{+} = f_p^{+}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$



: $\Phi_{c,p}^{+} = p(\pi\sqrt{-1}a)^2 + f_p^{+}(a, a, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$



: $\Phi_{c,p}^{-} = f_p^{-}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$



: $\Phi_{c,p}^{-} = -p(\pi\sqrt{-1}a)^2 + f_p^{-}(a, a, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$

Here, $w_{j_i}=\xi_N^{k_{j_i}}$, with $i=1,\dots,4$, and

$$f_p^+(a,b,w_{j_1},w_{j_2},w_{j_3},w_{j_4})=\frac{1}{p}\left\{\pi\sqrt{-1}p^2\frac{a+b}{2}\log\frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}}\right.$$
$$-p^2\log\frac{w_{j_2}}{w_{j_1}}\log\frac{w_{j_3}}{w_{j_2}}-\mathrm{Li}_2\left(e_a^p\frac{w_{j_4}^p}{w_{j_3}^p}\right)-\mathrm{Li}_2\left(e_b^p\frac{w_{j_4}^p}{w_{j_1}^p}\right)$$
$$\left.+\mathrm{Li}_2\left(\frac{w_{j_2}^p w_{j_4}^p}{w_{j_1}^p w_{j_3}^p}\right)+\mathrm{Li}_2\left(e_a^p\frac{w_{j_1}^p}{w_{j_2}^p}\right)+\mathrm{Li}_2\left(e_b^p\frac{w_{j_3}^p}{w_{j_2}^p}\right)-\frac{\pi^2}{6}\right\},$$

$$f_p^-(a,b,w_{j_1},w_{j_2},w_{j_3},w_{j_4})=\frac{1}{p}\left\{-\pi\sqrt{-1}p^2\frac{a+b}{2}\log\frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}}\right.$$
$$+p^2\log\frac{w_{j_3}}{w_{j_4}}\log\frac{w_{j_4}}{w_{j_1}}-\mathrm{Li}_2\left(e_a^p\frac{w_{j_1}^p}{w_{j_4}^p}\right)-\mathrm{Li}_2\left(e_b^p\frac{w_{j_3}^p}{w_{j_4}^p}\right)$$
$$\left.-\mathrm{Li}_2\left(\frac{w_{j_2}^p w_{j_4}^p}{w_{j_1}^p w_{j_3}^p}\right)+\mathrm{Li}_2\left(e_a^p\frac{w_{j_2}^p}{w_{j_3}^p}\right)+\mathrm{Li}_2\left(e_b^p\frac{w_{j_2}^p}{w_{j_1}^p}\right)+\frac{\pi^2}{6}\right\},$$

with $e_a=e^{\pi\sqrt{-1}a}$. Note that functions $\Phi_{c,p}^{\pm}$ have a modification term with respect to the Reidemeister move I for crossings between the same component. A potential function $\Phi_{D,p}(\boldsymbol{a},w_w,\dots,w_\nu)$ is a summation of local potential functions $\Phi_{c,p}$ with c running over all crossings of D. Here, $\boldsymbol{a}=(a_1,\dots,a_n)$ is an n-tuple of

$$a_k=\lim_{N\to\infty}\frac{i_k}{N}.$$

We can easily verify that $\Phi_{D,p}$ is obtained from $\Phi_{D,1}$. Therefore, we mainly consider $\Phi_{D,1}$ and put $\Phi_D=\Phi_{D,1}$. This potential function essentially coincides with Yoon's generalized potential function [17].

