SPECTRAL ANALYSIS APPROACH TO THE MAXIMAL REGULARITY FOR THE STOKES EQUTIONS AND FREE BOUNDARY PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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Abstract. In this note, spectral analysis of initial boundary value problem with non-homogeneous boundary data is investigated. By R-boundedness of solution operators for $1 < p < \infty$ and real interpolation methods for $p = 1$, we shall show a maximal L_p regularity for the initial boundary value problem with non-homogeneous boundary data. Especially, for $1 < p < \infty$, the transference theorem enable us to make a general framework of unique existence of time periodic solutions. As an application of our approach, the Stokes equations with non-homogeneous free boundary conditions and the free boundary problem for the Navier-Stokes equtions in the half-space are discussed.

1. A review of maximal regularity theorems

In this section, we consider a linear evolution equation:

$$\dot u(t) + Au(t) = f(t) \quad \text{for } t > 0, \quad u(0) = 0 \tag{1}$$

Here, $\dot u = \partial_t u$. For example,

$$\begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(x,0) &= 0 && \text{for } x \in \Omega. \end{aligned} \tag{2}$$

Here, Ω is a uniformly C^2 domain in N-dimensional Eucledian space $\mathbb{R}^N$ and $\partial\Omega$ denotes the boundary of Ω. The point here is a homogeneous boundary condition: $u|_{\partial\Omega} = 0$. Inhomogeneous boundary condition is one of main subjects in this note and it will be handled in the next section.

Let X be a Banach space with norm $\|\cdot\|_X$, and A a closed linear operator from $D(A)$ into X, where $D(A)$ is a subspace of X. For the example (2), typically we choose $A = -\Delta = \sum_{j=1}^N \partial_j^2$, $\partial_j = \partial/\partial x_j$, and $X = L_q(\Omega)$, $D(A) = \{u \in W^2_q(\Omega) \mid u|_{\partial\Omega} = 0\}$, or $X = B^s_{q,r}(\Omega)$, $D(A) = \{u \in B^{s+2}_{q,r}(\Omega) \mid u|_{\partial\Omega} = 0\}$. Here, $1 < q < \infty$ and $-1 + 1/q < s < 1/q$.

We assume that A is a sectorial operator, that is , there exists an $\epsilon \in (0, \pi/2)$ and $\gamma > 0$ such that the resolvent set $\rho(A)$ contains $\Sigma_\epsilon + \gamma$ and there exists a constant $C > 0$ such that

$$|\lambda|\|(\lambda\mathbf{I} + A)^{-1}f\|_X \leq C\|f\|_X \quad \text{for every } f \in X \text{ and } \lambda \in \Sigma_\epsilon + \gamma. \tag{3}$$

Here and in the sequel, we denote

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \pi - \epsilon\}, \quad \Sigma_\epsilon + \gamma = \{\lambda + \gamma \mid \lambda \in \Sigma_\epsilon\}.$$

Then, the operator A generates a continuous analytic semigroup, is denoted by $\{e^{-tA}\}_{t\geq 0}$ here. By using $\{e^{-tA}\}_{t\geq 0}$, a unique solution$u(t)$ of equations (1) is written as

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \tag{4}$$

Refer to Yosida [35] concerning the fundamental theory about continuous analytic semigroups.

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Let $1 \leq q \leq \infty$. If there exists a subspace Y of X such that for any $f \in L_q((0,T),Y)$ problem (1) admits a unique solution u possesses the estimate:

$$\int_0^T (\|\partial_t u(t)\|_Y + \|Au(t)\|_Y)^q \, \mathrm{d}t \leq C \int_0^T \|f(t)\|_Y^q \, \mathrm{d}t. \tag{5}$$

In this case, we say that the opearator A has the maximal L_q- Y regularity.

The basical maximal regularity result is the theorem due to Da Prato and Grisvard [11, Theorem 4.7], which can be seen as the first abstract result on maximal regularity in the mathematical litreture. Let

$$\mathcal{D}_A(\theta,q) = \{x \in X \mid [x]_{\theta,q} := \Big(\int_0^\infty \|t^{1-\theta}Ae^{-tA}x\|_X^q \frac{\mathrm{d}t}{t}\Big)^{1/q} < \infty\}$$

and $\|x\|_{\mathcal{D}_{\theta,q}} := \|x\|_X + [x]_{\theta,q}$. The space $\mathcal{D}_A(\theta,q)$ becomes a Banach space with norm $\|\cdot\|_{\theta,q}$. Then, Da Prato and Grisvard theorem tells us that problem (11) admits a unique solution u possessing the estimate (5) with $Y = \mathcal{D}_A(\theta,q)$. Here, $T > 0$ is finite time and C depends on $T > 0$. We know that $\mathcal{D}_A(\theta,q) = (X, D(A))_{\theta,q}$, where $(\cdot,\cdot)_{\theta,q}$ denotes real interpolation functor. For this fact, refer to Lunardi [25, Chapter 1.2] for example.

To prove the global well-posedness for small data, we need the theory for $T = \infty$. Namely. the estimate (5) holds for $T = \infty$. If we assume that $0 \in \rho(A)$, it may be possible to show that (5) holds for $T = \infty$, and so nowardays analysis allows us to prove the global in time unique existence theorem for small initial data for the corresponding nonlinear problem. But, in the unbounded domain case, usually $\rho(A) \not\ni 0$.

Danchin, Hieber, Much and Tolksdorf [6] treated the case where $T = \infty$ replacing $\mathcal{D}_A(\theta,q)$ with homogeneous space $\dot{\mathcal{D}}_A(\theta,q)$. Their theory is so interesting that I quote it here without fear of being mistaken. Refer to [6, Chapter 2] for detailes.

Assume that $\rho(A) \supset \Sigma_\epsilon$ and there exists a constant $C > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq C \quad (\lambda \in \Sigma_\epsilon). \tag{6}$$

In particular, we have

$$\|tAe^{-tA}\|_{\mathcal{L}(X)} \leq M \quad (t > 0). \tag{7}$$

Here and in the sequel, $\mathcal{L}(X)$ denotes the set of all bounded linear operators from X into itself and $\|\cdot\|_{\mathcal{L}(X)}$ denotes the norm of this space.

Assumption 1. *The operator A is injective and there exists a normed vector space Y (not necessarily complete) such that $D(A) \subset Y$ and there exist two constants C_1, $C_2 > 0$ such that*

$$C_1\|Ax\|_X \leq \|x\|_Y \leq C_2\|Ax\|_X \quad (x \in D(A)). \tag{8}$$

In the case where A stands for the Laplace operator on $\mathbb{R}^N_+ = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid x_N > 0\}$, a prominent example of a couple (X,Y) is $X = L_p(\mathbb{R}^N_+)$ and $Y = \dot{W}^2_p(\mathbb{R}^N_+)$ for $1 < p < \infty$. Here,

$$\dot{W}^2_p(\mathbb{R}^N_+) = \{f \mid \exists g \in \mathcal{S}'(\mathbb{R}^N)/P(\mathbb{R}^N) \text{ such that } g|_{\mathbb{R}^N_+} = f, \nabla^2 g \in L_p(\mathbb{R}^N)\},$$
$$\|f\|_{\dot{W}^2_p(\mathbb{R}^N_+)} = \inf\{\|\nabla^2 g\|_{L_p(\mathbb{R}^N)} \mid g \in \mathcal{S}'(\mathbb{R}^N)/P(\mathbb{R}^N) \text{ such that } g|_{\mathbb{R}^N_+} = f, \nabla^2 g \in L_p(\mathbb{R}^N)\},$$

and $\mathcal{P}$ denotes the set of all polynomials on $\mathbb{R}^N$.

Definition 2. If $-A$ satisfies Assumption 7, then define the domain of the homogeneous version $\dot{A}$ of A by

$$D(\dot{A}) := \{y \in Y \mid \text{there exists a sequence } (x_k)_{k\in\mathbb{N}} \subset D(A) \text{ with } \lim_{k\to\infty} x_k = y \text{ in } Y\}.$$

With this definition, define $\dot{A}$ operating to $y \in D(\dot{A})$ by

$$\dot{A}y = \lim_{k\to\infty} Ax_k.$$

Assume that X and Y are interpolation couple. That is, there exists a Hausdorff topological vector space Z such that X and Y are subspaces of Z.

Assumption 3. *The operator A and the normed vector space Y are such that*

$$D(\dot{A}) \cap X = D(A). \tag{9}$$

Let

$$\dot{\mathcal{D}}_A(\theta, q) := \{x \in X + D(\dot{A}) \mid \|x\|_{\dot{\mathcal{D}}_A(\theta,q)} := \Big(\int_0^\infty \|t^{1-\theta}\dot{A}e^{-tA}x\|_X^q \, \frac{\mathrm{d}t}{t}\Big)^{1/q} < \infty\}.$$

We see that for any $\theta \in (0, 1)$ and $1 \leq q \leq \infty$, there holds

$$(X, D(\dot{A}))_{\theta,q} = \dot{D}_A(\theta, q).$$

The homogeneous space version of Da Prato and Grisvard theorem reads as follows.

Theorem 4. *Let $\theta \in (0, 1)$, $1 \leq q \leq \infty$ and $0 < T \leq \infty$. Then, ther exists a constant $C > 0$ such that for all $f \in L_q((0, T), D_A(\theta, q))$ problem* (1) *admits a unique solution u defined by* (4) *such that $u(t) \in D(A)$ for almost every $t \in (0, T)$ and the homogeneous estimate:*

$$\|Au\|_{L_q((0,T),\dot{D}_A(\theta,q))} \leq C\|f\|_{L_q((0,T),\dot{D}_A(\theta,q))}.$$

Proof. Refer to [6, Chapter 2]. □

The Da Prato and Grisvard theorem holds for real interpolation space $D_A(\theta, q)$. For usual Sobolev space like L_p in space, we know L_q in time and L_p in space maximal regularity result for $1 < p, q < \infty$. I will explain the maximal regularity theory for $1 < q < \infty$ below.

For this, first we recall a Dore and Venni theory [12]. Assume that $\rho(A) \supset (0, \infty)$ and $t(t + A)^{-1}$ is bounded in $t > 0$, Moreover, we assume that the pure imaginary powers A^{is} are bounded linear operators and their operator norm is estimated by

$$\|A^{is}\| \leq Ke^{\theta|s|}, \quad s \in \mathbb{R}$$

with some $K \geq 1$ aned θ satisfying $0 \leq \theta\pi/2$, which is independent of s. If A has a bounded inverse, then for $1 < p < \infty$, the maximal L_p regularity holds, that is problem (1) admits a unique solution for given $f \in L_p((0, T), X)$, $0 < T < \infty$, $1 < p < \infty$ such that

$$\int_0^T \|\dot{u}(t)\|_X^p \, \mathrm{d}t + \int_0^T \|Au(t)\|_X^p \, \mathrm{d}t \leq C \int_0^T \|f(t)\|_X^p \, \mathrm{d}t \tag{10}$$

with $C = C(T, p, X)$ provided that X is a UMD space, that is to say such that the Hilbert transform is bounded on $L_p(\mathbb{R}, X)$ for some (all) $p \in (1, \infty)$. Later, Giga and Sohr [17] extended the Dore-Veni theorey to the case where A may not have a bounded inverse and the constant C in (10) is independent of T. Moreover, Weis [36] proved that the maximal L_p regularity holds if and only if A is an $\mathcal{R}$ sectorial operator, that is the set $\{\lambda(\lambda\mathbf{I} - A)^{-1} \mid |\arg\lambda| \leq \pi - \epsilon\}$ is $\mathcal{R}$-bounded for some $\epsilon > 0$. Here, the notion of $\mathcal{R}$ boundedness will be given in the next section. In this sense, for $1 < p < \infty$, the maximal L_p regularity is characterized completely, and it is applied to many problems in a mathematic fluid mechanics. For example, Giga and Sohr [17] proved regularity and large time behaviour of solutions to the Navier-Stokes equations with non-splip condition in exterior domains, which was first prove by Iwashita [21] by extetending the Fujita-Kato method ([22]) to the exterior domain case.

In (10), to take $T = \infty$ is an important problem for application to the nonlinear problem appearing in mathematical fluid mechanics. In the bounded domain case, usually we have $0 \in \rho(A)$, and so the exponential stability of the linearized equations are obtained. But, in the unbounded domain case, it is not the case that $0 \in \rho(A)$, one of the typical method is to combine the local maximal regularity result and some decay properties of semigroup $\{e^{-At}\}_{t\geq0}$ for the semilinear problem case [22, 21]. But, for the quasilinear problem case like free boundary problem for the Navier-Stokes equations, in general the maximal regularity theorem for the evolution equations with non-homogeneous boundary conditions which may not be covered in the maximal regularity theorem for continuous analytic semigroup theorem stated above. We treat the nonhomogeneous boundary condition case in the next chapter.

2. Abstract framework for the nonhomogeneous initial boundary value problem

Let X, Y and Z be three Banach spaces such that $X \subset Z \subset Y$ and the inclusions are continous. Let $A : X \to Y$, $B : X \to Z$, and $W : Z \to Y$ be bounded linear operators. In this section, we consider an evolution equation:

$$\partial_t U - AU = F, \quad BU = G \quad (t > 0), \quad U|_{t=0} = U_0. \tag{11}$$

Here, B is corresponding some boundary conditions for applications to PDE.

We consider the conditions to obtain maximal L_p regularity for the evolution equations (11), that is equations (11) admits a unique solution U having the regularity property:

$$U \in L_p((0,T), X) \cap W^1_p((0,T), Y) \tag{12}$$

as well as the estimate:

$$\begin{aligned} &\|U\|_{L_p((0,T),X)} + \|\partial_t U\|_{L_p((0,T),Y)} \\ &\quad \le C\{\|U_0\|_{(Y,X)_{1-1/p,p}} + \|F\|_{L_p((0,T),Y)} + \|G\|_{W^\alpha_p((0,T),Y)} + \|WG\|_{L_p((0,T),Y)})\}. \end{aligned} \tag{13}$$

Here, $(Y, X)_{\theta,p}$ denotes a real interpolation space, $L_p(((0,T), X)$ is a X valued Lebesgue space, and $W^1_p((0,T), Y)$ a Y valued Sobolev space, and

$$W^\alpha_p((0,T), Z) = B^\alpha_{p,p}((0,T), Z) = (L_p((0,T), Z), W^1_p((0,T), Z))_{\alpha,p} \quad \text{for } \alpha \in (0,1).$$

The L_p norm is defined by

$$\|f\|_{L_p((0,T),X)} = \Big(\int_0^T \|f(t)\|_X \, \mathrm{d}t\Big)^{1/p}, \quad \|e^{-\gamma t} f\|_{L_p((0,\infty),X)} = \Big\{\int_0^\infty (e^{-\gamma t}\|f(t)\|X)^p \, \mathrm{d}t\Big\}^{1/p}.$$

2.1. L_p maximal regularity for $1 < p < \infty$.

In the case where $1 < p < \infty$, we use $\mathcal{R}$-boundedness of solution operators $\mathcal{S}(\lambda)$. First, we give a definition of the $\mathcal{R}$-boundedness of operator families.

Definition 5. Let E and F be two Banach spaces. We say that an operator family $\mathcal{T} \subset \mathcal{L}(E, F)$ is $\mathcal{R}$ bounded if there exist constants $C > 0$ and $q \in [1, \infty)$ such that for any integer n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset E$, the inequality:

$$\int_0^1 \|\sum_{j=1}^n r_j(u) T_j f_j\|_F^q \, \mathrm{d}u \le C \int_0^1 \|\sum_{j=1}^n r_j(u) f_j\|_E^q \, \mathrm{d}u$$

is valid, where the Rademacher functions r_k, $k \in \mathbb{N}$, are given by $r_k : [0,1] \to \{-1, 1\}$; $t \mapsto \mathrm{sign}(\sin 2^k \pi t)$.

The smallest such C is called $\mathcal{R}$ bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(E,F)}\mathcal{T}$.

The detailed explanation of $\mathcal{R}$-boundedness is given in [9, 20].

