On triality relations of normal triality algebras and Lie algebras

– ある代数系の三対原理とリー代数–

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概要: この小論ではリー代数とジョルダン代数を含むある代数系 (called a normal triality algebra) の三対原理，つまり自己同型群と微分の一般化について論究します．そしてその代数系から root systems, Cartan matrix の concept を用いないで例外型の単純リー代数をいくつか構成する方法に応用します.

§0. Introduction (motivation), §1. Definitions of triality relations and normal triality algebras, §2. Examples of triality relations of algebras, §3. Lie algebras construction, Appendix (対称的合成代数の基底と三対原理), References.

この様な内容について，具体的実例を含め述べさせていただきます (又，これらの section は独立に読める様に心がけたつもりです).

§0. Introduction (motivation)

我々の三対原理の concept を複素数 $\mathbf{C}$ のとき考えます．$\mathbf{C}$ の普通の積を $*$，共役元を $\bar{x} = a - ib(x = a + ib,\ a, b$ は実数,$i = \sqrt{-1})$ とするとき，new product を $xy = \overline{x * y}$ によって定義します．この new product は非結合的代数系であり

$$g(xy) = (gx)(gy),\ d(xy) = (dx)y + x(dy)$$

なる自己同型 g と微分 d を考える．$\mathbf{C}$ を $re^{i\theta}$ で表示, 周期 2π で考察すると

$$Aut\ \mathbf{C} = \{e^{i\theta} | \theta = \frac{2\pi}{3}n,\ n : integer\} = \{1, \frac{-1+\sqrt{3}\ i}{2}, \frac{-1-\sqrt{3}\ i}{2}\} \cong S_3$$

$$Der\ \mathbf{C} = \{d \in End\ \mathbf{C} | d = \frac{2\pi}{3}in,\ n : integer\} = \{0, \frac{2\pi i}{3}, \frac{4\pi i}{3}\}$$

が成り立ちます．**product (積)** に注意してください.

$j = 0, 1, 2$ に対して (自然数 j を mod 3 で考えます), $End\ \mathbf{C}$ の元で,

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y),\ d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y)$$

となる $g_j,\ d_j$ を考えると，次のような $Aut\ \mathbf{C}, Der\ \mathbf{C}$ の一般化が得られます.

$$Trig\ \mathbf{C} = \{(g_0, g_1, g_2) |\ g_j = exp(\sqrt{-1}\ \alpha_j), \alpha_0 + \alpha_1 + \alpha_2 = 0,\ \alpha_j \in Re\ \mathbf{C}\},$$

$$Trid\ \mathbf{C} = \{(d_0, d_1, d_2) |\ d_j = \sqrt{-1}\ \alpha_j,\ \alpha_0 + \alpha_1 + \alpha_2 = 0,\ \alpha_j \in Re\ \mathbf{C}\}.$$

次に $n \times n$ 次の行列代数 A で考察すると，$j = 0, 1, 2$ に対して

$$\sigma_j(a)x = a_{j+1} * x *^t a_{j+2},\quad d_j(p)x = p_{j+1} * x - x * p_{j+2}.$$

ただし $a_j \in A_0^* := \{b \in A | b *^t b = Id_n\}$, $p_j \in Alt(A) := \{c \in A|\ {}^tc = -c\}$,

($x * y$ は standard product of matrix, ${}^t x$ は transpose of matrix),

とします. ここで $xy = {}^t(x * y)$ と **new product** を定義すると, この積 xy は非結合的代数ですが, 自己同型と微分の一般化が得られます;

$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$, $d_j(p)(xy) = (d_{j+1}(p)x)y + x(d_{j+2}(p)y)$, $p_j = (1 - a_j) * (1 + a_j)^{-1}$ (Cayley transformation) が成り立ちます.

Remark. 8 元数の三対原理については ([S],[T],[Tô] and [K.3] 等) を参照.

Remark. Note that for an algebra A, if $\sqrt{3} \in F$ (基礎体), and $dim_F A = 2$, then $AutA \cong S_3$ (３次の対称群), if $dim_F A = 1$, then $TrigA \cong K_4$ (Klein's 4 group) ([K-O.3], [K.2]). これらの概念を一般の非結合的代数で以下考えます.

§1. Definitions of triality relations and normal triality algebras

この章で非結合的代数系における一般的な triality relations (local and global の cases) を述べさせていただきます. (a correspondence of local and global triality relations)

Let A be a nonassociative algebra over $ch\ F \neq 2, 3$ (algebra は有限次元そして必ずしも単位元をもたない非結合的代数系の場合を考えます).

Following ([K-O.2] or [K-O.3]), suppose that a triple $g = (g_1, g_2, g_3) \in (Epi\ A)^3$ satisfies a global triality relation

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \tag{1.1}$$

where the index j is defined by modulo 3, so that $g_{j\pm 3} = g_j$ (this is said to be a triality group). We denote

$$Trig\ A = \{g = (g_1, g_2, g_3) \in (Epi\ A)^3 | g_j(xy) = (g_{j+1}x)(g_{j+2}y), \forall j = 1, 2, 3\} \tag{1.2}$$

This is a generalization of the automorphism group of A.