3 Geometric meanings

3.1 Thurston's Triangulation

We would like to establish the relationship between the potential function and geometry of the complement M of a hyperbolic link $L=L_1\cup\cdots\cup L_n$. The method we mainly use is Thurston's triangulation [13]. Namely, as shown in Figures 5 and 6, we put an octahedron between each crossing and decompose each octahedron into five ideal tetrahedra. Here, u_i and v_i, with $i=1,\dots,5$, are moduli of the ideal tetrahedra, and

$$z'=\frac{1}{1-z},\quad z''=1-\frac{1}{z}$$

for a complex number z. Following [1], we review how to determine the geometry of M from the triangulation. The link complement admits a hyperbolic structure if all tetrahedra are glued well. Assume that there are $k(e)$ tetrahedra around an edge e, and

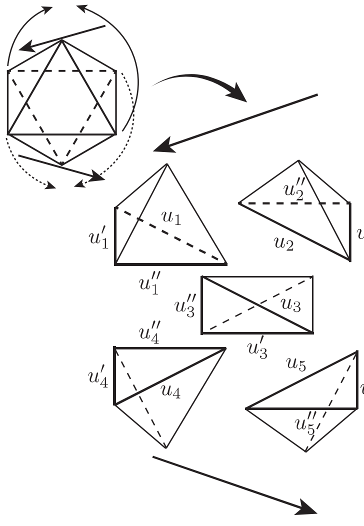


Figure 5: Ideal tetrahedra on a positive crossing.

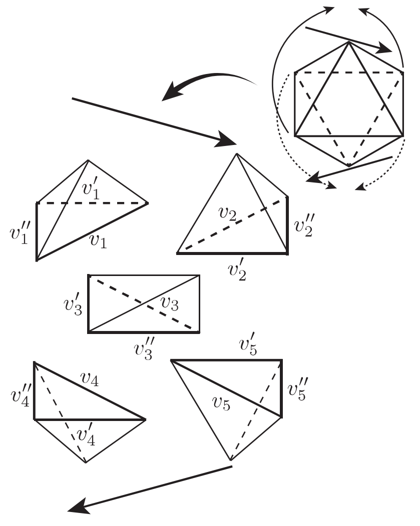


Figure 6: Ideal tetrahedra on a negative crossing.

let $z_1, \dots, z_{k(e)}$ be moduli of these tetrahedra. Then, all tetrahedra are glued well if and only if

$$z_1 \cdots z_{k(e)} = 1$$

holds for all edges e in the triangulation. We call this equation the 'gluing equation'. We can also determine completeness by the triangulation. In general, a component of the boundary of compactification of M admits a similarity structure, that is, a curve γ in the boundary induces the action of the form

$$\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}, \quad a, b \in \mathbb{C}.$$

We call the coefficient a the dilation component of γ and write $\delta(\gamma)$. M is complete if and only if components of the boundary admit a Euclidean structure. Namely, M is complete if and only if the dilation components of meridians and longitudes are all equal to 1. This is called a completeness condition. We can calculate the dilation component from cross-sections of truncated ideal tetrahedra. Cutting off ideal vertices of ideal tetrahedra with horospheres produces Euclidean triangles. A modulus of a vertex of the triangle is the modulus of the ideal tetrahedron's edge that contains the vertex. Each modulus of the triangle contributes to the dilation component of a curve γ in the manner shown in Figure 7.

3.2 The saddle point equation

First, we fix the parameters $\boldsymbol{a} = (a_1, \dots, a_n)$. If we put

$$u_1 = e_a \frac{w_{j_1}}{w_{j_2}}, \quad u_2 = e_a^{-1} \frac{w_{j_3}}{w_{j_4}}, \quad u_3 = \frac{w_{j_2} w_{j_4}}{w_{j_1} w_{j_3}}, \quad u_4 = e_b^{-1} \frac{w_{j_1}}{w_{j_4}}, \quad u_5 = e_b \frac{w_{j_3}}{w_{j_2}},$$

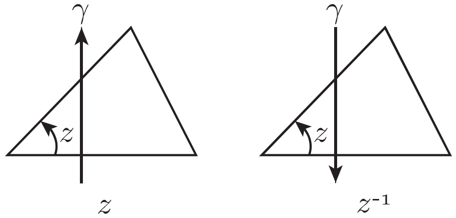


Figure 7: The contribution of each modulus of the triangle to the dilation component of γ.

$$v_1 = e_a^{-1}\frac{w_{j_4}}{w_{j_1}}, \quad v_2 = e_a\frac{w_{j_2}}{w_{j_3}}, \quad v_3 = \frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}}, \quad v_4 = e_b\frac{w_{j_2}}{w_{j_1}}, \quad v_5 = e_b^{-1}\frac{w_{j_4}}{w_{j_3}},$$

in Figures 5 and 6, then, the gluing equation automatically holds for oblique edges and interior edges.