The reason we introduce the $\mathcal{R}$-boundedness is to use Weis's operator valued Fourier multiplier theorem. For $m(\xi) \in L_\infty(\mathbb{R} \setminus \{0\}, \mathcal{L}(E, F))$, we set

$$T_m f = \mathcal{F}^{-1}_\xi[m(\xi)\mathcal{F}[f](\xi)] \quad f \in \mathcal{S}(\mathbb{R}, E),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}_\xi$ denote respective Fourier transformation and inverse Fourier transformation defined by

$$\mathcal{F}[f](\tau) = \mathcal{F}_\mathbb{R}[f](\tau) := \int_\mathbb{R} e^{i\tau t} f(t) \, \mathrm{d}t, \quad \mathcal{F}^{-1}[f](t) = \mathcal{F}^{-1}_\mathbb{R}[f](t) := \frac{1}{2\pi}\int_\mathbb{R} e^{-i\tau t} f(\tau) \, \mathrm{d}\tau.$$

To emphasize the Fourier transform and its inverse transform defined on $\mathbb{R}$, we use $\mathbb{R}$ as a subscript. T_m is called an operator valued Fourier multiplier.

To state Weis' theorem, we introduce an UMD space. A Banach space X is an **UMD space** if the Hilbert transform is bounded on $L_p(\mathbb{R}, X)$ for some $p \in (1, \infty)$ cf. [2, Sec.4.4] and [19, Chapter 4]. For example, for $1 < q < \infty$, the Lebesgue spaces L_q are UMD spaces. Since the subspaces of UMD spaces are also UMD spaces, and so for example the Sobolev spaces W^m_q are UMD spaces.

Theorem 6 (Weis's operator valued Fourier multiplier theorem)**.** *Let E and F be two UMD Banach spaces. Let $m(\xi) \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(E, F))$ and assume that*

$$\mathcal{R}_{\mathcal{L}(E,F)}(\{m(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) \leq r_b$$
$$\mathcal{R}_{\mathcal{L}(E,F)}(\{\xi m'(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) \leq r_b$$

with some constant $r_b > 0$. Then, for any $p \in (1, \infty)$, $T_m \in \mathcal{L}(L_p(\mathbb{R}, E), L_p(\mathbb{R}, F))$ and

$$\|T_m f\|_{L_p(\mathbb{R},F)} \leq C_p r_b \|f\|_{L_p(\mathbb{R},E)}$$

with some constant C_p depending solely on p.

Proof. For a proof, refer to L. Weis [36]. □

To obtain L_p maximal regularity for equations (11), we use $\mathcal{R}$ bounded solution operators of the corresponding generalized resolvent problem:

$$\lambda u - Au = f, \quad Bu = g. \tag{14}$$

For (14), we introduce the following assumption.

Assumption 7. *There exist constants $\epsilon \in (0, \pi/2)$ and $\gamma \geq 0$ such that for every $\lambda = \gamma + i\tau \in \Sigma_\epsilon + \gamma$, there exists an operator*

$$\mathcal{S}(\lambda) : Y \times Y \times Y \to X \quad (F_1, F_2, F_3) \mapsto \mathcal{S}(\lambda)(F_1, F_2, F_3)$$

satisfying the following three conditions:

(1) *$\mathcal{S}(\lambda)$ is an $\mathcal{L}(Y \times Y \times Y, X)$ valued holomorphic function defined on $\Sigma_\epsilon + \gamma$.*
(2) *For $\lambda \in \Sigma_\epsilon + \gamma$, $f \in Y$ and $g \in Z$, $u = \mathcal{S}(\lambda)(f, \lambda^\alpha g, Wg)$ is a unique solution of* (22)
(3) *$\mathcal{S}(\lambda)$ satisfies*

$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,X)}(\{(\tau\partial_\tau)^\ell \mathcal{S}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\} \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,Y)}(\{(\tau\partial_\tau)^\ell (\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\} \leq r_b,$$

for $\ell = 0, 1$ with some constant r_b.

$\mathcal{S}(\lambda)$ is called an $\mathcal{R}$ **bounded solution operator***, or $\mathcal{R}$* **solver***, for problem* (14).

Remark 8. For the concrete problem, the exponent α is related to the following requirement: For $f \in W^1_p((0, T), Y) \cap L_p((0, T), X)$, $f \in W^\alpha_p((0, T), Y)$ and

$$\|f\|_{W^\alpha_p((0,T),Z)} \leq C(\|f\|_{W^1_p((0,T),Y)} + \|f\|_{L_p((0,T),X)}).$$

Since the $\mathcal{R}$ boundedness implies the usual boundedness, we have

$$\|\lambda u\|_Y + \|u\|_X \leq r_b(\|f\|_Y + \|\lambda^\alpha g\|_Y + \|Wg\|_Y). \tag{15}$$

This estimate is corresponding to the Agranovich-Visik type esitmate for the mixed problem of the parabolic equations [3] and the Sakamoto type estimate for the mixed problem of the hyperbolic equations [28, 29]

The simple example to catch the situation, that comes to author's mind, is the generalized resolvent problem for the heat equation with Neumann boundary condition, which reads

$$\lambda u - \Delta u = f \quad \text{in } \Omega, \quad \nu \cdot \nabla u = g \quad \text{on } \partial\Omega.$$

Here, Ω is a C^2 domain in the N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ and $\partial\Omega$ is its boundary. ν denotes the unit outer normal to $\partial\Omega$ and $\nabla = (\partial_1, \ldots, \partial_N)$.

In this case, a prominent choice of solution spaces are

$$X = W^2_q(\Omega), \ \ Z = W^1_q(\Omega), \ \ Y = L_q(\Omega),$$

where $1 < q < \infty$. Moreover, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$, $B = \nu \cdot \nabla$, and $W = \nabla$.

We now prove the unique existence of solutions of equations (11) under Assumption 7.

First Step. Forget the initial conditions and consider the following equations:

$$\partial_t V - AV = F, \quad BV = G \quad (t \in \mathbb{R}). \tag{16}$$

Let γ be the constant appearing in Assumption 7. Let F and G satisfy the conditions: $e^{-\gamma t}F \in L_p(\mathbb{R}, Y)$ and $e^{-\gamma t}G \in W^\alpha_p(\mathbb{R}, Y) \cap L_p(\mathbb{R}, Z)$. Applying the Laplace transform with respect to time variable t implies that

$$\lambda v - Av = \hat F, \quad Bv = \hat G.$$

Here,

$$\hat H = \mathcal{L}[H](\lambda) = \mathcal{F}[e^{-\gamma t}H](\tau) = \int_{\mathbb{R}} e^{-\lambda t} H(t)\ \mathrm{d}t \quad (\lambda = \gamma + i\tau \in \Sigma_\epsilon + \gamma).$$

Thus, by Assumption 7, we have $v = \mathcal{S}(\lambda)(\hat F, \lambda^\alpha \hat G, W\hat G)$. Let $\mathcal{L}^{-1}$ denote the Laplace inverse transform defined by

$$\mathcal{L}^{-1}[J](t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\lambda t} J(\tau)\ \mathrm{d}\tau = e^{\gamma t}\mathcal{F}^{-1}[J](t).$$

Let Λ^α_γ be fractional derivative defined by

$$\Lambda^\alpha_\gamma G = \mathcal{L}^{-1}[\lambda^\alpha \mathcal{L}[G](\lambda)].$$

We know that

$$\|e^{-\gamma t}\Lambda^\alpha_\gamma G\|_{L_p(\mathbb{R}, Y)} \leq C\|e^{-\gamma t}G\|_{W^\alpha_p(\mathbb{R}, Y)}. \tag{17}$$

Applying Laplace inverse transformation, we define V by

$$V = \mathcal{L}^{-1}[\mathcal{S}(\lambda)(\hat F, \lambda^\alpha \hat G, W\hat G)] = \frac{\mathrm{e}^{\gamma t}}{2\pi}\int_{\mathbb{R}} e^{i\tau t}\mathcal{S}(\lambda)\mathcal{F}[\mathrm{e}^{-\gamma t}(F, \Lambda^\alpha G, WG)](\tau)\,\mathrm{d}\tau.$$

and so

$$e^{-\gamma t}V = \mathcal{F}^{-1}_\tau[\mathcal{S}(\gamma + i\tau)\mathcal{F}[\mathrm{e}^{-\gamma t}(F, \Lambda^\alpha G, WG)](\tau)].$$

Thus, applying the Weis operator valued Fourier multiplier theorem implies the following theorem

Theorem 9. *Let $1 < p < \infty$. Let Assumption 7 hold. Then, for any F and G satisfying the conditions:*

$$e^{-\gamma t}F \in L_p(\mathbb{R}, Y), \quad e^{-\gamma t}G \in L_p(\mathbb{R}, Z) \cap W^\alpha_p(\mathbb{R}, Y) \tag{18}$$

problem (16) admits a solution V satisfying the regularity condition:

$$e^{-\gamma t}V \in L_p(\mathbb{R}, X) \cap W^1_p(\mathbb{R}, Y)$$

as well as the estimate:

$$\begin{aligned} &\|e^{-\gamma t}\partial_t V\|_{L_p(\mathbb{R}, Y)} + \|e^{-\gamma t}V\|_{L_p(\mathbb{R}, X)} \\ &\leq C(\|e^{-\gamma t}F\|_{L_p(\mathbb{R}, Y)} + \|e^{-\gamma t}G\|_{W^\alpha_p(\mathbb{R}, Y)} + \|e^{-\gamma t}WG\|_{L_p(\mathbb{R}, Y)}) \end{aligned} \tag{19}$$

for some constant C independent of γ.

Second Step. Next step is to solve initial value problem:

$$\partial_t W + AW = 0, \quad BW = 0 \quad \text{for } t > 0, \quad W|_{t=0} = U_0 - V|_{t=0}. \tag{20}$$

Since $V \in W^1_p(\mathbb{R}, Y) \cap L_p(\mathbb{R}, X)$, by the trace method of the real interpolation we see that

$$\begin{aligned}\sup_{t\in\mathbb{R}} e^{-\gamma t}\|V(\cdot,t)\|_{(Y,X)_{1-1/p,p}} &\le C(\|e^{-\gamma t}\partial_t V\|_{L_p(\mathbb{R},Y)} + \|e^{-\gamma t}V\|_{L_p(\mathbb{R},X)}) \\ &\le C(\|e^{-\gamma t}F\|_{L_p(\mathbb{R},Y)} + \|e^{-\gamma t}G\|_{W^\alpha_p(\mathbb{R},Y)} + \|e^{-\gamma t}WG\|_{L_p(\mathbb{R},Y)}).\end{aligned} \tag{21}$$

For simplicity, we set $W_0 = U_0 - V|_{t=0}$. We consider the resolvent problem:

$$\lambda w + Aw = f, \quad Bw = 0. \tag{22}$$

By Assumption 7, we know the unique existence of solutions to equations (22), that is for any $\lambda \in \Sigma_\epsilon + \gamma$ and $f \in Y$, problem (22) admits a unique solution $w \in X$ satisfying the estimate:

$$\|\lambda w\|_Y + \|w\|_X \le C\|f\|_Y. \tag{23}$$

Let $D(\mathcal{A})$ and $\mathcal{A}$ be defined by

$$D(\mathcal{A}) = \{w \in X \mid Bw = 0\}, \quad \mathcal{A}w = Aw \quad \text{for } w \in D(\mathcal{A}). \tag{24}$$

By using $\mathcal{A}$, problem (20) is rewriten as

$$\partial_t W + \mathcal{A}W = 0, \quad W|_{t=0} = W_0. \tag{25}$$

In view of (23), the operator $\mathcal{A}$ generates a continuous analytic semigroup on Y such that

1. $T(t)W_0 \in C^0([0,\infty), Y) \cap C^1((0,\infty), Y) \cap C^0((0,\infty), D(A))$
2. $W = T(t)W_0$ is a unique solution of equations (25).
3. $\lim_{t\to 0+} \|T(t)W_0 - W_0\|_Y = 0$ for any $W_0 \in Y$.
4. $\|T(t)W_0\|_Y \le Ce^{\gamma t}\|W_0\|_Y$, for any $t > 0$ and $W_0 \in Y$.
5. $\|T(t)W_0\|_X + \|\partial_t T(t)W_0\|_Y \le Ce^{\gamma t}t^{-1}\|W_0\|_Y$ for any $t > 0$ and $W_0 \in Y$,
6. $\|T(t)W_0\|_X + \|\partial_t T(t)W_0\|_Y \le Ce^{\gamma t}\|W_0\|_Y$ for any $t > 0$ and $W_0 \in X$.

Combining estimates (5) and (6) above with real interpolation method implies that

$$\Big\{\int_0^\infty (e^{-\gamma t}(\|\partial_t W(t)\|_Y + \|W(t)\|_X)^p \, \mathrm{d}t\Big\}^{1/p} \le C\|W_0\|_{(Y,X)_{1-1/p,p}}. \tag{26}$$

Let $\mathcal{D}_(\mathcal{A})_p = (Y, D(\mathcal{A}))_{1-1/p,p}$. And then, we have the following maximal regularity theorem for equations (25).

Theorem 10. *Let $1 < p < \infty$. Assume that Assumption 7 hold. Then, for any $W_0 \in \mathcal{D}_p(\mathcal{A})$ problem* (25) *admits a unique solution W satisfying the estimate* (26).

If we set $U = V + W$, then by Theorems 9 and 10 U may be a solution of equations (11) provided that $W_0 = U_0 - V|_{t=0} \in \mathcal{D}_p(\mathcal{A})$. Since $BV|_{t=0} = G|_{t=0}$, the compatibility condition is $BU_0 - G|_{t=0} \in \mathcal{D}_p(\mathcal{A})$. Summing up, we have proved the following theorem.

Theorem 11. *Let $1 < p < \infty$ and assume that Assumption 7 hold. Then for any initial data $U_0 \in (Y,X)_{1-1/p,p}$ satisfying the compatibility condition: $BU_0 - G|_{t=0}\mathcal{D}_p(\mathcal{A})$ and right hand side F and G satisfying the conditions:*

$$e^{-\gamma t}F \in L_p(\mathbb{R}, Y), \quad e^{-\gamma t}G \in L_p(\mathbb{R}, Z) \cap W^\alpha_p(\mathbb{R}, Y)$$

problem (11) *admits a unique solution U satisfying the regularity condition:*

$$e^{-\gamma t}U \in L_p((0,\infty), X) \cap W^1_p((0,\infty), Y)$$

as well as the estimate:

$$\begin{aligned}&\|e^{-\gamma t}U\|_{L_p((0,\infty),X)} + \|e^{-\gamma t}\partial_t U\|_{L_p((0,\infty),Y)} + \sup_{t\in(0,\infty)} e^{-\gamma t}\|U(t)\|_{(Y,X)_{1-1/p,p}} \\ &\quad \le C(\|U_0\|_{(Y,X)_{1-1/p,p}} + \|e^{-\gamma t}F\|_{L_p(\mathbb{R},Y)} + \|e^{-\gamma t}G\|_{W^\alpha_p(\mathbb{R},Y)} + \|e^{-\gamma t}WG\|_{L_p(\mathbb{R},Y)}).\end{aligned}$$

2.2. 2π **periodic solutions.** We next consider 2π time peridic boundary problem

$$\partial_t w - Aw = F, \quad Bw = G \quad \text{for } t \in \mathbb{R}. \tag{27}$$

We assume that $F(t) = F(t + 2\pi)$ and $G(t) = G(t + 2\pi)$. Of couse, we can consider a general time period $T > 0$, but for the notational simplicity, we only consider the 2π period case.
Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{F}_{\mathbb{T}}$ and $\mathcal{F}_{\mathbb{T}}^{-1}$ be Fourier transform on $\mathbb{T}$ and its inverse transform defined by

$$\mathcal{F}_{\mathbb{T}}[f](\tau) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau t} f(t)\,dt, \quad \mathcal{F}_{\mathbb{T}}^{-1}[g](t) = \sum_{k\in\mathbb{Z}} e^{ikt} g(k).$$

To prove the L_p maximal regularity for periodic solutions, we shall use the operator valued de Leevw transference principle ([23]), which is stated as follows. Let

$$T_{m,\mathbb{R}} f = \mathcal{F}_{\mathbb{R}}^{-1}[m(\xi)\mathcal{F}_{\mathbb{R}}[f](\xi)]$$

be an operator valued Fourier multiplier on $\mathbb{R}$, where $m(\xi) \in \mathcal{L}(X, Y)$ for each $\xi \in \mathbb{R} \setminus \{0\}$. We consider the corresponding multiplier on $\mathbb{T}$ defined by

$$T_{(m_k)_{k\in\mathbb{T}}} f = \mathcal{F}_{\mathbb{T}}^{-1}[m(k)\mathcal{F}_{\mathbb{T}}[f](k)] = \sum_{k\in\mathbb{Z}} e^{ikt} m(k)\mathcal{F}_{\mathbb{T}}[f](k) \quad (k \in \mathbb{Z}).$$

Then, we have the following theorem.

