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Cubic fourfolds with eleven cusps and a related moduli space

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CUBIC FOURFOLDS WITH ELEVEN CUSPS AND A RELATED MODULI SPACE

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ABSTRACT. First we construct a cubic 4-fold whose singularities are 11 cusps and which has an action of the Mathieu group M_{11}, all over the ternary field $\mathbb{F}_3$. We next consider a certain moduli space of bundles on a supersingular K3 surface of Artin invariant one in characteristic 3. We show that it has 275 (-2) Mukai vectors which form the McLaughlin graph, and ask questions on it and on its relation with our M_{11}-cubic 4-fold.

We work over an algebraically closed field in characteristic 3, but varieties are mostly defined over $\mathbb{F}_3$ or $\mathbb{F}_9$. A general inseparable triple covering

$$V \to \mathbb{P}^4_{(vwxyz)}, \quad \tau^3 = G(v,w,x,y,z), \quad \deg G = 3. \tag{1}$$

of the projective 4-space is a cubic 4-fold in $\mathbb{P}^5_{(\tau vwxyz)}$ with 11 cusps, i.e., simple singularities of type A_2 (since $c_4(\Omega_{\mathbb{P}^4}(3)) = 11$). An example with high symmetry is obtained from the Segre cubic 3-fold

$$Seg^3 : \sum_{i=1}^{6} x_i = \sum_{1\le i<j<k\le 6} x_i x_j x_k = 0,$$

which has the maximal number (=10) of nodes (e.g., [6]).

Example 1. The inseparable triple covering

$$Seg^4 \to \mathbb{P}^4, \quad \tau^3 = \sum_{1\le i<j<k\le 6} x_i x_j x_k, \quad \text{with} \quad \mathbb{P}^4 : \sum_{i=1}^{6} x_i = 0 \subset \mathbb{P}^5_{(x)}, \tag{2}$$

with formal branch Seg^3 has 10 cusps over its nodes, and one more at $(x : \tau) = (111111 : -1)$.

The automorphism group $\mathfrak{S}_6$ of Seg^4 (and also of Seg^3) acts 11 cusps with two orbits of length 10 and 1. A little bit surprisingly there is a *more symmetric* cubic 4-fold with 11 cusps in the sense that the automorphism group acts transitively on the cusps.

The following is our main result of this note, and is regarded as a characteristic 3 analogue of the fact that the Fermat cubic 4-fold has an action of the Mathieu group M_{22} over $\mathbb{F}_4$ and the M_{22}-action on a set of 22 planes in it is (triply) transitive ([7], [4, p. 39]):

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Theorem 2. *The cubic 4-fold*

$$V : z^3 = \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}) \quad \text{in} \quad \mathbb{P}^5_{(xz)} \tag{3}$$

has an action of the Mathieu group M_{11} over $\mathbb{F}_3$ (via the extended ternary Golay code [12, 6, 6]). Moreover, V has cusps at 11 $\mathbb{F}_3$-points[1]*, on which the M_{11} acts (quadruply) transitively. V is smooth elsewhere.*

Two cubic 4-folds are closely related with a supersingular K3 surface of Artin invariant one, whose standard projective model is the Fermat quartic surface. Though it does not have an action of M_{11}, there is a chance for a suitable moduli space of bundles over it to have a birational action of M_{11}. In §3, we give an 8-dimensional candidate and ask two questions.

