Minimizing CM degree and specially K-stable varieties

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[1]

Abstract

We prove that the degree of the CM line bundle for a normal family over a curve with fixed general fibers is strictly minimized if the special fiber is either

- a smooth projective manifold with a unique cscK metric or
- "specially K-stable", which is a new class we introduce in this paper.

This phenomenon, as conjectured by Odaka (cf., [Oda20]), is a quantitative strengthening of the separatedness conjecture of moduli spaces of polarized K-stable varieties.

The above mentioned special K-stability implies the original K-stability and a lot of cases satisfy it e.g., K-stable log Fano, klt Calabi-Yau (i.e., $K_X \equiv 0$), lc varieties with the ample canonical divisor and uniformly adiabatically K-stable klt-trivial fibrations over curves (cf., [Hat22]).

1 Introduction

We work over $\mathbb{C}$ but all results in this paper except Corollary 3.10 or Theorem 3.21 also hold for any algebraically closed field of characteristic zero.

1.1 Separatedness of moduli spaces of K-stable varieties

To construct moduli spaces of polarized algebraic varieties, the following condition is one of the most important ingredients and guarantees "separatedness" in some sense (e.g. S-equivalence for K-semistable log Fano pairs cf. [BX19]).

♣ Let $(X, L) \to C$ and $(X', L') \to C$ be two proper flat normal families of n-dimensional polarized varieties of a certain class $(*)$ over a smooth curve. If the generic fibers of the two families coincide, then their special fibers (X_0, L_0) and (X'_0, L'_0) are "equivalent" in some sense over a closed point $0 \in C$.

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This is proved in a few cases, for example, when $(*)$ is the class of stable curves in [DM69]. For general K-ample (K_X is ample literally) slc pairs, ♣ similarly holds by the theory of MMP (cf., [KSB88], [Kol22]). They are known to be K-stable by [Oda12]. It is also proved in the unpublished note of Boucksom [Bou14b] that ♣ holds for klt minimal models in a similar way. On the other hand, it is easy to see that ♣ does not hold in general at least for K-unstable Fano varieties by [LX14]. In the recent studies of Fano varieties, K-stability plays an important role in construction of the moduli space of K-(poly)stable Fano varieties (so-called K-moduli cf., [Xu21]), and it is proved that ♣ holds when "equivalence" is S-equivalence for K-semistable Fano varieties by Blum and Xu [BX19]. However, to check whether ♣ holds or not for any class $(*)$ has still been one of the most challenging problems in algebraic geometry.

1.2 K-stability and CM minimization

K-stability was originally introduced by [Tia97], [Don02] in Kähler geometry to study when constant scalar curvature Kähler (for short cscK) metrics exist. Note that K-stability can be rephrasable as follows (cf., Definition 2.3). If the trivial test configuration minimizes the Donaldson-Futaki (DF) invariants of normal test configurations in the strict sense, then (X, L) is K-stable. Roughly speaking, this is one of the algebro-geometric counterparts of the result of [Mab86] on Fano manifolds, which states that the K-energy takes a local minimum at a Kähler-Einstein metric. We remark that it is known by [CC21] that if the K-energy is proper and $\mathrm{Aut}(X, L)$ is discrete, then it attains a unique global minimum at a unique cscK metric. So to speak, K-stability is characterized by "DF minimization" in the sense of Conjecture 1.1 below.

On the other hand, Paul and Tian [PT09] introduced the Chow-Mumford (CM) line bundle, which is a $\mathbb{Q}$-line bundle defined on the base of a flat family of polarized varieties. Note that the degree of the CM line bundle over a curve, which we call the CM degree, is a generalization of the DF invariants of test configurations (cf., [FR06, Lemma 2.5]).

Odaka proposed the following on CM degrees, which he called the *CM minimization conjecture*.

Conjecture 1.1 (CM minimization, cf., [Oda20, Conjecture 8.1])**.** *Let $\pi : (X, L) \to C$ be a polarized family over a smooth projective curve C such that (X_0, L_0) is K-semistable (cf., Definition 3.1). Let $\mathrm{CM}((X, L)/C)$ be the CM degree. Then*

$$\mathrm{CM}((X, L)/C) \leq \mathrm{CM}((X', L')/C)$$

for any polarized family $\pi' : (X', L') \to C$ such that there exists a C°-isomorphism $f^\circ : (X, L) \times_C C^\circ \cong (X', L') \times_C C^\circ$.

Furthermore, if (X_0, L_0) is K-stable and X' is normal, then equality holds if and only if f° can be extended to $f : (X, L) \cong (X', L')$ over C entirely.

Taking what we explained in the first paragraph of §1.2 into account, Conjecture 1.1 predicts that K-stability would be characterized not only by DF minimization but also by CM minimization. Conjecture 1.1 also predicts that if we chose $(*)$ to be the class of K-stable varieties in ♣, then we would immediately obtain separatedness. This conjecture was indeed proved for lc K-ample and klt Calabi-Yau ($K_X \equiv 0$) varieties by Wang and Xu [WX14] and by Odaka [Oda13c] respectively. Furthermore, for K-semistable Fano varieties, the above conjecture holds as shown by Xu [Xu21]. Thus, the results on separatedness in §1.1 except [Bou14b] follow from Conjecture 1.1 in special cases. In [WX14] and [Oda13c], Conjecture 1.1 is proved by the Hodge index theorem and by the observation of the log discrepancy. On the other hand, the proof of Conjecture 1.1 for K-stable Fano varieties relies heavily on the result of [LX14]. Unfortunately, their methods can not be applied to families of more general polarized varieties directly. On the other hand, Ohno [Ohn22] studied the opposite direction of Conjecture 1.1. That is, he proved that if the CM degree takes a minimum then the special fiber is necessarily slope K-semistable under a certain condition.

The aim of this paper is to confirm that Conjecture 1.1 holds for many cases. Our first result is to settle Conjecture 1.1 for the following case that seems to be quite meaningful to Kähler geometry.

Theorem 1.2 (= Corollary 3.10)**.** *Conjecture 1.1 holds if X_0 is smooth, (X_0, L_0) has a cscK metric and $\mathrm{Aut}(X_0, L_0)$ is discrete.*

According to this theorem, Conjecture 1.1 seems to be quite natural. Indeed, if (X_0, L_0) satisfies the assumption of Theorem 1.2, it is known that (X_0, L_0) is K-stable by [Sto09] and the Yau-Tian-Donaldson conjecture predicts that the converse would hold. A key ingredient to show Theorem 1.2 is Theorem 3.6, which is the technical heart of this paper. Theorem 3.6 essentially states that the following holds.

Theorem 1.3 (see Theorem 3.6)**.** *Notations as Conjecture 1.1. Then there exist $k \in \mathbb{Z}_{>0}$ and a good filtration $\mathscr{F}$ (cf. Definition 2.11) of (X_0, kL_0) such that*

$$\mathrm{DF}(\mathscr{F}) = \mathrm{CM}((X', L')/C) - \mathrm{CM}((X, L)/C).$$

For the concrete construction of $\mathscr{F}$, see Theorem 3.6.

On the other hand, we introduce a new class of K-stable varieties, *specially* K-*stable varieties* (cf., Definition 3.19). K-stable log Fano, slc K-ample and klt Calabi-Yau varieties are contained in this class. Furthermore, some polarized varieties confirmed to be uniformly K-stable in the previous works by the author (cf., [Hat21], [Hat22]) are also specially K-stable, for example, klt minimal models and uniformly adiabatically K-stable klt-trivial fibrations over curves (see Theorem 3.21).

Our third result confirms that Conjecture 1.1 also holds for specially K-stable varieties as follows.

Theorem 1.4 (see Theorem 3.22)**.** *In Conjecture 1.1, assume that K_X is $\mathbb{Q}$-Cartier. If (X_0, L_0) is specially K-semistable, then the following inequality holds*

$$\mathrm{CM}((X, L)/C) \leq \mathrm{CM}((X', L')/C).$$

If (X_0, L_0) is further specially K-stable and X' is normal, then equality holds if and only if f° extends to $f : (X, L) \cong (X', L')$ over C entirely.

We obtain that ♣ holds for specially K-stable varieties as an immediate corollary (i.e. Corollary 3.24), which is remarked in [BX19, Remark 3.6].

We can define special K-stability in an intrinsic way by using the δ-invariant (cf., [FO18], [BJ20]) and J-positivity (cf., [Che21], [DP21], [Son20, Definition 1.1], [Hat21] and Definition 3.19) rather than by using the DF invariants of test configurations. Thus, to check special K-stability is much easier than the original K-stability. Furthermore, thanks to Corollary 3.24, we could construct moduli spaces of certain classes of specially K-stable polarized varieties as Deligne-Mumford stacks if we knew openness and boundedness. In fact, the moduli spaces constructed by Hashizume and the author in [HH23] parametrize uniformly adiabatically K-stable klt-trivial fibrations over curves and Theorems 1.4 and 3.21 guarantee separatedness of these moduli spaces in a different way. Furthermore, we conclude that the separatedness of klt polarized minimal models (cf. [Bou14b]) follows from Theorem 1.4.

1.3 The technical heart of the proof of the main theorems

Let $\pi : (X, L) \to C$ and $\pi' : (X', L') \to C$ be two families generically isomorphic over C. As in Conjecture 1.1, let $0 \in C$ be a special point. We consider when X_0 is normal and irreducible and restrict C to an open neighborhood of 0. Then we define the following filtration as

$$\mathscr{F}^{-i}H^0(X_0, mL_0) = \mathrm{Im}\,(H^0(X', mL' + i(X'_0 - \hat{X}_0)) \to H^0(X_0, mL_0))$$

for $i \geq 0$. Otherwise, we set $\mathscr{F}^{-i}H^0(X_0, mL_0) = 0$. Here, $\hat{X}_0$ is the strict transform of X_0. This $\mathscr{F}$ is a filtration that appeared in Theorem 1.3. This filtration is firstly studied in [BX19, §5] when (X_0, L_0) and (X'_0, L'_0) are K-semistable Fano varieties, and Blum and Xu proved that $\mathscr{F}$ is finitely generated in this case. However, this filtration has not been fully considered yet in general cases. In this paper, we construct such filtrations in general settings. In contrast to the case treated in [BX19, §5], $\mathscr{F}$ might not be finitely generated. However, we see that $\mathscr{F}$ has the weight function $w_{\mathscr{F}}(m) = b_0m^{n+1} + b_1m^n + O(m^{n-1})$, where $n = \dim X$. Then, we define the DF invariant $\mathrm{DF}(\mathscr{F})$ of $\mathscr{F}$ in the same way we defined those of test configurations. Taking Theorem 1.3 into consideration, we want to compute $\mathrm{DF}(\mathscr{F})$ by approximating via finitely generated ones. However, there is a subtlety that b_0 is preserved when we take the limit but b_1 is not known to be so. This problem is called the conjecture of regularization of non-Archimedean entropy (cf., [BJ18, Conjecture 2.5], [Li22, Conjecture 1.8]). Fortunately, if (X_0, L_0) is smooth and has a cscK metric, and $\mathrm{Aut}(X_0, L_0)$ is discrete, then we can take a lower bound as $\mathrm{Chow}_\infty(\mathscr{F})$ of $\mathrm{DF}(\mathscr{F})$ by [Szé15, Proposition 11]. On the other hand, for specially K-stable varieties, we know by [Hat22, Appendix] that we can give a lower bound of the DF invariant of a test configuration as the sum of the log-twisted Ding invariant (cf., [Ber16]) and the non-Archimedean (for short., nA) J-functional introduced by [LS15]. For Ding invariants, as studied in [Fuj18] and [Fuj19a], filtrations play important roles. On the other hand, we see that nA J-functionals are compatible with taking the limit of finitely generated filtrations (cf., Proposition 2.18).

Then, we decompose CM degrees into log-twisted Ding degrees and J-degrees, which are generalizations of nA J-functionals, and we obtain a lower bound of the difference of two CM degrees.

1.4 Structure of this paper

In §2, we introduce *good filtrations*. A good filtration is defined to be a filtration such that its weight function is close to a polynomial with an error term $O(m^{n-1})$. We define the DF invariants of these filtrations in a different way from [Szé15] (cf., Definition 2.11, Remark 2.19). On the other hand, we have to consider the volumes of linear series on reducible or non-reduced schemes. There is a powerful tool, the Okounkov body (cf., [LM09], [BC11]), to discuss the volumes of linear series of varieties. However, the theory of Okounkov bodies might not work well for reducible or non-reduced schemes. For this, we work on the weight functions of filtered linear series of general schemes.

In §3, we first establish the formula as in Theorem 1.3. We also establish the log version of this formula in Corollary 3.8. Here, note that B_0, the fiber of the boundary over 0, might not be integral in general. Then, we apply the theory on the weight functions of filtrations of reducible or non-reduced polarized schemes constructed in §2.2.2 to obtain our formulae. Theorem 1.2 follows from Theorem 3.6 and from the result on Chow_∞ of [Szé15].

The proof of Theorem 1.4 is more complicated than that of Theorem 1.2. In §3.5, we prove Theorem 1.4 in three steps. We first decompose the CM degree into the log-twisted Ding degree and the J-degree of a family.

In §3.3, we consider "J-minimization". As studied in [Hat21], nA J-functionals are not affected by singularities. This is a difference between J-stability and K-stability in the sense of [Oda13b]. For this, no problem like regularization of nA entropy occurs when we consider nA J-functionals. Thus we define the nA J-functional of a non finitely generated filtration by taking the limit of a sequence of those of finitely generated filtrations (cf., Definition 2.17, Proposition 2.18). With this in mind, we prove J-minimization (Theorem 3.14) by applying Corollary 3.8.

Next, we consider "Ding minimization" in §3.4. For this, we construct the following new method to prove the implication $\delta(X,-K_X)\ge 1 \Rightarrow$ Ding-semistability of Fano varieties more directly than [FO18] (cf., [Fuj19a, Theorem 5.1]) without applying the result of [LX14]. The reason why we need the new method is that there is a subtle problem that we can not make use of MMP directly for general log twisted Fano pairs as [LX14] or [BLZ22] since the twist term can be anti-ample. Let us explain the method briefly for test configurtaions. Let $(\mathcal{X},\mathcal{L})$ be a semiample test configuration for a Fano manifold $(X,-K_X)$ and $\mathfrak{a}=\sum t^i\mathfrak{a}_i$ be an ideal (which is called a flag ideal in [Oda13a]) such that $(\mathcal{X},\mathcal{L})$ is the blow up of $\mathfrak{a}$. Here t is the canonical coordinate of $\mathbb{A}^1$. As in the proof of [Fuj19a, Theorem 4.1], we have

$$\mathrm{Ding}(\mathcal{X},\mathcal{L})=\mathrm{lct}(X\times\mathbb{A}^1,\mathfrak{a};X\times\{0\})-1-\frac{\mathcal{L}^{n+1}}{(n+1)L^n}.$$

Then we relate $\frac{\mathcal{L}^{n+1}}{(n+1)L^n}$ to the asymptotic behavior of the δ_k-invariant for sufficiently large k. Thus, we deduce Ding-semistability from $\delta(X,-K_X) \geq 1$. To show Ding minimization, we generalize our method to any family over a curve in the log-twisted setting.

Finally, we combine the results on J-minimization and Ding minimization to obtain Theorem 1.4 and explain its applications in §3.5.

2 Preliminaries

We assume that a polarized scheme (X,L) is proper over $\mathbb{C}$, connected and equidimensional. If X is a variety, we further assume that X is irreducible and reduced. We denote the support of a coherent sheaf $\mathscr{F}$ on X by $\operatorname{Supp}\mathscr{F}$. Unless otherwise stated, we understand L to be a $\mathbb{Q}$-line bundle, i.e., rL is an ample line bundle for some $r \in \mathbb{Z}_{\geq 0}$. We denote the intersection product as $L^m \cdot H^{n-m}$ and we understand $mL = L^{\otimes m}$.

2.1 K-stability and test configurations

First, we recall some basic concepts.

Definition 2.1. Let (X,L) be a polarized reduced scheme. Suppose that X is smooth or normal crossing in codimension one points and satisfies Serre's condition S_2. Let B be an effective $\mathbb{Q}$-Weil divisor on X such that no irreducible component of $\operatorname{Supp}B$ is contained in the singular locus of X and K_X+B is $\mathbb{Q}$-Cartier. Then we call (X,B,L) a *polarized deminomal pair*. We denote its automorphism group by $\operatorname{Aut}(X,B,L)$.

If X is further a normal variety, then we call (X,B,L) a polarized normal pair.

We recall log discrepancies, δ-invariants and singularities of pairs as follows.