Theorem 12. *Let X and Y be two Banach spaces, and $1 < p < \infty$. Let $m \in L_\infty(\mathbb{R}, \mathcal{L}(X, Y))$ be a Fourier multiplier from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Suppose that for all $x \in X$ the point $k \in \mathbb{Z}$ is a Lebesgue point of $\xi \to m(\xi)x$, and set $m_k x := m(k)x$. Then, $(m_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $L_p(\mathbb{T}, X)$ to $L_p(\mathbb{T}, Y)$, and in fact*

$$\|T_{(m_k)_{k\in\mathbb{Z}}}\|_{\mathcal{L}(L_p(\mathbb{T},X), L_p(\mathbb{T},Y))} \le \|T_m\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))}.$$

Proof. For a proof, refer to [20, Proposition 5.7.1]. □

We assume that Assuption 7 holds. Let $\lambda_0 > 0$ be a large number such that

$$\{i\xi \mid |\xi| \ge \lambda_0, \xi \in \mathbb{R}\} \subset \Sigma_\epsilon + \gamma,$$

where $\Sigma_\epsilon + \gamma$ is the same set as in Assumption 7 (2). Let $\varphi(t) \in C^\infty(\mathbb{R})$ which equals 1 for $|t| \ge \lambda_0 + 1$ and 0 for $|t| \le \lambda_0 + 1/2$ and set

$$W_1 = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi(\xi)\mathcal{S}(i\xi)\mathcal{F}_{\mathbb{R}}[(F, \Lambda^\alpha G, WG)](\xi),$$
$$w_1 = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(\tau)\mathcal{S}(i\tau)\mathcal{F}_{\mathbb{T}}[(F, \Lambda^\alpha G, WG](i\tau)].$$

From Assumption 7, we have

$$\|\partial_t W_1\|_{L_p(\mathbb{R},Y)} + \|W_1\|_{L_p(\mathbb{R},X)} \le C(\|F_\varphi\|_{L_p(\mathbb{R},Y)} + \|\Lambda_\varphi^\alpha G\|_{L_p(\mathbb{R},Y)} + \|WG_\varphi\|_{L_p(\mathbb{R},Y)}).$$

Here, we have set $H_\varphi = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi(\xi)\mathcal{F}_{\mathbb{R}}[H](\xi)]$, and

$$\Lambda_\varphi^\alpha G = \mathcal{F}_{\mathbb{R}}[\varphi(\xi)\mathcal{F}_{\mathbb{R}}[\Lambda^\alpha G](\xi)] = \mathcal{F}_{\mathbb{R}}[\lambda^\alpha \mathcal{F}_{\mathbb{R}}[\mathcal{F}_{\mathbb{R}}^{-1}[\varphi\mathcal{F}_{\mathbb{R}}[G]]] = \Lambda^\alpha G_\varphi.$$

Then, by Theorem 12 we have

$$\|\partial_t w_1\|_{L_p(\mathbb{T},Y)} + \|w_1\|_{L_p(\mathbb{T},X)} \le C(\|F_\varphi\|_{L_p(\mathbb{T},Y)} + \|G_\varphi^\alpha\|_{L_p(\mathbb{T},Y)} + \|WG_\varphi\|_{L_p(\mathbb{T},Y)}).$$

Here, we have set $H_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(\tau)\mathcal{F}_{\mathbb{T}}[H](i\tau)]$, and

$$G_\varphi^\alpha = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(k)\mathcal{F}_{\mathbb{T}}[\Lambda^\alpha G](ik)] = \mathcal{F}_{\mathbb{T}}^{-1}[\lambda^\alpha \mathcal{F}_{\mathbb{T}}[\mathcal{F}_{\mathbb{T}}^{-1}[\varphi\mathcal{F}_{\mathbb{T}}[\mathbb{G}]]] = \Lambda^\alpha G_\varphi.$$

Thus, the problem is reduced to show the existence of finite number of solutions v_k of equations:

$$i\sigma w + Aw = \mathcal{F}_{\mathbb{T}}[F](i\sigma), \quad Bw = \mathcal{F}_{\mathbb{T}}[G](i\sigma). \tag{28}$$

And then,

$$w = w_1 + \sum_{|k| \leq \lambda_0 + 1/2} e^{ikt} v_k$$

is a solution of (27).

Summing up, we have obtained the following theorem.

Theorem 13. *Let X, Y, and Z be UMD spaces and let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(X, Z) \cap \mathcal{L}(Z, Y)$. Assume that Assumption 7 holds. Let $\Sigma_\epsilon + \gamma$ be the set in Assumption 7 and let $\lambda_0 > 0$ be a number such that $\{i\xi \mid |\xi| \geq \lambda_0, \xi \in \mathbb{R}\} \subset \Sigma_\epsilon + \gamma$..*

Moreover, for $k \in \mathbb{Z}$ with $|k| \leq \lambda_0$, let $\mathcal{X}_k$, $\mathcal{Y}_k$ and $\mathcal{Z}_k$ be Banach spaces such that $(ik\mathbf{I}+A)^{-1} \in \mathcal{L}(\mathcal{X}_k, \mathcal{Y}_k)$, $B \in \mathcal{L}(\mathcal{X}_k, \mathcal{Z}_k)$, and such that for all $(f, g) \in \mathcal{Y}_k \times \mathcal{Z}_k$ there exists a unique solution $w \in \mathcal{X}_k$ to (28) with $\sigma = k$ such that

$$\|w\|_{\mathcal{X}_k} \leq C_k(\|f\|_{\mathcal{Y}_k} + \|g\|_{\mathcal{Z}_k})$$

for some constant $C_k > 0$.

Then, for any $p \in (1, \infty)$ and (F, G) defined by

$$F(t) = \sum_{k=-k_0}^{k_0} F_k e^{ikt} + F_\varphi(t), \quad G(t) = \sum_{k=-k_0}^{k_0} G_k e^{ikt} + G_\varphi(t)$$

with $(F_k, G_k) \in \mathcal{Y}_k \times \mathcal{Z}_k$ for $k \in \mathbb{Z}$, $|k| \leq k_0$, and $(F_\varphi, G_\varphi) \in L_p(\mathbb{T}, Y) \times (L_p(\mathbb{T}, Z) \cap W^\alpha_p(\mathbb{T}, Y))$ such that $(\mathcal{F}_\mathbb{T}[F_\varphi](k), \mathcal{F}_\mathbb{T}[G_\varphi](k)) = 0$ for all $|k| \leq k_0$, there exists a unique element

$$(u_{-k_0}, \ldots, u_{k_0}, u_\varphi) \in \mathcal{X}_{-k_0} \times \cdots \times \mathcal{X}_{k_0} \times (L_p(\mathbb{T}, X) \cap W^1_p(\mathbb{T}, Y))$$

with $\mathcal{F}_\mathbb{T}[u_\varphi](k) = 0$ for $|k| \leq k_0$, such that

$$u(t) := \sum_{k=-k_0}^{k_0} u_k e^{ikt} + u_\varphi$$

is a unique solution to time-periodic problem (27), and

$$\|u_k\|_{\mathcal{X}_k} \leq C_k(\|F_k\|_{\mathcal{Y}_k} + \|G_k\|_{\mathcal{Z}_k}),$$
$$\|u_\varphi\|_{L_p(\mathbb{T},X) \cap W^1_p(\mathbb{R},Y)} \leq Cr_b(\|F_\varphi\|_{L_p(\mathbb{T},Y)} + \|G_\varphi\|_{W^\alpha_p(\mathbb{T},Y)} + \|WG_\varphi\|_{L_p(\mathbb{T},Y)})$$

for some constants C_k and C.

Remark 14. I do not give any concreat example for periodic solutions to Stokes equations and Navier-Stokes equations in the following sections. I want to mention my joint papers, co-authored with Thomas Either and Mads Kyed, and co-authored solely with Thomas Either, about periodic solutions for the initial-boundary problems for the Navier-Stokes equations.

(1) In [13], we proved the unique existence of time periodic solutions of the one-phase and the two phase problem for the Navier-Stokes equations in bounded domains. We took the surface tension into account. We used the coordinate system whose center is the center of gravity. The incompressiblity guarantees that the center of gravity does not move, and so the free boundary can be writtne as unknown functions in this coordinate system and the surface tension gives us enough regularity of the functions describing the free surface. Thus, we can use our $\mathcal{R}$-solver approach to this problem.

A difficult problem for us to solve among time periodic problems is the free boundary problem without surface tension. In fact, if we represent the unknown surfce by using some functions, we do not obtain enough regularty of this functions, and so far, we could not prove the existence of periodic solutions in our $\mathcal{R}$-solver approach.

By the way, in the evolution problem case for the free boundary problem without surface tension, we use the Lagrange transformation which will be explained later sections below. In this case, the free surface is transformed to the boundary of the reference domain, and so we do not have such difficulty. But, so far we find some difficulty to use the Lagrange transformation to treat the free surface problem without surface tension, which should be solved in the future work.

(2) In [14], we proved the existence of time periodic solutions for the Navier-Stokes equations with non-slip conditions in bounded domains and exterior domains by using the $\mathcal{R}$ solver approach.

(3) In [15], we proved the unique existence theorem of solutions for the boundary value problems for the Navier-Stokes equations with non-slip boundary conditions in bounded or exterior domains, whose boundary is time periodically moving, by using the $\mathcal{R}$ solver approach.

In [14] and [15], our essential contribution was that we treated the exterior domains. In this case, our $\mathcal{R}$ solver approach is quite effective, and we can reduce the difficulty to analysis of the finite number of spectral problems in exterior domains.

2.3. **L_1-maximal regularity.** In this subsection, we discuss the L_1 maximal regularity for equations (11). Unlike the L_p case, we can not use the operator valued Fourier multiplier theorem, and so instead of the operator valued Fourier multiplier with respect to time variable, we use some combination of complex and real interpolation methods.

To obtain L_1 maximal regularity for equations (11), we also consider the corresponding generalized resolvent problem:

$$\lambda u - Au = f, \quad Bu = g. \tag{29}$$

For (29), we introduce the following assumption.

Assumption 15. *There exist constants $\epsilon \in (0, \pi/2)$ and $\gamma \geq 0$ such that for every $\lambda = \gamma + i\tau \in \Sigma_\epsilon + \gamma$, there exists an operator*

$$\mathcal{S}(\lambda) : Y \times Y \times Y \to X \quad (F_1, F_2, F_3) \mapsto \mathcal{S}(\lambda)(F_1, F_2, F_3)$$

satisfying the following four conditions:

(1) *$\mathcal{S}(\lambda)$ is an $\mathcal{L}(Y \times Y \times Y, X)$ valued holomorphic function defined on $\Sigma_\epsilon + \gamma$.*

(2) *For $\lambda \in \Sigma_\epsilon + \gamma$, $f \in Y$ and $g \in Z$, $u = \mathcal{S}(\lambda)(f, \lambda^\alpha g, Wg)$ is a unique solution of (22)*

(3) *$\mathcal{S}(\lambda)$ satisfies the generalized resolvent estimate:*

$$\|\lambda \mathcal{S}(\lambda) F\|_Y + \|\mathcal{S}(\lambda) F\|_X \leq C\|F\|_{Y \times Y \times Y} \tag{30}$$

for every $\lambda \in \Sigma_\epsilon + \gamma$ with some constant $C > 0$.

Moreover, there exist two small numbers $\sigma_i \in (0, 1)$ and two triples of Banach sapces $Y_{\sigma_i} \times Y_{\sigma_i} \times Y_{\sigma_i}$ $(i = 1, 2)$ such that

$$\|\lambda \mathcal{S}(\lambda) F\|_Y + \|\mathcal{S}(\lambda) F\|_X \leq C|\lambda|^{-\sigma_1}\|F\|_{Y_{\sigma_1} \times Y_{\sigma_1} \times Y_{\sigma_1}} \quad \text{for } F \in Y_{\sigma_1}^3, \tag{31}$$

$$\|\partial_\lambda(\lambda \mathcal{S}(\lambda) F)\|_Y + \|\partial_\lambda \mathcal{S}(\lambda) F\|_X \leq C|\lambda|^{-\sigma_2}\|F\|_{Y_{\sigma_2} \times Y_{\sigma_2} \times Y_{\sigma_2}} \quad \text{for } F \in Y_{\sigma_2}^3 \tag{32}$$

with some constant C.

(4) *Let $\theta \in (0, 1)$ be a number satisfying the relation : $1 = (1 - \theta)(1 - \sigma_1) + \theta(2 - \sigma_2)$. Then, we assume that $Y = (Y_{\sigma_1}, Y_{\sigma_2})_{\theta,1}$.*

Here, we write $Z^3 = Z \times Z \times Z$ for $Z \in \{Y, Y_{\sigma_1}, Y_{\sigma_2}\}$ for the notational simplicity.

If we consider a generalized resolvent problem for the heat equation with Neumann boundary condition, which reads

$$\lambda u - \Delta u = f \quad \text{in } \Omega, \quad \nu \cdot \nabla u = g \quad \text{on } \partial\Omega,$$

then we choose $Y = B^s_{q,1}$, and $X = B^{s+2}_{q,1}$ for $1 < q < \infty$ and $-1+1/q < s < 1/q$. If Ω is a domain in $\mathbb{R}^N$ whose boundary is a compact C^3 hypersurface, a half-space, or a compactly perturbed half-space, layer, perturbed layer and so on, then we can show the existence of a solver $\mathcal{S}(\lambda)$ satisfying (30). Moreover, for any small positive number σ such that $-1 + 1/q < s - \sigma < s + \sigma < 1/q$, we have (31) with $Y_{\sigma_1} = B^{s+\sigma}_{q,1}$ and $\sigma_1 = \sigma/2$, and (32) with $Y_{\sigma_2} = B^{s-\sigma}_{q,1}$ and $\sigma_2 = 1 - \sigma/2$. Notice that the requirement for the domains comes from the existence of a particion of unity consisting of finite number of smooth functions.

We now prove the unique existence of solutions of equations (11) under Assumption 15.

First Step. As was seen in the L_p case, first we forget the initial conditions and consider the following equatuions:

$$\partial_t V - AV = F, \quad BV = G \quad (t \in \mathbb{R}). \tag{33}$$

We assume that F and G satisfy the conditions: $e^{-\gamma t}F \in L_1(\mathbb{R}, Y)$ and $e^{-\gamma t}G \in W^\alpha_1(\mathbb{R}, Y) \cap L_1(\mathbb{R}, Z)$. Here, $W^\alpha_1(\mathbb{R}, Y) = (L_1(\mathbb{R}, Y), W^1_1(\mathbb{R}, Y))_{\alpha,1}$ and

$$W^\alpha_1((0,T), Y) = \{f \mid \exists\, g \in W^\alpha_1(\mathbb{R}, Y) \text{ such that } g|_{(0,T)} = f\},$$
$$\|f\|_{W^\alpha_1((0,T),Y)} = \inf\{\|g\|_{W^\alpha_1(\mathbb{R},Y)} \mid g \in W^\alpha_1(\mathbb{R}, Y) \text{ such that } g|_{(0,T)} = f\}.$$

We can show that

$$\|e^{-\gamma t}\Lambda^\alpha f\|_{L_1(\mathbb{R},Y)} \leq C\|e^{-\gamma t} f\|_{W^\alpha_1(\mathbb{R},Y)}. \tag{34}$$

where $\Lambda^\alpha f = \mathcal{L}^{-1}[\lambda^\alpha \mathcal{L}[f]]$.