Remark 3.1. Thurston's triangulation is, in fact, a triangulation of a link complement minus two points, and we can verify that a boundary of a neighborhood of each puncture is a sphere. However, this contradicts that Euclidean triangles are glued. Therefore, a boundary of each puncture is collapsed and we can treat the triangulation as that of the link complement.

Derivatives with parameters assigned to regions of the diagram correspond to the gluing equations for the remaining horizontal edges in Figures 5 and 6. For example,

$$w_{j_1}\frac{\partial \Phi_c^+}{\partial w_{j_1}} = \pi\sqrt{-1}\frac{a-b}{2} + \log\left(1 - e_a\frac{w_{j_1}}{w_{j_2}}\right)^{-1}\left(1 - e_b^{-1}\frac{w_{j_1}}{w_{j_4}}\right)^{-1}\left(1 - \frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}}\right)$$

holds. Therefore, at a glance, the derivative with respect to w_i is

$$w_i\frac{\partial \Phi_D}{\partial w_i} = \frac{\pi\sqrt{-1}}{2}r(a_1,\ldots,a_n) + \log G_i,$$

where $r(a_1,\ldots,a_n)$ is a linear polynomial with respect to $a_1,\ldots,a_n$, and G_i is a product of moduli of tetrahedra around the edge lying on the region with w_i assigned. Considering the contribution of each color to $r(a_1,\ldots,a_n)$, however, we can varify that $r(a_1,\ldots,a_n) = 0$. Therefore, the system of equations

$$\exp\left(w_i\frac{\partial \Phi_D}{\partial w_i}\right) = 1, \quad i = 1,\ldots,\nu$$

coincides with the gluing equation for the horizontal edges in Figures 5 and 6. Hence a saddle point $(\sigma_1(\boldsymbol{a}),\ldots,\sigma_\nu(\boldsymbol{a}))$ determines a hyperbolic structure of the link complement that is not necessarily complete. Here, we choose the saddle point such that $(\sigma_1(\mathbf{1}),\ldots,\sigma_\nu(\mathbf{1}))$ gives a hyperbolic structure with the volume $\mathrm{Vol}(M)$, where $\mathbf{1} = (1,\ldots,1)$. Let $M_{\boldsymbol{a}}$ be a manifold with this hyperbolic structure. The dilation component of the meridian m_j of the link component L_j with a_j labeled is $\delta(m_j) = e^{-2\pi\sqrt{-1}a_j}$. This means that the action of the meridian generally rotates faces $2\pi(1-a_j)$ and makes a singular set. Therefore, the

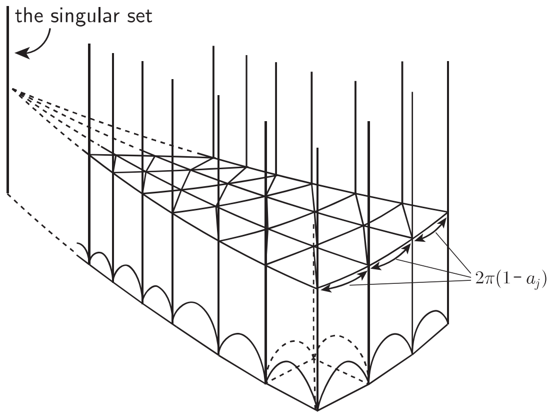


Figure 8: The developing image of $M_{\boldsymbol{a}}$ in $\mathbb{H}^3$.

developing image of $M_{\boldsymbol{a}}$ is as shown in Figure 8. This implies that $M_{\boldsymbol{a}}$ is a cone-manifold with cone-angles $2\pi(1-a_j)$ around L_j, with $j=1,\dots,n$. When we regard a_j, with $j=1,\dots,n$, as variables,

$$\exp\left(\frac{1}{\pi\sqrt{-1}}\frac{\partial\Phi_D}{\partial a_j}\right)=\delta(\tilde{l}_j) \tag{1}$$

holds, where $\tilde{l}_j$ is the longitude of the component L_j with $\mathrm{lk}(\tilde{l}_j,L_j)=0$. Namely, the saddle point equation with respect to a_j coincides with the completeness equation. This implies the following theorem:

Theorem 1.4. Let D be a diagram of a hyperbolic link with n components, and let $\mathbf{1}$ be $(1,\dots,1)\in\mathbb{Z}^n$. The point $(\mathbf{1},\sigma_1(\mathbf{1}),\dots,\sigma_\nu(\mathbf{1}))$ is a saddle point of the function $\Phi_D(a_1,\dots,a_n,w_1,\dots,w_\nu)$ and gives a complete hyperbolic structure to the link complement.

The idea of proof of (1) is as follows: For example, the derivative of Φ_c^+ with respect to a is

$$\frac{\partial\Phi_c^+}{\partial a}=\frac{\pi\sqrt{-1}}{2}\log\left(1-e_a\frac{w_{j_4}}{w_{j_3}}\right)\left(1-e_a\frac{w_{j_1}}{w_{j_2}}\right)^{-1}\left(1-e_a^{-1}\frac{w_{j_3}}{w_{j_4}}\right)\left(1-e_a^{-1}\frac{w_{j_2}}{w_{j_1}}\right)^{-1}.$$

The ingredient of log is a product of four moduli shown in Figure 9. Therefore, the derivative of the potential function Φ_D with respect to a_j corresponds to two parallel longitudes of L_j that are canceled with $\frac{1}{2}$.

3.3 Witten-Reshetikhin-Turaev invariant

Let $M_{f_1,\dots,f_n}$ be the hyperbolic manifold obtained by Dehn surgery on a link $L=L_1\cup\cdots\cup L_n$ with a framing f_j on L_j, where $j=1,\dots,n$. Let $\Phi(\alpha_1,\dots,\alpha_n,w_1,\dots,w_\nu)$ be the potential function of the Witten-Reshetikhin-Turaev invariant of $M_{f_1,\dots,f_n}$, where

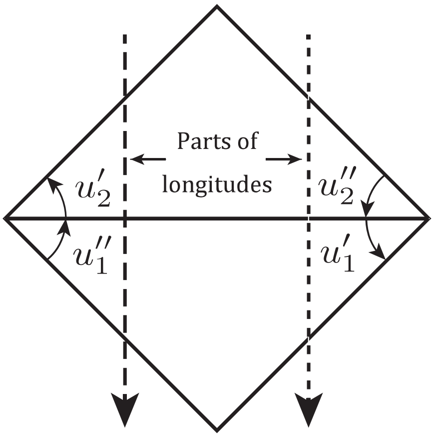


Figure 9: Upper side of a positive crossing.

$\alpha_j = e^{\pi\sqrt{-1}a_j}$. So far, a_j is a real number hence α_j is in the unit circle, but hereinafter, we regard each α_j as a complex parameter that is not necessarily in the unit circle. Using the formula for the Witten-Reshetikhin-Turaev invariant in [9], we can verify that the derivative of Φ with respect to α_j is

$$\exp\left(\alpha_j \frac{\partial \Phi}{\partial \alpha_j}\right) = \alpha_j^{-2f_j}\delta(\tilde{l}_j).$$

Since $\delta(m_j) = \alpha_j^{-2}$, the saddle point equation implies that

$$\delta(m_j)^{-f_j} = \delta(\tilde{l}_j).$$

Assuming that $f_j > 0$ and $|\alpha_j| < 1$, the developing image would be as shown in Figure 10. When we shift the region X in Figure 10 in the direction of the longitude once, it

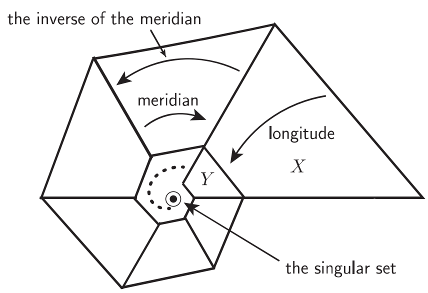


Figure 10: The schematic diagram of the developing image in the case of $f_j = 6$.

reaches the region Y. On the other hand, when we shift the region X in the direction of the inverse of the meridian f_j times, it again reaches the region Y.