Definition 2.2. First, let (X,B,L) be a polarized normal pair. For any prime divisor F over X, we define the *log discrepancy* $A_{(X,B)}(F)$ with respect to F as follows. Choose a projective birational morphism $\pi : Y \to X$ from a normal variety Y on which F is a prime divisor defined. Then

$$A_{(X,B)}(F) = 1 + \operatorname{ord}_F(K_Y - \pi^*(K_X+B)).$$

This is independent from the choice of π. Then we say that (X,B) is

- *klt* if $A_{(X,B)}(F) > 0$ for any F,
- *lc* if $A_{(X,B)}(F) \geq 0$ for any F.

We remark that if B is noneffective, we define the log discrepancy in the same way and we say (X,B) is sublc if $A_{(X,B)}(F) \geq 0$ for any F.

For any effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, we define the *log canonical threshold* of (X, B) with respect to D

$$\mathrm{lct}(X, B; D) = \sup\{t \in \mathbb{Q} | (X, B + tD) \text{ is a sublc pair}\}.$$

Next, we define the δ-invariant of a polarized lc pair (X, B, L) as follows. If (X, B) is not klt, then set $\delta(X, B, L) = 0$. Otherwise, take $r_0 \in \mathbb{Z}_{>0}$ such that $r_0 L$ is an ample Cartier divisor. For $m \in \mathbb{Z}_{>0}$, we call D an $r_0 m$-basis type divisor if $D = \frac{1}{r_0 m h^0(X, r_0 mL)} \sum_{i=1}^{h^0(X, r_0 mL)} D_i$ where $\{D_i\}_{i=1}^{h^0(X, r_0 mL)}$ forms a basis of $H^0(X, r_0 mL)$. Then, set

$$\delta_{r_0 m}(X, B, L) = \inf_{D: r_0 m\text{-basis}} \mathrm{lct}(X, B; D).$$

It is known by [BJ20, Theorem A] that $\lim_{m \to \infty} \delta_{r_0 m}(X, B, L)$ exists and we call this the δ-invariant of (X, B, L) and denote this by $\delta(X, B, L)$.

On the other hand, let (X, B, L) be a polarized deminormal pair. Let $\nu : \overline{X} \to X$ be the normalization and let $\mathfrak{cond}_{\overline{X}} \subset \overline{X}$ be the conductor subscheme defined by the ideal $\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\overline{X}}, \mathcal{O}_X)$. Then $\mathfrak{cond}_{\overline{X}}$ is known to be a reduced Weil divisor [Kol15, §5.1]. We say that (X, B) is *slc* if $(\overline{X}, \nu_*^{-1} B + \mathfrak{cond}_{\overline{X}})$ is lc and then set $\delta(X, B, L) = 0$ for non normal polarized slc pairs.

If $B = \sum_{i=1}^r a_i D_i$ is a $\mathbb{Q}$-divisor on n-dimensional deminormal scheme X, where each D_i is an irreducible component of B, then we set

$$\chi(B, mH|_B) := \sum_{i=1}^r a_i \chi(D_i, mH|_{D_i})$$

for any line bundle H on X, where $\chi(D_i, mH|_{D_i})$ is the Hilbert polynomial. We remark that if B is a Weil divisor, although $\chi(B, mH|_B)$ defined as above does not coincide in general with the Hilbert polynomial of $(B, mH|_B)$ in the usual sense, their leading terms are the same (cf., [KM98, §1.5]). In this paper, we are only interested in the leading term of $\chi(B, mH|_B)$.

Definition 2.3. Let (X, B, L) be a deminormal polarized pair of dimension n. A pair $(\mathcal{X}, \mathcal{L})$ is called a *semiample test configuration* for (X, L) if the following conditions hold.

1. $\mathcal{X}$ is a scheme and $\mathcal{L}$ is a semiample $\mathbb{Q}$-line bundle on $\mathcal{X}$ such that $\mathbb{G}_m$ acts on $(\mathcal{X}, \mathcal{L})$ in the sense of [MFK94, §1.4],
2. There exists a projective, flat, and $\mathbb{G}_m$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{A}^1$, where $\mathbb{A}^1$ admits a natural $\mathbb{G}_m$-action by multiplication,
3. $(\pi^{-1}(1), \mathcal{L}|_{\pi^{-1}(1)}) = (X, L)$.

If $\mathcal{L}$ is ample, then we call $(\mathcal{X},\mathcal{L})$ an ample test configuration. For any semiample test configuration $(\mathcal{X},\mathcal{L})$, we get the canonical compactification over $\mathbb{P}^1$ denoted by $(\overline{\mathcal{X}},\overline{\mathcal{L}})$, whose restriction to $\mathbb{P}^1\setminus\{0\}$ coincides with $(X\times\mathbb{A}^1, L\times\mathbb{A}^1)$ such that $\mathbb{G}_m$ trivially acts on the first component X.

We denote a test configuration $X\times\mathbb{A}^1$, with the trivial $\mathbb{G}_m$-action on the first component, by $X_{\mathbb{A}^1}$. For any ample test configuration $(\mathcal{X},\mathcal{L})$, there exists another semiample test configuration $(\mathcal{Y},\gamma^*\mathcal{L})$ such that there exist two $\mathbb{G}_m$-equivariant morphisms $\gamma:\mathcal{Y}\to\mathcal{X}$ and $\rho:\mathcal{Y}\to X_{\mathbb{A}^1}$. Here, γ and ρ induce the identity morphism over $\mathbb{A}^1\setminus\{0\}$. We may also assume that $\mathcal{Y}$ is deminormal by [Fuj19b, Proposition 3.2] and [Oda13a]. Let $\mathcal{B}$ be the closure of $B\times(\mathbb{A}^1\setminus\{0\})$ in $\overline{\mathcal{X}}$. On the other hand, let H be an $\mathbb{R}$-line bundle on X. Then we define the following functional on $(\mathcal{X},\mathcal{L})$ after [LS15]

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X},\mathcal{L})=(L^n)^{-1}\left(\overline{\rho^*(H\times\mathbb{A}^1)}\cdot\overline{\gamma^*\mathcal{L}}^n-\frac{nH\cdot L^{n-1}}{(n+1)L^n}\overline{\mathcal{L}}^{n+1}\right).$$

We call this the *non-Archimedean (nA) J^H-functional* of $(\mathcal{X},\mathcal{L})$. It is easy to check that $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ does not depend on the choice of γ. If $\mathcal{X}$ is deminormal, we also define

$$\mathrm{DF}_B(\mathcal{X},\mathcal{L})=(L^n)^{-1}\left((K_{\overline{\mathcal{X}}/\mathbb{P}^1}+\mathcal{B})\cdot\overline{\mathcal{L}}^n-\frac{n(K_X+B)\cdot L^{n-1}}{(n+1)L^n}\overline{\mathcal{L}}^{n+1}\right).$$

We call this the (log) *Donaldson-Futaki (DF) invariant* of $(\mathcal{X},\mathcal{L})$ (cf., [Oda12], [Wan12]).

We say that (X,B,L) is

- uniformly K-stable (resp., K-semistable) if there exists a rational constant $\epsilon>0$ (resp., $\epsilon=0$) such that

$$\mathrm{DF}_B(\mathcal{X},\mathcal{L})\ge(\mathcal{J}^{\epsilon L})^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$

- uniformly J^H-stable (resp., J^H-semistable) if there exists a rational constant $\epsilon>0$ (resp., $\epsilon=0$) such that

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X},\mathcal{L})\ge(\mathcal{J}^{\epsilon L})^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$$

for any ample deminormal test configuration $(\mathcal{X},\mathcal{L})$ (cf., [BHJ17], [Hat21]). Here, $(\mathcal{J}^L)^{\mathrm{NA}}$ is nothing but the $I^{\mathrm{NA}}-J^{\mathrm{NA}}$-norm in [BHJ17, §7] or the minimum norm in [Der16]. It is well-known (cf. [BHJ17, 7.8 and 7.9], [Der16, 4.7]) that $(\mathcal{J}^L)^{\mathrm{NA}}$ is indeed a norm in some sense. See also Lemma 3.13.

To consider K-stability of polarized deminormal pairs, we may restrict to slc pairs by [Oda13b], [OS15, Theorem 6.1], [BHJ17, §9].

2.2 Filtrations and DF invariant

We assume that for any polarized scheme (X,L), $\dim X=n$ and L is $\mathbb{Z}$-Cartier throughout this section. First, we prepare the fundamental terminology common to linear series of general polarized schemes.

Definition 2.4. Let $R=\bigoplus_{m\in\mathbb{Z}_{\geq 0}} R_m$ be a graded algebra over $\mathbb{C}$ with a unit element 1. $\mathscr{F}=\mathscr{F}^\bullet R$ is called a *linearly bounded multiplicative $\mathbb{Z}$-filtration* of R if $\mathscr{F}$ satisfies the following.

1. For $\lambda>\lambda'\in\mathbb{Z}$ and $m\in\mathbb{Z}$, $\mathscr{F}^\lambda R_m\subset\mathscr{F}^{\lambda'}R_m$,
2. $\mathscr{F}^\lambda R_m\cdot\mathscr{F}^{\lambda'}R_{m'}\subset\mathscr{F}^{\lambda+\lambda'}R_{m+m'}$ for any $\lambda,\lambda'\in\mathbb{Z}$ and $m,m'\in\mathbb{Z}_{\geq 0}$,
3. There exists a positive constant C such that for sufficiently large $m\in\mathbb{Z}_{\geq 0}$, $\mathscr{F}^\lambda R_m=0$ for $\lambda\geq Cm$ and $\mathscr{F}^\lambda R_m=R_m$ for $\lambda<-Cm$,
4. $1\in\mathscr{F}^0R_0$.

In this paper, we call $\mathscr{F}$ a filtration for simplicity. Moreover, if $\bigoplus_{m\in\mathbb{Z}_{\geq 0},\lambda\in\mathbb{Z}}\mathscr{F}^\lambda R_m$ forms a finitely generated bigraded $\mathbb{C}$-algebra, then we say that $\mathscr{F}$ is finitely generated.

Suppose that $\dim R_m$ is finite. We define $w_{\mathscr{F}}(m)$ the *weight function* of $\mathscr{F}$ as

$$w_{\mathscr{F}}(m)=\sum_{\lambda\in\mathbb{Z}}\lambda\dim(\mathscr{F}^\lambda R_m/\mathscr{F}^{\lambda+1}R_m).$$

Let $N\in\mathbb{Z}$. The *weight N-shift* $\mathscr{F}_{(N\text{-shift})}$ of $\mathscr{F}$ is a filtration defined by

$$\mathscr{F}^\lambda_{(N\text{-shift})}R_k=\mathscr{F}^{\lambda+Nk}R_k.$$

This indeed satisfies the conditions of Definition 2.4.

For any $x\in\mathbb{R}$, we set $\mathscr{F}^xR_m=\mathscr{F}^{\lceil x\rceil}R_m$ and $R^{(x)}:=\bigoplus_{m\geq 0}R^{(x)}_m$, where $R^{(x)}_m:=\mathscr{F}^{mx}R_m$. Then $R^{(x)}$ is a graded subalgebra of R and it holds that

$$w_{\mathscr{F}}(m)=\int_{-Cm}^{\infty}\dim\mathscr{F}^\lambda R_m d\lambda-Cm\dim R_m \tag{1}$$

if m and C satisfy the condition (3) above (see [BHJ17, §5] or [Fuj19a, Prop. 2.12 (2)]).

Example 2.5. There exists the *trivial* filtration $\mathscr{F}_{\mathrm{triv}}$, which is defined by $\mathscr{F}^\lambda_{\mathrm{triv}}R_k=R_k$ if $\lambda\leq 0$. Otherwise $\mathscr{F}^\lambda_{\mathrm{triv}}R_k=0$.

The following is the most important case in this paper. If R is a graded subalgebra of $\bigoplus_{m\geq 0}H^0(X,mL)$ for some polarized scheme (X,L), then we call R a *linear series* of (X,L). We define the *volume* of R as

$$\mathrm{vol}(R)=n!\limsup_{m\to\infty}\frac{\dim R_m}{m^n}.$$

If $\mathscr{F}$ is further a filtration of $R=\bigoplus_{m\geq 0}H^0(X,mL)$, then we call $\mathscr{F}$ a *filtration* of (X,L).

Example 2.6. Let (X, L) be a polarized scheme and D be a closed subscheme. If $\mathscr{F}$ is a filtration of $R = \bigoplus_{m\geq 0} H^0(X, mL)$, then we define a new filtration $\mathscr{F}_D$ of $\bigoplus_{m\geq 0} H^0(D, mL|_D)$ by

$$\mathscr{F}_D^\lambda H^0(D, mL|_D) = \mathrm{Im}\left(\mathscr{F}^\lambda H^0(X, mL) \to H^0(D, mL|_D)\right).$$

We call $\mathscr{F}_D$ the *restriction* of $\mathscr{F}$ to D or the *induced* filtration. We also denote $R^{(\lambda)}|_D = \bigoplus_{m\geq 0}(R^{(\lambda)}|_D)_m = \bigoplus_{m\geq 0} \mathscr{F}_D^\lambda H^0(D, mL|_D)$. It is easy to see that $\mathscr{F}_D$ is linearly bounded and multiplicative by the Serre vanishing theorem.

2.2.1 Volumes of linear series on varieties

We recall the results on volumes of linear series of varieties (cf., [BC11]). Let (X, L) be a polarized *variety* and R be a linear series of (X, L). We say R *contains an ample series* if the following hold.

1. $R_k \neq 0$ for sufficiently large $k > 0$,
2. There exist an integer $m \in \mathbb{Z}_{>0}$ and an ample line bundle A_m such that $mL - A_m$ is effective and
$$H^0(X, A_m) \subset R_m \subset H^0(X, mL).$$

If (1) and (2) hold for some m, it is known that then (2) hold for any $m \in \mathbb{Z}_{>0}$ (see [LM09, 2.10]).

Recall the definition of the Okounkov body of R in [LM09] and [BC11, §1]. Suppose that R contains an ample series. Fix a smooth closed point $x \in X$ and a regular system of parameters $(z_1, \cdots, z_n)$ at x. Let ord_x be the canonical valuation of rational rank n defined by the lexicographic order with respect to $(z_1, \cdots, z_n)$. We have the following map with the image whose cardinality is $\dim R_m$

$$\frac{1}{m}\mathrm{ord}_x : R_m \setminus \{0\} \to \mathbb{Q}^n_{\geq 0}.$$

Let Δ_R be the closed convex hull of $\bigcup_{m\geq 0} \frac{1}{m}\mathrm{ord}_x(R_m \setminus \{0\})$ and we call Δ_R the *Okounkov body* of R. It is known that Δ_R is bounded and if ρ is the Lebesgue measure on $\mathbb{R}^n$, then $\mathrm{vol}(\Delta_R) = \int_{\Delta_R} d\rho = \frac{1}{n!}\mathrm{vol}(R)$ by [Bou14a, 1.12]. It follows from this fact that $\lim_{m\to\infty} \frac{\dim R_m}{m^n}$ exists.

Next, we consider when $R = \bigoplus_{m\geq 0} H^0(X, mL)$ and $\mathscr{F}$ is a filtration of R. Set

$$e_{\max}(R, \mathscr{F}) = \limsup_{k\to\infty} \frac{\sup\{t \in \mathbb{R} | \mathscr{F}^t R_k \neq 0\}}{k}$$

and then $R^{(t)}$ contains an ample series for $t < e_{\max}(R, \mathscr{F})$ by [BC11, Lemma 1.6]. On the other hand, it is easy to see that $\mathrm{vol}(R^{(t)}) = 0$ for $t > e_{\max}(R, \mathscr{F})$. Then set

$\Delta_t = \Delta_{R^{(t)}}$ for $t < e_{\max}(R, \mathscr{F})$. For any $t < s < e_{\max}(R, \mathscr{F})$, we have $\Delta_s \subset \Delta_t$. Set $\Delta := \Delta_R$. Then we define the *concave transformation* $G : \Delta \to \mathbb{R}$ *associated with* $\mathscr{F}$ by

$$G(p) = \sup\{t \in \mathbb{R} | p \in \Delta_t\}$$

for any $p \in \Delta$. It is well-known that G is concave and upper semicontinuous (cf., [BJ20]).

Remark 2.7. Our notation of linearly bounded multiplicative $\mathbb{Z}$-filtrations is different from one of Székelyhidi [Szé15] in sign.