Applying the Laplace transform to (33) in time variable t implies that

$$\lambda v - Av = \hat{F}, \quad Bv = \hat{G}. \tag{35}$$

Here,

$$\hat{H} = \mathcal{L}[H](\lambda) = \mathcal{F}[e^{-\gamma t}H](\tau) = \int_{\mathbb{R}} e^{-\lambda t} H(t)\, \mathrm{d}t \quad (\lambda = \gamma + i\tau \in \Sigma_\epsilon + \gamma).$$

Thus, by Assumption 15, we have $v = \mathcal{S}(\lambda)(\hat{F}, \lambda^\alpha \hat{G}, W\hat{G})$. Let $\mathcal{L}^{-1}$ denote the Laplace inverse transform defined by

$$\mathcal{L}^{-1}[J](t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-R}^{R} e^{\lambda t} J(\tau)\, \mathrm{d}\tau.$$

Let

$$T(t)F = \mathcal{L}^{-1}[\mathcal{S}(\lambda)F](t) \quad \text{for } F = (F_1, F_2, F_3) \in Y^3. \tag{36}$$

Let $\Gamma = \Gamma_+ \cup \Gamma_-$ be a contour in the complex plane $\mathbb{C}$ defined be

$$\Gamma_\pm = \{z = re^{i\pm(\pi-\epsilon)} \mid r \in (0, \infty)\}.$$

Employing the same argument as in the holomorphic semigroup theory ([35]), by (2) in the Assumption 15, we have

$$\begin{aligned} T(t)F &= \frac{1}{2\pi i} \int_{\Gamma+\gamma} e^{\lambda t} \mathcal{S}(\lambda)F\, \mathrm{d}\lambda \quad \text{for } t > 0, \\ T(t)F &= 0 \quad \text{for } t < 0. \end{aligned} \tag{37}$$

Moreover, by (2) in Assumption 15, we have

$$\|\partial_t T(t)F\|_Y + \|T(t)F\|_X \leq Ce^{\gamma t} t^{-1+\sigma_1} \|F\|_{Y^3_{\sigma_1}}. \tag{38}$$

Integration by parts gives

$$T(t)F = -\frac{1}{2\pi i t} \int_{\Gamma+\gamma} e^{\lambda t} \partial_\lambda \mathcal{S}(\lambda)F\, \mathrm{d}\lambda, \quad \partial_t T(t)F = -\frac{1}{2\pi i t} \int_{\Gamma+\gamma} e^{\lambda t} \partial_\lambda(\lambda \mathcal{S}(\lambda)F)\, \mathrm{d}\lambda.$$

Thus, by (3) in Assumtion 15, we have

$$\|\partial_t T(t)F\|_Y + \|T(t)F\|_X \leq Ce^{\gamma t} t^{-2+\sigma_2} \|F\|_{Y^3_{\sigma_2}}. \tag{39}$$

In view of real interpolation theory, by (38), (39) and $Y = (Y_{\sigma_1}, Y_{\sigma_2})_{\theta,1}$, we have

$$\int_0^\infty e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X)\, \mathrm{d}t \leq C\|F\|_{Y^3}. \tag{40}$$

In fact, we write

$$
\begin{aligned}
&\int_0^\infty e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X)\ \mathrm{d}t \\
&\quad = \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X)\ \mathrm{d}t \\
&\quad \le \sum_{j\in\mathbb{Z}} (2^{j+1} - 2^j) \sup_{t\in(2^j,2^{j+1})} e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X) \\
&\quad = \sum_{j\in\mathbb{Z}} 2^j a_j,
\end{aligned}
$$

where we have set

$$
a_j = \sup_{t\in(2^j,2^{j+1})} e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X).
$$

By (38) and (39)

$$
2^{(1-\sigma_1)j} a_j \le C\|F\|_{Y^3_{\sigma_1}}, \quad 2^{(2-\sigma_2)j} a_j \le C.
$$

Let ℓ^m_p be the set of all sequences $(a_j)_{j\in\mathbb{Z}}$ such that

$$
\begin{aligned}
&\|(a_j)_{j\in\mathbb{Z}}\|_{\ell^m_p} = \Big\{\sum_{j\in\mathbb{Z}} (2^{mj} a_j)^p\Big\}^{1/p} \quad \text{for } 1 \le p < \infty, \\
&\|(a_j)_{j\in\mathbb{Z}}\|_{\ell^m_\infty} = \sup_{j\in\mathbb{Z}} 2^{mj} a_j.
\end{aligned}
$$

We know that $\ell^m_p = (\ell^{m_1}_q, \ell^{m_2}_q)_{\theta,p}$ for $1 \le p, q, \le \infty$, $-\infty < m_1 < m < m_2 < \infty$ and $m = (1-\theta)m_1 + \theta m_2$ cf. [5, 5.6.1. Theorem]. Thus, $\ell^1_1 = (\ell^{1-\sigma_1}_\infty, \ell^{2-\sigma_2}_\infty)_{\theta,1}$, where $\theta \in (0,1)$ is satisfied a relation: $1 = (1-\theta)(1-\sigma_1) + \theta(2-\sigma_2)$. From this it follows that

$$
\int_0^\infty e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X)\ \mathrm{d}t \ \le C\|F\|_{(Y_{\sigma_1}, Y_{\sigma_2})^3_{\theta,1}}.
$$

By (4) in Assumption 15, we have $(Y_{\sigma_1}, Y_{\sigma_2})_{\theta,1} = Y$, we have

$$
\int_0^\infty e^{-\gamma t}(\|\partial_t T(t)F\|_Y + \|T(t)F\|_X)\ \mathrm{d}t \le C\|F\|_{Y^3}. \tag{41}
$$

We now consider equations (33). Then, by (35) and Assumption 15 (2), problem (16) admits a solution V defined by

$$
V = \mathcal{L}^{-1}[\mathcal{S}(\lambda)(\mathcal{L}[F], \lambda^\alpha \mathcal{L}[G], W\mathcal{L}[G])] = \mathcal{L}^{-1}[\mathcal{S}(\lambda)(\mathcal{L}[F], \mathcal{L}[\Lambda^\alpha G], \mathcal{L}[WG]]).
$$

Thus, by (36) and $T(t) = 0$ for $t < 0$,

$$
\begin{aligned}
V &= \mathcal{L}^{-1}[\mathcal{S}(\lambda) \int_{\mathbb{R}} e^{-\lambda\tau}(F, \Lambda^\alpha G, WG)\ \mathrm{d}\tau] \\
&= \int_{\mathbb{R}} \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-R}^{R} e^{\lambda(t-\tau)}(F, \Lambda^\alpha G, WG)\ \mathrm{d}\tau \\
&= \int_{-\infty}^{t} T(t-\tau)(F, \Lambda^\alpha G, WG)(\cdot, \tau)\ \mathrm{d}\tau.
\end{aligned}
$$

Thus, by (41) and Fubini's theorem we have

$$\begin{aligned}
&\int_0^\infty e^{-\gamma t}\|V(\cdot,t)\|_X\,\mathrm{d}t\\
&\quad\le \int_0^\infty e^{-\gamma t}\Big\{\int_{-\infty}^t \|T(t-\tau)(F,\Lambda^\alpha G, WG)(\cdot,\tau)\|_X\,\mathrm{d}\tau\Big\}\,\mathrm{d}t\\
&\quad= \int_{-\infty}^\infty \Big\{\int_\tau^\infty e^{-\gamma t}\|T(t-\tau)(F,\Lambda^\alpha G, WG)(\cdot,\tau)\|_X\,\mathrm{d}t\Big\}\,\mathrm{d}\tau\\
&\quad= \int_{-\infty}^\infty e^{-\gamma\tau}\Big\{\int_0^\infty e^{-\gamma t}\|T(t)(F,\Lambda^\alpha G, WG)(\cdot,\tau)\|_X\,\mathrm{d}t\Big\}\,\mathrm{d}\tau\\
&\quad= \int_{-\infty}^\infty e^{-\gamma\tau}\|(F,\Lambda^\alpha G, WG)(\cdot,\tau)\|_Y\,\mathrm{d}\tau.
\end{aligned}$$

To estimate the time derivative, using equations (33) and the assumtion that $A : X \to Y$ is a bounded linear operator, that is $\|Av\|_Y \le C\|v\|_X$ for some constant $C > 0$, we have

$$\begin{aligned}
&\int_0^\infty e^{-\gamma t}\|\partial_t V(\cdot,t)\|_Y\,\mathrm{d}t\\
&\quad\le \int_0^\infty e^{-\gamma t}\|F(\cdot,t)\|_Y\,\mathrm{d}t + \int_0^\infty e^{-\gamma t}\|AV(\cdot,t)\|_Y\,\mathrm{d}t\\
&\quad\le C\int_0^\infty e^{-\gamma t}(\|F(\cdot,t)\|_Y + \|V(\cdot,t)\|_X)\,\mathrm{d}t\\
&\quad\le C\int_{-\infty}^\infty e^{-\gamma t}\|(F(\cdot,t),\Lambda^\alpha G(\cdot,t), WG(\cdot,t))\|_Y\,\mathrm{d}t.
\end{aligned}$$

Summing up, we have proved the following theorem.

Theorem 16. *Assume that Assumption 15 holds. Then, for any F and G satisfying the conditions:*

$$e^{-\gamma t}F \in L_1(\mathbb{R},Y), \quad e^{-\gamma t}G \in L_1(\mathbb{R},Z)\cap W^\alpha_p(\mathbb{R},Y) \tag{42}$$

problem (33) *admits a solution V satisfying the regularity condition:*

$$e^{-\gamma t}V \in L_1(\mathbb{R},X)\cap W^1_p(\mathbb{R},Y)$$

as well as the estimate:

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t V\|_{L_1((0,\infty),Y)} + \|e^{-\gamma t}V\|_{L_1((0,\infty),X)}\\
&\quad\le C(\|e^{-\gamma t}F\|_{L_1(\mathbb{R},Y)} + \|e^{-\gamma t}G\|_{W^\alpha_1(\mathbb{R},Y)} + \|e^{-\gamma t}WG\|_{L_1(\mathbb{R},Y)})
\end{aligned} \tag{43}$$

for some constant C independent of γ.

Second Step. Next step is to solve initial problem:

$$\partial_t W + AW = 0, \quad BW = 0 \quad \text{for } t > 0, \quad W|_{t=0} = U_0 - V|_{t=0}. \tag{44}$$

Since $V \in W^1_1(\mathbb{R},Y)\cap L_1(\mathbb{R},X)$, we see easily that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} e^{-\gamma t}\|V(\cdot,t)\|_Y \le C\|e^{-\gamma t}\partial_t V\|_{L_p(\mathbb{R},Y)}\\
&\qquad\qquad \le C(\|e^{-\gamma t}F\|_{L_1(\mathbb{R},Y)} + \|e^{-\gamma t}G\|_{W^\alpha_p(\mathbb{R},Y)} + \|e^{-\gamma t}WG\|_{L_p(\mathbb{R},Y)}).
\end{aligned} \tag{45}$$

For simplicity, we set $W_0 = U_0 - V|_{t=0}$. We consider the resolvent problem:

$$\lambda w + Aw = f, \quad Bw = 0. \tag{46}$$

By Assumption 15, problem (46) admits a unique solution $w = \mathcal{S}(\lambda)(f,0,0)$ for any $\lambda \in \Sigma_\epsilon + \gamma$ and $f \in Y$, which satisfies the estimate:

$$\|\lambda w\|_Y + \|w\|_X \le C\|f\|_Y. \tag{47}$$

Let $D(\mathcal{A})$ and $\mathcal{A}$ be defined by

$$D(\mathcal{A}) = \{w \in X \mid Bw = 0\}, \quad \mathcal{A}w = Aw \quad \text{for } w \in D(A). \tag{48}$$

As is well-known, $w = (\lambda\mathbf{I}+\mathcal{A})^{-1}f$ for any $\lambda \in \Sigma_\epsilon+\gamma$, and in reality, $(\lambda\mathbf{I}+\mathcal{A})^{-1}f = \mathcal{S}(\lambda)(f, 0, 0)$. Let Γ be the contour given in (37). From the well-known theory of holomorphic semigroup [35], the operator $\mathcal{A}$ generates a continous analytic semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, which is defined by

$$\mathcal{T}(t)f = \int_{\Gamma+\gamma} (\lambda\mathbf{I} + \mathcal{A})^{-1}f \, d\lambda.$$

By (37), we have $\mathcal{T}(t)f = T(t)(f, 0, 0)$. In particular, by (41), we have

$$\int_0^\infty e^{-\gamma t}(\|\partial_t\mathcal{T}(t)f\|_Y + \|\mathcal{T}(t)F\|_X) \, dt \leq C\|f\|_Y. \tag{49}$$

Moreover, by the theory of holomorphic semigroup, we know that $\{\mathcal{T}(t)\}_{t\geq 0}$ satisfies the following properties:

(1) $\mathcal{T}(t)f \in C^0([0, \infty), Y) \cap C^1((0, \infty), Y) \cap C^0((0, \infty), D(A))$.
(2) $\partial_t\mathcal{T}(t)f + \mathcal{A}\mathcal{T}(t)f = 0$ for any $t > 0$ and $f \in Y$.
(3) $\lim_{t\to 0+} \|\mathcal{T}(t)f - f\|_Y = 0$ for any $f \in Y$.
(4) $\|\mathcal{T}(t)f\|_Y \leq Ce^{\gamma t}\|f\|_Y$ for any $t > 0$ and $f \in Y$.
(5) $\|\partial_t\mathcal{T}(t)f\|_Y + \|\mathcal{T}(t)f\|_X \leq Ce^{\gamma t}t^{-1}\|f\|_Y$ for any $t > 0$ and $f \in Y$.
(6) $\|\partial_t\mathcal{T}(t)f\|_Y + \|\mathcal{T}(t)f\|_X \leq Ce^{\gamma t}\|f\|_X$ for any $t > 0$ and $f \in D(\mathcal{A})$.

In particular, $W = \mathcal{T}(t)(U_0 - V|_{t=0})$ satisfies equations (44) as well as estimates:

$$\int_0^\infty e^{-\gamma t}(\|\partial_t W(\cdot, t)\|_Y + \|W(\cdot, t)\|_X) \, dt \leq C\|U_0 - V|_{t=0}\|_Y.$$

Set $U = V + W$. Using (45), (49) and the first step we have the following L_1 maximal regularity theorem for equations (11).

Theorem 17. *Assume that Assumption 15 holds. Then for any initial data $U_0 \in Y$ and right hand side F and G satisfying the conditions:*

$$e^{-\gamma t}F \in L_1(\mathbb{R}, Y), \quad e^{-\gamma t}G \in L_1(\mathbb{R}, Z) \cap W^\alpha_p(\mathbb{R}, Y)$$

problem (11) *admits a unique solution U satisfying the regularity condition:*

$$e^{-\gamma t}U \in L_1((0, \infty), Y) \cap W^1_1((0, \infty), Y)$$

as well as the estimate:

$$\begin{aligned}
&\|e^{-\gamma t}U\|_{L_1((0,\infty),X)} + \|e^{-\gamma t}\partial_t U\|_{L_1((0,\infty),Y)} + \sup_{t\in(0,\infty)} e^{-\gamma t}\|U(t)\|_Y \\
&\quad \leq C(\|U_0\|_Y + \|e^{-\gamma t}F\|_{L_1(\mathbb{R},Y)} + \|e^{-\gamma t}G\|_{W^\alpha_1(\mathbb{R},Y)} + \|e^{-\gamma t}WG\|_{L_p(\mathbb{R},Y)}).
\end{aligned}$$

3. Free boundary problem for the Navier Stokes equations in the L_p-L_q maximal regularity framework

In this section and the next section, we consider free boundary problem for the Navier-Stokes equations. The mathematical problem for the free boundary problem of the Navier-Stokes equations is to find a time dependent domain Ω_t, t being time variable, in the N-dimensional Euclidean space $\mathbb{R}^N$, the velocity field $\mathbf{v}(x, t) = (v_1(x, t), \ldots, v_N(x, t))$, and the pressure field

$\mathfrak{p}(x,t)$ satisfying the NavierStokes equations in Ω_t with free boundary conditions, which reads

$$
(50)\qquad \begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t - \mathfrak{p}_0\mathbf{n} & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ V_n = \mathbf{v}\cdot\mathbf{n}_t & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \Omega_t|_{t=0} = \Omega \end{cases}
$$

Here, Γ_t is the boundary of Ω_t, $\mathbf{n}_t = (n_{t1},\ldots,n_{tN})$ the unit outer normal to Γ_t, $\partial_t = \partial/\partial t$, $\mathbf{v}_0 = (v_{01},\ldots,v_{0N})$ a given initial data, Ω the reference domain, $\mathbf{D}(\mathbf{v}) = (D_{ij}(\mathbf{v})) = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top$ the doubled deformation tensor, $\mathbf{I}$ the $N\times N$ identity matrix, $H(\Gamma_t)$ the $N-1$ fold mean curvature of Γ_t, which is given by $H(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t} x$ with $x\in\Gamma_t$, Δ_{Γ_t} being the Laplace-Beltrami operator on Γ_t, V_n the evolution speed of free surface Γ_t in the direction $\mathbf{n}_t$, $\mathfrak{p}_0$ the outside pressure, and μ and σ are positive constants representing respective the viscous coefficient and the coefficient of the surface tension. Moreover, for any matrix field $\mathbf{K} = (K_{ij})$, $\mathrm{Div}\,\mathbf{K}$ denotes the N-vector of functions whose i^{th} component is $\sum_{j=1}^N D_j K_{ij}$, $D_j = \partial/\partial x_j$. For any N-vector of function $\mathbf{v}$, $\mathrm{div}\,\mathbf{v} = \sum_{j=1}^N D_j v_j$ and $\mathbf{v}\cdot\nabla\mathbf{v}$ denotes the N vector of functions whose i^{th} component is $\sum_{j=1}^N v_j D_j v_i$.