4 AJ conjecture

4.1 A-polynomial

A parametrized potential function is conjectured to lead the A-polynomial [7, 16]. Let K be a hyperbolic knot. A factor of the A-polynomial $A_K(l, m)$ is conjectured to be obtained from the system of equations

$$\begin{cases} \exp\left(w_i \dfrac{\partial \Phi_D}{\partial w_i}\right) = 1, \quad (i = 1, \dots, \nu) \\ \exp\left(\alpha \dfrac{\partial \Phi_D}{\partial \alpha}\right) = l^2 \end{cases} \tag{2}$$

by eliminating $w_1, \dots, w_\nu$. The other factor of $A_K(l, m)$ is $l-1$ that corresponds to abelian representations.

4.2 A_q-polynomial and the AJ conjecture

An A_q-polynomial [5] $A_q(K)$ for a knot K is the polynomial defined as an annihilator of $J_K(n) = J_n(K; q)$. If we have a recursion relation of $J_K(n)$

$$\sum_{i=0}^{d} c_i(q, q^n) J_K(n+i) = 0,$$

where $c_i(q, q^n) \in \mathbb{Z}[q, q^n]$, we rewrite it as

$$\left(\sum_{i=0}^{d} c_i(q, Q) E^i\right) J_K(n) = 0,$$

where E and Q are operators defined by

$$(EJ_K)(n) = J_K(n+1), \text{ and } (QJ_K)(n) = q^n J_K(n).$$

E and Q generate a noncommutative algebra

$$\mathcal{A} = \left\{ \sum_{i=0}^{d} c_i(q, Q) E^i \,\middle|\, \begin{array}{l} d \in \mathbb{Z}_{\geq 0} \\ c_i(q, Q) \in \mathbb{Z}[q, Q] \\ EQ = qQE \end{array} \right\}.$$

A set of all annihilating polynomials $I_K = \{P \in \mathcal{A} \mid PJ_K(n) = 0\}$ is a left ideal in $\mathcal{A}$. $\mathcal{A}$ is not a principal ideal domain but a localization $\mathcal{A}_{\mathrm{loc}}$ of $\mathcal{A}$ defined by

$$\mathcal{A}_{\mathrm{loc}} = \left\{ \sum_{k=0}^{\infty} a_k E^k \mid a_k \in \mathbb{Q}(q, Q), \ a_k = 0 \text{ for sufficiently large } k \right\}$$

is a principal ideal domain. Here, the multiplication of monomials in $\mathcal{A}_{\mathrm{loc}}$ is given by

$$aE^k \cdot bE^l = a\sigma^k(b) E^{k+l},$$

where σ is the automorphism of the field $\mathbb{Q}(q, Q)$ defined by

$$\sigma(f)(q, Q) = f(q, qQ).$$

An A_q-polynomial $A_q(K)(E, Q)$ for a knot K is a generator of the annihilating ideal of $J_K(n)$ in $\mathcal{A}_{\mathrm{loc}}$ with the smallest E-degree and coprime coefficients. Garoufalidis [5] proposed the AJ conjecture, which relates the A-polynomial and the colored Jones polynomial.