Let w be the weight function of $\mathscr{F}$ and take a constant C such that $R_m^{(-C)} = R_m$ for sufficiently large $m \in \mathbb{Z}_{>0}$. Then it follows from the equation (1) for such m and from the Lebesgue dominated convergence theorem that

$$n! \lim_{m\to\infty} \frac{w(m)}{m^{n+1}} = \int_{-C}^{\infty} \mathrm{vol} R^{(x)} dx - C\mathrm{vol}(R). \tag{2}$$

On the other hand, it follows from [BC11, Theorem 1.11] that

$$\lim_{m\to\infty} \frac{w(m)}{m^{n+1}} = \int_{\Delta} G d\rho. \tag{3}$$

In [BHJ17, §5], $\frac{1}{\mathrm{vol}(\Delta)} \left(\int_\Delta G d\rho\right)$ is called the *barycenter* of (the Duistermaat-Heckman measure associated with) $\mathscr{F}$. Taking Remark 2.7 into account, we define the norm of $\mathscr{F}$ as [Szé15]

$$\|\mathscr{F}\|_2 = \sqrt{\int_\Delta G^2 d\rho - \frac{1}{\mathrm{vol}(\Delta)} \left(\int_\Delta G d\rho\right)^2}.$$

2.2.2 The weights of filtrations of polarized reducible or non-reduced schemes

As we saw in §2.2.1, the theory of Okounkov bodies is useful to calculate the volumes of linear series of varieties but we cannot apply this to general polarized schemes directly. In §3, we consider families over curves and compare the intersection numbers with the weight functions of filtrations of central fibers. However, such fibers may be reducible or non-reduced in general. In this subsection, we discuss the weight functions of filtered linear series of reducible or non-reduced schemes.

Let (X, L) be a polarized scheme, $R_m = H^0(X, mL)$ and $\mathscr{F}$ be a linearly bounded multiplicative filtration of $R = \bigoplus_{m\ge0} R_m$. Let $w_{\mathscr{F}}$ be the weight function of $\mathscr{F}$. As §2.2.1, we define the *barycenter* of $\mathscr{F}$ to be

$$\mathcal{B}_{\mathscr{F}} = \limsup_{m\to\infty} \frac{w_{\mathscr{F}}(m)}{m^{n+1}}.$$

Then, we show the following to deduce Lemma 3.9 below.

Proposition 2.8. *Notations as above. Let $\{X_i\}_{i=1}^r$ be the set of irreducible components of X. We define the scheme structure of X_i by the scheme-theoretic image of the canonical morphism $\operatorname{Spec}\mathcal{O}_{X,\eta_i} \to X$, where η_i is the generic point of X_i (cf. [Har77, II, Exercise 3.11 (d)]). Let m_i be the multiplicity of X_i, i.e., the length of $\mathcal{O}_{X,\eta_i}$. Let also $X_{i,\mathrm{red}}$ be the reduced structure of X_i and $\mathscr{F}_{i,\mathrm{red}} = \mathscr{F}_{X_{i,\mathrm{red}}}$ (cf. Example 2.6).*

Then $\mathcal{B}_{\mathscr{F}} \geq \sum_{i=1}^r m_i \mathcal{B}_{\mathscr{F}_{i,\mathrm{red}}}$. If X is further reduced, then $\lim_{m\to\infty} \frac{w_{\mathscr{F}}(m)}{m^{n+1}}$ exists and

$$\lim_{m\to\infty} \frac{w_{\mathscr{F}}(m)}{m^{n+1}} = \mathcal{B}_{\mathscr{F}} = \sum_{i=1}^r \mathcal{B}_{\mathscr{F}_{i,\mathrm{red}}}. \tag{4}$$

To show Proposition 2.8, set for any linear series R' of (X, L),

$$\underline{\mathrm{vol}}(R') = n! \liminf_{m\to\infty} \frac{\dim R'_m}{m^n}.$$

Set $e_i = e_{\max}(R|_{X_{i,\mathrm{red}}}, \mathscr{F}_{X_{i,\mathrm{red}}})$. The following is a key step to show Proposition 2.8.

Lemma 2.9. *Let $t \in \mathbb{R} \setminus \{e_1, \ldots, e_r\}$. Then $\underline{\mathrm{vol}}(R^{(t)}) \geq \sum_{i=1}^r m_i \mathrm{vol}(R^{(t)}|_{X_{i,\mathrm{red}}})$. If X is further reduced, then*

$$\mathrm{vol}(R^{(t)}) = \underline{\mathrm{vol}}(R^{(t)}) = \sum_{i=1}^r \mathrm{vol}(R^{(t)}|_{X_i}). \tag{5}$$

Remark 2.10. $\underline{\mathrm{vol}}(R^{(t)}) \geq \sum_{i=1}^r m_i \mathrm{vol}(R^{(t)}|_{X_{i,\mathrm{red}}})$ can be strict in general. Consider $X = \mathbb{P}^1 \times_{\mathbb{C}} \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ and $L = \mathcal{O}_X(1)$, and set for $\lambda \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$

$$\mathscr{F}^\lambda R_m := H^0\left(\mathbb{P}^1, \mathcal{O}\left(m - \frac{\lambda + |\lambda|}{2}\right)\right) \oplus \epsilon \mathscr{F}_{\mathrm{triv}}^{\lambda - m} H^0(\mathbb{P}^1, \mathcal{O}(m)).$$

One can check that $\underline{\mathrm{vol}}(R^{(t)}) = 2 - t$ and $\mathrm{vol}(R^{(t)}|_{\mathbb{P}^1}) = 1 - t$ for any $0 < t < 1$.

On the other hand, for general linear series R, $\underline{\mathrm{vol}}(R) \geq \sum_{i=1}^r m_i \mathrm{vol}(R|_{X_{i,\mathrm{red}}})$ does not hold. Fix a closed point $0 \in \mathbb{P}^1$. Let $X = \mathbb{P}^1 \cup_0 \mathbb{P}^1$ be a reducible curve with two irreducible components $\mathbb{P}^1$ intersecting transversally at 0. Let R_m be the diagonal of $H^0(\mathbb{P}^1, \mathcal{O}(m)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(m))$. Then let $R = \bigoplus_{m\geq 0} R_m$ and we have $\underline{\mathrm{vol}}(R) = \mathrm{vol}(R|_{\mathbb{P}^1}) = 1$ for two components.

Proof of Lemma 2.9. First, we may assume that the canonical morphism $\mathcal{O}_X \to \prod \mathcal{O}_{X_i}$ is injective by replacing X with the closed subscheme defined by the ideal $\mathrm{Ker}(\mathcal{O}_X \to \prod \mathcal{O}_{X_i})$. Let $\mathfrak{c} \subset \mathcal{O}_X$ be the inverse image of $\mathrm{Hom}_{\mathcal{O}_X}(\prod \mathcal{O}_{X_i}, \mathcal{O}_X)$ under the natural map $\mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}(\prod \mathcal{O}_{X_i}, \prod \mathcal{O}_{X_i})$. Then, we claim the following.

Claim 1. *Let t be a real number such that $t < e_i$. Then there exist an integer $m_i \in \mathbb{Z}_{>0}$ and $s_i \in R^{(t)}_{m_i} \cap H^0(X, m_i L \otimes \mathfrak{c} \cdot \mathrm{Ann}(\mathscr{I}_{X_i}))$ such that s_i is a unit at the generic point η_i of X_i where $\mathrm{Ann}(\mathscr{I}_{X_i})$ is the annihilator of the ideal $\mathscr{I}_{X_i}$ corresponding to X_i.*

Proof of Claim 1. There exists a section $s' \in H^0(X, lL \otimes \mathfrak{c} \cdot \mathrm{Ann}(\mathscr{I}_{X_i}))$ such that s' is a unit at η_i for some $l \in \mathbb{Z}_{>0}$. Regard s' as an element of $R_l^{(-C)}$ for some $C > 0$. Next, take a sufficiently small constant $\epsilon > 0$ that $t + \epsilon < e_i$. Then $s' \cdot R_p^{(t+\epsilon)} \subset R_{l+p}^{(\frac{p(t+\epsilon)-Cl}{l+p})}$ for any $p \in \mathbb{Z}_{>0}$. If we take p sufficiently large that $p\epsilon \geq (C+t)l$, then $s' \cdot R_p^{(t+\epsilon)} \subset R_{l+p}^{(t)}$. Furthermore, there exists a section $s'' \in R_p^{(t+\epsilon)}$ such that the restriction $s''|_{X_{i,\mathrm{red}}} \neq 0$ (cf. [BC11, Lemma 1.6]). By letting $s_i = s's''$ and $m_i = l + p$, we complete the proof of Claim 1. □

Next, we claim that $\underline{\mathrm{vol}}(R^{(t)}) \geq \sum_{i=1}^r \underline{\mathrm{vol}}(R^{(t)}|_{X_i})$ for any $t \in \mathbb{R} \setminus \{e_1, \dots, e_r\}$. Let $0 \leq r' \leq r$ be an integer such that $t < e_i$ if and only if $i \leq r'$. By Claim 1, there exist sections $s_i \in R_{m_i}^{(t)} \cap H^0(X, m_iL \otimes \mathfrak{c} \cdot \mathrm{Ann}(\mathscr{I}_{X_i}))$ such that s_i is a unit at η_i for any $i \leq r'$. Replacing s_i by s_i^k for some $k \in \mathbb{Z}_{>0}$, we may assume that $m = m_i$ for $i \leq r'$. Recall that the restriction map $R_k \to (R|_{X_i})_k$ is surjective for any sufficiently large $k \in \mathbb{Z}_{>0}$ and i by the Serre vanishing theorem and consider the following $\mathbb{C}$-linear map

$$h : \prod_{i \leq r'} (R|_{X_i})_k \ni (t_i|_{X_i}) \mapsto \sum_{i \leq r'} s_i t_i \in R_{k+m}.$$

We see that h is well-defined and $\mathrm{Ker}\, h \subset \bigoplus H^0(X_i, mL|_{X_i} \otimes \mathrm{Ker}\, h')$ where $h' : \prod_{i \leq r'} \mathcal{O}_{X_i} \to \mathcal{O}_X$ is the map induced by s_i's. Since $\mathrm{Supp}\,\mathrm{Ker}\, h'$ is nowhere-dense in $\bigcup_{i \leq r'} X_i$, we have for $i \leq r'$,

$$\lim_{m \to \infty} \frac{h^0(X_i, mL|_{X_i} \otimes \mathrm{Ker}\, h')}{m^n} = 0.$$

Thus we have $\underline{\mathrm{vol}}(R^{(t)}) \geq \sum_{i=1}^{r'} \underline{\mathrm{vol}}(R^{(t)}|_{X_i})$.

In this paragraph, we show the first assertion of Lemma 2.9. By the previous paragraph, we may assume that X is irreducible and $t < e_{\max}(R|_{X_{\mathrm{red}}}, \mathscr{F}_{X_{\mathrm{red}}})$. Let m_0 be the multiplicity of X. We prove the assertion by induction on m_0. Suppose that $m_0 > 1$ and let $\mathcal{O}_{X,\eta}$ be the local ring at the generic point η. Since $\mathcal{O}_{X,\eta}$ is Artinian, there exists an element $f \in \mathcal{O}_{X,\eta}$ that generates a non-zero minimal ideal. Here, we identify $L|_U$ with $\mathcal{O}_U$ for some non-empty open subset U. Since L is ample, there exists a section $s \in H^0(X, mL)$ such that the germ s_η at η of s generates the minimal ideal $f \cdot \mathcal{O}_{X,\eta}$ for sufficiently large $m > 0$. On the other hand, there exists $s' \in R_{m'}^{(\tau)}$ for some $m' \in \mathbb{Z}_{>0}$ and $t < \tau < e_{\max}(R|_{X_{\mathrm{red}}}, \mathscr{F}_{X_{\mathrm{red}}})$ such that s' is a unit at η by Claim 1. Then we see that $ss'^k \in R^{(t)}$ for sufficiently large $k \in \mathbb{Z}_{>0}$. Thus, by replacing m and s with ss'^k, we may assume that $s \in R_m^{(t)}$ and generates $f \cdot \mathcal{O}_{X,\eta}$. For any $p \in \mathbb{Z}_{\geq 0}$, we consider a surjective map $r_p : R_p^{(t)} \to (R^{(t)}|_{X'})_p$ where X' is the closed subscheme defined by the ideal generated by s. Since the multiplicity of X' is $m_0 - 1$, it follows that

$$\underline{\mathrm{vol}}(R^{(t)}|_{X'}) \geq (m_0 - 1)\mathrm{vol}(R^{(t)}|_{X_{\mathrm{red}}})$$

from the induction hypothesis. Note that $\mathrm{Ker}\, r_{p+m}$ contains $s \cdot R_p^{(t)}$. Let X'' be the closed subscheme defined by $\mathrm{Ann}(s)$. Then $s \cdot R_p^{(t)} \cong (R^{(t)}|_{X''})_p$. Thus we have

$$\underline{\mathrm{vol}}(R^{(t)}) \geq \underline{\mathrm{vol}}(R^{(t)}|_{X'}) + \underline{\mathrm{vol}}(R^{(t)}|_{X''}).$$

Since X'' is generically reduced,

$$\underline{\mathrm{vol}}(R^{(t)}|_{X''}) \geq \mathrm{vol}(R^{(t)}|_{X_{\mathrm{red}}}).$$

Hence, we have $\underline{\mathrm{vol}}(R^{(t)}) \geq m_0 \mathrm{vol}(R^{(t)}|_{X_{\mathrm{red}}})$ and the first assertion holds.

Finally, suppose that X is reduced. Then we obtain by the restriction map $R^{(t)} \to \prod_{i=1}^{r} R^{(t)}|_{X_i}$ to each component that

$$\mathrm{vol}(R^{(t)}) \leq \sum_{i=1}^{r} \mathrm{vol}(R^{(t)}|_{X_i}).$$

The last assertion of Lemma 2.9 follows from this and the first assertion. □

Proof of Proposition 2.8. The first assertion follows from Lemma 2.9, Fatou's lemma and the equations (1) and (2) that

$$\begin{aligned} n!\mathcal{B}_{\mathscr{F}} &\geq n! \liminf_{m\to\infty} \frac{w_{\mathscr{F}}(m)}{m^{n+1}} \geq \int_{-C}^{\infty} \underline{\mathrm{vol}} R^{(x)} dx - C\mathrm{vol}(R) \\ &\geq \sum_{i=1}^{r} m_i \left(\int_{-C}^{\infty} \mathrm{vol}(R^{(t)}|_{X_{i,\mathrm{red}}}) dx - C\mathrm{vol}(R|_{X_{i,\mathrm{red}}}) \right) = n! \sum_{i=1}^{r} m_i \mathcal{B}_{\mathscr{F}_{i,\mathrm{red}}}. \end{aligned}$$

Next, suppose that X is reduced. Note that the equation (5) holds for any $t \in \mathbb{R} \setminus \{e_1, \ldots, e_r\}$ by Lemma 2.9. Thus, we obtain the last assertion by applying the dominated convergence theorem to the equation (1). □

2.2.3 The Donaldson-Futaki invariants of good filtrations

In this subsection, we define good filtrations and their DF invariants.

Definition 2.11. Let (X, L) be an n-dimensional polarized deminormal scheme and $\mathscr{F}$ be a linearly bounded multiplicative $\mathbb{Z}$-filtration of $R = \bigoplus_{m\geq 0} H^0(X, mL)$.

Let $w(r)$ be the weight function of $\mathscr{F}$. Suppose that $w(r) = b_0 r^{n+1} + b_1 r^n + O(r^{n-1})$. Then we call $\mathscr{F}$ a *good* filtration of R and we define the *DF invariant* of $\mathscr{F}$ as

$$\mathrm{DF}(\mathscr{F}) = 2\frac{b_0 a_1 - b_1 a_0}{a_0^2},$$

where $\chi(X, rL) = a_0 r^n + a_1 r^{n-1} + O(r^{n-2})$. On the other hand, we define the r-th Chow weight as

$$\mathrm{Chow}_r(\mathscr{F}) = 2\left(\frac{r b_0}{a_0} - \frac{w(r)}{\chi(X, rL)} \right).$$

If (X, B, L) is a polarized deminormal pair, then we define the log DF invariant of a good filtration $\mathscr{F}$ as follows. Let $B = \sum c_i D_i$ be the irreducible decomposition, $\mathscr{F}_{D_i}$ be the restriction of $\mathscr{F}$ to D_i and its weight function be w_i. Then we set

$$\mathrm{DF}_B(\mathscr{F}) = \mathrm{DF}(\mathscr{F}) - \frac{b_0 \tilde{a}_0 - \tilde{b}_0 a_0}{a_0^2}.$$

Here, $\chi(B, mL) = \tilde{a}_0 m^{n-1} + O(m^{n-2})$ and $\tilde{b}_0 := \lim_{m\to\infty} \sum \frac{c_i w_i(m)}{m^n}$.

Next, we prepare the useful condition below.

Condition 2.12. Let (X, L) be a polarized *reduced* scheme and $X = \bigcup_{i=1}^{r} X_i$ be the irreducible decomposition. Here, let $R = \bigoplus_{m\geq 0} H^0(X, mL)$ and assume that $R|_{X_i} = \bigoplus_{m\geq 0} H^0(X_i, mL|_{X_i})$ holds. Assume also that $H^0(X, L)$ generates R and $R|_{X_i}$ for all i. If $\mathscr{F}$ is a filtration of (X, L), assume that there exists $N \in \mathbb{Z}_{>0}$ such that $R^{(-N)} = R$.