In constrast to the algebra triality relations (1.1), we may also consider a local triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y). \tag{1.3}$$

Analogously to (1.2) if $[t_j, t_k]$ is closed, we introduce

$$Trid\ A = \{t = (t_1, t_2, t_3) \in (End\ A)^3 | t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y), \forall j = 1, 2, 3\}. \tag{1.4}$$

Then, it defines a Lie algebra with component wise commutation relation. Also if $(t_1, t_2, t_3) \in Trid\ A$, it is easy to verify that we have for any $\alpha_j \in F, \alpha_{j\pm 3} = \alpha_j$

$$t' = (t'_1, t'_2, t'_3) \in Trid\ A, \text{ where } t'_j = \sum_{k=1}^{3} \alpha_{j-k} t_k \ (j = 1, 2, 3).$$

Furtheremore, if the exponential map $t_j \to \xi_j$ is given by

$$\xi_j = exp\ t_j = \sum_{n=0}^{\infty} \frac{1}{n!}(t_j)^n \tag{1.5}$$

is well-defined, then we can show that

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y), \tag{1.6}$$

provided that $t = (t_1, t_2, t_3) \in Trid\ A$ and vice-versa.

Next we introduce multiplication operators of A, $L(x), R(x) \in End\ A$ by

$$L(x)y = xy \text{ and } R(x)y = yx.$$

Def.1.1. Let $d_j(x, y) \in End\ A$, for $x, y \in A$ $(j = 1, 2, 3)$ be to satisfy
(i)

$$d_1(x, y) = R(y)L(x) - R(x)L(y) \tag{1.7a}$$

$$d_2(x, y) = L(y)R(x) - L(x)R(y). \tag{1.7b}$$

(ii) The explicit form for $d_3(x, y)$ is unspecified except for

$$d_3(x, y) = -d_3(y, x), \tag{1.7c}$$

(iii)

$$(d_1(x, y), d_2(x, y), d_3(x, y)) \in Trid\ A.$$

We call the algebra A satisfying these conditions to be a *regular triality algebra.*

Remark. Any Lie (resp. Jordan) algebra with product $[x, y]$ (resp. xy) is an example of the regular (in particular, normal) triality algebra with respect to the $L([x, y]) = d_j(x, y)$ (resp.$[L(x), L(y)]$) for $j = 1, 2, 3$.

Proposition 1.2 ([K-O.3]). *Let A be a regular triality algebra satisfying either the condition (B) or (C); (B) $AA = A$, (C) if some $b \in A$ satisfies either $L(b) = 0$ or $R(b) = 0$, then $b = 0$. Then we obtain the following.*
(i) For any $t = (t_1, t_2, t_3) \in Trid\ A$, we have

$$[t_j, d_k(x, y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y) \tag{1.8a}$$

Especially, if we choose $t_j = d_j(x, y)$ it yields

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y). \tag{1.8b}$$

(ii) For any $g = (g_1, g_2, g_3) \in Trig\ A$, we have

$$g_j d_k(x, y)g_j^{-1} = d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y). \tag{1.8c}$$

Let A be a regular triality algebra with either (B) or (C), and set

$$\mathfrak{D} = span < d_j(x, y), \forall x, y \in A,\ j = 1, 2, 3 > . \tag{1.9}$$

Then $\mathfrak{D}$ is a Lie algebra by (1.8b). Moreover, it is an ideal of the large Lie algebra $Trid\ A$ by (1.8a), denoted by $\mathfrak{D} \triangleleft Trid\ A$. We call an "inner triality derivation"(naming of the author) this $\mathfrak{D}$ as a generalization of derivations.

Def.1.3. If a regular triality algebra satisfies Eqs.(1.7) as well as

$$d_3(x, y)z + d_3(y, z)x + d_3(z, x)y = 0, \tag{1.10a}$$

$$[d_j(u,v), d_k(x,y)] = d_k(d_{j-k}(u,v)x, y) + d_k(x, d_{j-k}(u,v)y), \tag{1.10b}$$

then we call a *pre normal triality algebra* ([K-O.1]). Furthermore, if we have

$$Q(x,y,z) := d_1(z,xy) + d_2(y,zx) + d_3(x,yz) = 0, \tag{1.11}$$

then A is called a *normal triality algebra* ([K-O.2]). Next we introduce the second bilinear product in the same vector space A with involution $x \to \overline{x}$ by

$$x * y = \overline{xy} = \overline{y}\ \overline{x}. \tag{1.12}$$

Then the resulting algebra $(A, x*y)$ is said to be a *conjugation algebra* of A, for the new product $x*y$, by means of $\overline{Qx} = \overline{Q}\ \overline{x}$ and $Q \in End\ A$, we have

$$\overline{g}_j(x*y) = (g_{j+1}x)*(g_{j+2}y),\ \overline{d}_j(x*y) = (d_{j+1}x)*y + x*(d_{j+2}y). \tag{1.13}$$

Remark ([K-O.1]). The conjugation algebra of a structurable algebra which contains an alternative algebra is a normal triality algebra.