1. Preliminary

The Mathieu group has a presentation

$$\langle a, b, c, \mid a^{11} = b^5 = c^4 = (ac)^3 = 1, a^b = a^4, b^c = b^2\rangle$$

with 3 generators ([4, p.18]). The following is our starting point:

Lemma 3. *(*[1, Lemma 3.1]*) The Klein's cubic form $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1}$ is preserved by the linear transformations $A' = \mathrm{diag}[\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5]$ and $B' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$. It is not preserved by $C' = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & a_4 & a_0 \\ a_4 & a_0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 & a_0 & a_1 \end{pmatrix}$ but transformed under C' to $\sum_{i\in\mathbb{Z}/5\mathbb{Z}}(y_i^2y_{i+1} + y_i^3)$, where $a_1 = -(\zeta+1)^2$, $a_3 = a_1^3$, $a_0 = a_3^3$, $a_2 = a_0^3$ and $a_4 = a_2^3$. In particular, $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1}$ is invariant under the action of $M_{11} = \langle A', B', C'\rangle$ modulo cubes of linear forms.*

Remark 4. The cubic 4-fold $\tau^3 - \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1} = 0 \subset \mathbb{P}^5_{(\tau y)}$ is interesting over the complex number field $\mathbb{C}$ in the sense that its automorphism group $PSL(2, 11)$ is maximal among all finite groups with a symplectic action on a smooth cubic 4-fold ([9]). Similar holds for for Klein's cubic 3-fold $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1} = 0 \subset \mathbb{P}^4_{(y)}$ ([13]).

[1] More precisely, the 11 cusps locate at

$$(x_0 : \ldots : x_4 : z) = (10000; 0), (01000; 0), (00100; 0), (00010; 0), (00001; 0), (-1-1-1-1-1; 0),$$
$$(01-1-11; 1), (101-1-1; 1), (-1101-1; 1), (-1-1101; 1), (1-1-110; 1),$$

which are the 11 points $1, 2, 3, 4, 5, 6$ and a, b, c, d, e in the notation of Coxeter-Todd ([12]).

We make the statement in the lemma into that over $\mathbb{F}_3$ by the following change of variables:

$$
\begin{aligned}
y_0 &= \zeta x_0 + \zeta^9 x_1 + \zeta^4 x_2 + \zeta^3 x_3 + \zeta^5 x_4 \\
y_1 &= \zeta^9 x_0 + \zeta^4 x_1 + \zeta^3 x_2 + \zeta^5 x_3 + \zeta x_4 \\
y_2 &= \zeta^4 x_0 + \zeta^3 x_1 + \zeta^5 x_2 + \zeta x_3 + \zeta^9 x_4 \\
y_3 &= \zeta^3 x_0 + \zeta^5 x_1 + \zeta x_2 + \zeta^9 x_3 + \zeta^4 x_4 \\
y_4 &= \zeta^5 x_0 + \zeta x_1 + \zeta^9 x_2 + \zeta^4 x_3 + \zeta^3 x_4
\end{aligned} \tag{4}
$$

In fact, Klein's cubic form and the generators A', B', C' of M_{11} are transformed to

$$
\sum_{i \in \mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}), \tag{5}
$$

$$
A = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Thus we have the following reformulation over $\mathbb{F}_3$ of Adler's Lemma 3:

Lemma 5. *The cubic form* (5) *of 5 variables is preserved by the linear transformations A and B. It is not preserved by C but transformed under C to*

$$
\sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}).
$$

The original Lemma 3 is nothing but the expression by taking the eigenvectors of C as basis.

2. Cubic 4-folds with action of M_{11} over $\mathbb{F}_3$

2.1. **Golay cocodes.** Here, following [4] and [5], we understand the Mathieu group M_{11} is the group of linear transformations of the ternary Golay code $[11, 6, 5]$, which is a certain 6-dimensional vector subspace $\mathcal{C}_{11} \subset \bigoplus_{i \in \mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i$ of the 11-dimensional vector space with basis $\{e_i \mid i \in \mathbb{F}_{11}\}$. M_{11} acts on the basis as signed permutations, and is generated by the following three transformations A, B and C:

$$
\begin{aligned}
&A : e_i \mapsto e_{i+1}, \quad B : e_i \mapsto e_{3i}, (i \in \mathbb{F}_{11}) \\
&C : e_1 \mapsto -e_5 \mapsto e_9 \mapsto -e_4 \mapsto e_1;\; e_3 \mapsto -e_3; \\
&e_6 \mapsto e_{10} \mapsto e_8 \mapsto e_7 \mapsto e_6;\; e_2 \mapsto e_2.
\end{aligned} \tag{6}
$$