Conjecture 4.1 (the AJ conjecture)**.** *For any knot K, $A_K(l, m)$ is equal to $\varepsilon A_q(K)(l, m^2)$ up to multiplication by an element in $\mathbb{Q}(m)$, where ε is the evaluation map at $q = 1$.*

4.3 Creative Telescoping and the potential function

The A_q-polynomial can be calculated by creative telescoping. Let us recall this process by following [5, 6]. Let $F(n, k_1, \ldots, k_\nu)$ be a multi-$\mathbb{Z}$-variable discrete function. Creative telescoping is a method to calculate an annihilating polynomial of

$$G(n) = \sum_{\boldsymbol{k}} F(n, \boldsymbol{k}), \quad \boldsymbol{k} = (k_1, \ldots, k_\nu)$$

from $F(n, k_1, \ldots, k_\nu)$. First, we define operators Q, E, Q_i, and E_i by

$$\begin{aligned}
(QF)(n, k_1, \ldots, k_\nu) &= q^n F(n, k_1, \ldots, k_\nu),\\
(EF)(n, k_1, \ldots, k_\nu) &= F(n+1, k_1, \ldots, k_\nu),\\
(Q_iF)(n, k_1, \ldots, k_\nu) &= q^{k_i} F(n, k_1, \ldots, k_\nu),\\
(E_iF)(n, k_1, \ldots, k_\nu) &= F(n, k_1, \ldots, k_i + 1, \ldots, k_\nu).
\end{aligned}$$

These operators generate the noncommutative algebra $\mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle$ with the following relations:

$$Q_iQ_j = Q_jQ_i, \;\; E_iE_j = E_jE_i, \;\; E_iQ_j = q^{\delta_{ij}} Q_jE_i,$$

where $i, j \in \{0, \ldots, \nu\}$, $E_0 = E$, and $Q_0 = Q$.

Definition 4.2. *$F : \mathbb{Z}^{\nu+1} \to \mathbb{Q}(q)$ is called q-hypergeometric if $E_iF/F \in \mathbb{Q}(q, q^n, q^{k_1}, \ldots, q^{k_\nu})$ holds for all $i = 0, \ldots, \nu$.*

Definition 4.3. *A q-hypergeometric discrete function $F(n, \boldsymbol{k})$ is called proper if it is of the form*

$$F(n, \boldsymbol{k}) = \frac{\prod_s (A_s; q)_{a_s n + \boldsymbol{b}_s\cdot\boldsymbol{k} + c_s}}{\prod_t (B_t; q)_{u_t n + \boldsymbol{v}_t\cdot\boldsymbol{k} + w_t}} q^{A(n,\boldsymbol{k})} \xi^{\boldsymbol{k}},$$

where $A_s, B_t \in \mathbb{Q}(q)$, a_s, u_t are integers, $\boldsymbol{b}_s, \boldsymbol{v}_t$ are vectors of r integers, c_s, w_t are variables, $A(n, \boldsymbol{k})$ is a quadratic form, ξ is an r vector of elements in $\mathbb{Q}(q)$, and

$$(A; q)_n = \prod_{i=0}^{n-1} (1 - Aq^i).$$

It is known that a proper q-hypergeometric function has a good recurrence relation.

Theorem 4.4 ([14]). *Every proper q-hypergeometric function $F(n, \boldsymbol{k})$ has a $\boldsymbol{k}$-free recurrence*

$$\sum_{(i,\boldsymbol{j})\in S} \sigma_{i,\boldsymbol{j}}(q^n)F(n+i, \boldsymbol{k}+\boldsymbol{j}) = 0,$$

where S is a finite set, and $\sigma_{i,\boldsymbol{j}}$ are $\mathbb{Q}(q)$-coefficient polynomials.

Theorem 4.4 implies the existence of an annihilating polynomial of $F(n, \boldsymbol{k})$ of the form $P(E, Q, E_1, \ldots, E_\nu) \in \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle$. Expanding $P(E, Q, E_1, \ldots, E_\nu)$ at $(E_1, \ldots, E_\nu) = \mathbf{1}^\nu = (1, \ldots, 1) \in \mathbb{Z}^\nu$, we have

$$P_0(E, Q) + \sum_{i=1}^{\nu}(E_i - 1)R_i(E, Q, E_1, \ldots, E_\nu),$$

where $P_0(E, Q) = P(E, Q, \mathbf{1}^\nu)$, and $R_i \in \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle$. Putting $G_i = R_iF$, we have

$$\begin{aligned} &P_0(E, Q)F(n, \boldsymbol{k}) \\ &= -\sum_{i=1}^{\nu}(G_i(n, k_1, \ldots, k_i + 1, \ldots, k_\nu) - G_i(n, k_1, \ldots, k_\nu)). \end{aligned}$$