Note that the vector space $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ with 182 dimension ([S]) appeared Tits second construction of the Lie algebra of type E_8 is a normal triality algebra (see next section for the details).

Theorem 1.4 ([K-O.2]). *The symmetric composition algebra, Lie and Jordan algebras are a normal triality algebra.*

Theorem 1.5. *For a normal triality algebra A, if we define*

$$\xi_j = exp\ d_j\ \ (j=1,2,3),\ \ (\ assuming\ \ the\ well-defined)$$

then we have

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y),\ \text{ that is,}(\xi_j, \xi_{j+1}, \xi_{j+2}) \in Trig\ A,$$

$$[\frac{d}{dt}((exp\ td_j)d_k(exp\ td_j)^{-1})]_{t=0} = [d_j, d_k] \in Trid\ A.$$

That is, this means that $([d_j,d_k],[d_{j+1},d_{k+1}],[d_{j+2},d_{k+2}]) \in Trid\ A$.

Corollary. *For the pseudo octonion or para Hurwitz algebras, the same result in Theorem 1.5 holds, as these algebras are a symmetric composition algebra and so a normal triality algebra.*

Remark. In the normal triality algebra A, if we define an endomorphism by $D(x,y) := d_1(x,y)+d_2(x,y)+d_3(x,y)$, then we have the relations $D(x,y) = -D(x,y)$, $D(xy,z)+D(yz,x)+D(zx,y) = 0$ and $D(x,y)$ is a derivation satisfying $[D(x,y), D(u,v)] = D(D(x,y)u,v) + D(u, D(x,y)v)$, thus this algebra A is a generalized structurable algebra ([K.1]).

Remark([K-O.1]). The conjugation algebra $(A, x*y)$ of normal triality algebra (A, xy) with a para unit e (i.e., $ex = xe = \overline{x}$) is a structurable algebra with the unit $e*x = x*e = x$, since $x*y = \overline{xy}$ and $\overline{\overline{x}} = x$.

Remark. Let $(A, x*y)$ be an associative algebra. Then $(A, x\cdot y)$ is a Jordan algebra with new product and involution difined by $x \cdot y = x*y + y*x$ and $\overline{x} = x$, since they satisfy the identities $x\cdot y = y \cdot x$ and $(x\cdot y)\cdot x^2 = x\cdot(y\cdot x^2)$.

Remark. Let A be a normal triality alg. For $(\xi_1, \xi_2, \xi_3) = (exp\ d_1, exp\ d_2, exp\ d_3) \in exp\ \mathfrak{D}$, provided that the exponential map is well-defined,

$$\forall g = (g_1, g_2, g_3) \in Trig\ A \implies$$

$$g_j \xi_k g_j^{-1} = g_j (exp\ d_k) g_j^{-1} = exp(g_j d_k g_j^{-1}) \in exp\ \mathfrak{D}. \text{ (by (1.8) and (1.9))}$$

Therefore $G_0 =< \xi_1, \xi_2, \xi_3 >_{span}$ is an invariant subgroup of $Trig\ A$. We call an "inner triality group"(naming of the author) and we obtain $\mathfrak{D} \longleftrightarrow G_0$.

For the definitions of this section, we would like to refer ([K-O.1],[K-O.2]and [K-O.3]), that is, for the concept of normal triality algebras and related topics.

It seems that this concept (called a normal triality algebra) of a generalization of the derivation may be regarded as a generalization in the "*principle of triality*" of the octonion (or para octonion) algebra due to Tits ([K.3]).

§2. Examples of triality relations of algebras

For several examples of normal triality algebras, in particular, to construct Lie algebras, we will exhibit them in this section.

First, it is known that a symmetric composition algebra A over a field $F (ch\ F \neq 2)$ with a symmetric bilinear form $< x, y >$ satisfying the relations

$$(xy)x = x(yx) =< x, x > y,\ < x, y >\ \ is\ non-degenerate,$$

is a normal triality algebra and the symmetric composition algebra is either a para-Hurwitz algebra (that is, the conjugation algebra is a Hurwitz algebra which contains the Cayley number if $ch\ F = 0$), or an eight dimensional pseudo octonion algebra due to M.Gell-mann ([G],[K-O.2],[K-O.3], [K3] and [O]).

This symmetric composition algebra A has a triality derivation $(d_0, d_1, d_2) \in Trid\ A$ defined by $d_0(x, y) = 2\{[L(x), L(y)] - R([x, y])\}$,

$$d_1(x, y) = R(y)L(x) - R(x)L(y),\ and\ d_2(x, y) = L(y)R(x) - L(x)R(y). \quad (\spadesuit)$$

The case of the symmetric composition algebra;

$$< d_j(a, b) >_{span} \cong D_4\ (Lie\ algebra\ of\ 28\ dimension),$$

since $< d_j(a, b)x, y > + < x, d_j(a, b)y >= 0,\ (\forall\ j = 0, 1, 2)$.