In particular, the i^{th} component of equations (50) reads as

$$
(51)\qquad \begin{cases} \partial_t v_i + \sum_{j=1}^N v_j D_j v_i - \sum_{j=1}^N \mu D_j D_{ij}(\mathbf{v}) + D_i\mathfrak{p} = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ \sum_{j=1}^N D_j v_j = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ \sum_{j=1}^N \mu D_{ij}(\mathbf{v}) n_{tj} - \mathfrak{p} n_{ti} = \sigma H(\Gamma_t)\mathbf{n}_{ti} - \mathfrak{p}_0 n_{ti} & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ V_n = \sum_{j=1} v_j n_{tj} = 0 & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ v_i|_{t=0} = v_{0i} \quad \text{in } \Omega, \quad \Omega_t|_{t=0} = \Omega. \end{cases}
$$

Concerning the outside pressure $\mathfrak{p}_0$, the equilibrium state is that $\mathbf{v} = 0$ and so from the first equation it follows that $\nabla\mathfrak{p} = 0$, that is $\mathfrak{p}$ is constant. Moreover, $\mathbf{n}_t = \mathbf{n}$ and $\sigma H(\Gamma_t) = \sigma H(\Gamma)$ for any $t \geq 0$. Here, Γ is the boundary of the reference domain Ω and $\mathbf{n}$ is the unit outer normal to Γ. Thus,

$$
\mathfrak{p}_0 = \sigma H(\Gamma) + \mathfrak{p}.
$$

In this note, we consider the simplest problem, that is the $\sigma = 0$ case, without surface tension problem. In this case, we set $\mathfrak{p} - \mathfrak{p}_0 = \mathfrak{q}$, which is the new pressure field. Namely, we consider the following problem:

$$
(52)\qquad \begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = 0 & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ V_n = \mathbf{v}\cdot\mathbf{n}_t & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \Omega_t|_{t=0} = \Omega & \end{cases}
$$

Since Ω_t is unknown, the first step to solve (52) is to transform Ω_t to some known domain. To this end, we use the Lagrange transform. Let $y = (y_1, \dots, y_N)$ be Lagrange coordinates and let $X = X(y,t)$ be a solution to the ordinary differential equtions:

$$
\frac{dX}{dt} = \mathbf{v}(X,t) \quad \text{for } t > 0, \quad X(y,0) = y.
$$

If we define $\mathbf{u}(y,t)$, the Lagrange velocity field, by $\mathbf{u}(y,t) = \mathbf{v}(X(y,t),t)$, then

$$
(53)\qquad x = \mathbf{X}_{\mathbf{u}}(y,t) = y + \int_0^t \mathbf{u}(y,s)\, ds.
$$

which is called the Lagrange transform and this map gives the correspondence between Euler coordinate system $x = (x_1, \dots, x_N) \in \Omega_t$ and Lagrange coordinate system $y = (y_1, \dots, y_N) \in \Omega$. Since $\dfrac{dX}{dt}\cdot\mathbf{n}_t = \mathbf{v}\cdot\mathbf{n}_t$, the kinematic condition: $V_n = \mathbf{v}\cdot\mathbf{n}_t$ is automatically satisfied. And

$$
\Omega_t = \{x = X(y,t) \mid y \in \Omega\}, \quad \Gamma_t = \{x = X(y,t) \mid y \in \Gamma\}.
$$

Let $\mathfrak{q}(y,t) = \mathfrak{p}((X(y,t),t)$ and we are going to find $\mathbf{u}(y,t)$ and $\mathfrak{q}(y,t)$. To find equations satisfied by $\mathbf{u}$ and $\mathfrak{q}$, we consider the inverse map: $y \to x$ which should exist under the condition that

$$
(54)\qquad \sup_{0<t<T} \int_0^t \|\nabla\mathbf{u}(y,s)\|_{L_\infty(\Omega)}\, ds \le c_0 << 1.
$$

In fact, if $\mathbf{u}$ exists, then $\mathbf{u}(y_1,t) - \mathbf{u}(y_2,t) = \nabla\mathbf{u}(y_2 + \theta(y_1 - y_2))\cdot(y_1 - y_2)$ as follows from the mean value theorem, and so by (54)

$$
|\mathbf{X}_{\mathbf{u}}(y_1,t) - \mathbf{X}_{\mathbf{u}}(y_2,t)| \ge |y_1 - y_2| - \int_0^T \|\nabla(\mathbf{u}(y_1,s) - \mathbf{u}(y_2,s))\|_{L_\infty(\Omega)} \ge (1 - c_0)|y_1 - y_2|,
$$

which, combined with $c_0 < 1$, implies that the map $x = X(y,t)$ is injective. Thus, the Lagrange map is bijective from Ω onto Ω_t under the assumption (54).

In the sequel, we consider the case where $\Omega = \mathbb{R}^N_+$ only, which is a model problem. Since the Jacobian matrix of the transformation $\mathbf{X}_{\mathbf{u}}(y,t)$ is given by

$$
(55)\qquad \nabla_y \mathbf{X}_{\mathbf{u}}(y,t) = \mathbf{I} + \int_0^t \nabla_y \mathbf{u}(y,\tau)\, d\tau,
$$

the invertibility of $\mathbf{X}_{\mathbf{u}}(y,t)$ in (53) is guaranteed for all $t \in (0,T)$ if $\mathbf{u}$ satisfies

$$
(56)\qquad \sup_{t\in(0,T)} \Big\| \int_0^t \nabla_y \mathbf{u}(\,\cdot\,,\tau)\, d\tau \Big\|_{L_\infty(\mathbb{R}^N_+)} \le \sup_{0<t<T} \int_0^t \|\nabla_y \mathbf{u}(y,s)\|_{L_\infty(\Omega)}\, ds \le c_0 << 1,
$$

which may be achieved by a Neumann-series argument. By virtue of (56), we may write

$$
(57)\qquad \mathbf{A}_{\mathbf{u}}(y,t) := (\nabla\mathbf{X}_{\mathbf{u}}(y,t))^{-1} = \sum_{l=0}^\infty \Big(-\int_0^t \nabla_y \mathbf{u}(y,\tau)\, d\tau \Big)^l.
$$

With the above notation, for $T>0$ Problem (52) in Lagrangian coordinates reads as

$$
\left\{\begin{aligned}
\partial_t \mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}) &= \mathbf{F}(\mathbf{u}) && \text{in } \mathbb{R}^N_+ \times (0,T),\\
\mathrm{div}\,\mathbf{u} &= G_{\mathrm{div}}(\mathbf{u}) = \mathrm{div}\,\mathbf{G}(\mathbf{u}) && \text{in } \mathbb{R}^N_+ \times (0,T),\\
(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q})\mathbf{n} &= \mathbf{H}(\mathbf{u})\mathbf{n}_0 && \text{on } \partial\mathbb{R}^N_+ \times (0,T),\\
\mathbf{u}|_{t=0} &= \mathbf{a} && \text{in } \mathbb{R}^N_+,
\end{aligned}\right. \tag{58}
$$

where $\mathbf{n}_0 = (0,\dots,0,-1)$. The right-hand members $\mathbf{F}(\mathbf{u})$, $G_{\mathrm{div}}(\mathbf{u})$, $\mathbf{G}(\mathbf{u})$, and $\mathbf{H}(\mathbf{u})$ represent nonlinear terms given as follows:

$$
\begin{aligned}
\mathbf{F}(\mathbf{u}) &:= \Big(\int_0^t \nabla\mathbf{u}\,\mathrm{d}\tau\Big)\Big(\partial_t\mathbf{u} - \mu\Delta_y\mathbf{u}\Big) + \mu\Big(\mathbf{I} + \int_0^t \nabla\mathbf{u}\,\mathrm{d}\tau\Big)\mathrm{div}_y\Big((\mathbf{A}_\mathbf{u}\mathbf{A}_\mathbf{u}^\top - \mathbf{I})\nabla_y\mathbf{u}\Big)\\
&\qquad + \mu\nabla_y\Big((\mathbf{A}_\mathbf{u}^\top - \mathbf{I})\colon \nabla_y\mathbf{u}\Big),\\
G_{\mathrm{div}}(\mathbf{u}) &:= (\mathbf{I} - \mathbf{A}_\mathbf{u}^\top)\colon \nabla_y\mathbf{u},\\
\mathbf{G}(\mathbf{u}) &:= (\mathbf{I} - \mathbf{A}_\mathbf{u})\mathbf{u},\\
\mathbf{H}(\mathbf{u}) &:= \mu((\nabla_y\mathbf{u} + (\mathbf{A}_\mathbf{u}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_\mathbf{u})(\mathbf{I} - \mathbf{A}_\mathbf{u}^\top)\\
&\qquad + \mu((\mathbf{I} - (\mathbf{A}_\mathbf{u}^\top)^{-1})[\nabla_y\mathbf{u}]^\top\mathbf{A}_\mathbf{u} + [\nabla_y\mathbf{u}]^\top(\mathbf{I} - \mathbf{A}_\mathbf{u})).
\end{aligned} \tag{59}
$$

Here, $K^\top$ denotes the transposed K for any vector K or matrix K. Recall that for $N\times N$ matrices $\mathbf{A} = (A_{j,k})$ and $\mathbf{B} = (B_{j,k})$, we write $\mathbf{A}\colon\mathbf{B} = \sum_{j,k}^N A_{j,k}B_{j,k}$. For the detailed derivation of (58) and (59), refer [30, Section 3.3.3]. Notice that all the nonlinear terms in (58) and (59) do not contain the pressure term $\mathfrak{q}$.

Since Lagrange transformation (53) gives a C^1 diffeomorphism under the assumption (56) in our functional space settings in this section and the next section, instead of equations (52), we consider equations (58) in the sequel.

For the reader's convenience, we provide here how to derive (59). To this end, we use the following well-known formulas:

$$
\nabla_x = \mathbf{A}_\mathbf{u}^\top\nabla_y, \qquad \mathrm{div}_x(\,\cdot\,) = \mathbf{A}_\mathbf{u}^\top\colon \nabla_y(\,\cdot\,) = \mathrm{div}_y(\mathbf{A}_\mathbf{u}\,\cdot\,), \tag{60}
$$

$$
\mathbf{n} = \frac{\mathbf{A}_\mathbf{u}^\top\mathbf{n}_0}{|\mathbf{A}_\mathbf{u}^\top\mathbf{n}_0|}, \qquad \nabla_x\mathrm{div}_x(\,\cdot\,) = \mathbf{A}_\mathbf{u}^\top\nabla_y\mathrm{div}_y(\,\cdot\,) + \mathbf{A}_\mathbf{u}^\top\nabla_y((\mathbf{A}_\mathbf{u}^\top - \mathbf{I})\colon \nabla_y\,\cdot\,), \tag{61}
$$

In fact, as it was proved in [33], there holds $\det\mathbf{A}_\mathbf{u} = 1$ as follows from the divergence free condition, which yields the first formula $\nabla_x = \mathbf{A}_\mathbf{u}^\top\nabla_y$. By using these formulas, it is easy to verify the representations of $G_{\mathrm{div}}(\mathbf{u})$ and $\mathbf{G}(\mathbf{u})$. Hence, it suffices to derive the representations of $\mathbf{F}(\mathbf{u})$ and $\mathbf{H}(\mathbf{u})$.

By a direct calculation, we observe

$$
\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = \mu\Delta_x\mathbf{v} + \mu\nabla_x\mathrm{div}_x\mathbf{v} - \nabla_x\mathfrak{p}. \tag{62}
$$

We see that

$$
\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = \partial_t\mathbf{u}, \tag{63}
$$

$$
\Delta_x\mathbf{v} = \mathrm{div}_x\nabla_x\mathbf{v} = \mathrm{div}_y(\mathbf{A}_\mathbf{u}\mathbf{A}_\mathbf{u}^\top\nabla_y\mathbf{u}) = \mathrm{div}_y((\mathbf{A}_\mathbf{u}\mathbf{A}_\mathbf{u}^\top - \mathbf{I})\nabla_y\mathbf{u}) + \Delta_y\mathbf{u}, \tag{64}
$$

$$
\nabla_x\mathrm{div}_x\mathbf{v} = \mathbf{A}_\mathbf{u}^\top\nabla_y(\mathbf{A}_\mathbf{u}^\top : \nabla_y\mathbf{u}) = \mathbf{A}_\mathbf{u}^\top\nabla_y((\mathbf{A}_\mathbf{u}^\top - \mathbf{I}) : \nabla_y\mathbf{u}) + \mathbf{A}_\mathbf{u}^\top\nabla_y\mathrm{div}_y\mathbf{u}, \tag{65}
$$

$$
\nabla_x\mathfrak{p} = \mathbf{A}_\mathbf{u}^\top\nabla_y\mathfrak{q}. \tag{66}
$$

Since $\mathbf{A}_{\mathbf{u}}^\top$ is invertible and $(\mathbf{A}_{\mathbf{u}}^\top)^{-1} = \mathbf{I} + \int_0^t \nabla\mathbf{u}\,d\tau$, the first equation in equation (52) is transformed into

$$
\begin{aligned}
&\partial_t\mathbf{u} - \mu\Delta_y\mathbf{u} - \mu\nabla_y \mathrm{div}\,_y\mathbf{u} + \nabla_y\mathfrak{q} \\
&= \Big(\int_0^t \nabla\mathbf{u}\,d\tau\Big)\Big(\partial_t\mathbf{u} - \mu\Delta_y\mathbf{u}\Big) + \mu(\mathbf{I} + \int_0^t \nabla\mathbf{u}\,d\tau)\mathrm{div}\,_y\Big((\mathbf{A}_{\mathbf{u}}\mathbf{A}_{\mathbf{u}}^\top - \mathbf{I})\nabla_y\mathbf{u}\Big) \\
&\quad + \mu\nabla_y\Big((\mathbf{A}_{\mathbf{u}}^\top - \mathbf{I})\colon \nabla_y\mathbf{u}\Big).
\end{aligned}
$$

Combined this formula with

$$
\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mu\Delta_y\mathbf{u} + \mu\nabla_y\mathrm{div}\,_y\mathbf{u} - \nabla_y\mathfrak{q}, \tag{67}
$$

we have the representation of $\mathbf{F}(\mathbf{u})$. Note that $\mathbf{F}(\mathbf{u})$ does not contain the pressure $\mathfrak{q}$.