The representation space V_5 of Lemma 5 is the space of cocodes, namely, the quotient space $\left(\bigoplus_{i \in \mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i\right)/\mathcal{C}_{11}$. Since $\mathcal{C}_{11}$ is a cyclic code with generating polynomial $X^5 + X^4 - X^3 + X^2 - 1 = 0$, the space of cocodes is naturally identified with the finite field $\mathbb{F}_{243}$ as $\mathbb{F}_3$-vector space. Though a standard choice of basis is $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$ as taken in [1], we take here $\{\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5\}$ as basis, instead.

2.2. **From 5 variables to 6.** We extend the cubic form in the previous section into that of 6 variables which is truly invariant under M_{11}. For that purpose, we must extend the (linear) representation V_5 of M_{11} in the previous section to a 6-dimensional one V_6 so that it contains an M_{11}-invariant vector ν_∞ and we have an exact sequence

$$0 \to \mathbb{F}_3 \cdot \nu_\infty \to V_6 \to V_5 \to 0 \tag{7}$$

of M_{11}-modules (which does not split).

Our construction of the V_6 is again a space of cocodes. Now we consider the extended Golay code $[12, 6, 6]$, which is the extension of (perfect) Golay code $[11, 6, 5]$ by zero-sum condition. We consider it as a 6-dimensional subspace

$$\mathcal{C}_{12} \subset \bigoplus_{i \in \mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i$$

and take V_6 as the quotient vector space $\left(\bigoplus_{i\in\mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i\right)/\mathcal{C}_{12}$. The code $\mathcal{C}_{12}$ is self-dual and hence the space of cocodes and $\mathcal{C}_{12}$ are dual to each other. The set $\mathbb{P}^1(\mathbb{F}_{11})$ of indices are divided into two parts:

$$Q = \{0, 1, 3, 4, 5, 9\} \quad \text{and} \quad N = \{\infty, 2, 6, 7, 8, X\}, \quad X := 10,$$

according as quadratic residue or non-residue. We denote the image of e_i in the space of cocodes by ν_i. We take $\nu_i, i \in Q \setminus \{0\}$ and ν_∞ as its basis. M_{11} coincides with the stabilizer group of M_{12} at ν_∞.

The ternary Golay codes $\mathcal{C}_{12}$ contains 12 elemants $w_j \in \mathbb{P}^1(\mathbb{F}_{11})$, called *total words*, including

$$w_\infty = -\sum_{i\in\mathbb{P}^1(\mathbb{F}_{11})} e_i \quad \text{and} \quad w_0 = \sum_{i\in Q} e_i - \sum_{i \in N} e_i. \tag{8}$$

(Other total words w_j's are just the translations of w_0.) Then M_{11} acts transitively on the set $w_j \in \mathbb{P}^1(\mathbb{F}_{11})$ of 12 total words. Hence we have the exact sequence (7) of M_{11}-modules by setting

$$V_6 = \left(\bigoplus_{i\in\mathbb{P}^1(\mathbb{F}_{11})} \mathbb{F}_3 \cdot e_i\right)/\mathcal{C}_{12}, \quad V_5 = \left(\bigoplus_{i\in\mathbb{F}_{11}} \mathbb{F}_3 \cdot e_i\right)/\mathcal{C}_{11}$$