Secondarily, the Lie algebra $(\varepsilon = -1)$ and Jordan algebra $(\varepsilon = 1)$ are a normal triality algebra equipped with the product $L(x)y = xy = \varepsilon yx = \varepsilon R(x)y$ and the triality derivation $d(x, y) := d_0(x, y) = d_1(x, y) = d_2(x, y) = -\varepsilon[L(x), L(y)]$.

The other examples are the following:

Example a) ([K-O.2]). The vector space $O \otimes O$ with 64 dimension induced from two para octonion algebras O (para octonion is the conjucation algebra of octonion $\mathbf{O}$) is a normal triality algebra with respect to the $D_j(a \otimes x, b \otimes y) := d_j^{(1)}(a, b) \otimes < x, y > id + < a, b > id \otimes d_j^{(2)}(x, y)$, where $d_j^{(1)}$ and $d_j^{(2)}$ with the triality derivation defined by $(\spadesuit)$ respectively. Note that $D = D_0 + D_1 + D_2$ construct a Lie algebra of type $G_2 \oplus G_2$. This implies that $Der\ O \cong G_2$(Lie algebra of 14 dim) in particular.

As we will show in next section, this case is relevant for a construction of so-called Freudenthal's magic square.

Remark. A vector of the tensor product $O_p \otimes O_p$ induced from the pseudo octonion algebra O_p is a normal triality algebra with $D_j(a \otimes x, b \otimes y) = d_j^{(1)}(a,b) \otimes < x, y > id + < a, b > id \otimes d_j^{(2)}(x,y)$ as well as the tensor product of the para octonion algebra O. However note that $D = D_0 + D_1 + D_2$ construct a Lie algebra of type $A_2 \oplus A_2$. Hence this means that $Der\ O_p \cong A_2$ (Lie algebra of 8 dim).

Example b) ([K-O.1], [K-O.3]). For the octonion algebra $\mathbf{O}$ and a para Zorn vector matrix with 56 dimension induced from the vector space

$$\begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \qquad (\clubsuit)$$

is a normal triality algebra, where $\forall\ \mathbf{a}, \mathbf{b} \in H_3(\mathbf{O})(= \mathfrak{J})$ is a exceptional Jordan algebra of degree three with 27 dimension over a field F, and $\alpha, \beta \in F$ (the base field). For the details, see section 5 in ([K-O.3]). This concept was appeared the meta symplectic geometry due to H. Freudenthal (for example, see ([K-O.4])).

Example c). The vector space $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ with 182 dimension appeared Tits second construction of E_8([S],[Tô]) is a normal triality algebra with the (triality) derivation $D_0 = D_1 = D_2 (= D)$. Here this derivation of $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ means that $D = (d_0 + d_1 + d_2) \otimes < x, y > id + < a, b > id \otimes [R(x), R(y)]$, where d_j is the triality derivation of the normal triality algebra $\mathfrak{A}$ (as the para octonion algebra) and $[R(x), R(y)]$ is the derivation of the normal triality algebra $\mathfrak{J}$ (as the exceptional Jordan algebra with $dim\ \mathfrak{J} = 27$). Here denote by $\mathfrak{A}_0 = \{a \in \mathfrak{A} | trace\ (a) = 0\}$, $\mathfrak{J}_0 = \{x \in \mathfrak{J} | Trace\ (x) = 0\}$, and $dim\ \mathfrak{A}_0 = 7$, $dim\ \mathfrak{J}_0 = 26$ respectively.

§3. Lie algebras construction

In this section, we will discuss a construction of Lie algebras associated with the normal triality algebras of Examples a), b) and c) in section 2.

Following (]K-O.2]), let A be a normal triality algebra and consider linear maps:

$$\rho_j : A \to V, \text{and } T_j : A \otimes A \to V \qquad (3.1)$$

for $j = 0, 1, 2$, where V is an unspecified algebras with skew symmetric bi-linear product $[\circ, \circ]$. We set now

$$T(A, A) = span < T_j(x, y), \forall x, y \in A >, \forall j = 0, 1, 2. \qquad (3.2)$$

$$L(A) = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A, A). \qquad (3.3)$$

Let (i, j, k) be a cyclic permutation of indices $(0, 1, 2)$, and assume the following anti-communtative multiplication relations:

$$[\rho_i(x), \rho_i(y)] = -[\rho_i(y), \rho_i(x)] = T_{3-i}(x, y) \qquad (3.4a)$$

$$[\rho_i(x), \rho_j(y)] = -[\rho_j(y), \rho_j(x)] = -\rho_k(xy) \qquad (3.4b)$$

$$[T_l(x, y), \rho_j(z)] = -[\rho_j(z), T_l(x, y)] = \rho_j(d_{l+j}(x, y)z) \qquad (3.4c)$$

$$[T_l(u,v), T_m(x,y)] = T_m(d_{l-m}(u,v)x, y) + T_m(x, d_{l-m}(u,v)y)$$
$$= -T_l(d_{m-l}(x,y)u, v) - T_l(u, d_{m-l}(x,y)v) \tag{3.4d}$$

for $l, m = 0, 1, 2$. Hence, we introdcuce the Jacobian in $L(A)$ by

$$J(X,Y,Z) = [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \tag{3.5}$$

for $X, Y, Z \in L(A)$.