Condition 2.12 is also assumed in [Szé15, §3] to define an approximation in Definition 2.13 below.

Definition 2.13. Under Condition 2.12, take sufficiently large $N \in \mathbb{Z}_{>0}$ that $R^{(-N)} = R$. Suppose that $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ is a sequence of finitely generated filtrations of R generated by $\mathscr{F}^\bullet R_k$ and $\mathscr{F}^\bullet_{\mathrm{triv},(-N\text{-shift})}R$ as in [Szé15, §3.2] for all $k \in \mathbb{Z}_{>0}$. Note that $\mathscr{F}^\lambda_{(k)} R_m \subset \mathscr{F}^\lambda R_m$. Then we call $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ an *approximation* to $\mathscr{F}$.

We see that $\lim_{k\to\infty} \mathcal{B}_{\mathscr{F}_{(k)}}$ (cf., §2.2.2) is independent of the choice of N by the following.

Lemma 2.14. *In the situation of Definition 2.13, it holds that*

$$\mathcal{B}_{\mathscr{F}} = \lim_{k\to\infty} \mathcal{B}_{\mathscr{F}_{(k)}}.$$

Proof. The assertion when X is irreducible follows from [Szé15, Lemma 6] and the equation (3). Thus the assertion follows in the general case from the equation (4) in Proposition 2.8. $\square$

Let $\mathscr{F}$ be a filtration and $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ be an approximation. For any reduced closed subscheme $D \subset X$, we may assume by replacing L with cL for some $c \in \mathbb{Z}_{>0}$ that $\{(\mathscr{F}_{(k)})_D\}_{k\in\mathbb{Z}_{>0}}$ is an approximation to $\mathscr{F}_D$ (cf. Example 2.6). Indeed, by replacing L by cL, we may assume that $\mathscr{F}_D$ satisfies Condition 2.12 and there exists $N \in \mathbb{Z}_{>0}$ such that $(\mathscr{F}_{(k)})_D$ contains $\mathscr{F}_{\mathrm{triv},(-N\text{-shift})}$ for all k by the Serre vanishing theorem.

We remark the following important result of Székelyhidi that we make use of in the proof of Corollary 3.10.

Theorem 2.15 ([Don01], [Szé15])**.** *Let (X, L) be a polarized smooth variety with a cscK metric in $\mathrm{c}_1(L)$ such that $\mathrm{Aut}(X, L)$ is discrete. If $\mathscr{F}$ is a good filtration of (X, L), then*

$$\mathrm{DF}(\mathscr{F}) \geq 0.$$

Furthermore, if $\|\mathscr{F}\|_2 > 0$, then $\mathrm{DF}(\mathscr{F}) > 0$.

Proof. First, we may replace L by cL for some $c \in \mathbb{Z}_{>0}$ and assume that (X, L) and $\mathscr{F}$ satisfy Condition 2.12. Take an approximation $\{\mathscr{F}_{(r)}\}_{r\in\mathbb{Z}_{>0}}$ to $\mathscr{F}$. Then we define

$$\mathrm{Chow}_\infty(\mathscr{F}) = \liminf_{r\to\infty} \mathrm{Chow}_r(\mathscr{F}_{(r)})$$

after [Szé15]. Suppose that $\dim X = n$. Let $w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$ and $w_r(k) = b_0^{(r)} k^{n+1} + O(k^n)$ be the weights of $\mathscr{F}^\bullet R_k$ and $\mathscr{F}^\bullet_{(r)} R_k$ respectively. If $\mathrm{DF}(\mathscr{F}) \geq \mathrm{Chow}_\infty(\mathscr{F})$ holds, then the assertion follows from [Don01, Corollary 4] and from [Szé15, Proposition 11].

For this, we show $b_0^{(r)} \leq b_0$. Let Δ be the Okounkov body of (X, L) and ρ be the Lebesgue measure of Δ. Let also G (resp., $G^{(r)}$) be the concave transformation with respect to $\mathscr{F}$ (resp., $\mathscr{F}^{(r)}$). By the equation (3),

$$b_0 = \int_\Delta G\, d\rho,\ b_0^{(r)} = \int_\Delta G^{(r)} d\rho$$

and $G \geq G^{(r)}$ (cf., [Szé15, Lemma 6] and Remark 2.7). Thus we have $b_0^{(r)} \leq b_0$.

On the other hand, $w_r(r) = w(r)$ since $\mathscr{F}^\bullet R_r = \mathscr{F}^\bullet_{(r)} R_r$. Thus, we have

$$\mathrm{DF}(\mathscr{F}) = \lim_{r\to\infty} 2\left(\frac{rb_0}{a_0} - \frac{w(r)}{\chi(X, rL)}\right) \geq \liminf_{r\to\infty} 2\left(\frac{rb_0^{(r)}}{a_0} - \frac{w_r(r)}{\chi(X, rL)}\right) = \mathrm{Chow}_\infty(\mathscr{F}).$$

We complete the proof. □

Finally, we introduce the nA J-functionals of filtrations. To do this, we consider the following. Let (X, L) be an n-dimensional polarized deminormal scheme, $\mathscr{F}$ be a linearly bounded multiplicative filtration of (X, L) and H be an ample $\mathbb{Q}$-line bundle on X. Assume that $\mathscr{F}$ and (X, L) satisfy Condition 2.12. Take an approximation $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ to $\mathscr{F}$. We define a semiample test configuration $(\mathcal{X}^{(k)}, \mathcal{L}^{(k)})$ that dominates $X_{\mathbb{A}^1}$ as follows. Let $\mathfrak{a}_{(k)}$ be the image of the following evaluation map, where t is the canonical coordinate of $\mathbb{A}^1$

$$\bigoplus_\lambda t^{-\lambda} \mathscr{F}^\lambda H^0(X, kL) \otimes \mathcal{O}_{X_{\mathbb{A}^1}}(-kL \times \mathbb{A}^1) \to \mathcal{O}_X[t, t^{-1}]. \tag{6}$$

$\mathfrak{a}_{(k)}$ is a $\mathbb{G}_m$-invariant fractional ideal of $\mathcal{O}_{X_{\mathbb{A}^1}}$ and called a *flag ideal* (cf., [Oda13a, 3.1], [BHJ17, §2.6]). Then, let $\mu_k : \mathcal{X}^{(k)} \to X_{\mathbb{A}^1}$ be the blow up along $\mathfrak{a}_{(k)}$ and set

$$\mathcal{L}^{(k)} := \mu_k^*(L \times \mathbb{A}^1) - \mu_k^{-1}(\mathfrak{a}_{(k)}).$$

We call $(\mathcal{X}^{(k)}, \mathcal{L}^{(k)})$ the *semiample test configurtion induced by* $\mathscr{F}_{(k)}$. For sufficiently divisible $c \in \mathbb{Z}_{>0}$, cH is a very ample line bundle and there exists a non-empty open subset consisting of $D \in |cH|$ such that the support of $\mu_k^* D_{\mathbb{A}^1}$ contains no μ_k-exceptional divisor. Then we define the following.

Definition 2.16. Notations as the previous paragraph. We say that an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D is *compatible* with $\mathscr{F}_{(k)}$ if the support of $\mu_k^* D_{\mathbb{A}^1}$ contains no μ_k-exceptional divisor. If D is further compatible with all $\mathscr{F}_{(k)}$, then we call D *compatible* with $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$.

Since $\mathbb{C}$ is uncountable, there exist $c \in \mathbb{Z}_{>0}$ and an effective $\mathbb{Q}$-Cartier divisor D such that $cD \sim cH$ and D is compatible with $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$.

Definition 2.17. Let $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ be an approximation to $\mathscr{F}$ and $D \sim_{\mathbb{Q}} H$ be a compatible divisor with $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$. We know by (4) in Proposition 2.8 that there exists a constant $\tilde{b}_{0,i}$ for any irreducible component D_i of D such that

$$\tilde{b}_{0,i} = \lim_{m\to\infty} \frac{\tilde{w}_{\mathscr{F}_{D_i}}(m)}{m^n}$$

(cf., Example 2.6). Then we set the *nA J^H-functional* of $\mathscr{F}$ as

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}) = \frac{\tilde{b}_0 a_0 - b_0 \tilde{a}_0}{a_0^2}. \tag{7}$$

Here, $\tilde{a}_0 = \lim_{m\to\infty} \frac{\chi(D,mL)}{m^{n-1}}$ and $\tilde{b}_0 = \sum m_i \tilde{b}_{0,i}$, where $D = \sum m_i D_i$.

Proposition 2.18. *Let (X, L) be a polarized deminormal scheme and $\mathscr{F}$ be its filtration satisfying Condition 2.12. Let $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ be an approximation to $\mathscr{F}$ and $D \sim_{\mathbb{Q}} H$ be a compatible divisor with $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$. If $(\mathcal{X}^{(k)}, \mathcal{L}^{(k)})$ is the semiample test configuration induced by $\mathscr{F}_{(k)}$ for every $k \in \mathbb{Z}_{>0}$, then*

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}_{(k)}) = (\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}^{(k)}, \mathcal{L}^{(k)}) \tag{8}$$
$$\lim_{k\to\infty} (\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}_{(k)}) = (\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}). \tag{9}$$

In particular, the following hold.

(i) $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F})$ is independent from the choice of a compatible divisor D.

(ii) If (X, L) is J^H-semistable, then $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}) \geq 0$ for any filtration.

Proof. Note that it immediately follows from the equations (8) and (9) that (i) and (ii) hold. Thus, it suffices to show the first assertion.

Suppose that $\dim X = n$. Let $\mathfrak{a}_{(k)}$ be the flag ideal defined by the map (6) and $\mu_k \colon \mathcal{X}^{(k)} \to X_{\mathbb{A}^1}$ be the canonical morphism. Let $D = \sum m_i D_i$ be the irreducible decomposition and note that $(\mu_k)_*^{-1}(D \times \mathbb{A}^1) = \mu_k^*(D \times \mathbb{A}^1)$. Let $\tilde{w}_i^{(k)}(m) = \tilde{b}_{0,i}^{(k)} m^n + O(m^{n-1})$ (resp. $w^{(k)}(m) = b_0^{(k)} m^{n+1} + O(m^n)$) be the weight function of $(\mathscr{F}_{(k)}^{\bullet})_{D_i}$ (resp. $\mathscr{F}_{(k)}^{\bullet} H^0(X, mL)$). In this paragraph, we show

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}^{(k)}, \mathcal{L}^{(k)}) = -\frac{b_0^{(k)} \tilde{a}_0 - \sum m_i \tilde{b}_{0,i}^{(k)} a_0}{a_0^2},$$

which is equivalent to (8). To see this, we may assume by (4) in Proposition 2.8 that X is integral, $D = D_1$ and $m_1 = 1$. If necessary, we may assume that $\mathscr{F}^i H^0(X, mL) = 0$ for $i > 0$ by replacing $\mathscr{F}$ by $\mathscr{F}_{(N\text{-shift})}$ for a suitable $N \in \mathbb{Z}$. Note that $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}) =$

$(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}_{(N\text{-shift})})$ and $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}^{(k)}, \mathcal{L}^{(k)})$ does not change. Then, it suffices to show (cf. Definition 2.3)

$$b_0^{(k)} = \frac{1}{(n+1)!}(\overline{\mathcal{L}^{(k)}})^{n+1}, \quad \text{and} \quad \tilde{b}_{0,1}^{(k)} = \frac{1}{n!}((\mu_k)_*^{-1}(D\times\mathbb{P}^1))\cdot(\overline{\mathcal{L}^{(k)}})^n.$$

Since $\mathfrak{a}_{(k)}|_{D\times\mathbb{A}^1}$ is generated by $\sum t^{-\lambda}(\mathscr{F}^\lambda_{(k)})_D H^0(D, kL|_D)$ and $(\mu_k)_*^{-1}(D\times\mathbb{A}^1)$ is the blow up along $\mathfrak{a}_{(k)}|_{D\times\mathbb{A}^1}$, the latter equation follows from [Mum77, Proposition 2.6]. The former follows in the same way. Thus, (8) holds.

Next, we claim the following. If we set $\tilde{b}_0$ as (7), then

$$\lim_{k\to\infty} b_0^{(k)} = b_0, \quad \text{and} \quad \lim_{k\to\infty}\sum m_i \tilde{b}_{0,i}^{(k)} = \tilde{b}_0. \tag{10}$$

Indeed, we may assume that $\{(\mathscr{F}^\bullet_{(k)})_{D_i}\}_{k\in\mathbb{Z}_{>0}}$ is an approximation to $\mathscr{F}_{D_i}$ by replacing L by lL for sufficiently divisible $l\in\mathbb{Z}_{>0}$. Thus, (10) follows from Lemma 2.14. We conclude that (9) holds by (10). $\square$

Remark 2.19. By Proposition 2.18, we can define $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F})$ of a non finitely generated filtration to be $\lim_{k\to\infty}(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}_{(k)})$. Note that $\mathscr{F}_{(k)}$ is good and we can define $\mathrm{DF}(\mathscr{F}_{(k)})$ for all $k\in\mathbb{Z}_{>0}$. Székelyhidi defined the Futaki invariant of a non finitely generated filtration $\mathscr{F}$ to be

$$\mathrm{Fut}(\mathscr{F}) = \liminf_{k\to\infty}\mathrm{DF}(\mathscr{F}_{(k)})$$

in [Szé15]. There is a subtle but nontrivial problem that if $\mathscr{F}$ is good, we do not know whether $\mathrm{DF}(\mathscr{F}) \geq \liminf_{k\to\infty}\mathrm{DF}(\mathscr{F}_{(k)})$ or not in contrast to Proposition 2.18. This problem is closely related to [BJ18, Conjecture 2.5]. This is why we applied [Szé15, Proposition 11] instead of [*loc.cit.*, Theorem 10] to deduce Theorem 2.15.

3 Proof of the main theorems

3.1 Construction of a good filtration

Before proving our main results, we first define CM degrees of polarized families over curves.

Definition 3.1. Let $\pi : (X, L) \to C$ be a projective flat morphism from a normal variety X with a $\mathbb{Q}$-line bundle L to a smooth curve C. Fix a closed point $0\in C$ and $\dim X = n+1$. Let $C^\circ = C\setminus\{0\}$ and assume that $\pi_*\mathcal{O}_X \cong \mathcal{O}_C$. If L is (semi)ample, then we call π a *polarized (resp., semiample) family* over a curve C. We say that B is a *horizontal* $\mathbb{Q}$-divisor if any irreducible component D is flat over C. If there further exists an open subset $U\subset X$ such that $U\cap X_0$ is smooth and contains the generic points of all irreducible components of $\pi|_D^{-1}(0)$ for any irreducible component D of B, we call B *restrictable* to X_0. Then we set the restricted $\mathbb{Q}$-divisor B_0 of B to X_0 as the closure of $B|_{U\cap X_0}$. We note that if X_0 is normal, then any horizontal $\mathbb{Q}$-divisor is restrictable since any horizontal prime divisor D is Cartier at any codimension one point of X_0.

Furthermore, let $\pi : (X, B, L) \to C$ be a morphism such that $\pi : (X, L) \to C$ is a polarized (resp., semiample) family over C and B is an effective horizontal $\mathbb{Q}$-divisor. Then we call this a (log) polarized (resp., semiample) family over C. If $K_X + B$ is further $\mathbb{Q}$-Cartier, we call this a $\mathbb{Q}$-Gorenstein family. If B is restrictable, we denote the fiber of (X, B, L) over 0 by (X_0, B_0, L_0).

Let $\pi : (X, L) \to C$ and $\pi' : (X', L') \to C$ be semiample families. $f : (X, L) \to (X', L')$ is called a *C-isomorphism* if f is an isomorphism between X and X' preserving the structure morphisms to C such that $L \sim_{\mathbb{Q},C} f^*L'$. Let $\pi : (X, B, L) \to C$ and $\pi' : (X', B', L') \to C$ be log semiample families over C. We say that $f : (X, B, L) \to (X', B', L')$ is a C-isomorphism of log semiample families, if f is a C-isomorphism from (X, L) to (X', L') as semiample families such that $f_*B = B'$. We define a C°-isomorphism of semiample families between $(X, B, L) \times_C C^\circ := (X \times_C C^\circ, B \times_C C^\circ, L|_{X\times_C C^\circ})$ and $(X', B', L') \times_C C^\circ$ in the same way.

Definition 3.2. Suppose that $\pi : (X, L) \to C$ is a semiample family over a proper smooth curve C. We define the *CM degree*

$$\mathrm{CM}((X, L)/C) = 2\frac{b_0a_1 - b_1a_0}{a_0^2} + 2(1 - g(C)),$$

where $g(C)$ is the genus of C, $\chi(X, kL) = b_0k^{n+1} + b_1k^n + O(k^{n-1})$ and $\chi(X_0, kL_0) = a_0k^n + a_1k^{n-1} + O(k^{n-2})$ for sufficiently divisible $k \in \mathbb{Z}_{>0}$ (cf., [Ohn22]). This is a positive multiple of the degree of the CM line bundle (cf., [FR06]) and we note that if $L \sim_{\mathbb{Q},C} rL'$ then $\mathrm{CM}((X, L)/C) = \mathrm{CM}((X, L')/C)$ for any $r \in \mathbb{Q}_{>0}$.