It remains to deal with $\mathbf{H}(\mathbf{u})$. It is easy to find that

$$
\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I} = \mu\Big(\mathbf{A}_{\mathbf{u}}^\top\nabla_y\mathbf{u} + [\nabla\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big) - \mathfrak{q}\mathbf{I}. \tag{68}
$$

From the third line of equations (52), it follows that

$$
\mu\Big(\mathbf{A}_{\mathbf{u}}^\top\nabla_y\mathbf{u} + [\nabla\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)\frac{\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0}{|\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0|} - \mathfrak{q}\frac{\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0}{|\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0|} = 0 \qquad \text{on } \partial\mathbb{R}^N_+. \tag{69}
$$

Multiplying this equation by $|\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0|(\mathbf{A}_{\mathbf{u}}^\top)^{-1}$ yields

$$
\mu\Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0 - \mathfrak{q}\mathbf{n}_0 = 0 \qquad \text{on } \partial\mathbb{R}^N_+. \tag{70}
$$

We consider only the velocity field, and then

$$
\begin{aligned}
&\Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)\mathbf{A}_{\mathbf{u}}^\top\mathbf{n}_0 \\
&= \Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)(\mathbf{A}_{\mathbf{u}}^\top - \mathbf{I})\mathbf{n}_0 \\
&+ \Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)\mathbf{n}_0 \\
&= \Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)(\mathbf{A}_{\mathbf{u}}^\top - \mathbf{I})\mathbf{n}_0 \\
&+ \Big(\nabla_y\mathbf{u} + ((\mathbf{A}_{\mathbf{u}}^\top)^{-1} - \mathbf{I})[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}} + [\nabla_y\mathbf{u}]^\top(\mathbf{A}_{\mathbf{u}} - \mathbf{I}) + [\nabla_y\mathbf{u}]^\top\Big)\mathbf{n}_0 \\
&= \Big(\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}}\Big)(\mathbf{A}_{\mathbf{u}}^\top - \mathbf{I})\mathbf{n}_0 \\
&+ \Big(((\mathbf{A}_{\mathbf{u}}^\top)^{-1} - \mathbf{I})[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}} + [\nabla_y\mathbf{u}]^\top(\mathbf{A}_{\mathbf{u}} - \mathbf{I})\Big)\mathbf{n}_0 + (\nabla_y\mathbf{u} + [\nabla_y\mathbf{u}]^\top)\mathbf{n}_0
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&(\mathbf{D}(\mathbf{u}) - \mathfrak{q})\mathbf{n}_0 \\
&= \mu\Big((\nabla_y\mathbf{u} + (\mathbf{A}_{\mathbf{u}}^\top)^{-1}[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}})(\mathbf{I} - \mathbf{A}_{\mathbf{u}}^\top) + (\mathbf{I} - (\mathbf{A}_{\mathbf{u}}^\top)^{-1})[\nabla_y\mathbf{u}]^\top\mathbf{A}_{\mathbf{u}} + [\nabla_y\mathbf{u}]^\top(\mathbf{I} - \mathbf{A}_{\mathbf{u}})\Big)\mathbf{n}_0
\end{aligned}
$$

Hence, we obtain the representation of $\mathbf{H}(\mathbf{u})$.

3.1. **Stokes equations with free boundary conditions.** In this section, we shall discuss the Stokes equations with free boundary conditions which reads as

$$
\begin{cases}
\partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mathbf{F} & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\
\mathrm{div}\,\mathbf{u} = G_{\mathrm{div}} = \mathrm{div}\,\mathbf{G} & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\
(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\mathbf{n} = \mathbf{H}\mathbf{n}_0 & \text{on } \partial\mathbb{R}^N_+ \times (0,\infty), \\
\mathbf{u}|_{t=0} = \mathbf{a} & \text{in } \mathbb{R}^N_+,
\end{cases} \tag{71}
$$

Here, $\mathbf{F}$, $G_{\rm div}$, $\mathbf{G}$ and $\mathbf{H}$ are given functions, and

$$\partial\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.$$

The corresponding generalized resolvent problem reads as

$$\begin{cases} \lambda\mathbf{v} - {\rm Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ {\rm div}\,\mathbf{v} = g_{\rm div} = {\rm div}\,\mathbf{g} & \text{in } \mathbb{R}^N_+, \\ (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n} = \mathbf{h}\mathbf{n}_0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{72}$$

First of all, we shall state the existence of $\mathcal{R}$ bounded solution operators for equations (72). To this end, we introduce variables $F = (F_1, F_2, F_3, F_4, F_5, F_6)$, where $F_1 \in L_q(\mathbb{R}^N_+)^N$, $F_2 \in L_q(\mathbb{R}^N_+)$, $F_3 \in L_q(\mathbb{R}^N_+)^N$, $F_4 \in L_q(\mathbb{R}^N_+)^N$, $F_5 \in L_q(\mathbb{R}^N_+)^N$ and $F_6 \in L_q(\mathbb{R}^N_+)^{N^2}$, and F_1, F_2, F_3, F_4, F_5 and F_6 are corresponding variables to $\mathbf{f}$, $\lambda^{1/2}g_{\rm div}$, $\nabla g_{\rm div}$, $\lambda\mathbf{G}$, $\lambda^{1/2}\mathbf{h}$ and $\nabla\mathbf{h}$, respectively. Set $M_N = 4N + 1 + N^2$, namely, $F \in L_q(\mathbb{R}^N_+)^{M_N}$. Let $\widehat{W}^1_{q,0}(\mathbb{R}^N_+)$ denote a homogeneous space defined by

$$\widehat{W}^1_{q,0}(\mathbb{R}^N_+) = \{u \in L_{q,{\rm loc}}(\mathbb{R}^N_+) \mid \nabla u \in L_q(\mathbb{R}^N_+), \quad u|_{x_N=0} = 0\},$$

and $1 < q < \infty$ and $q' = q/(q-1)$.

We have the following theorem concerning the $\mathcal{R}$ bounded solution operators for equations (72)

Theorem 18. *Let $1 < q < \infty$ and $\epsilon \in (0, \pi/2)$. Then, there exist oprators $\mathcal{S}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{S}(\lambda) \in {\rm Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, W^2_q(\mathbb{R}^N_+)^N)),$$
$$\mathcal{P}(\lambda) \in {\rm Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, W^1_q(\mathbb{R}^N_+) + \hat{W}^1_{q,0}(\mathbb{R}^N_+)))$$

such that the following two assertions hold:

(1) *For any $\lambda \in \Sigma_\epsilon$ and $\mathbf{f} \in L_q(\mathbb{R}^N_+)$, $g_{\rm div} \in W^1_q(\mathbb{R}^N_+)$, $\mathbf{g} \in L_q(\mathbb{R}^N_+)^N$, and $\mathbf{h} \in W^1_q(\mathbb{R}^N_+)^N$, problem (72) admits unique solutions $\mathbf{u} \in W^2_q(\mathbb{R}^N_+)^N$ and $\mathfrak{p} \in W^1_q(\mathbb{R}^N_+) + \widehat{W}^1_{q,0}(\mathbb{R}^N_+)$ such that $\mathbf{u} = \mathcal{S}(\lambda)\mathbf{F}$ and $\nabla\mathfrak{p} = \mathcal{P}(\lambda)\mathbf{F}$, where*

$$\mathbf{F} = (\mathbf{f}, \lambda^{1/2}g_{\rm div}, \nabla g_{\rm div}, \lambda\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}).$$

(2) *There hold*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b$$

for $\ell = 0, 1$ with some constant r_b depending solely on ϵ and q.

In the sequel, we give the sketch of a proof of Theorem 18. One of basical tools to solve equations (71) is the unique solvablity of the weak Dirichlet problem which reads as

$$(\nabla u, \nabla\varphi) = (\mathbf{f}, \nabla\varphi) \quad \text{for any } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+). \tag{73}$$

We know the following result cf [31].

Lemma 19. *Let $1 < q < \infty$. Then, for any $\mathbf{f} \in L_q(\mathbb{R}^N_+)^N$, problem (73) admits a unique solutoin $u \in \widehat{W}^1_{q,0}(\mathbb{R}^N_+)$ satisfying the estimate:*

$$\|\nabla u\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}.$$

Let $\mathcal{K}$ be an operator defined by $u = \mathcal{K}\mathbf{f}$.

Later, we introduce Stokes semigroup, and to this end at this point the Helmholtz decomposition is introduced. Let $J_q(\mathbb{R}^N_+)$ be Solenoidal space defined by

$$J_q(\mathbb{R}^N_+) = \{\mathbf{g} \in L_q(\mathbb{R}^N_+) \mid (\mathbf{g}, \nabla\varphi) = 0 \quad \text{for every } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+)\}. \tag{74}$$

According to Lemma 19, for any $\mathbf{f} \in L_q(\mathbb{R}^N_+)$, there exists a unique $u \in \widehat{W}^1_{q,0}(\mathbb{R}^N_+)$ such that equation (73) holds. Thus, setting $\mathbf{g} = \mathbf{f} - \nabla u$, we see that $\mathbf{g} \in J_q(\mathbb{R}^N_+)$, and setting $G_q(\mathbb{R}^N_+) = \{\nabla u \mid u \in \widehat{W}^1_{q,0}(\mathbb{R}^N_+)\}$, we have

$$L_q(\mathbb{R}^N_+)^N = J_q(\mathbb{R}^N_+) \oplus G_q(\mathbb{R}^N_+) \quad (\oplus \text{ means the direct sum}). \tag{75}$$

This called the second Helmholtz decomposition. We have the following lemma.

Lemma 20. *Let $1 < q < \infty$. Then, for $\mathbf{u} \in L_q(\mathbb{R}^N_+)$, what $\operatorname{div}\mathbf{u} = 0$ in the distribusion sense is equivalent to what $\mathbf{u} \in J_q(\mathbb{R}^N_+)$.*

Now, we shall discuss solution formulas of problem (72). Let $\epsilon \in (0, \pi/2)$ and recall that

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon\}.$$

3.2. **Solution formulas.** We shall give solution formulas of equations (72)

Step 1: Reductions. Let $0 < \epsilon < \pi/2$, $\gamma \ge 0$ and $\lambda \in \Sigma_\epsilon + \gamma$. Let $\mathbf{f}$, $\mathbf{g} \in L_q(\mathbb{R}^N_+)^N$, $g_{\rm div} \in W^1_q(\mathbb{R}^N_+)$, and $h_N \in W^1_q(\mathbb{R}^N_+)$. Assume that there holds $g_{\rm div} = \operatorname{div}\mathbf{g}$. According to Corollary 19, the weak Dirichlet problem,

$$\begin{cases} (\nabla\mathfrak{q}_1, \nabla\varphi) = (\mathbf{f} - \lambda\mathbf{g} + 2\mu\nabla g_{\rm div}, \nabla\varphi) & \text{for all } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+), \\ \mathfrak{q}_1|_{\partial\mathbb{R}^N_+} = (-h_N + 2\mu g_{\rm div})|_{\partial\mathbb{R}^N_+}, \end{cases} \tag{76}$$

admits a unique solution $\mathfrak{q}_1 \in W^1_q(\mathbb{R}^N_+) + \widehat{W}^1_{q,0}(\mathbb{R}^N_+)$. In fact, $\mathfrak{q}_1$ is defined by

$$\mathfrak{q}_1 = -h_N + 2\mu g_{\rm div} + \mathcal{K}(\mathbf{f} - \lambda\mathbf{g} + \nabla h_N).$$

In addition, by Lemma 20 $\mathfrak{q}_1$ satisfies the estimate

$$\|\nabla\mathfrak{q}_1\|_{L_q(\mathbb{R}^N_+)} \le C\Big(\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla(g_{\rm div}, h_N)\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}\Big). \tag{77}$$

Let $\mathbf{u}_1 = (u_{1,1}, \ldots, u_{1,N}) \in W^2_q(\mathbb{R}^N_+)^N$ be a solution to the following elliptic system:

$$\begin{cases} (\lambda - \mu\Delta)\mathbf{u}_1 = \mathbf{f} - \nabla\mathfrak{q}_1 + \mu\nabla g_{\rm div} & \text{in } \mathbb{R}^N_+, \\ u_{1,j}|_{\partial\mathbb{R}^N_+} = 0, & j = 1, \ldots, N-1, \\ \partial_N u_{1,N}|_{\partial\mathbb{R}^N_+} = g_{\rm div}|_{\partial\mathbb{R}^N_+}. \end{cases} \tag{78}$$

Notice that the solution $\mathbf{u}_1 \in W^2_q(\mathbb{R}^N_+)^N$ to equations (78) necessarily satisfies the divergence conditions:

$$\operatorname{div}\mathbf{u}_1 = g_{\rm div} = \operatorname{div}\mathbf{g} \qquad \text{in } \mathbb{R}^N_+. \tag{79}$$

In fact, for any $\varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+)$, we may write

$$(\lambda\mathbf{u}_1 - \mu\Delta\mathbf{u}_1, \nabla\varphi) = (\mathbf{f} - \nabla\mathfrak{q}_1 + \mu\nabla g_{\rm div}, \nabla\varphi) = (\lambda\mathbf{g} - \mu\nabla g_{\rm div}, \nabla\varphi). \tag{80}$$

In addition, there holds

$$(\Delta\mathbf{u}_1, \nabla\varphi) = (\nabla\operatorname{div}\mathbf{u}_1, \nabla\varphi) \qquad \text{for all } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+). \tag{81}$$

Combining (80) and (81) gives

$$(\lambda(\mathbf{u}_1 - \mathbf{g}), \nabla\varphi) - \mu(\nabla(\operatorname{div}\mathbf{u}_1 - g_{\rm div}), \nabla\varphi) = 0 \qquad \text{for all } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+). \tag{82}$$

Noting that $W^1_{q',0}(\mathbb{R}^N_+) \subset \widehat{W}^1_{q'0}(\mathbb{R}^N_+)$ and that $\operatorname{div} \mathbf{g} = g_{\rm div}$, we may show that

$$\lambda(\operatorname{div} \mathbf{u}_1 - g_{\rm div}, \varphi) + \mu(\nabla(\operatorname{div} \mathbf{u}_1 - g_{\rm div}), \nabla\varphi) = 0 \quad \text{for all } W^1_{q',0}(\mathbb{R}^N_+). \tag{83}$$

Moreover, it holds $\operatorname{div} \mathbf{u}_1 - g_{\rm div} = 0$ on $\partial\mathbb{R}^N_+$ due to the boundary conditions $(78)_{2,3}$. Thus, from the uniqueness of solutions to the resolvent problem for the weak Dirichlet problem we deduce that $\operatorname{div} \mathbf{u}_1 = g_{\rm div} = \operatorname{div} \mathbf{g}$. Thus, we arrive at (79).

Since $\mathfrak{q}_1 = -h_N + 2\mu g_{\rm div}$ on $\partial\mathbb{R}^N_+$, we have $(2\mu\partial_N u_{1,N} - \mathfrak{q}_1) = h_N$ on $\partial\mathbb{R}^N_+$. Therefore, $\mathbf{u}_1$ and $\mathfrak{q}_1$ necessarily satisfy the following Stokes system:

$$\begin{cases} \lambda\mathbf{u}_1 - \operatorname{Div}(\mu\mathbf{D}(\mathbf{u}_1) - \mathfrak{q}_1\mathbf{I}) = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \mathbf{u}_1 = g_{\rm div} = \operatorname{div} \mathbf{g} & \text{in } \mathbb{R}^N_+, \\ u_{1,j} = 0, & \text{on } \partial\mathbb{R}^N_+, \\ 2\mu\partial_N u_{1,N} - \mathfrak{q}_1 = h_N|_{\partial\mathbb{R}^N_+} & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{84}$$

where j runs from 1 through $N-1$.

We now set $\mathbf{u}_2 := \mathbf{u} - \mathbf{u}_1$ and $\mathfrak{q}_2 := \mathfrak{q} - \mathfrak{q}_1$ with $\mathbf{u}_2 = (u_{2,1}, \ldots, u_{2,N})$. Then $(\mathbf{u}_2, \mathfrak{q}_2)$ solves

$$\begin{cases} \lambda\mathbf{u}_2 - \operatorname{Div}(\mu\mathbf{D}(\mathbf{u}_2) - \mathfrak{q}_2\mathbf{I}) = 0 & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \mathbf{u}_2 = 0 & \text{in } \mathbb{R}^N_+, \\ \mu(\partial_N u_{2.j} + \partial_j u_{2,N}) = (h_j - \mu(\partial_N u_{1,j} + \partial_j u_{1,N}))\,|_{\mathbb{R}^N_+} & \text{on } \partial\mathbb{R}^N_+ \\ (2\mu\partial_N u_{2,N} - \mathfrak{q}_2) = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{85}$$

for $j = 1, \ldots, N-1$. Clearly, $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$ are solutions to (72).