Lemma 3.1. *$T(A,A)$ and $T_j(A,A)$ for $j = 0,1,2$ are Lie algebras. Also $T_j(A,A)$ is an ideal of $T(A,A)$, where $T_j(A,A) = span < T_j(x,y), \forall x, y \in A >$.*

Condition (D) Suppose that we have $\rho_j(x) = 0$ for some $x \in A$ and for some value of $j = 0,1,2$, we then have $x = 0$.

Proposition 3.2. *Let A be a pre-normal triality algebra. If we have*

$$J(x,y,z) = T_0(x,yz) + T_1(z,xy) + T_2(y,zx) = 0, \tag{3.6}$$

then $L(A)$ is a Lie algebra. Moreover, if the condition (D) holds, then A is a normal triality algebra. Conversly, if $L(A)$ is a Lie algebra and if the condtion (D) holds, then A is a normal triality algebra with the validity of Eq.(3.6).

Theorem 3.3. *Let A be a normal triality algebra. Then, the quotient algebra $\tilde{L}(A) = L(A)/J$ is a Lie algebra, where $J = span < J(x,y,z) >$.*

この定理 3.3 を適用して 5-graded $g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ の例外型リー代数を構成します. 特に normal triality algebra A の inner triality derivation d_j を T_j とみなす.

$$L(A) = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A) \oplus T(A,A), \ \textit{and}$$

$$T(A,A) =< T_j(a,b), \forall j = 0,1,2, \forall a, b \in A >_{span},$$

and we identify $T_j(a,b)$ with the triple $T_j(a,b) = (d_j(a,b), d_{j+1}(a,b), d_{j+2}(a,b))$.

§2 の Example a),b),c) の場合に付随した Lie algebras を考察する. 以後, 基礎体 F は標数 0 の代数閉体とする.

a) $A = \mathbf{O} \otimes \mathbf{O}$ の場合 (tensor product case , and $dim\ g_{-2} = dim\ g_2 = 14$): $A = A_1 \otimes A_2$, $dim\ A_1, dim\ A_2$ なる記号を用いると, それぞれ construct される Lie algebra は以下の様になります, ただし $\mathbf{O}$ は octonion algebra.

$$L(A) \cong E_8, \ L_j = \rho_j(A) \oplus T_{3-j}(A,A) \cong D_8, T_j(A,A) \cong D_4 \oplus D_4.$$

$dimA_2 \backslash dimA_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

E_8 を Extended Dynkin diagram で表すと $L(A)/L_j(A)$ は 128 次元の対称空間;

$\circledcirc - \circ - \circ - \circ - \circ - \circ - \circ - \boxed{\circ}$ $\quad \boxed{\circ}$ omitted $\cong$ D_8, and $\circledcirc$ is hieghest root.

$|$

$\circ$

b) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ の場合 (balanced case, and $dim\ g_{-2} = dim\ g_2 = 1$):

$L(A) \cong E_8,\ L_j = \rho_j(A) \oplus T_{3-j} \cong E_7 \oplus A_1,\ T_j(A,A) \cong E_6 \oplus gl(1) \oplus gl(1)$. ここで $H_3(\mathbf{O}) \to H_3(\mathfrak{A})(= B)$ なる記号に置き換える．ただし $\mathfrak{A}$ は Hurwitz algebras over F. $\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} F & B \\ B & F \end{pmatrix} = A$ とすると $dimB$ に対応する Lie algebra は以下の様になります.

$dimB$	1	6	9	15	27
$dimA$	4	14	20	32	56
$dimL(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

この場合の construct された E_8 を Extended Dynkin diagram で表すと $L(A)/L_j(A)$ は 112 次元の対称空間;

$\circledcirc - \boxed{\circ} - \circ - \circ - \circ - \circ - \circ - \circ$ $\quad \boxed{\circ}$ omitted $\cong\ A_1 \oplus E_7$.
$\qquad\qquad\qquad\qquad\qquad\quad |$
$\qquad\qquad\qquad\qquad\qquad\quad \circ$

c) $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$ の場合: Here denote $\mathfrak{A}_0 = \{x \in \mathbf{O} | trace\ x = 0\}, \mathfrak{J}_0 = \{x \in H_3(\mathbf{O}) | Trace\ x = 0\}$, $(dim\ A = 7 \times 26,\ dim\ g_0 = 66)$:

$L(A) = Der(A) \oplus A \cong E_8,\ Der(A) = T(A,A) = Der\mathfrak{A} \oplus Der\mathfrak{J} \cong G_2 \oplus F_4 =< D(X,Y) >_{span}$. この場合の $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ は $X \circ Y = (a * b) \otimes (x * y)$ and $[X,Y] = D(X,Y) + X \circ Y$ による generalized structurable algebra からのリー代数の構成です ([K.1]). b) の場合と同様に $\mathfrak{J} = H_3(\mathbf{O}) \to H_3(\mathfrak{A})$ $(= B)$ と置く.