Furthermore, let B be an effective horizontal $\mathbb{Q}$-divisor on X. Note that for any general $t \in C$ such that $X_t := \pi^{-1}(t)$ is normal, B is restrictable to X_t. Suppose that $\chi(B, mL) = \tilde{b}_0m^n + O(m^{n-1})$ and $\chi(B|_{X_t}, mL_t) = \tilde{a}_0m^{n-1} + O(m^{n-2})$ for sufficiently divisible $m \in \mathbb{Z}_{>0}$ (cf., §2.1). Here we abusively denote $L|_B$ by L. Then, we set the *log CM degree* as

$$\mathrm{CM}((X, B, L)/C) = \mathrm{CM}((X, L)/C) - \frac{b_0\tilde{a}_0 - \tilde{b}_0a_0}{a_0^2}.$$

We remark that $\tilde{a}_0$ is independent of the choice of $t \in C$. Indeed, let D be a horizontal prime divisor and D^*_t be the scheme theoretic fiber of D over $t \in C$. Then we see that

$$\chi(D^*_0, mL_0) = \chi(D^*_t, mL_t) = \chi(D|_{X_t}, mL_t) + O(m^{n-1})$$

by [B01, Corollary 1.12]. It is well-known that the following holds as [Oda13a] and [Wan12] for any general $t \in C$

$$\mathrm{CM}((X, B, L)/C) = \frac{1}{L_0^n}\left((K_{X/C} + B) \cdot L^n - \frac{n}{n+1}\frac{(K_{X_t} + B|_{X_t}) \cdot L_t^{n-1}}{L_t^n}L^{n+1}\right). \quad (11)$$

We remark that we can define the intersection number $(K_{X/C} + B) \cdot L^n$ since $K_X + B$ is Cartier in codimension one (cf., [Oda13a, Lemma 3.5]).

As DF invariants, the following hold by the equation (11) and by the same argument as in [BHJ17, §7]. The proof is easy and left to the reader.

Lemma 3.3. *Let $\pi : (X, B, L) \to C$ be as above. Then the following hold.*

1. *For any finite morphism $f : C' \to C$ of smooth curves of degree r, let (X', B', L') be the normalization of $(X, B, L) \times_C C'$. Then we have*

$$\mathrm{CM}((X', B', L')/C') \leq r\,\mathrm{CM}((X, B, L)/C).$$

2. *For any proper birational morphism $\mu : X' \to X$ from a normal variety that is isomorphic over C°, let B' be the strict transform of B. Then*

$$\mathrm{CM}((X', B', \mu^* L)/C) = \mathrm{CM}((X, B, L)/C).$$

Odaka proposed the following conjecture.

Conjecture 3.4 (CM minimization, cf., [Oda20, Conjecture 8.1])**.** *Let $\pi : (X, B, L) \to C$ be a $\mathbb{Q}$-Gorenstein polarized family such that (X_0, B_0, L_0) is K-semistable. Then*

$$\mathrm{CM}((X, B, L)/C) \leq \mathrm{CM}((X', B', L')/C)$$

for any polarized family $\pi' : (X', B', L') \to C$ such that there exists a C°-isomorphism $f^\circ : (X, B, L) \times_C C^\circ \cong (X', B', L') \times_C C^\circ$. Furthermore, if (X_0, B_0, L_0) is K-stable, then equality holds if and only if f° extends to $f : (X, B, L) \cong (X', B', L')$ over C entirely.

Remark 3.5. In Conjecture 3.4, we assume that X and X' are normal. If we do not assume so, (X, B, L) and (X', B', L') are not isomorphic entirely in general even if (X_0, B_0, L_0) is specially K-stable (cf., Definition 3.19 below). This phenomenon was observed for test configurations by [LX14]. Thus, if (X_0, B_0, L_0) is K-stable but X' is not normal, then (X, B, L) and (X', B', L') are conjectured to be isomorphic only in codimension one.

Conjecture 3.4 is proved in the Calabi-Yau case by [Oda13c], in the K-ample case by [WX14] and in the K-(semi)stable log Fano case by [Xu21]. Note that their results follow from Theorems 3.21 and 3.22.

We prove Conjecture 3.4 when (X_0, B_0, L_0) is

- a cscK manifold such that $\mathrm{Aut}(X_0, L_0)$ is discrete in §3.2,
- specially K-stable (cf., Definition 3.19) in §3.5.

To do this, we first prove that the difference of CM degrees is the DF invariant of a certain good filtration.

Theorem 3.6. *Let $\pi : (X, L) \to C$ be a polarized family and $\pi' : (X', L') \to C$ be a semiample family over a proper smooth curve such that there exists a projective birational morphism $\mu \colon X' \to X$ such that $\mu|_{X'\times_C C^\circ} : (X', L') \times_C C^\circ \cong (X, L) \times_C C^\circ$ is a C°-isomorphism. Suppose that L' is μ-ample and there exists an effective $\mathbb{Q}$-divisor E on X' such that $L' = \mu^* L - E$.*

Then there exist $k \in \mathbb{Z}_{>0}$ and an affine open neighborhood U of $0 \in C$ satisfying the following. Set

$$F^i H^0(\pi^{-1}U, mkL) := \begin{cases} t^i H^0(\pi'^{-1}U, mkL') \cap H^0(\pi^{-1}U, mkL) & \text{for } i \leq 0 \\ 0 & \text{for } i > 0, \end{cases}$$

and $\mathscr{F}^i H^0(X_0, mkL_0) := \mathrm{Im}\,(F^i H^0(\pi^{-1}U, mkL) \to H^0(X_0, mkL_0))$. Then $\mathscr{F}$ is a filtration of (X_0, kL_0) and the weight $w_{\mathscr{F}}(m)$ of $\mathscr{F}^\bullet H^0(X_0, mkL_0)$ satisfies that

$$\begin{aligned} w_{\mathscr{F}}(m) &= -\dim H^0(\pi^{-1}U, mkL)/H^0(\pi'^{-1}U, mkL') \\ &= \chi(X', mkL') - \chi(X, mkL) + O(m^{n-1}) \end{aligned} \tag{12}$$

for any sufficiently large $m \in \mathbb{Z}_{>0}$. In particular,

$$\mathrm{DF}(\mathscr{F}) = \mathrm{CM}((X', L')/C) - \mathrm{CM}((X, L)/C).$$

Proof. We may assume that L and L' are $\mathbb{Z}$-Cartier by replacing L by kL for some $k \in \mathbb{Z}_{>0}$. We first prove that there exists $k \in \mathbb{Z}_{>0}$ such that

$$\chi(X', mkL') - \chi(X, mkL) = -\dim H^0(\pi^{-1}U, mkL)/H^0(\pi'^{-1}U, mkL') + O(m^{n-1}) \tag{13}$$

holds for any sufficiently large m and any sufficiently small affine open neighborhood U of $0 \in C$. We may assume that L is ample (resp., L' is semiample) since $\chi(X', mL') - \chi(X, mL)$ does not change when we replace L and L' by $L + cX_0$ and $L' + cX'_0$ for some $c \in \mathbb{Z}_{>0}$ respectively. We also see that $H^0(\pi^{-1}U, mL)$ and $H^0(\pi'^{-1}U, mL')$ do not change either if we choose U small enough. By the Serre vanishing theorem, $h^i(X, mL) = 0$ for $i > 0$ and sufficiently large m. On the other hand, we prove the following claim.

Claim 2. *It holds that*

$$\begin{aligned} h^i(X', mL') &= O(m^{n-1}), \text{ for } i > 0, \\ \chi(X', mL') &= h^0(X', mL') + O(m^{n-1}) \end{aligned}$$

for sufficiently large m.

Proof of Claim 2. Indeed, set $X'_{\mathrm{amp}} := \mathrm{Proj}_C(\oplus_{m\geq 0} \pi'_* \mathcal{O}_X(mL'))$. Let $\xi : X' \to X'_{\mathrm{amp}}$ be the canonical morphism and $L'_{\mathrm{amp}} := \mathcal{O}_{X'_{\mathrm{amp}}}(1)$. Note that L'_{amp} is relatively ample over C and $\xi^* L'_{\mathrm{amp}} \sim L'$. Note that ξ^{-1} is defined at all codimension one points of X'_{amp} and [Har77, III Theorem 11.1] implies that $\mathrm{codim}_{X'_{\mathrm{amp}}} \mathrm{Supp}\, R^j \xi_* \mathcal{O}_{X'} \geq 2$ for all $j > 0$. By the Leray spectral sequence $H^i(X'_{\mathrm{amp}}, R^j \xi_* \mathcal{O}_{X'} \otimes L'^m_{\mathrm{amp}}) \Rightarrow H^{i+j}(X', L'^m)$ and the Serre vanishing theorem, we obtain $h^i(X', mL') = O(m^{n-1})$ for $i > 0$ and hence $\chi(X', mL') = \chi(X'_{\mathrm{amp}}, mL'_{\mathrm{amp}}) + O(m^{n-1})$. Note also that $h^0(X', mL') = h^0(X'_{\mathrm{amp}}, mL'_{\mathrm{amp}})$. Thus we have Claim 2 by the Serre vanishing theorem applied to L_{amp}. □

Let $\mathfrak{a}$ be an ideal on X such that μ is the blow up of $\mathfrak{a}$ and there exists an integer $k \in \mathbb{Z}_{>0}$ such that $\mu^{-1}\mathfrak{a} = \mathcal{O}(-kE)$. Indeed, the $\mathcal{O}_X$-algebra $\bigoplus_{l\geq 0}\mu_*\mathcal{O}(-lE)$ is finitely generated and hence $\mu_*\mathcal{O}(-kE)$ generates $\bigoplus_{l\geq 0}\mu_*\mathcal{O}(-lE)$ for some $k \in \mathbb{Z}_{>0}$. We may assume that $k=1$ by replacing L by kL. Then, we obtain the following exact sequence,

$$0 \to H^0(X, mL\otimes \mathfrak{a}^m) \to H^0(X, mL) \xrightarrow{\alpha} H^0(X, mL\otimes (\mathcal{O}_X/\mathfrak{a}^m)) \to H^1(X, mL\otimes \mathfrak{a}^m).$$

By [Laz04, Lemma 5.4.24], $H^i(X', mL') = H^i(X, mL\otimes\mathfrak{a}^m)$ for sufficiently large $m>0$. Since $h^1(X', mL') = O(m^{n-1})$ by Claim 2, we obtain

$$\begin{aligned}\chi(X', mL') - \chi(X, mL) &= -\dim \operatorname{Im}\alpha + O(m^{n-1})\\ &= -h^0(X, mL\otimes(\mathcal{O}_X/\mathfrak{a}^m)) + O(m^{n-1}).\end{aligned}$$

Next, take an affine open neighborhood U of $0\in C$ small enough such that there exists $t \in H^0(U, \mathcal{O}_U)$ such that $\mathcal{O}_U/t = \mathcal{O}_{C,0}/\mathfrak{m}_0$. Here, $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}_{C,0}$. Note that the support of $\mathcal{O}_X/\mathfrak{a}$ is contained in X_0. Let $\beta\colon H^0(\pi^{-1}(U), mL) \to H^0(X, mL\otimes(\mathcal{O}_X/\mathfrak{a}^m))$ be the canonical morphism. Then, α factors through β and hence

$$\operatorname{Im}\alpha \subset \operatorname{Im}\beta \subset H^0(X, mL\otimes(\mathcal{O}_X/\mathfrak{a}^m)).$$

Since

$$h^0(X, mL\otimes(\mathcal{O}_X/\mathfrak{a}^m)) = \dim\operatorname{Im}\alpha + O(m^{n-1}),$$

we conclude that

$$\chi(X', mL') - \chi(X, mL) = -\dim\operatorname{Im}\beta + O(m^{n-1}).$$

Note also that

$$\operatorname{Im}\beta \cong H^0(\pi^{-1}(U), mL)/H^0(\pi'^{-1}(U), mL').$$

Thus we see that (13) holds for any sufficiently large m.

Next, we investigate the properties of $\mathscr{F}$. $F^{-i}H^0(\pi^{-1}U, mL)$ satisfies that

$$t^iF^{-i}H^0(\pi^{-1}U, mL) = H^0(\pi'^{-1}U, mL') \cap t^iH^0(\pi^{-1}U, mL) \tag{14}$$

as submodules of $H^0(X, mL)$ for $i\in\mathbb{Z}_{\geq 0}$. Then, $F^\bullet H^0(\pi^{-1}U, mL)$ defines a linearly bounded multiplicative filtration of $H^0(\pi^{-1}U, mL)$. Indeed, if we take $i>0$ such that $iX'_0 - E$ is effective then for $m>0$,

$$t^{mi}H^0(\pi^{-1}U, mL) = H^0(\pi'^{-1}U, mL' + mE - miX'_0) \subset H^0(\pi'^{-1}U, mL'). \tag{15}$$

Hence, $\mathscr{F}^\bullet H^0(X_0, mL_0)$ is also a linearly bounded multiplicative filtration. Then we claim the following.

Claim 3. *For any $m>0$ such that $H^1(X_0, mL_0) = 0$, there exists an isomorphism*

$$H^0(\pi^{-1}U, mL)/H^0(\pi'^{-1}U, mL') \cong \bigoplus_{i\geq 0} t^i(H^0(X_0, mL_0)/\mathscr{F}^{-i}H^0(X_0, mL_0))$$

as $\mathbb{C}$-vector spaces.

Proof of Claim 3. For this, note that $H^0(\pi^{-1}U, mL) \to H^0(X_0, mL_0)$ is surjective and

$$H^0(\pi'^{-1}U, mL') = \sum_{i\geq 0} t^i F^{-i} H^0(\pi^{-1}U, mL) \subset H^0(\pi^{-1}U, mL).$$

For any inclusion $\mathcal{V}' \subset \mathcal{V}$ of coherent $\mathcal{O}_U$-modules, there is the following commutative diagram whose rows and columns are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{V}' \cap t\mathcal{V} \arrow[r] \arrow[d] & t\mathcal{V} \arrow[r] \arrow[d] & t\mathcal{V}/\mathcal{V}' \cap t\mathcal{V} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}' \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}/\mathcal{V}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}'/\mathcal{V}' \cap t\mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}/t\mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}/(\mathcal{V}' + t\mathcal{V}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Suppose that $\mathrm{Supp}\,(\mathcal{V}/\mathcal{V}') = \{0\}$ and there exists $N \in \mathbb{Z}_{\geq 0}$ such that $t^N\mathcal{V} \subset \mathcal{V}'$. Then we show by induction on N that

$$\mathcal{V}/\mathcal{V}' \cong \bigoplus_{i\geq 0} t^i\mathcal{V}/(t^{i+1}\mathcal{V} + \mathcal{V}' \cap t^i\mathcal{V}). \tag{16}$$

By the above diagram, we obtain an isomorphism

$$\mathcal{V}/\mathcal{V}' \cong t\mathcal{V}/\mathcal{V}' \cap t\mathcal{V} \oplus \mathcal{V}/(\mathcal{V}' + t\mathcal{V})$$

as $\mathbb{C}$-vector spaces. Note that $t^{N-1}\cdot(t\mathcal{V}) \subset \mathcal{V}' \cap t\mathcal{V}$. By applying the induction hypothesis to $\mathcal{V}' \cap t\mathcal{V} \subset t\mathcal{V}$, we obtain that

$$t\mathcal{V}/\mathcal{V}' \cap t\mathcal{V} \cong \bigoplus_{i\geq 0} t^{i+1}\mathcal{V}/(t^{i+2}\mathcal{V} + \mathcal{V}' \cap t^{i+1}\mathcal{V}).$$

Thus, we have (16). When $\mathcal{V} = H^0(\pi^{-1}U, mL)$ and $\mathcal{V}' = H^0(\pi'^{-1}U, mL')$, by the equation (14), we have that

$$t^i\mathcal{V}/(t^{i+1}\mathcal{V} + \mathcal{V}' \cap t^i\mathcal{V}) \cong t^i(H^0(X_0, mL_0)/\mathscr{F}^{-i}H^0(X_0, mL_0)).$$

Hence we obtain by (16) and (15) that

$$H^0(\pi^{-1}U, mL)/H^0(\pi'^{-1}U, mL') \cong \bigoplus_{i\geq 0} t^i(H^0(X_0, mL_0)/\mathscr{F}^{-i}H^0(X_0, mL_0)).$$

Thus, we finish the proof of Claim 3. $\square$

Then, we obtain

$$w_{\mathscr{F}}(m) = -\dim H^0(\pi^{-1}U, mL)/H^0(\pi'^{-1}U, mL')$$

by Claim 3 and [Mum77, Lemma 2.14]. By (13), we also have that

$$w_{\mathscr{F}}(m) = \chi(X', mL') - \chi(X, mL) + O(m^{n-1}).$$

On the other hand, if $\chi(X, mL) = b_0 m^{n+1} + b_1 m^n + O(m^{n-1})$ and $\chi(X', mL') = b'_0 m^{n+1} + b'_1 m^n + O(m^{n-1})$, then $w_{\mathscr{F}}(m) = (b'_0 - b_0)m^{n+1} + (b'_1 - b_1)m^n + O(m^{n-1})$. Therefore,

$$\mathrm{DF}(\mathscr{F}) = \mathrm{CM}((X', L')/C) - \mathrm{CM}((X, L)/C).$$

We complete the proof. □

Remark 3.7. If X_0 is irreducible, $\hat{X}_0$ is the strict transform of X_0 in X', and E is μ-exceptional, then

$$\mathscr{F}^{-i} H^0(X_0, mL_0) = \mathrm{Im}\,(H^0(\pi'^{-1}U, mL' + i(X'_0 - \hat{X}_0)) \to H^0(X_0, mL_0))$$

for $i \in \mathbb{Z}_{\geq 0}$ in the above proof. If (X, L) and (X', L') are $\mathbb{Q}$-Fano families with K-semistable fibers, then $\mathscr{F}$ coincides with the filtration constructed in [BX19, §5].