Step 2: Solution formulas. We next derive the boundary symbol for the systems (78) and (85). To this end, in the sequel, let $\epsilon \in (0, \pi/2)$. For each $\xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ we define $A = |\xi'| = (\sum_{j=1}^{N-1} \xi_j^2)^{1/2}$. In addition, for each $\xi' \in \mathbb{R}^{N-1}$ we define complex functions $B = B(\lambda, \xi')$ and $M_{x_N} = M(\lambda, \xi', x_N)$ in the following way. Let $B = B(\lambda, \xi')$ be the principal branch of the square root of $\mu^{-1}\lambda + |\xi'|^2$, i.e., $B = \sqrt{\mu^{-1}\lambda + |\xi'|^2}$ for $\lambda \in \Sigma_\epsilon$ with $\operatorname{Re} B > 0$; and for $x_N > 0$ let $M_{x_N} = M(\lambda, \xi', x_N)$ be defined by

$$M_{x_N} = M(\lambda, \xi', x_N) = \frac{e^{-Bx_N} - e^{-Ax_N}}{B - A}, \tag{86}$$

which is called a Stokes kernel. We also define

$$D_{A,B} = B^3 + AB^2 + 3A^2B - A^3, \tag{87}$$

which is the determinant of the Lopatinskii matrix.

To derive the boundary symbol for Systems (78) and (85), it suffices to consider the following systems:

$$\begin{cases} \lambda\mathbf{w}_1 - \mu\Delta\mathbf{w}_1 = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ w_{1,j} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ \partial_N w_{1,N} = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{88}$$

$$\begin{cases} \lambda\mathbf{w}_2 - \mu\Delta\mathbf{w}_2 = 0 & \text{in } \mathbb{R}^N_+, \\ w_{2,j} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ \partial_N w_{2,N} = g_{\rm div}|_{\partial\mathbb{R}^N_+} & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{89}$$

$$
(90)\quad \begin{cases} \lambda \mathbf{w}_3 - \operatorname{div}(\mu \mathbf{D}(\mathbf{w}_3) - Q_3\mathbf{I}) = 0 & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \mathbf{w}_3 = 0 & \text{in } \mathbb{R}^N_+, \\ \mu(\partial_N w_{3,j} + \partial_j w_{3,N}) = h_j|_{\partial\mathbb{R}^N_+} & \text{on } \partial\mathbb{R}^N_+, \\ 2\mu\partial_N w_{3,N} - Q_3 = 0 & \text{on } \partial\mathbb{R}^N_+ . \end{cases}
$$

Here, j runs from 1 through $N-1$. Indeed, by replacing h_j by $h_j - \mu(\partial_N u_{1,j} + \partial_j u_{1,N})$ for every $j = 1, \dots, N-1$, we see that $\mathbf{w}_1 + \mathbf{w}_2$ and $(\mathbf{w}_3, Q_3)$ are solutions to (78) and (85), respectively, i.e., $\mathbf{u}_1 = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{u}_2 = \mathbf{w}_3$, and $\mathfrak{q}_2 = Q_3$. The functions $\mathbf{w}_1 = (w_{1,1}, \dots, w_{1,N})$ and $\mathbf{w}_2 = (w_{2,1}, \dots, w_{2,N})$ given by

$$
(91)\qquad w_{1,j} = \mathcal{F}^{-1}_\xi \left[\frac{\mathcal{F}[f_j^o](\xi)}{\lambda + \mu|\xi|^2} \right]\Bigg|_{\mathbb{R}^N_+}, \qquad w_{1,N} = \mathcal{F}^{-1}_\xi \left[\frac{\mathcal{F}[f_N^e](\xi)}{\lambda + \mu|\xi|^2} \right]\Bigg|_{\mathbb{R}^N_+},
$$

$$
(92)\qquad w_{2,j} = 0, \qquad w_{2,N} = -\mathcal{F}^{-1}_{\xi'} \left[\frac{e^{-Bx_N}}{B} \mathcal{F}'[g_{\mathrm{div}}](\xi') \right]
$$

solve (88) and (89), respectively. Here, j runs from 1 through $N-1$, and f^e and f^o denote respective the even extension of f to $x_N < 0$ and the odd extension of f to $x_N < 0$, which are defined by setting

$$
f^e(x) = \begin{cases} f(x) & (x_N > 0), \\ f(x', -x_N) & (x_N < 0), \end{cases} \qquad f^e(x) = \begin{cases} f(x) & (x_N > 0), \\ -f(x', -x_N) & (x_N < 0). \end{cases}
$$

To derive the formulas for $\mathbf{w}_3$ and Q_3, we apply the Fourier transform in the tangential direction x' with covariable ξ' and solve the transformed problem (i.e., a boundary value problem for an ordinary differential equation on $\mathbb{R}_+$). Following the computations in [31, Subsec. 3.5.3], we observe that $(\mathbf{w}_3, Q_3)$ is given by

$$
\begin{aligned}
w_{3,j}(x) &= \mathcal{F}^{-1}_{\xi'} \left[\frac{e^{-Bx_d}}{\mu B} \widehat{h}_j - \frac{i\xi_j M_{x_N}}{\mu D_{A,B}} (2Bi\xi' \cdot \widehat{h}') + \frac{i\xi_j e^{-Bx_N}}{\mu B D_{A,B}} (3B - A)(i\xi' \cdot \widehat{h}') \right](x'), \\
(93)\qquad w_{3,N}(x) &= \mathcal{F}^{-1}_{\xi'} \left[\frac{A M_{x_N}}{\mu D_{A,B}} (2Bi\xi' \cdot \widehat{h}') + \frac{e^{-Bx_N}}{\mu D_{A,B}} (B - A)(i\xi' \cdot \widehat{h}') \right](x'), \\
Q_3(x) &= -\mathcal{F}^{-1}_{\xi'} \left[\frac{(A+B)e^{-Ax_N}}{D_{A,B}} 2Bi\xi' \cdot \widehat{h}' \right](x'),
\end{aligned}
$$

where $j = 1, \dots, N-1$ and we have set $i\xi' \cdot \widehat{h}' = \sum_{j=1}^{N-1} i\xi_j \widehat{h}_j(\xi', 0)$.

3.3. **Estimate of multipliers.** In the sequel, $\epsilon \in (0, \pi/2)$ and $1 < q < \infty$.

The whole space case

Proposition 21. *Let $m(\lambda, \xi)$ be a function defined on $\Sigma_\epsilon \times (\mathbb{R}^N \setminus \{0\})$ such that for any $\lambda \in \Sigma_\epsilon$ and for any multi-index $\alpha \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exists a constant C_α depending on α and ϵ such that*

$$
(94)\qquad |\partial_\xi^\alpha m(\lambda, \xi)| \le C_\alpha |\xi|^{-|\alpha|}
$$

for any $(\lambda, \xi) \in \Sigma_\epsilon \times (\mathbb{R}^N \setminus \{0\})$. And, for any $\xi \in \mathbb{R}^N \setminus \{0\}$, $m(\lambda, \xi')$ is a holomorphic function with respect to $\lambda \in \Sigma_\epsilon$. Let K_λ be an operator defined by

$$
K_\lambda f = \mathcal{F}^{-1}_\xi [m(\lambda, \xi) \mathcal{F}[f](\xi)] = \int_{\mathbb{R}^N} e^{ix\cdot\xi} m(\lambda, \xi) \mathcal{F}[f](\xi)\, \mathrm{d}\xi.
$$

Then, the set $\{K_\lambda \mid \lambda \in \Sigma_\epsilon\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$
(95)\qquad \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \Sigma_\epsilon\}) \le C_{q,N} \max_{|\alpha| \le N+2} C_\alpha
$$

with some constant $C_{q,N}$ which depends solely on q and N.

Proof. For a proof, refer to [16, Section 3]. □

Since $\mathcal{R}$ boundedness implies the boundedness, we have

$$\|K_\lambda f\|_{L_q(\mathbb{R}^N)} \le (C_{q,N} \max_{|\alpha|\le N+2} C_\alpha)\, \|f\|_{L_q(\mathbb{R}^N)}, \tag{96}$$

which directly follows from the Mikhlin-Hölmander Fourier multiplier theorem.

Lemma 22. *Let $\epsilon \in (0, \pi/2)$. Then, there holds*

$$|\lambda + \mu|\xi^2| \ge \sin\frac{\epsilon}{2}(|\lambda| + \mu|\xi|^2).$$

Half-space case.

Let $m(\lambda, \xi')$ be a function defined on $\Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$, which is holomorphic with respect to $\lambda = \gamma + i\tau \in \Sigma_\epsilon$ for any fixed $\xi \in \mathbb{R}^{N-1} \setminus \{0\}$ and C^∞ with respect to $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$ for any fixed $\lambda \in \Sigma_\epsilon$. We say that $m(\lambda, \xi')$ is an $\mathbb{M}_{m,1}$ symbol if for any $\alpha' \in \mathbb{N}_0^{N-1}$, there hold

$$|\partial_{\xi'}^{\alpha'}((\tau\partial_\tau)^\ell m(\lambda, \xi'))| \le C_{\alpha'}(|\lambda|^{1/2} + |\xi'|)^{m-|\alpha'|} \quad (\ell = 0, 1).$$

And also, we say that $m(\lambda, \xi')$ is an $\mathbb{M}_{m,2}$ symbol if for any $\alpha' \in \mathbb{N}_0^{N-1}$, there hold

$$|\partial_{\xi'}^{\alpha'}((\tau\partial_\tau)^\ell m(\lambda, \xi'))| \le C_{\alpha'}(|\lambda|^{1/2} + |\xi'|)^{m}|\xi|^{-|\alpha'|} \quad (\ell = 0, 1).$$

Let

$$\|m\|_{\mathbb{M}_{m,i}} = \max_{|\alpha'|\le N} C_{\alpha'}.$$

Lemma 23. *Let $B = \sqrt{\lambda + |\xi'|^2}$ and $D_{A,B} = B^3 + AB^2 + 2A^2B - A^3$. Then, for any $\nu \in \mathbb{R}$, B^ν is a $\mathbb{M}_{\nu,1}$ symbol and $D_{A,B}^\nu$ a $\mathbb{M}_{3\nu,2}$ symbol.*

Proof. The proposition follows from [31, Lemma 3.5.9]. □

Proposition 24. *Let $0 < \epsilon < \pi/2$ and $1 < q < \infty$. Given multipliers $n_1 \in \mathbb{M}_{-2,1}$ and $n_2 \in \mathbb{M}_{-1,2}$, let operators $T_i(\lambda)$ $(i = 1, 2)$ acting on $h = h(x', x_N) \in W_q^1(\mathbb{R}_+^N)$ be defined by*

$$T_1(\lambda)f = \mathcal{F}_{\xi'}^{-1}[Be^{-Bx_N} n_1(\lambda, \xi')\mathcal{F}[h](\xi', 0)](x'),$$
$$T_2(\lambda)f = \mathcal{F}_{\xi'}^{-1}[AM_{x_N} n_2(\lambda, \xi')\mathcal{F}[h](\xi', 0)](x').$$

Then, there exist operator families $\mathcal{T}_i(\lambda) \in \mathrm{Hol}(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}_+^N)^{N+1}, W_q^2(\mathbb{R}_+^N)))$ such that for any $\lambda \in \Sigma_\epsilon$ and $h \in W_q^1(\mathbb{R}_+^N)$, $T_i(\lambda)h = \mathcal{T}_i(\lambda)(\lambda^{1/2}h, \nabla h)$, and there exists a constant $r_b^i > 0$ depending on $\|n_i\|_{\mathbb{M}_{-2,i}}$ such that

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^{N+1}, L_q(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{T}_i(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b^i,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^{N+1}, L_q(\mathbb{R}_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{T}_i(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b^i,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^{N+1}, L_q(\mathbb{R}_+^N)^{N^2})}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{T}_i(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b^i,$$

for $\ell = 0, 1$ and $i = 1, 2$.

Proof. For a proof, see Lemma 3.5.13 in [31]. □

Proposition 25. *Let $0 < \epsilon < \pi/2$ and $1 < q < \infty$. Given multiplier $n_3 \in \mathbb{M}_{-1,1}$, let operator $T_3(\lambda)$ acting on $h = h(x', x_N) \in W_q^1(\mathbb{R}_+^N)$ be defined by*

$$T_3(\lambda)h = \mathcal{F}_{\xi'}^{-1}[Ae^{-Ax_N} n_1(\lambda, \xi')\mathcal{F}[h](\xi', 0)](x').$$

Then, there exists an operator family $\mathcal{T}_3(\lambda) \in \mathrm{Hol}(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{N+1}, W^1_q(\mathbb{R}^N_+)))$ *such that for any* $\lambda \in \Sigma_\epsilon$ *and* $h \in W^1_q(\mathbb{R}^N_+)$, $T_3(\lambda)h = \mathcal{T}_3(\lambda)(\lambda^{1/2}h, \nabla h)$, *and there exists a constant* $r^3_b > 0$ *depending on* $\|n_i\|_{\mathbb{M}_{-1,2}}$ *such that*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N+1}, L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell \mathcal{T}_3(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \le r^3_b,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N+1}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell (\nabla \mathcal{T}_3(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r^3_b,$$

for $\ell = 0, 1$ *and* $i = 1, 2$.

3.4. **Existence of $\mathcal{R}$-solvers, A proof of Theorem 18.** Before starting with the proof of Theorem 18, we give a lemma which tells us that the $\mathcal{R}$ norm has the same property as the usual norm has.

Lemma 26. (1) *Let* X *and* Y *be Banach spaces, and let* $\mathcal{T}$ *and* $\mathcal{S}$ *be* $\mathcal{R}$ *bounded families in* $\mathcal{L}(X, Y)$. *Then,* $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ *is also an* $\mathcal{R}$*-bounded family in* $\mathcal{L}(X, Y)$ *and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let* X, Y, *and* Z *be Banach spaces, and let* $\mathcal{T}$ *and* $\mathcal{S}$ *be* $\mathcal{R}$*-bounded families in* $\mathcal{L}(X, Y)$ *and* $\mathcal{L}(Y, Z)$, *respectively. Then,* $\mathcal{S}\mathcal{T} = \{ST \mid S \in \mathcal{S},\ T \in \mathcal{T}\}$ *also an* $\mathcal{R}$*-bounded family in* $\mathcal{L}(X, Z)$ *and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(ST) \le \mathcal{R}_{\mathcal{L}(X,Y)}(S)\mathcal{R}_{\mathcal{L}(Y,Z)}(T).$$

Proof. For a proof, refer to [9, p.28, Proposition 3.4]. □

We start a proof of Theorem 18. Let $\mathfrak{q}_1$ be defined by

$$\mathfrak{q}_1 = -h_N + 2\mu g_{\mathrm{div}} + \mathcal{K}(\mathbf{f} - \lambda\mathbf{g} + \nabla h_N),$$

then $\mathfrak{q}_1$ satisfies equations (76) as well as the estimate:

$$\|\nabla \mathfrak{q}_1\|_{L_q(\mathbb{R}^N_+)} \le C_0 \|(\mathbf{f}, \nabla g_{\mathrm{div}}, \lambda\mathbf{g}, \nabla h_N)\|_{L_q(\mathbb{R}^N_+)}. \tag{97}$$

for some constant $C_0 > 0$. Thus, we define $\mathcal{P}_1(\lambda)$ by $\mathcal{P}_1(\lambda)F = -F_{6N} + 2\mu F_3 + \nabla\mathcal{K}(F_1 - F_4 + F_{6N})$, where $F_6 = (F_{61}, \ldots, F_{6N})$ and $F_{6j} \in L_q(\mathbb{R}^N_+)^N$ are the corresponding variables to ∇h_i for $\mathbf{h} = (h_1, \ldots, h_N)$. Obvisouly,

$$\mathcal{P}_1(\lambda)\mathbf{F} = \nabla\mathfrak{q}_1, \quad \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{P}_1(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \le C_0 \tag{98}$$

for $\ell = 0, 1$. Here, $\mathbf{F} = (\mathbf{f}, \lambda^{1/2} g_{\mathrm{div}}, \nabla g_{\mathrm{div}}, \lambda\mathbf{G}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$.