	$dimB = 1$	$dimB = 6$	$dimB = 9$	$dimB = 15$	$dimB = 27$
$dim\mathfrak{A} = 1$	0	A_1	A_2	C_3	F_4
$dim\mathfrak{A} = 2$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$dim\mathfrak{A} = 4$	A_1	C_3	A_5	D_6	E_7
$dim\mathfrak{A} = 8$	G_2	F_4	E_6	E_7	E_8

この場合の construct された E_8 を Dynkin diagram で表すと, $L(A)/(G_2 \oplus F_4)$ は 182 次元の reductive homogeneous space であり, 次のように予想されます.

$\boxed{\circ} - \circ - \circ - \boxed{\circ} - \circ - \circ - \circ$ $\quad \boxed{\circ}$ omitted $\cong G_2 \oplus F_4$.
$\qquad\qquad\qquad\quad |$
$\qquad\qquad\qquad\quad \circ$

ここでは　Examples a), b) and c) の (extended) Dynkin diagram に関して, E_8 についてだけ具体的に述べましたが, G_2, F_4, E_6, E_7 の場合も同様に表せます.

For the triality relations of Lie algebras associated with triple systems (or structurable algebras) without using root systems, to see our references.

つまり, 上記の normal triality (super)algebras の場合と三項系代数からのリー代数 (リー超代数) の構成が存在します. 三項系 (Freudenthal-Kantor triple systems) からの構成に関し, super 化を含め ([K-O.1]∼[K-O.4]) の文献を参照.

以上の結果は $g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ の 5-graded リー（超) 代数の構成には Jordan and alternative algebras を含む normal triality algebras とその super 化の概念が重要であることを示しています.

Appendix (対称的合成代数の基底と三対原理).

Let A be a symmetric composition algebra over the field F and $\sqrt{3} \in F$.

$$\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | a_j a_{j+1} = a_{j+2},\ < a_j, a_j >= 1,\ \forall j = 1, 2, 3\}. \quad (A.1)$$

For a given $a = (a_1, a_2, a_3) \in \Sigma$, we introduce a notation

$$\Lambda(a) = \{p = (p_1, p_2, p_3) \in A^3 |$$

$$a_j p_{j+1} + p_j a_{j+1} = p_{j+2}, < p_j, a_j >= 0,\ \forall j = 1, 2, 3\} \qquad (A.2)$$

Note that $\Lambda(a)$ is a vector space over F.

Moreover, we define $q_j \in A$ by $q_j = a_{j+1} p_{j+2} = p_j - p_{j+1} a_{j+2}$.

From (Th.2.5 and Th.3.2 in [K-O.3]) and the notation being as avobe, we have the following theorems for global and local triality relations.

Theorem A.1. *For $\forall a, b \in \Sigma$, we have $\sigma_j(a)$ and $\theta_j(a) \in Trig\ A$, $G =< \theta_j(a)\sigma_j(b)$ and $\sigma_j(a)\theta_j(b) >_{span}$ is an invariant subgroup of $TrigA$, and $\sigma_{j+2}(a)\sigma_{j+1}(a)\sigma_j(a) = Id$, $\theta_{j+2}(a)\theta_{j+1}(a)\theta_j(a) = Id$, (inner triality group) where $\sigma_j(a) = R(a_{j+1})R(a_{j+2})$ and $\theta_j(a) = L(a_{j+2})L(a_{j+1})$.*

Theorem A.2. *For any $a \in \Sigma$ and $p \in \Lambda(a)$, if we introduce $D_j(a, p) \in End\ A$ by $D_j(a, p)x = (p_{j+1}x)a_{j+1} + a_j(xq_j)$, then this $D_j(a, p)$satisfies*

$$D_j(a, p)(xy) = (D_{j+1}(a, p)x)y + x(D_{j+2}(a, p)y)\ (\textit{inner triality derivation}).$$

Furtheremore, we obtain that $T_j(A, A) =< D_j(a, p) >_{span}$ is an ideal of $Trid\ A$.

また 8 次元の symmetric composision algebra の場合, u, v を適当に選ぶと

$D_j(a, p) < - - - - > d_j(u, v)$ (normal triality algebra における local triality derivation) の対応が知られています ([K-O.3]). Hence, $< d_j(u, v) >_{span} \cong D_4$ (28 次元のリー代数) なので $D_j(a, p)$ を構成する 28 個の $a = (a_1, a_2, a_3)$ *and* $p = (p_1, p_2, p_3)$ を具体的に以下列挙させていただきます. これらの基底に関する乗積表は para and pseudo octonion algebras はそれぞれ次の表で, 内積は $x = \sum_j x_j e_j$, $y = \sum_j y_j e_j$ のとき $< x, y >= \sum_j x_j y_j$ で定義されます.