Proof of Theorem 1.3. Take a proper morphism $\mu' \colon X'' \to X'$ such that μ' is isomorphic over $X' \times_C C^\circ$ and there exists a morphism $\mu \colon X'' \to X$ such that μ is isomorphic over $X \times_C C^\circ$. We see that there exists an effective $\mathbb{Q}$-divisor E such that $L'' = \mu^* L - E$ if we set $L'' := \mu'^* L' + cX''_0$ for some integer c. Let $\xi \colon X'' \to X''_{\mathrm{amp}} := \mathrm{Proj}_X(\oplus \mu_* \mathcal{O}_{X''}(mr_0 L''))$ be the canonical birational morphism, where r_0 is a positive integer such that $r_0 L''$ is $\mathbb{Z}$-Cartier, and L''_{amp} be a $\mathbb{Q}$-line bundle such that $\xi^* L''_{\mathrm{amp}} \sim_{\mathbb{Q}} L''$. By Lemma 3.3 (2), we see that

$$\mathrm{CM}((X', L')/C) = \mathrm{CM}((X'', L'')/C) = \mathrm{CM}((X''_{\mathrm{amp}}, L''_{\mathrm{amp}})/C).$$

Then the assertion holds by applying Theorem 3.6 to $(X''_{\mathrm{amp}}, L''_{\mathrm{amp}})$. □

We obtain the log version of Theorem 1.3 as follows.

Corollary 3.8. *Let $\pi : (X, B, L) \to C$ and $\pi' : (X', B', L') \to C$ be polarized log families over a proper smooth curve C. Suppose that X_0 is deminormal, B is restrictable to X_0 (cf. Definition 3.1), and B_0 is a $\mathbb{Q}$-divisor and there exists a C°-isomorphism $(X, B, L) \times_C C^\circ \to (X', B', L') \times_C C^\circ$.*

Then there exist $k \in \mathbb{Z}_{>0}$ and a good filtration $\mathscr{F}$ of $\bigoplus_{m \geq 0} H^0(X_0, mkL_0)$ such that

$$\mathrm{DF}_{B_0}(\mathscr{F}) \leq \mathrm{CM}((X', B', L')/C) - \mathrm{CM}((X, B, L)/C).$$

Proof. As the proof of Theorem 1.3, we may replace (X', B', L') by a semiample family and may assume that there exists a birational contraction $\mu : X' \to X$ and $L' = \mu^* L - E$ where E is an effective divisor such that $-E$ is μ-ample. Then we take $\mathscr{F}$ as Theorem 3.6. It is easy to see that the assertion follows from Lemma 3.9 below applied to an irreducible component D of B. □

Lemma 3.9. *In the situation of Theorem 3.6, let D be a restrictable prime divisor to X_0 on X such that D_0 is a Weil divisor and $D_0 = \sum m_i \Gamma_i$ is the irreducible decomposition, and let $D' \subset X'$ be the strict transform of D. Then*

$$\frac{1}{n!}(D' \cdot L'^n - D \cdot L^n) \geq \sum m_i \mathcal{B}_{\mathscr{F}_{1,i}},$$

where $\mathscr{F}_{1,i}$ is the induced filtration on $\bigoplus H^0(\Gamma_i, mkL|_{\Gamma_i})$ by $\mathscr{F}$ and $w_{\mathscr{F}_{1,i}}(m)$ is the weight function such that $\lim_{m\to\infty} \frac{w_{\mathscr{F}_{1,i}}(m)}{m^n} = \mathcal{B}_{\mathscr{F}_{1,i}}$.

Proof. Let U be an affine open neighborhood of $0 \in C$ and $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal as in the proof of Theorem 3.6. Replacing L by kL, we may assume that $k = 1$ in Theorem 3.6. Instead of Claim 2, we see that for sufficiently large m,

$$\begin{aligned}\chi(D', mL') &= h^0(D', mL') + O(m^{n-1}) \\ \chi(D, mL) &= h^0(D, mL) = h^0(D', m\mu^*L) + O(m^{n-1}) = \chi(D', m\mu^*L) + O(m^{n-1})\end{aligned}$$

by [Mum77, Lemma 2.7]. Note that D' is the blow up of D along $\mathfrak{a}|_D$. Hence, $H^i(D', mL') = H^i(D, mL \otimes \mathfrak{a}|_D^m)$ for sufficiently large $m > 0$ by [Laz04, Lemma 5.4.24]. Thus,

$$\chi(D', mL') - \chi(D, mL) = -\dim\left(H^0(D|_U, mL)/H^0(D|_U, mL \otimes \mathfrak{a}^m|_{D|_U})\right) + O(m^{n-1})$$

holds as Theorem 3.6. Here, $D|_U := \pi^{-1}(U) \cap D$.

Define a filtration $G^\bullet$ on $\bigoplus H^0(D|_U, mL|_{D|_U})$ by

$$t^i G^{-i} H^0(D|_U, mL) = H^0(D'|_U, mL') \cap t^i H^0(D|_U, mL)$$

if $i \geq 0$. Otherwise, $G^{-i}H^0(D|_U, mL) = 0$. Then, let $\mathscr{G}^i H^0(D_0^*, mL_0)$ be the image of $G^i H^0(D|_U, mL) \to H^0(D_0^*, mL_0)$. Here, D_0^* is the scheme theoretic fiber of D over 0. Let $\mathscr{F}_{1,*}$ be the induced filtration of $\bigoplus H^0(D_0^*, mL|_{D_0^*})$ by $\mathscr{F}$. By Proposition 2.8,

$$\mathcal{B}_{\mathscr{F}_{1,*}} = \limsup_{m\to\infty} \frac{w_{\mathscr{F}_{1,*}}(m)}{m^n} \geq \sum m_i \mathcal{B}_{\mathscr{F}_{1,i}}. \tag{17}$$

It is easy to see that $\mathscr{F}^i_{1,*} H^0(D_0^*, mL_0) \subset \mathscr{G}^i H^0(D_0^*, mL_0)$. Therefore,

$$\mathcal{B}_{\mathscr{G}} = \limsup_{m\to\infty} \frac{w_{\mathscr{G}}(m)}{m^n} \geq \mathcal{B}_{\mathscr{F}_{1,*}}. \tag{18}$$

On the other hand, we see that

$$\begin{aligned} w_{\mathscr{G}}(m) &= -\dim\left(H^0(D|_U, mL)/H^0(D|_U, mL \otimes \mathfrak{a}^m|_{D|_U})\right) \\ &= \chi(D', mL') - \chi(D, mL) + O(m^{n-1}) \end{aligned} \tag{19}$$

by Claim 3. Thus, we obtain the assertion by (17), (18) and (19). □

3.2 Minimizing CM degree for cscK manifolds with its automorphism group discrete

Due to Theorem 3.6, we can prove Conjecture 3.4 for certain cscK manifolds. The following is Theorem 1.2.

Corollary 3.10. *Let $\pi : (X, L) \to C$ and $\pi' : (X', L') \to C$ be polarized families over a proper smooth curve C such that there exists a C°-isomorphism $f^\circ : (X, L) \times_C C^\circ \cong (X', L') \times_C C^\circ$. If X_0 is smooth, (X_0, L_0) has a cscK metric and $\mathrm{Aut}(X_0, L_0)$ is discrete, then*

$$\mathrm{CM}((X', L')/C) \geq \mathrm{CM}((X, L)/C).$$

Equality holds if and only if the birational map f° can be extended to a C-isomorphism $(X, L) \cong (X', L')$.

Proof. The first assertion immediately follows from Theorems 1.3 and 2.15. As the proof of Theorem 1.3, we may assume that there exists a semiample family $(X'', L'') \to C$ satisfying the following.

- There exists a birational projective morphism $\xi\colon X'' \to X'$ such that $L'' \sim_{C,\mathbb{Q}} \xi^* L'$,
- $\mathrm{CM}((X', L')/C) = \mathrm{CM}((X'', L'')/C)$ and
- there exists a projective morphism $\mu\colon X'' \to X$ such that μ is isomorphic over $X \times_C C^\circ$ and there exists an effective $\mathbb{Q}$-divisor E such that $-E$ is μ-ample and $L'' = \mu^* L - E$.

If necessary, we may further assume that L'' is semiample and E is μ-exceptional by replacing E with $E + cX''_0$ for some $c \in \mathbb{Q}$. Applying Theorem 3.6 to $(X'', L'') \to C$, we obtain a good filtration $\mathscr{F}$ of (X_0, kL_0) satisfying (12). Replacing L by kL, we may assume that $k = 1$. By Theorem 2.15, to see the last assertion, it suffices to show that $\|\mathscr{F}\|_2 > 0$ when we assume that $E \neq 0$. Let G be the concave transformation (§2.2.1) with respect to $\mathscr{F}$. On Δ°, which is the interior of the Okounkov body Δ of (X_0, L_0), note that G is continuous. Then it suffices to show $G|_{\Delta^\circ}$ is not a constant function by [Szé15, Lemma 7].

Let $L_s = \mu^* L - sE$ for $0 \leq s \leq 1$ and $f(s) = L_s^{n+1} - L^{n+1}$. Then $f(0) = 0$ and

$$\frac{d}{ds} f(s) = -(n+1) E \cdot L_s^n \leq 0.$$

$\frac{d}{ds} f(s) < 0$ for $0 < s < 1$ since L_s is π-ample and $E \neq 0$ is effective. Hence,

$$f(1) = L''^{n+1} - L^{n+1} < 0. \tag{20}$$

We claim that then $G|_{\Delta^\circ}$ is not a constant function. Indeed,

$$\mathscr{F}^0 H^0(X_0, mL_0) = \mathrm{Im}\left(H^0((\pi \circ \mu)^{-1} U, mL') \to H^0(X_0, mL_0)\right)$$

holds now by construction (cf. Remark 3.7). This contains the image of $H^0(X'', mL'') \to H^0(\hat{X}_0, mL''|_{\hat{X}_0})$ and the kernel of this map is $H^0(X'', mL'' - \hat{X}_0)$ where $\hat{X}_0$ is the strict transform of X_0. Since L'' is semiample, $H^0(X'', mL'' - \hat{X}_0) \neq H^0(X'', mL'')$ for sufficiently large m and hence there exists at least one rational point $p \in \Delta$ such that $G(p) = 0$. Note that $G \leq 0$. By the concavity of G, for any small $\epsilon > 0$, there exists a point $q \in \Delta^\circ$ such that $G(q) > -\epsilon$. If $G|_{\Delta^\circ}$ is constant, $G|_{\Delta^\circ} \equiv 0$. From Theorem 3.6, it follows that $L''^{n+1} = L^{n+1}$. This contradicts to the inequality (20). Thus, $G|_{\Delta^\circ}$ is non constant and we complete the proof. □

3.3 Minimization for J-degree

In this subsection, we deal with the minimization problem for J-degree (Theorem 3.14). This is the first ingredient to show Theorem 3.22 below. First, we introduce the following generalization of nA J-functionals.

Definition 3.11. Let $\pi : (X, L) \to C$ be a polarized family over a proper smooth curve and $\pi' : (X', L') \to C$ be another semiample family such that $(X, L) \times_C C^\circ \cong (X', L') \times_C C^\circ$. Suppose that H is a $\mathbb{Q}$-line bundle on X and there exists a canonical birational morphism $\mu : X' \to X$. Then, we define the J^H-degree of (X', L') as

$$\mathcal{J}^H(X', L') = \frac{1}{L_0^n}\left(\mu^* H \cdot L'^n - \frac{n}{n+1}\frac{H_0 \cdot L_0^{n-1}}{L_0^n} L'^{n+1}\right).$$

Note that this degree is pullback invariant (i.e. Lemma 3.3 holds for J^H-degrees) and for general $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the support of $D' = \mu^* D$ contains no component of X'_0, we deduce from [Laz04, Corollary 1.4.41] that

$$\mathcal{J}^H(X', L') = \frac{1}{a_0^2}\left(a_0\left(\lim_{m\to\infty}\frac{\chi(D', mL')}{m^n}\right) - b'_0\left(\lim_{m\to\infty}\frac{\chi(D_0, mL_0)}{m^{n-1}}\right)\right).$$

Here, $a_0 = \lim_{m\to\infty}\frac{\chi(X_0, mL_0)}{m^n}$ and $b'_0 = \lim_{m\to\infty}\frac{\chi(X', mL')}{m^{n+1}}$. If this is the case, we call D a *compatible* divisor with μ. Here, we remark that a compatible divisor D is not necessarily effective.

Remark 3.12. We note that $\mathcal{J}^H(X', L')$ depends on (X, L) as well as H. By definition, if we fix a polarized family (X, L), then $\mathcal{J}^H(X', L')$ is linear with respect to $\mathbb{Q}$-line bundles H on X.

Note also that $\mathcal{J}^H(X', L')$ is independent from the relative linear equivalence class of L' over C. The following is a generalization of [BHJ17, 7.8 and 7.9] and [Der16, 4.7].

Lemma 3.13. *Notations as above. If $H \equiv L$, then*

$$\mathcal{J}^H(X', L') \geq \mathcal{J}^H(X, L).$$

Equality holds if and only if $\mu^* L \sim_{\mathbb{Q},C} L'$.

Proof. Let $L' = \mu^*L + E$ where the support of E is contained in X'_0. Let $L'_s = \mu^*L + sE$ and $f(s) := \mathcal{J}^H(X', L'_s) - \mathcal{J}^H(X, L)$ for $s \in [0,1]$. Then by [LX14, Lemma 1]

$$\frac{d}{ds}f(s) = -\frac{ns}{(L_0)^n}(E^2) \cdot (L_s'^{n-1}) \geq 0.$$

Moreover, this derivative is positive when $E \not\sim_{C,\mathbb{Q}} 0$ for any $s \in (0,1)$. □

By Proposition 2.18, if (X, L) is J^H-semistable then $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}) \geq 0$ for any filtration. With this in mind, we prove the following, so-called J-minimization.

Theorem 3.14. *Let $\pi : (X, L) \to C$ be a polarized family over a proper smooth curve and $\pi' : (X', L') \to C$ be another semiample family such that there exists a C°-isomorphism $f^\circ : (X, L) \times_C C^\circ \cong (X', L') \times_C C^\circ$. Let H be a $\mathbb{Q}$-line bundle on X such that H_0 is nef. Suppose that (X_0, L_0) is a polarized J^{H_0}-semistable deminormal scheme and there exists a birational morphism $\mu : X' \to X$ over C such that $\mu|_{X' \times_C C^\circ} = f^{\circ -1}$.*

Then the following inequality holds.

$$\mathcal{J}^H(X', L') \geq \mathcal{J}^H(X, L). \tag{21}$$

Furthermore, if H_0 is ample and (X_0, L_0) is uniformly J^{H_0}-stable, then equality holds if and only if $\mu^*L \sim_{\mathbb{Q},C} L'$.

Proof. First, we treat the case when H_0 is ample and (X_0, L_0) is uniformly J^{H_0}-stable and show (21). We may assume that L and L' are $\mathbb{Z}$-Cartier. By Definition 3.11, we may replace X' with $\mathrm{Proj}_X(\oplus_{m\geq 0}\mu_*\mathcal{O}_{X'}(mL'))$ and assume that L' is μ-ample and there exists an effective $\mathbb{Q}$-divisor E on X' such that $L' = \mu^*L - E$. Note that $\mu^*D = \mu_*^{-1}D$ holds now. As Theorem 3.6, replacing L by kL for sufficiently divisible $k \in \mathbb{Z}_{>0}$, take a good filtration $\mathscr{F}$ of (X_0, L_0) satisfying (12). Now, we may assume that (X, L) and $\mathscr{F}$ satisfy Condition 2.12 and take its approximation $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ as Definition 2.13. Replacing H by cH for some $c \in \mathbb{Z}_{>0}$, we may assume that H is $\mathbb{Z}$-Cartier, H_0 is very ample and there exists an open neighborhood $0 \in U \subset C$ such that H is ample over U by [KM98, Proposition 1.41]. Since we work over $\mathbb{C}$, we pick a very general divisor $D \sim H$ up satisfying the following.