The extended transformation $\tilde{A}, \tilde{B}$ is defined by putting $e_\infty \mapsto e_\infty$, that is, just the trivial extension. Only the extension $\tilde{C}$ of C is non-trivial, and it is induced by the permutation of the 12 total words:

$$(0\infty)(12345)(6789X).$$

Proof of Theorem 2. The invariance under A and B are clear from that under A' and B'. The linear transformation C interchanges two total words w_0 and w_∞ in (8). It acts on x_0, x_1, x_2, x_3, x_4 by permutation composed with negative by (6). Hence, by Lemma 5, it transforms

$$(\tau - x_0 - x_1 - x_2 - x_3 - x_4)^3 + \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2})$$

into

$$(\tau + x_0 + x_1 + x_2 + x_3 + x_4)^3 + (x_0 + x_1 + x_2 + x_3 + x_4)^3 + \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}),$$

which means the desired invariance. □

3. Conjectural symplectic 8-fold as moduli of bundles on Fermat quartic

3.1. **Two questions.** The Fermat quartic surface $Fer_4 : \sum_1^4 x_i^4 = 0 \subset \mathbb{P}^3_{(x)}$, has an action of the finite unitary group $PGU_4(3)$. The action of a subgroup of index 4, namely, of $U_4(3) := PSU_4(3)$ is symplectic. Though $U_4(3)$ does not contain M_{11} as a subgroup, the moduli space $\overline{M}_{Fer}(v)$ of (semi-)stable sheaves on the Fermat quartic Fer_4 might have a birational action of M_{11}, or even a much larger finite simple group, for suitable Mukai vector $v = (r, *, s) \in \mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$. A hopeful candidate, in view of symmetry of the Leech lattice, is 8-dimensional, i.e., $\langle v^2 \rangle = 6$, and the group containing M_{11} should be the McLaughlin group McL.

Question 6. Does the moduli space $\overline{M}_{Fer}(3, \alpha, -3)$ have a birational action of McL, where α is a (-12)-divisor class attached to Segre's hemisystem (see §3.2)?

McL contains simple groups $U_4(3)$ and M_{11} as maximal subgroups, and hence is generated by these two subgroups. The action of the former on the moduli is not surprising since its $\mathbb{Q}$-twisted expression is $\overline{M}_{Fer}(3, 0, -1)$ (Proposition 9). Seeking after an action of the latter, we pose the following

Question 7. Is $\overline{M}_{Fer}(3, \alpha, -3)$ birational to the conjectural LLSvS 8-fold (see [10] but only over $\mathbb{C}$ under some condition) associated with the M_{11}-cubic 4-fold V?

3.2. **Segre's hemisystem and the McLaughlin graph in a Picard lattice.** The Fermat quartic surface Fer_4 has 280 $\mathbb{F}_9$-(rational) points, with weight distribution 2: 24, 3: 64 and 4: 192. For each $\mathbb{F}_9$-point p, the tangent plane T_p cuts out the union of 4 lines passing through p from Fer_4. Since every line has 10 $\mathbb{F}_9$-points, the number of lines in Fer_4 is $280 \times 4/10 = 112$. The Picard lattice is generated by these line classes. Its discriminant group $\mathrm{Disc}(Fer_4)$ is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ (see e.g. [8]).

Segre's hemisystem is a set H of 56 lines, among the 112, which covers $Fer_4(\mathbb{F}_9)$ doubly, that is, every $\mathbb{F}_9$-point is contained in exactly two members of H. There are 648 hemisystems and they are divided into 4 orbits of length 162 by the action of $U_4(3)$. These 4 orbits corresponds to the four elements of norm $2/3$ modulo $2\mathbb{Z}$ in the discriminant group $\mathrm{Disc}(Fer_4)$ as we will see below. We chose one of them. Then the intersection size $|H \cap H'|$ of two among our 162 hemisystems are either 20 or 32 ([3, §10.34]).