(∗) para octonion の基底: $e_1, \cdots, e_7$ の積の例, $e_1 e_2 = -e_3, e_7 e_1 = -e_6, e_6 e_7 = -e_1$.

para case	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	$-e_3$	e_2	$-e_5$	e_4	$-e_7$	e_6
e_2	e_3	-1	$-e_1$	e_6	$-e_7$	$-e_4$	e_5
e_3	$-e_2$	e_1	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	e_3	$-e_6$	e_7	-1	$-e_1$	e_2	$-e_3$
e_5	$-e_4$	e_7	e_6	e_1	-1	$-e_3$	$-e_2$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	$-e_5$	$-e_4$	e_3	e_2	e_1	-1

(AI) Examples of triality group and derivation in the para octonion algebra (the conjugation algebra of Cayley algebra), この場合 $e_0 = 1$ なので省略.

$a = (a_1, a_2, a_3), p = [p_1, p_2, p_3]$ と表示する.

(AI):(para octonion algebra), with respect to basis $e_1, \cdots, e_7$.

(1) $a = (e_1, e_2, -e_3),\ p = [e_2, -e_1, 0]$,

Remark. $a=(a_1,a_2,a_3)\in\Sigma$ に対して $p_j=a_{j+1}p_{j+2}+p_{j+1}a_{j+2}$ を満たす $p=[p_1,p_2,p_3]$ の選び方はいくつか存在する．例えば, $p=[e_3,0,-e_1]$. $p=[0,e_3,-e_3]$ がある (これはある関数における微分係数の様に理解できます).
$(2)(e_1,e_4,-e_5),[e_4,-e_1,0]$ $(3)(e_1,e_6,-e_7),[e_6,-e_1,0]$ $(4)(e_2,e_4,e_6),[e_4,-e_2,0]$ $(5)(e_2,e_5,-e_7),[e_5-e_2,0]$ $(6)(e_3,e_4,-e_7),[e_4,-e_3,0]$ $(7)(e_3,e_5,-e_6),[e_5,-e_3,0]$ $(8)(\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2,\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2,-1,),[\frac{\sqrt{3}}{2}e_1-\frac{1}{2}e_2,\frac{\sqrt{3}}{2}e_1-\frac{1}{2}e_2,0]$, $(i,j)=(1,2)$ を除いて (9)〜(28): $i\neq j,(\frac{1}{2}e_i+\frac{\sqrt{3}}{2}e_j,\frac{1}{2}e_i+\frac{\sqrt{3}}{2}e_j,-1),[\frac{\sqrt{3}}{2}e_i-\frac{1}{2}e_j,\frac{\sqrt{3}}{2}e_i-\frac{1}{2}e_j,0]$ つまり $(\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7,\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7,-1),[\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7,\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7,0]$ etc. (8)〜(28): ${}_7C_2=21$ 通り. where $e_0e_i=e_ie_0=\overline{e}_i=-e_i$ $(i\neq 0), e_0$ *is a para unit.*

Note that the product $[e_j,e_k]:=e_je_k-e_ke_j$, $(j,\ k=1,\cdots,7)$ makes a Malcev algebra with seven dimension ([O]).

(**) pseudo octonion の基底: $e_1,\cdots,e_8$ の積の例, $e_1e_2=e_3,\ e_8e_1=e_1,e_8e_8=-e_8$.

pseudo case	e_1	e_2	e_3	e_4
e_1	e_8	e_3	$-e_2$	$\frac{\sqrt{3}}{2}e_6+\frac{1}{2}e_7$
e_2	$-e_3$	e_8	e_1	$\frac{1}{2}e_6-\frac{\sqrt{3}}{2}e_7$
e_3	e_2	$-e_1$	e_8	$\frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5$
e_4	$\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7$	$-\frac{1}{2}e_6-\frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5$	$\frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_8$
e_5	$\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7$	$\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	$-\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8$
e_6	$\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5$	$\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6+\frac{1}{2}e_7$	$\frac{\sqrt{3}}{2}e_1-\frac{1}{2}e_2$
e_7	$\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5$	$-\frac{1}{2}e_6-\frac{\sqrt{3}}{2}e_7$	$-\frac{1}{2}e_1-\frac{\sqrt{3}}{2}e_2$
e_8	e_1	e_2	e_3	$-\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$

pseudo case	e_5	e_6	e_7	e_8
e_1	$-\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5$	$-\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	e_1
e_2	$\frac{\sqrt{3}}{2}e_6+\frac{1}{2}e_7$	$-\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5$	e_2
e_3	$-\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7$	$\frac{1}{2}e_6-\frac{\sqrt{3}}{2}e_7$	e_3
e_4	$\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8$	$\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2$	$\frac{1}{2}e_1-\frac{\sqrt{3}}{2}e_2$	$-\frac{1}{2}e_4-\frac{\sqrt{3}}{2}e_5$
e_5	$\frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_8$	$-\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2$	$\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2$	$\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5$
e_6	$\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2$	$-\frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_8$	$-\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8$	$-\frac{1}{2}e_6-\frac{\sqrt{3}}{2}e_7$
e_7	$\frac{\sqrt{3}}{2}e_1-\frac{1}{2}e_2$	$\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8$	$-\frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_8$	$\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7$
e_8	$-\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5$	$-\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7$	$-\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7$	$-e_8$

(AII) Examples of triality group and derivation in the pseudo octonion algebra (Gell-Mann's eight dimensional matrix) この代数は単位元をもたない.