- $D \cap \pi^{-1}U$ is effective, reduced and irreducible,
- D has connected fibers and is horizontal over U,
- D is compatible with μ,
- The restriction $D_0 \in |H_0|$ of D to X_0 is compatible with $\{\mathscr{F}_{(k)}\}_{k\in\mathbb{Z}_{>0}}$ (see Definitions 2.16 and 2.17).

Let $a_0 = \lim_{m\to\infty} \frac{\chi(X_0,mL_0)}{m^n}$, $b'_0 = \lim_{m\to\infty} \frac{\chi(X',mL')}{m^{n+1}}$ and $b_0 = \lim_{m\to\infty} \frac{\chi(X,mL)}{m^{n+1}}$. Then we have

$$\begin{aligned}\mathcal{J}^H(X',L') - \mathcal{J}^H(X,L) = \frac{-1}{a_0^2}\Bigg((b'_0 - b_0)\left(\lim_{m\to\infty} \frac{\chi(D_0,mL_0)}{m^{n-1}}\right) \\ - a_0\left(\lim_{m\to\infty}\left(\frac{\chi(D',mL')}{m^n} - \frac{\chi(D,mL)}{m^n}\right)\right)\Bigg),\end{aligned}$$

In the above equation, the first term of the right hand side is calculated in Theorem 3.6. To calculate the second term, let D_1 be the Zariski closure of $D \cap \pi^{-1}U$ in X. Then

$$\left(\frac{\chi(\mu^*D,mL')}{m^n} - \frac{\chi(D,mL)}{m^n}\right) = \left(\frac{\chi(\mu_*^{-1}D_1,mL')}{m^n} - \frac{\chi(D_1,mL)}{m^n}\right)$$

holds and we apply Lemma 3.9 to an effective divisor D_1 instead of D. Then, we obtain the inequality

$$\mathcal{J}^H(X',L') - \mathcal{J}^H(X,L) \geq (\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}).$$

We have $(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}) \geq 0$ by Proposition 2.18 and the assumption that (X_0,L_0) is J^{H_0}-semistable.

On the other hand, (X_0,L_0) is uniformly $\mathrm{J}^{(H_0-\epsilon L_0)}$-stable and $H_0 - \epsilon L_0$ is ample for sufficiently small $\epsilon > 0$. Therefore, if equality holds in (21), so does $\mathcal{J}^L(X',L') = \mathcal{J}^L(X,L)$. Then it follows that $\mu^*L \sim_{\mathbb{Q},C} L'$ from Lemma 3.13.

Finally, suppose that H_0 is nef and (X_0,L_0) is J^{H_0}-semistable. For any $\epsilon > 0$, $H_0 + \epsilon L_0$ is ample and (X_0,L_0) is uniformly $\mathrm{J}^{(H_0+\epsilon L_0)}$-stable. We have already shown that $\mathcal{J}^{H+\epsilon L}(X',L') \geq \mathcal{J}^{H+\epsilon L}(X,L)$. Thus,

$$\mathcal{J}^H(X',L') - \mathcal{J}^H(X,L) = \lim_{\epsilon\to 0}(\mathcal{J}^{H+\epsilon L}(X',L') - \mathcal{J}^{H+\epsilon L}(X,L)) \geq 0.$$

We complete the proof. □

Remark 3.15. If the base field k is a countable algebraically closed field, Theorem 3.14 also holds. As in the proof, we take an approximation $\{\mathscr{F}_{(r)}\}_{r\in\mathbb{Z}_{>0}}$ to $\mathscr{F}$ and uncountable algebraically closed field k' containing k. Note that in this case, there exists no compatible divisor with $\{\mathscr{F}_{(r)}\}_{r\in\mathbb{Z}_{>0}}$. However, if we change the base field to k', then there exists a compatible divisor D. Denote $\mathscr{F}'$ be the filtration of $R \otimes_k k'$ defined by $\mathscr{F}^\bullet R \otimes_k k'$ for example. Let $h : X \times_k k' \to X$ be the canonical morphism. Then it is easy to see that $(\mathcal{J}^{H_0})^{\mathrm{NA}}(\mathscr{F}_{(r)}) = (\mathcal{J}^{h_0^*H_0})^{\mathrm{NA}}((\mathscr{F}_{(r)})')$ and $\mathcal{J}^H(X,L) = \mathcal{J}^{h^*H}(X \times_k k', h^*L)$. The same equation holds for (X',L'). Thus we have similarly

$$\mathcal{J}^H(X',L') - \mathcal{J}^H(X,L) \geq \lim_{r\to\infty}(\mathcal{J}^H)^{\mathrm{NA}}(\mathscr{F}_{(r)}).$$

3.4 Minimization for Ding degree

In this section, we consider the following generalization of the log-twisted Ding invariant of a semiample test configuration and prove Theorem 3.18, which is the second ingredient to show Theorem 3.22 below.

Definition 3.16. Let $\pi : (X, B, L) \to C$ be a $\mathbb{Q}$-Gorenstein polarized family over a proper smooth curve where X_0 is irreducible (cf. Definition 3.1). Suppose that there exists a $\mathbb{Q}$-line bundle H on X such that $L|_{X \times_C C^\circ} \sim_{\mathbb{Q},C^\circ} -(K_X + B + H)|_{X \times_C C^\circ}$. Let $\pi' : (X', B', L') \to C$ be another semiample family such that $(X, B, L) \times_C C^\circ \cong (X', B', L') \times_C C^\circ$. Suppose that there exists a birational C-morphism $\mu : X' \to X$ that induces the C°-isomorphism. Then we define the *log-twisted Ding degree* of (X', B', L') with respect to H

$$\begin{aligned}\mathrm{Ding}_{(B',H)}(X', L') = \sup\{t \in \mathbb{Q} | (X', B' + D_{(X',B',H,L')} + tX'_0) \text{ is sublc around } X'_0\} \\ - 1 - \frac{L'^{n+1}}{(n+1)L_0^n},\end{aligned}$$

where $D_{(X',B',H,L')}$ is a unique $\mathbb{Q}$-divisor whose support is contained in X'_0 such that $D_{(X',B',H,L')} \equiv -(K_{X'/C} + B' + L' + \mu^* H)$ holds. Indeed, uniqueness of $D_{(X',B',H,L')}$ follows from the fact that there exists a very general curve $C' \subset X'$ that is disjoint from any irreducible component of X'_0 but the strict transform of X_0 and from [KM98, Lemma 3.39]. Note that $K_{X'} + B' + D_{(X',B',H,L')}$ is $\mathbb{Q}$-Cartier and we can check whether $(X', B' + D_{(X',B',H,L')} + tX'_0)$ is sublc around X'_0 or not (see Definition 2.2). We denote the first term of the right hand side of the above equation by $\mathrm{lct}(X', B' + D_{(X',B',H,L')}; X'_0)$. It is easy to see that if $L'' \sim_{\mathbb{Q},C} L'$, then

$$\mathrm{Ding}_{(B',H)}(X', L') = \mathrm{Ding}_{(B',H)}(X', L'').$$

In the above definition, if $H = 0$ and $B' = 0$, then the log-twisted Ding degree coincides with the degree of the Ding line bundle introduced by [Ber16, (3.5)]. The log-twisted Ding degree has similar properties to those of the log-twisted Ding invariant.

Lemma 3.17. *Notations as above. Then the following hold.*

1. *Let $\pi'' : (X'', B'', L'') \to C$ be another semiample family over C such that there exists a birational morphism $\mu'' : X'' \to X'$ such that μ'' induces a C°-isomorphism $(X', B') \times_C C^\circ \cong (X'', B'') \times_C C^\circ$ and $L'' = \mu''^* L'$. Then*

$$\mathrm{Ding}_{(B'',H)}(X'', L'') = \mathrm{Ding}_{(B',H)}(X', L').$$

2. *Let $f : C' \to C$ be a finite morphism of smooth curves and $\pi'' : (X'', B'', L'') \to C'$ be the normalized base change of $\pi' : (X', B', L') \to C$. Let $\mu'' : X'' \to X'$ be the induced morphism and $r = \deg f$. Then*

$$\mathrm{Ding}_{(B'',H)}(X'', L'') = r\, \mathrm{Ding}_{(B',H)}(X', L').$$

3. *We have the following inequality*

$$\mathrm{Ding}_{(B',H)}(X', L') \leq \mathrm{CM}_{(B',H)}(X', L') := \mathrm{CM}((X', B', L')/C) + \mathcal{J}^H(X', L').$$

Equality holds if $(X', B' + X'_0)$ is lc around X'_0 and $K_{X'/C} + B' + L' + \mu^* H \sim_{\mathbb{Q},C} 0$.

The proof is similar to [Hat22, Proposition A.12] and left to the reader. Here, we prove the following generalization of [BX19, Theorem 3.1] to the log twisted Fano case.

Theorem 3.18. *Let $\pi : (X, B, L) \to C$ be a polarized $\mathbb{Q}$-Gorenstein family over a proper smooth curve. Let H be a $\mathbb{Q}$-line bundle such that $H \sim_{\mathbb{Q},C} -(K_{X/C} + B + L)$. Suppose that $\delta(X_0, B_0, L_0) \geq 1$. Let $\pi' : (X', B', L') \to C$ be a semiample family such that there exists a birational morphism $\mu : X' \to X$ that induces a C°-isomorphism $(X, B, L) \times_C C^\circ \cong (X', B', L') \times_C C^\circ$. Then*

$$\mathrm{Ding}_{(B',H)}(X', L') \geq \mathrm{Ding}_{(B,H)}(X, L).$$

Proof. First, we consider the case when $\delta(X_0, B_0, L_0) > 1$. We may assume that $L' = \mu^* L - E$ where E is an effective exceptional $\mathbb{Q}$-divisor. Take sufficiently divisible $r_0 \in \mathbb{Z}_{>0}$ that $r_0 L$ and $r_0 L'$ are $\mathbb{Z}$-Cartier. We may also assume by Lemma 3.17 (1) that $-E$ is μ-ample and that there exists an ideal $\mathfrak{a} \subset \mathcal{O}_X$ such that $\mu^{-1}\mathfrak{a} = \mathcal{O}(-r_0 E)$. By Theorem 3.6, we may further assume that there exists a good filtration $\mathscr{F}$ satisfying (12) on $R = \bigoplus_{m \geq 0} R_m$ where $R_m = H^0(X_0, m r_0 L_0)$. Thus, we see that $\lim_{k\to\infty} \frac{w_{\mathscr{F}}(k)}{k^{n+1}} = \frac{r_0^{n+1}}{(n+1)!}(L'^{n+1} - L^{n+1})$. Thus, we have that

$$\mathrm{Ding}_{(B',H)}(X', L') - \mathrm{Ding}_{(B,H)}(X, L) = \mathrm{lct}(X, B + \mathfrak{a}; X_0) - 1 - \lim_{k\to\infty} \frac{w_{\mathscr{F}}(k)}{r_0 k \dim R_k}.$$

For the definition of

$$\mathrm{lct}(X, B + \mathfrak{a}; X_0) = \sup\{t \in \mathbb{Q} | (X, B + \mathfrak{a} + tX_0) \text{ is sublc around } X_0\},$$

we refer to [Fuj18, Definition 2.6]. Since $\mathcal{O}_{C,0}$ is a discrete valuation ring, there exist free bases $\{s_1, \cdots, s_{N_k}\}$ of $\pi_* \mathcal{O}(k r_0 L) \otimes_{\mathcal{O}_C} \mathcal{O}_{C,0}$ where $N_k = \dim R_k$ and $\{s'_1, \cdots, s'_{N_k}\}$ of $\pi'_* \mathcal{O}(k r_0 L') \otimes_{\mathcal{O}_C} \mathcal{O}_{C,0}$ such that $s'_i = t^{\lambda_i} s_i$ for some $\lambda_i \in \mathbb{Z}_{\geq 0}$. Here, t is a generator of the maximal ideal of $\mathcal{O}_{C,0}$ and recall that any homomorphism of free $\mathcal{O}_{C,0}$-modules is represented by a diagonal matrix. By Theorem 3.6, $w_{\mathscr{F}}(k) = -\sum_{i=1}^{N_k} \lambda_i$. For sufficiently large k, we may assume that $\delta_{r_0 k}(X_0, B_0, L_0) > 1$ by [BJ20, Theorem A]. Let $\mathrm{div}(s_i) = D_i$. Then, regard s'_i as an element of $\pi_* \mathcal{O}(k r_0 L) \otimes_{\mathcal{O}_C} \mathcal{O}_{C,0}$ and we have

$$\mathrm{lct}(X, B + \mathfrak{a}; X_0) \geq \mathrm{lct}\left(X, B + \frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} \mathrm{div}(s'_i); X_0\right).$$

Here, $(X_0, B_0 + \frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} D_{i,0})$ is klt since $\frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} D_{i,0}$ is an $r_0 k$-basis type divisor with respect to L_0. By the inversion of adjunction [KM98, Theorem 5.50], $(X, B + \frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} D_i + X_0)$ is lc around X_0 for sufficiently divisible k. On the other hand,

$$\frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} \mathrm{div}(s'_i) + \left(1 - \frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} \lambda_i\right) X_0 = \frac{1}{r_0 k N_k} \sum_{i=1}^{N_k} D_i + X_0.$$

Hence we obtain

$$\mathrm{lct}(X, B+\mathfrak{a}; X_0) - 1 + \sum_{i=1}^{N_k} \frac{\lambda_i}{r_0 k N_k} \geq 0$$

for such k. Therefore, it follows that $\mathrm{Ding}_{(B',H)}(X', L') \geq \mathrm{Ding}_{(B,H)}(X, L)$.

Finally, we treat the case when $\delta(X_0, B_0, L_0) = 1$. We have shown that

$$\mathrm{Ding}_{(B',H+\epsilon L)}(X', (1-\epsilon)L') \geq \mathrm{Ding}_{(B,H+\epsilon L)}(X, (1-\epsilon)L)$$

for sufficiently small $\epsilon > 0$ since $\delta(X_0, B_0, (1-\epsilon)L_0) = (1-\epsilon)^{-1} > 1$. Then the coefficients of $D_{(X',B',H+\epsilon L,(1-\epsilon)L')}$ depend on ϵ continuously and so does $\mathrm{lct}(X', B' + D_{(X',B',H+\epsilon L,(1-\epsilon)L')}; X'_0)$. Indeed, by [KM98, Corollary 2.31] (see also the last paragraph of [BHJ17, §1.5]), there is a set $\{F_i\}$ of finitely many prime divisors over X' such that

$$\mathrm{lct}(X', B' + D_{(X',B',H+\epsilon L,(1-\epsilon)L')}; X'_0) = \min_{F_i} \frac{A_{(X',B'+D_{(X',B',H+\epsilon L,(1-\epsilon)L')})}(F_i)}{\mathrm{ord}_{F_i}(X'_0)}.$$

Thus, we have

$$\begin{aligned} &\mathrm{Ding}_{(B',H)}(X', L') - \mathrm{Ding}_{(B,H)}(X, L) \\ &= \lim_{\epsilon \to 0} (\mathrm{Ding}_{(B',H+\epsilon L)}(X', (1-\epsilon)L') - \mathrm{Ding}_{(B,H+\epsilon L)}(X, (1-\epsilon)L)) \geq 0. \end{aligned}$$

We complete the proof. □

3.5 CM minimization and specially K-stable varieties

We introduce special K-stability as follows.

Definition 3.19 (Specially K-stable varieties)**.** Let (X, B, L) be a polarized slc pair. If (X, B, L) is uniformly $\mathrm{J}^{K_X+B+\delta(X,B,L)L}$-stable and $K_X + B + \delta(X, B, L)L$ is ample, then we say that (X, B, L) is *specially* K-*stable*.

Similarly, if (X, B, L) is $\mathrm{J}^{K_X+B+\delta(X,B,L)L}$-semistable and $K_X + B + \delta(X, B, L)L$ is nef, then we say that (X, B, L) is *specially* K-*semistable*.

Remark 3.20. We see that special K-stability implies uniform K-stability in Corollary 3.23 below.