Proposition 8. ([3, §10.61]) *The graph with the following three types of vertices and a suitable adjacency is a strongly regular graph* $srg(275, 112, 30, 56)$*, isomorphic to the McLaughlin graph: (i)* ∞*, (ii) the 112 lines in* Fer_4 *and (iii) the 162 hemisystems.*

We realize this graph inside the extended Picard lattice $U(-1) \oplus \mathrm{Pic}\, Fer_4$ of the Fermat quartic surface, or more precisely, in the sublattice $(3, \alpha, -3)^\perp$, which is expected to be the Picard lattice of the conjectural moduli symplectic 8-fold ([11], [14], [15] but only over $\mathbb{C}$). Here U denotes the standard hyperbolic lattice of rank 2. The intersection pairing $(D.D')$ on the Picard lattice extends to the orthogonal sum $\mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$ obviously but with changing the sign of U, namely,

$$\langle (r, D, s), (r', D', s') \rangle = -rs' + (D.D') - sr', \quad (r, s), (r', s') \in U(-1). \tag{9}$$

Now we define a divisor class for a hemisystem H. Consider the sum $\sum_{m \in H} m$ of its all members in the Picard grup $\mathrm{Pic}\, Fer_4$. Then we have

$$(\sum_{m \in H} m.\, l) = \begin{cases} 8 & \text{if } l \in H, \\ 20 & \text{otherwise.} \end{cases} \tag{10}$$

In particular, $\sum_{m \in H} m$ is divisible by 4 in the Picard group. So we define

$$\alpha_H := 2h - \frac{1}{4} \sum_{m \in H} m \in \mathrm{Pic}\, Fer_4,$$

where h is the hyperplane section class of Fer_4. Since $(\alpha_H.l)$ is divisible by 3 for all lines l, $\alpha_H/3$ defines an element in the discriminant group, whose norm is $2/3$ since $(\alpha_H^2) = -12$. The following is equivalent to the preceding proposition:

Proposition 9. *The graph on the following three types of (-2)-vectors in $(3, 0, -1)^\perp \otimes \mathbb{Q}$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3, h, 1)$,
- $(0, l, 0)$ *for the 112 lines l in Fer_4 and*
- $(1, -\frac{\alpha_H}{3}, \frac{1}{3})$ *for the 162 hemisystems H chosen as above.*

Geometrically, these are the Mukai vectors of the rank 3 bundle $T_{\mathbb{P}^3}(-1)$ restricted to Fer_4, torsion sheaves supported on lines and $\mathbb{Q}$-line bundles on Fer_4, respectively.

Now we fix a hemisystem F among our 162, put $\alpha = \alpha_F$ and take twist by tensor product of the $\mathbb{Q}$-line bundle $\mathcal{O}_{Fer}(\frac{\alpha}{3})$. Then all the vertices in the proposition become integral. Since the the tensor of a line bundle preserves the inner product (9), we have

Corollary 10. *The graph on the following three types of (-2) Mukai vectors in $(3, \alpha, -3)^\perp$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3, h + \alpha, -3)$,
- $(0, l, *)$ *for the 112 lines l in Fer_4 and*
- $(1, \frac{\alpha - \alpha_H}{3}, **)$ *for the 162 hemisystems H,*

where $*$ is equal to 0 if $l \in F$ and 1 otherwise, and $**$ is equal to $1, -1, -2$ according as $H = F, |H \cap F| = 20$ and $|H \cap F| = 32$.

Proof. $\alpha - \alpha_H$ is divisible by 3 since both $\alpha/3$ and $\alpha_H/3$ defines the same element in $\mathrm{Disc}(Fer_4)$. Hence the vertices are Mukai vectors of a rank 3 bundles, torsion sheaves and the 162 line bundles $\mathcal{O}_{Fer}(\frac{\alpha - \alpha_H}{3})$. □

Remark 11. The McLaughlin graph is realized in the Leech lattice Λ using a triangle of type 223 ([4, p.100]). Hence the corollary can be proved using the construction of Λ from the Niemeier lattice of type $12A_2$ in the way of Borcherds [2].

Remark 12. Two more strongly regular graphs are similarly realized by taking (-2) Mukai vectors as their vertices in characteristic 2 and 5, which will be discussed elsewhere.

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