(AII):(pseudo octonion algebra), with respect to basis $e_1,\cdots e_8$.
$(1)(e_1,e_2,e_3),[e_2,-e_1,0]$, $(2)(e_1,e_4,\frac{\sqrt{3}}{2}e_6+\frac{1}{2}e_7),p=[e_4,\frac{1}{2}e_1-\frac{\sqrt{3}}{2}e_2,0]$, $(3)(e_1,e_5,-\frac{1}{2}e_6+\frac{\sqrt{3}}{2}e_7),[e_5,\frac{1}{2}e_1-\frac{\sqrt{3}}{2}e_2,0]$, $(4)(e_1,e_6,\frac{1}{2}e_5+\frac{\sqrt{3}}{2}e_4),[e_6,\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2,0]$, $(5)(e_1,e_7,\frac{\sqrt{3}}{2}e_5-\frac{1}{2}e_4),[e_7,\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2,0]$, $(6)(e_2,e_4-\frac{\sqrt{3}}{2}e_7+\frac{1}{2}e_6)$, $[e_4,\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2,0]$, $(7)(e_2,e_5,\frac{1}{2}e_7+\frac{\sqrt{3}}{2}e_6),[e_5,\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2,0]$, $(8)(e_2,e_6,\frac{\sqrt{3}}{2}e_5-\frac{1}{2}e_4),[e_6,-\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2,0]$ $(9)(e_2,e_7,-\frac{\sqrt{3}}{2}e_4-\frac{1}{2}e_5),[e_7,-\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2,0]$,
$(10)(e_3,e_4,\frac{\sqrt{3}}{2}e_4+\frac{1}{2}e_5),[e_4,\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8,0]$, $(11)(e_3,e_5,\frac{\sqrt{3}}{2}e_5-\frac{1}{2}e_4),[e_5,\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_8,0]$, $(12)(e_3,e_6,-\frac{\sqrt{3}}{2}e_6-\frac{1}{2}e_7),[e_6,\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8,0]$, $(13)(e_3,e_7,-\frac{\sqrt{3}}{2}e_7+\frac{1}{2}e_6)$, $[e_7,\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8,0]$, $(14)(e_4,e_4,-\frac{1}{2}e_8+\frac{\sqrt{3}}{2}e_3),[e_3,-\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_8,0]$

(15)$(e_4, e_5, \frac{\sqrt{3}}{2}e_8 + \frac{1}{2}e_3), [e_5, -e_4, 0]$, (16)$(e_4, e_6, \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2), [e_6, \frac{\sqrt{3}}{2}e_5 + \frac{1}{2}e_4, 0]$, (17)$(e_4, e_7, -\frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_1), [e_7, \frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5, 0]$, (18)$(e_4, e_8, -\frac{\sqrt{3}}{2}e_5 - \frac{1}{2}e_4)[e_5, -e_3, 0]$, (19)$(e_5, e_5, \frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8), [e_3, \frac{\sqrt{3}}{2}e_8 - \frac{1}{2}e_3, 0]$, (20)$(e_5, e_6, \frac{\sqrt{3}}{2}e_2 - \frac{1}{2}e_1), [e_6, \frac{1}{2}e_5 - \frac{\sqrt{3}}{2}e_4, 0]$, (21)$(e_5, e_7, \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2), [e_7, -\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5, 0]$,
(22)$(e_5, e_8, \frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5), [e_4, e_3, 0]$, (23)$(e_6, e_6, -\frac{1}{2}e_8 - \frac{\sqrt{3}}{2}e_3), [e_3, -\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8, 0]$, (24)$(e_6, e_7, \frac{\sqrt{3}}{2}e_8 - \frac{1}{2}e_3), [e_7, -e_6, 0]$, (25)$(e_6, e_8, -\frac{\sqrt{3}}{2}e_7 - \frac{1}{2}e_6), [e_7, e_3, 0]$,
(26)$(e_7, e_7, \frac{1}{2}e_8 + \frac{\sqrt{3}}{2}e_3), [e_3, -\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8, 0]$, (27)$(e_7, e_8. -\frac{1}{2}e_7 + \frac{\sqrt{3}}{2}e_6), [e_6, -e_3, 0]$, (28)$(-e_8, -e_8, -e_8), [e_1, -e_1, 0]$. where $e_8e_8 = -e_8$. (e_8 が単位元的な役割です). Note that if we define a product by $[\frac{1}{2}e_j, \frac{1}{2}e_k] = \sum_{l=1}^{8} f_{jkl}(\frac{1}{2}e_l)$, where the notation f_{jkl} refer ([G], [O], [K.3]), then it makes the Lie algebra of type A_2.

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