On the other hand, K-stable varieties are not specially K-stable in general. Indeed, let $(X, -\epsilon K_X + L)$ be a polarized ruled surface where L is the fiber class for $\epsilon > 0$. It is well-known that $(X, -\epsilon K_X + L)$ is K-stable when a corresponding vector bundle is stable by [ACGTF11]. Furthermore, $\lim_{\epsilon\to 0} \delta(X, -\epsilon K_X + L) = 2$ by [Hat22, Theorem D]. However, $K_X + \delta(X, -\epsilon K_X + L)(-\epsilon K_X + L)$ is not nef for sufficiently small $\epsilon > 0$.

The following are known to be specially K-stable.

Theorem 3.21. *Suppose that a polarized lc pair (X, Δ, L) satisfies one of the following.*

1. *(K-ample, Calabi-Yau and log Fano pairs, [Oda12], [BJ20], [LXZ22]) There exists a constant $c \in \mathbb{Q}$ such that $K_X + \Delta \sim_{\mathbb{Q}} cL$ and (X, Δ, L) is K-stable.*
2. *(Klt minimal models, [Hat21]) (X, Δ) is klt, $K_X + \Delta$ is nef and $L = K_X + \Delta + \epsilon H$ for H is ample and sufficiently small $\epsilon > 0$.*
3. *(K-ample fibrations over curves, [Hat21]) There exists a morphism $f : X \to C$ such that (C, A) is a polarized smooth curve, $f_*\mathcal{O}_X \cong \mathcal{O}_C$, $K_X + \Delta$ is ample, and $L = \epsilon(K_X + \Delta) + f^*A$ for sufficiently small $\epsilon > 0$.*
4. *(Uniformly adiabatically K-stable klt-trivial fibrations over curves, [Hat22]) There exists a polarized klt-trivial fibration $f : (X, \Delta, H) \to (C, A)$ such that (C, A) is a polarized smooth curve, (C, B, M, A) is K-stable and $L = f^*A + \epsilon H$ for H and for sufficiently small $\epsilon > 0$. Here, B is the discriminant and M is the moduli divisor.*

Then (X, Δ, L) is specially K-stable.

It is easy to see that if there exists a constant $c \in \mathbb{Q}$ such that $K_X + B \equiv cL$ and (X, B, L) is K-semistable, then (X, B, L) is specially K-semistable. On the other hand, there exists a polarized lc minimal model (X, B, H) such that $(X, B, \epsilon H + K_X + B)$ is K-unstable for sufficiently small $\epsilon > 0$ by [Hat22, Remark 3.5].

In [Hat22, Appendix], we see that the sum of the nA J-functional and the log-twisted Ding invariant is a lower bound of the DF invariant of a normal semiample test configuration. In light of Theorem 3.6, it is quite natural to consider to give a lower bound of a CM degree in a similar way. With this in mind, we combine Theorems 3.14 and 3.18 to show Conjecture 3.4 when (X_0, B_0, L_0) is specially K-(semi)stable. The following is a generalization of Theorem 1.4.

Theorem 3.22 (CM minimization for special K-stability)**.** *Let $\pi : (X, B, L) \to C$ and $\pi' : (X', B', L') \to C$ be two polarized log families over a proper smooth curve such that there exists a C°-isomorphism $f^\circ : (X, B, L) \times_C C^\circ \cong (X', B', L') \times_C C^\circ$. Suppose that B is restrictable (cf. Definition 3.1), $K_X + B$ is $\mathbb{Q}$-Cartier and (X_0, B_0, L_0) is a polarized specially K-semistable deminormal pair.*

Then

$$\mathrm{CM}((X', B', L')/C) \geq \mathrm{CM}((X, B, L)/C).$$

Furthermore, if (X_0, B_0, L_0) is specially K-stable, then equality holds if and only if f° can be extended to a C-isomorphism $f : (X, B, L) \cong (X', B', L')$.

Proof. Case 1. (X_0, B_0) is klt. Suppose first that (X_0, B_0, L_0) is specially K-stable. There exist proper birational morphisms $\mu \colon X'' \to X$ and $\mu' \colon X'' \to X'$ of normal varieties such that μ (resp. μ') is isomorphic over $X \times_C C^\circ$ (resp. $X' \times_C C^\circ$) and $\mu \circ \mu'|^{-1}_{X' \times_C C^\circ} = f^{\circ -1}$. Set $B'' := \mu_*^{-1} B$ and $L'' := \mu'^* L'$. Replacing L by $(\delta(X_0, B_0, L_0) - \epsilon)L$ for sufficiently small $\epsilon > 0$ such that $\delta(X_0, B_0, L_0) - \epsilon \in \mathbb{Q}$, we may also assume that

$\delta(X_0, B_0, L_0) \geq 1$, (X_0, B_0, L_0) is uniformly $\mathrm{J}^{K_{X_0}+B_0+L_0}$-stable and $K_{X_0} + B_0 + L_0$ is ample. Let $H = -(K_X + B + L)$. Then,

$$\begin{aligned}
\mathrm{CM}((X', B', L')/C) &= \mathrm{CM}((X'', B'', L'')/C) + \mathcal{J}^H(X'', L'') - \mathcal{J}^H(X'', L'') \\
&= \mathrm{CM}_{(B'',H)}(X'', L'') - \mathcal{J}^H(X'', L'') \\
&\geq \mathrm{Ding}_{(B'',H)}(X'', L'') + \mathcal{J}^{K_X+B+L}(X'', L''), \\
\mathrm{CM}((X, B, L)/C) &= \mathrm{CM}_{(B,H)}(X, L) - \mathcal{J}^H(X, L) \\
&= \mathrm{Ding}_{(B,H)}(X, L) + \mathcal{J}^{K_X+B+L}(X, L).
\end{aligned}$$

Here, we applied Lemma 3.17. By Theorem 3.18, we have that $\mathrm{Ding}_{(B'',H)}(X'', L'') \geq \mathrm{Ding}_{(B,H)}(X, L)$. On the other hand, $\mathcal{J}^{K_X+B+L}(X'', L'') \geq \mathcal{J}^{K_X+B+L}(X, L)$ and hence we have the desired inequality by Theorem 3.14. Furthermore, equality holds if and only if $\mu^*L \sim_{\mathbb{Q},C} \mu'^*L'$, which is equivalent to the existence of f as the assertion.

When (X_0, B_0, L_0) is specially K-semistable, it is easy to see that

$$\mathrm{CM}_{(B',\epsilon L)}(X', L') \geq \mathrm{CM}_{(B,\epsilon L)}(X, L)$$

for $\epsilon > 0$ by the same argument as above. By taking the limit $\epsilon \to 0$, we obtain the desired inequality.

Case 2. General case. We may assume that (X_0, B_0, L_0) is not klt but slc. Now, we may replace π' with a semiample family and assume that there exists a projective morphism $\mu : X' \to X$ such that $\mu|_{X' \times_C C^\circ} = f^{\circ -1}$ as the proof of Case 1. Applying Theorem 3.14 to the case when $H = K_{X/C} + B$, it suffices to show that

$$(K_{X'} + B' - \mu^*(K_X + B)) \cdot L'^n \geq 0.$$

Indeed, L' is semiample over X and $K_{X'} + B' - \mu^*(K_X + B)$ is effective since $(X, B + X_0)$ is lc around X_0 by [Kaw07]. □

Finally, we obtain the following corollaries. First, we apply Theorem 3.22 to test configurations and obtain:

Corollary 3.23. *Let (X, B, L) be an slc polarized pair. If (X, B, L) is specially K-stable (resp. specially K-semistable), then (X, B, L) is uniformly K-stable (resp., K-semistable).*

We prove ♣ for specially K-stable pairs in §1.1, which is remarked in [BX19, Remark 3.6].

Corollary 3.24 (Separatedness of $\mathbb{Q}$-Gorenstein specially K-stable families)**.** *Let $\pi : (X, B, L) \to C$ and $\pi' : (X', B', L') \to C$ be two polarized $\mathbb{Q}$-Gorenstein families over a smooth affine curve such that there exists a C°-isomorphism $f^\circ : (X, B, L) \times_C C^\circ \cong (X', B', L') \times_C C^\circ$. Suppose that B and B' are restrictable and (X_0, B_0, L_0) (resp. (X'_0, B'_0, L'_0)) is a polarized specially K-stable (resp. specially K-semistable) demi-normal pair.*

Then f° can be extended to a C-isomorphism $f : (X, B, L) \cong (X', B', L')$.

Proof. Note that C is not proper in general. However, by properness of Hilbert schemes, we compactify the family $\pi : (X, B, L) \to C$ to a polarized family over a smooth proper curve $\overline{C}$. If necessary, take a suitable blow up of the compactification of X and we may assume that this is $\mathbb{Q}$-Gorenstein. On the other hand, for any $0' \in \overline{C} \setminus C$, we compactify π' in the same way as π around $0'$. Thus, we may assume that C is proper. Then we apply Theorem 3.22 and obtain the assertion. □

Corollary 3.25. *Let* (X, B, L) *be a specially K-stable lc pair. Then* $\mathrm{Aut}\,(X, B, L)$ *is a finite group. Moreover, the identity component* $\mathrm{Aut}_0\,(X, B)$ *of* $\mathrm{Aut}\,(X, B)$ *is an Abelian variety.*

Proof. The first assertion follows from Corollary 3.24 and the same argument of [BX19, Corollary 3.5]. The rest also follows as [Oda12, Corollary 1.6] from the fact that the action of the linear algebraic group $\mathrm{Aut}_0\,(X, B)$ on $\mathrm{Pic}^0(X)$ defines a quasi-finite morphism $\mathrm{Aut}_0\,(X, B) \to \mathrm{Pic}^0(X)$ by the first assertion. □

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References

[ACGTF11] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and C. W. Tønnesen-Friedman. *Extremal Kähler metrics on projective bundles over a curve.* Adv. Math. **227** (2011), 2385–2424.

[B01] L. Bădescu. *Algebraic Surfaces.* Springer-Verlag, New York-Heidelberg, 2001, Universitext.

[Ber16] R. J. Berman, K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics, Invent. Math. **203** (2016), 973–1025.

[BJ20] H. Blum, M. Jonsson. *Thresholds, valuations, and K-stability*, Adv. Math. **365** (2020).

[BLZ22] H. Blum, Y. Liu and C. Zhou. *Optimal destabilization of K-unstable Fano varieties via stability thresholds,* Geom. Topol. **26** (2022), 2507–2564

[BX19] H. Blum and C. Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Ann. of Math. **190** (2019), no. 2, 609–656.

[Bou14a] S. Boucksom. Corps d'Okounkov. Exp. No. 1059. Astérisque **361** (2014), 1–41.

[Bou14b] S. Boucksom, *Remarks on minimal models of degenerations*, unpublished note (2014).

[BC11] S. Boucksom, H. Chen, *Okounkov bodies of filtered linear series*, Compos. Math. **147** (2011), no. 4, 1205–1229.

[BHJ17] S. Boucksom, T. Hisamoto, M. Jonsson. *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs,* Ann. Inst. Fourier (Grenoble) **67**(2), 743–841, 2017.

[BJ18] S. Boucksom, M. Jonsson. *A non-Archimedean approach to K-stability.* arXiv:1805.11160

[Che21] G. Chen, *The J-equation and the supercritical deformed Hermitian-Yang-Mills equation*, Invent. Math. **225** (2021), 529–602.

[CC21] X. Chen, J. Cheng, On the constant scalar curvature Kähler metrics (II)–Existence results, J. Amer. Math. Soc. **34** (2021), 937–1009.

[DP21] V. Datar, V. Pingali. *A numerical criterion for generalised Monge-Ampère equations on projective manifolds.* Geom. Funct. Anal. **31** (2021), 767–814.

[DM69] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHÉS. **36** (1969), 75–109.

[Der16] R. Dervan, *Uniform stability of twisted constant scalar curvature Kähler metrics*, Int. Math. Res. Not. IMRN **15** (2016), 4728–4783.

[Don01] S. K. Donaldson, *Scalar Curvature and Projective Embeddings I.* J. Differential Geom. **59**(3) (2001), 479–522.

[Don02] S. K. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. **62** (2002), no. 2, 289–349.

[FR06] J. Fine, J. Ross. *A note on positivity of the CM line bundle.* Int. Math. Res. Not. IMRN, Article ID 95875, 14pp, 2006.

[Fuj18] K. Fujita. *Optimal bounds for the volumes of Kähler-Einstein Fano manifolds.* Amer. J. Math., **140** no. 2, (2018), 391–414.

[Fuj19a] K. Fujita, *A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties.* J. reine angew. Math. **751** (2019), 309–338.

[Fuj19b] K. Fujita. *Openness results for uniform K-stability*, Math. Ann. **373** (2019), 1529–1548.

[FO18] K. Fujita, Y. Odaka, *On the K-stability of Fano varieties and anticanonical divisors*. Tohoku Math. J. **70** (2018), 511–521.

[Har77] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[HH23] K. Hashizume, M. Hattori, On boundedness and moduli spaces of K-stable Calabi-Yau fibrations over curves. To appear.

[Hat21] M. Hattori, *A decomposition formula for J-stability and its applications*, arXiv:2103.04603

[Hat22] M. Hattori, *On K-stability of Calabi-Yau fibrations*, arXiv:2203.11460

[Kaw07] M. Kawakita, *Inversion of adjunction on log canonicity*. Invent. Math. **167**, 129–133 (2007).

[Kol15] J. Kollár. *Singularities of the Minimal Model Program.* Cambridge Tracts in Mathematics **200**. Cambridge University Press, Cambridge, 2015.

[Kol22] J. Kollár. *Families of varieties of general type.* to appear.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties.* Cambridge Tracts in Mathematics **134**. Cambridge University Press, Cambridge, 1998

[KSB88] J. Kollár, and N. Shepherd-Barron. Threefolds and deformation of surface singularities. Invent. Math. **91** (1988), 299–338.

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry I.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer. (2004).

[LM09] R. Lazarsfeld, M. Mustaţă. *Convex bodies associated to linear series.* Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 783–835.

[LS15] M. Lejmi and G. Székelyhidi, *The J-flow and stability*, Adv. Math., **274** (2015), 404–431.

[Li22] C. Li, Geodesic rays and stability in the cscK problem, Ann. Sci. Éc. Norm. Supér. (4) **55** (2022), no.6, 1529-1574.

[LX14] C. Li, C. Xu. *Special test configuration and K-stability of Fano varieties.* Ann. of Math. **180** (2014), no. 1, 197–232.

[LXZ22] Y. Liu, C. Xu, and Z. Zhuang. *Finite generation for valuations computing stability thresholds and applications to K-stability*. Ann. of Math. **196** (2022), 507–566.

[Mab86] T. Mabuchi, K-energy maps integrating Futaki invariants, Tohoku Math. J. **38** (1986), 575–593.

[Mum77] D. Mumford, *Stability of projective varieties*, Enseignement Math. **23** (1977), 39–110.

[MFK94] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd. edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, **34**, Springer-Verlag (1994).

[Oda12] Y. Odaka, *The Calabi conjecture and K-stability*, Int. Math. Res. Not. IMRN **10** (2012), 2272–2288.

[Oda13a] Y. Odaka, *A generalization of Ross-Thomas slope theory*, Osaka J. Math. **50** (2013), 171–185.

[Oda13b] Y. Odaka, *The GIT stability of polarized varieties via discrepancy*, Ann. of Math. **177** (2013), 645–661.

[Oda13c] Y. Odaka, *On the moduli of Kähler-Einstein Fano manifolds*, Proceeding of Kinosaki algebraic geometry symposium 2013, available at arXiv:1211.4833.

[Oda20] Y. Odaka, On K-stability and moduli of varieties - their connection with Kähler-Einstein metrics - Sugaku **72** (2020), no.4. Published by Math Society of Japan.

[OS15] Y. Odaka, S. Sun, *Testing log K-stability by blowing up formalism.* Ann. Fac. Sci. Toulouse Math., **24** (2015), 505–522.

[Ohn22] K. Ohno. *Minimising CM degree and slope stability of projective varieties.* Mathematical Proceedings of the Cambridge Philosophical Society, **172**(1) (2022), 125–137.

[PT09] S. Paul, G. Tian. *CM stability and the generalized Futaki invariant II.* Astérisque No. **328** (2009), 339–354.

[Son20] J. Song. *Nakai-Moishezon criterion for complex hessian equations.* arXiv:2012.07956v1, 2020.

[Sto09] J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221**, no. 4 (2009), 1397–1408.

[Szé15] G. Székelyhidi, *Filtrations and test-configurations*, Math. Ann. **362** (2015), 451–484.

[Tia97] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. **130** (1997), no. 1, 1–37.

[Wan12] X. Wang, Heights and GIT weights, Math. Res. Letters vol. **19**, (2012).

[WX14] X. Wang, C. Xu, Non existence of asymptotic GIT compactification, Duke Math. J. **163** (2014), 2217–2241.

[Xu21] C. Xu, K-stability of Fano varieties: an algebro-geometric approach, EMS Surv. Math. Sci. **8** (2021), no. 1-2, 265–354.