Note that the right-hand side is in the form of a telescoping sum. Namely, most of the right-hand side terms are canceled when summing up with respect to $\boldsymbol{k}$. This method to create a form of telescoping sum is called creative telescoping. Summing up the equality, we verify that $P_0(E, Q)G(n)$ is a sum of multisums of proper q-hypergeometric functions with one variable less. Repeating this process, we obtain $P_1(E, Q)P_0(E, Q)G(n) = 0$ for a polynomial $P_1(E, Q)$. Then, how can we obtain $P(E, Q, E_1, \ldots, E_\nu)$? Note that

$$P(E, Q, E_1, \ldots, E_\nu) \in \mathrm{Ann}(F) \cap \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle,$$

where $\mathrm{Ann}(F) = \{P \in \mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle \mid PF = 0\}$ is an annihilating ideal of F. Moreover, it is known that if we put

$$\frac{E_iF}{F} = \left.\frac{R_i}{S_i}\right|_{Q=q^n,\ Q_j=q^{k_j}}$$

for $R_i, S_i \in \mathbb{Z}[q, Q, Q_{\boldsymbol{k}}]$, then, $\mathrm{Ann}(F)$ is generated by $\{S_iE_i - R_i \mid i = 0, \ldots, \nu\} \subset \mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle$. Combining these facts, we would be able to obtain $P(E, Q, E_1, \ldots, E_\nu)$ from

$$S_iE_i - R_i = 0, \quad i = 0, \ldots, \nu$$

by eliminating $Q_1, \ldots, Q_\nu$. Therefore, we would be able to obtain $\varepsilon P_0(E, Q)$ by eliminating $Q_1, \ldots, Q_\nu$ from

$$\begin{cases} \varepsilon(S_iE_i - R_i)\mid_{E_i=1} = 0 \quad (i = 1, \ldots, \nu), \\ \varepsilon(SE - R) = 0, \end{cases}$$

where $S = S_0,\ R = R_0$. In this context, the following proposition holds:

Proposition 4.5 ([12])**.** *Following equalities hold:*

$$\exp\left(w_i \frac{\partial \Phi}{\partial w_i}\right) = \varepsilon \left. \frac{E_i F}{F} \right|_{\substack{q^{k_j}=w_j \\ q^m=\alpha}},$$

$$\exp\left(\alpha \frac{\partial \Phi}{\partial \alpha}\right) = \varepsilon \left. \frac{E_m F}{F} \right|_{\substack{q^{k_i}=w_i \\ q^m=\alpha}},$$

where E_m is an operator that shifts m to $m+1$.

Namely, the system of equations (2) coincide with

$$\begin{cases} \varepsilon \left. \dfrac{E_j F}{F} \right|_{\substack{q^{k_i}=w_i \\ q^m=\alpha}} = 1, \qquad (j = 1, \dots, \nu) \\ \varepsilon \left. \dfrac{E_m F}{F} \right|_{\substack{q^{k_i}=w_i \\ q^m=\alpha}} = E^2, \end{cases}$$

under the correspondence $l = E$. After finite times of creative telescoping, all indices $\boldsymbol{k}$ vanish and we obtain

$$P'(E,Q)P_0(E,Q)J_K(n) + f(q,q^n) = 0,$$

where $P'(E,Q) \in \mathcal{A}_{\text{loc}}$ and $f(q,q^n) \in \mathbb{Q}(q,q^n)$. We can cancel $f(q,q^n)$ by multiplying $(E-1) \cdot f(q,Q)^{-1}$ from the left. This factor corresponds to $l-1$ in the A-polynomial.

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Department of Pure and Applied Mathematics
School of Fundamental Science and Engineering
Waseda University
3-4-1 Okubo, Shinjuku
Tokyo 169-8555 JAPAN
E-mail address: `sa-shun.1729ttw@asagi.waseda.jp`

早稲田大学大学院基幹理工研究科数学応用数理専攻 沢辺 俊