In view of Proposition 21, Lemma 22, and (91), there exists an operator $\mathcal{W}_1(\lambda)$ with

$$\mathcal{W}_1(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, L_q(\mathbb{R}^N_+)^N, W^2_q(\mathbb{R}^N_+)^N)))$$

such that for any $\lambda \in \Sigma_\epsilon$ and $\mathbf{f} \in L_q(\mathbb{R}^N_+)^N$, $\mathbf{w}_1 = \mathcal{W}_1(\lambda)\mathbf{f}$ is a solution of equations (88) and there hold

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell (\lambda\mathcal{W}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell (\lambda^{1/2}\nabla\mathcal{W}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N, L_q(\mathbb{R}^N_+)^{N^3})}(\{(\tau\partial_\tau)^\ell (\nabla^2\mathcal{W}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b
\end{aligned} \tag{99}$$

for $\ell = 0, 1$ with some constant r_b depending on ϵ and q. By Proposition 24, Lemma 23, and (92), there exists an operator $\mathcal{W}_2(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{N+1}, W^2_q(\mathbb{R}^N_+)^N))$ such that for any $\lambda \in \Sigma_\epsilon$ and $g_{\mathrm{div}} \in W^1_q(\mathbb{R}^N_+)$, $\mathbf{w}_2 = \mathcal{W}_2(\lambda)(\lambda^{1/2}g_{\mathrm{div}}, \nabla g_{\mathrm{div}})$ is a unique solution of equations (89), and there holds

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell (\lambda\mathcal{W}_2(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell (\lambda^{1/2}\nabla\mathcal{W}_2(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^{N^3})}(\{(\tau\partial_\tau)^\ell (\nabla^2\mathcal{W}_2(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \le r_b
\end{aligned} \tag{100}$$

for $\ell = 0, 1$ with some constant r_b depending on ϵ and q. Thus, if we set $\mathfrak{q}_1 = -h_N + 2\mu g_{\rm div} + \mathcal{K}(\mathbf{f} - \lambda\mathbf{g} + \nabla h_N)$, and $\mathbf{u}_1 = \mathcal{W}_1(\lambda)(\mathbf{f} - \nabla\mathfrak{q}_1 + \mu\nabla g_{\rm div}) + \mathcal{W}_2(\lambda)(\lambda^{1/2}g_{\rm div}, \nabla g_{\rm div})$, then $\mathbf{u}_1$ and $\mathfrak{q}_1$ are solutions of (84). Thus, we define an $\mathcal{R}$ bounded solution operator $\mathcal{U}_1(\lambda)$ with

$$\mathcal{U}_1(\lambda) \in {\rm Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{M_N}, W^2_q(\mathbb{R}^N_+)^N))$$

by $\mathcal{U}_1(\lambda)F = \mathcal{W}_1(\lambda)(F_1 - \mathcal{P}_1(\lambda)F + \mu F_3) + \mathcal{W}_2(\lambda)(F_2, F_3)$. From the definition of $\mathcal{U}_1(\lambda)$ obviously it follows that $\mathbf{u}_1 = \mathcal{U}_1(\lambda)\mathbf{F}$ with $\mathbf{F} = (\mathbf{f}, \lambda^{1/2}g_{\rm div}, \nabla g_{\rm div}, \lambda\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$, and by (99), (100) and Lemma 26, we have

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{U}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{U}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, L_q(\mathbb{R}^N_+)^{N^3})}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{U}_1(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b
\end{aligned} \tag{101}$$

Likewise, by Propositions 24 and 25, Lemma 23, and (93), there exists an operator $\mathcal{W}_3(\lambda) \in {\rm Hol},(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, W^2_q(\mathbb{R}^N_+)^N))$ and $\mathcal{Q}_3(\lambda) \in {\rm Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, L_q(\mathbb{R}^N_+)^N))$ such that for any $\lambda \in \Sigma_\epsilon$ and $\mathbf{h}' = (h_1, \ldots, h_{N-1}) \in W^1_q(\mathbb{R}^N_+)^{N-1}$, $\mathbf{w}_3 = \mathcal{W}_3(\lambda)(\lambda^{1/2}, \mathbf{h}', \nabla\mathbf{h}')$ and $\nabla Q_3 = \mathcal{Q}_3(\lambda)(\lambda^{1/2}, \mathbf{h}', \nabla\mathbf{h}')$ satisfy equations (90) and there hold

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{W}_3(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{W}_3(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, L_q(\mathbb{R}^N_+)^{N^3})}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{W}_3(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{N^2-1}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{Q}_3(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq r_b
\end{aligned} \tag{102}$$

for $\ell = 0, 1$ with some constant r_b depending on ϵ and q. Thus, setting $\mathbf{u}_1 = (u_{11}, \ldots, u_{1N})$ and $\mathbf{H}' = \mu(\partial_N u_{11} + \partial_1 u_{1N}, \ldots, \partial_N u_{1N-1} + \partial_{N-1}u_{1N}) = \mu(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_0 - (\mathbf{D}(\mathbf{u}_1)\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0$, we define $\mathbf{v}$ and $\nabla\mathfrak{p}$ by

$$\begin{aligned}
\mathbf{v} &= \mathbf{u}_1 + \mathcal{W}_3(\lambda)(\lambda^{1/2}\mathbf{h}', \nabla\mathbf{h}') - \mathcal{W}_3(\lambda)(\lambda^{1/2}\mathbf{H}', \nabla\mathbf{H}'),\\
\nabla\mathfrak{p} &= \mathcal{P}_1(\lambda)\mathbf{F} + \mathcal{Q}_3(\lambda)(\lambda^{1/2}\mathbf{h}', \nabla\mathbf{h}') - \mathcal{Q}_3(\lambda)(\lambda^{1/2}\mathbf{H}', \nabla\mathbf{H}').
\end{aligned}$$

Then, $\mathbf{v}$ and $\nabla\mathfrak{p}$ satisfy equations (72). Thus, we define $\mathcal{S}(\lambda)$ and $\mathcal{P}(\lambda)$ by

$$\begin{aligned}
\mathcal{S}(\lambda)F &= \mathcal{U}_1(\lambda)F + \mathcal{W}_3(\lambda)(F_5', F_6') - \mathcal{W}_3(\lambda)(\mu\lambda^{1/2}(\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0 - (\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0),\\
&\qquad \nabla(\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0 - (\mathcal{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0)),\\
\mathcal{P}(\lambda) &= \mathcal{P}_1(\lambda)F + \mathcal{Q}_3(\lambda)(F_5', F_6') - \mathcal{Q}_3(\lambda)(\lambda^{1/2}(\mathcal{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0 - (\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0),\\
&\qquad \nabla(\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0 - (\mathbf{D}(\mathcal{U}_1(\lambda)F)\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0)).
\end{aligned}$$

Obviously, $\mathcal{S}(\lambda)\mathbf{F} = \mathbf{v}$ and $\mathcal{P}(\lambda)\mathbf{F} = \nabla\mathfrak{p}$.

Moreover, by Lemma 26, (98), (101) and (102), $\mathcal{S}(\lambda)$ and $\mathcal{P}(\lambda)$ satisfy (2) of Theorem 18. This completes the proof of Theorem 18. □

To estimate lower order derivatives of solutions to equations (72), we shall use the following lemma.

Lemma 27. (1) *Let $1 < p, q < \infty$ and let D be a domain in $\mathbb{R}^N$. Let $m = m(\lambda)$ be a bounded function defined on a subset of $\mathbb{C}$ and let $M_m(\lambda)$ be an operator defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in U\}) \leq C_{N,q,D}\|m\|_{L_\infty(U)}$.*
(2) *Let $n = n(\tau)$ be a C^1 function defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions: $|n(\tau)| \leq \gamma$ and $|\tau n'(\tau)| \leq \gamma$ with some constant γ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let T_n be an operator-valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ for every f with $\mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, L_q(D))$. Then, T_n is*

extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R},L_q(D)))} \leq C_{p,q,D}\gamma.$$

Proof. For a proof, refer [9, p.27, Remark 3.2]. □

Combining Theorem 18 and Lemma 27 (1), we have the following corollary.

Corollary 28. *Let $1 < q < \infty$ and $\epsilon \in (0, \pi/2)$. Let $\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N_+)^{M(N)}, W^2_q(\mathbb{R}^N_+)^N))$ be an solution operators for problem (72) given in Theorem 18. Then, for any $\gamma > 0$ there hold*

$$(103)\qquad \begin{aligned} \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M(N)}, L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{S}(\lambda) \mid \lambda \in \Sigma_\epsilon + \gamma\}) &\leq \gamma^{-1} r_b, \\ \mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^{M(N)}, L_q(\mathbb{R}^N_+)^{N^2})}(\{(\tau\partial_\tau)^\ell (\nabla\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_\epsilon + \gamma\}) &\leq \gamma^{-1/2} r_b \end{aligned}$$

for $\ell = 0, 1$ with some constant r_b depending on ϵ and q.

3.5. **L_p-L_q maximal regularity theorem for Stokes equations** (71)**.** According to Theorem 18 and Corollary 28, we know the existence of $\mathcal{R}$-bounded solution operators for problem (71). First, we consider the following evolution equations for whole time interval:

$$(104)\qquad \begin{cases} \partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mathbf{F} & \text{in } \mathbb{R}^N_+ \times \mathbb{R}, \\ \mathrm{div}\,\mathbf{u} = G_{\mathrm{div}} = \mathrm{div}\,\mathbf{G} & \text{in } \mathbb{R}^N_+ \times \mathbb{R}, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\mathbf{n} = \mathbf{H}\mathbf{n}_0 & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}. \end{cases}$$

According to the argument as in the First Step of subsection 2.1, we have the following proposition.

Proposition 29. *Let $1 < p, q < \infty$. Let $\gamma > 0$ be any number. Assume that $\mathbf{F}$, G_{div}, $\mathbf{G}$ and $\mathbf{H}$, the right member of equations (104), satisfy the conditions:*

$$\begin{aligned} &e^{-\gamma t}\mathbf{F} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}G_{\mathrm{div}} \in L_p(\mathbb{R}, W^1_q(\mathbb{R}^N_+)) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)), \\ &e^{-\gamma t}\mathbf{G} \in W^1_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}\mathbf{H} \in L_p(\mathbb{R}, W^1_q(\mathbb{R}^N_+)^N) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N). \end{aligned}$$

Then problem (104) admits a solution $\mathbf{u}$ and $\mathfrak{q}$ such that

$$e^{-\gamma t}\mathbf{u} \in L_p(\mathbb{R}, W^2_q(\mathbb{R}^N_+)^N) \cap W^1_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}\nabla\mathfrak{q} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N),$$

as well as

$$\begin{aligned} &\|e^{-\gamma t}\mathbf{u}\|_{L_p(\mathbb{R}, W^2_q(\mathbb{R}^N_+))} + \|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|e^{-\gamma t}\nabla\mathfrak{q}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} \\ &\quad \leq C(\|e^{-\gamma t}(\mathbf{F}, \partial_t\mathbf{G})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|e^{-\gamma t}\nabla(G_{\mathrm{div}}, \mathbf{H})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} \\ &\qquad + \|e^{-\gamma t}(G_{\mathrm{div}}, \mathbf{H})\|_{W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+))}). \end{aligned}$$

Moreover, if $\mathbf{F}$, G_{div}, $\mathbf{G}$ and $\mathbf{H}$ satisfy the conditions:

$$\begin{aligned} &\mathbf{F} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad G_{\mathrm{div}} \in L_p(\mathbb{R}, W^1_q(\mathbb{R}^N_+)) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)), \\ &\mathbf{G} \in W^1_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad \mathbf{H} \in L_p(\mathbb{R}, W^1_q(\mathbb{R}^N_+)^N) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \end{aligned}$$

then

$$\partial_t\mathbf{u} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad \nabla^2\mathbf{u} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^{N^3}), \quad \nabla\mathfrak{q} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N)$$

as well as

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|\partial_t\mathbf{u}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|\nabla\mathfrak{q}\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} \\ &\quad \leq C(\|(\mathbf{F}, \partial_t\mathbf{G})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|\nabla(G_{\mathrm{div}}, \mathbf{H})\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} + \|(G_{\mathrm{div}}, \mathbf{H})\|_{\dot{W}^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+))}). \end{aligned}$$

To prove the existence of C_0 analytic semigroup associated with problem (71), we consider the following equations:

$$
(105)\qquad \begin{cases} \partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\mathbf{n} = 0 & \text{on } \partial\mathbb{R}^N_+ \times (0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{a} & \text{in } \mathbb{R}^N_+, \end{cases}
$$

Since $\mathfrak{q}$ does not have time evolution, we have to eliminate $\mathfrak{q}$.

We consider the corresponding resolvent problem to equations (105) which reads as

$$
(106)\qquad \begin{cases} \lambda\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } \mathbb{R}^N_+, \\ (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}
$$

Noting that $\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v})-\mathfrak{p}) = \mu\mathrm{Div}\,\mathbf{D}(\mathbf{v})-\nabla\mathfrak{p}$, we consider the second Helmholtz decomposition of $\mu\mathrm{Div}\,\mathbf{D}(\mathbf{v})$. Namely, for $\mathbf{v} \in W^2_q(\mathbb{R}^N_+)^N$, let $u = K(\mathbf{v}) \in \widehat{W}^1_{q,0}(\mathbb{R}^N_+)$ be a solution to the weak Dirichlet problem:

$$
(\nabla u, \nabla\varphi) = (\mu\mathrm{Div}\,\mathbf{D}(\mathbf{v}), \nabla\varphi) \quad \text{for every } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+)
$$

subject to $u = (\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n})$ on $\partial\mathbb{R}^N_+$. We see that $\mathbf{v} \in J_q(\mathbb{R}^N_+)$ is equivalent to $\mathrm{div}\,\mathbf{v} = 0$. If $\mathbf{f} \in J_q(\mathbb{R}^N_+)$, then $\mathfrak{p} = K(\mathbf{v})$. In fact, for any $\varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+)$, we have

$$
0 = (\mathbf{f}, \nabla\varphi) = (\lambda\mathbf{v}, \nabla\varphi) - (\mu\mathrm{Div}\,\mathbf{D}(\mathbf{v}), \nabla\varphi) + (\nabla\mathfrak{p}, \nabla\varphi) = (\nabla(\mathfrak{p} - u), \nabla\varphi).
$$

Moreover, on the boundary, $\mathfrak{p} = (\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}) = K(\mathbf{v})$, and so by the uniqueness of solutions implies that $\mathfrak{p} = u$.

Thus, from (106) it follows that for $\mathbf{f} \in J_q(\mathbb{R}^N_+)$, $\mathbf{v} \in W^2_q(\mathbb{R}^N_+)^N$ satisfies equations:

$$
(107)\qquad \begin{cases} \lambda\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v})\mathbf{I}) = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ (\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v}))\mathbf{n} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}
$$

Since $(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v}))\mathbf{n}, \mathbf{n}) = (\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}) - K(\mathbf{v}) = 0$ is automatically satisfied, the actual boundary conditions are $\mu\mathbf{D}(\mathbf{v})\mathbf{n} - (\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n})\mathbf{n} = 0$, that is the tangential component of $\mu\mathbf{D}(\mathbf{v})\mathbf{n}$ vanishes on the boundary.

Let the domain $\mathcal{D}(\mathcal{A})$ and an operator $\mathcal{A}$ be defined by

$$
(108)\qquad \begin{aligned} \mathcal{D}(\mathcal{A}) &= \{\mathbf{f} \in J_q(\mathbb{R}^N_+) \cap W^2_q(\mathbb{R}^N_+) \mid \mu\mathbf{D}(\mathbf{v})\mathbf{n} - (\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n})\mathbf{n} = 0 \text{ on } \partial\mathbb{R}^N_+\}, \\ \mathcal{A}\mathbf{v} &= -\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v})\mathbf{I}) \quad \text{for } \mathbf{v} \in \mathcal{D}(\mathcal{A}). \end{aligned}
$$

We can write (107) as

$$
(\lambda\mathbf{I} + \mathcal{A})\mathbf{v} = \mathbf{f} \quad \text{for } \mathbf{f} \in J_q(\mathbb{R}^N_+) \text{ and } \mathbf{v} \in \mathcal{D}(\mathcal{A}).
$$

From Theorem 18 and Corollary 28, $\mathbf{v} = \mathcal{S}(\lambda)(\mathbf{f}, 0, \cdots, 0)$ and $\nabla K(\mathbf{v}) = \nabla\mathfrak{p} = \mathcal{P}(\lambda)(\mathbf{f}, 0, \cdots, 0)$ and

$$
(109)\qquad \|\lambda\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} \le C\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}
$$

for any $\lambda \in \Sigma_\epsilon$. Thus, $(\lambda\mathbf{I} + \mathcal{A})^{-1}$ exists and satisfies

$$
\|\lambda(\lambda\mathbf{I} + \mathcal{A})^{-1}\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} \le C\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}
$$

for any $\mathbf{f} \in \dot{J}_q(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_\epsilon$. From this it follows the generation of C_0 analytic semigroup $\{T(t)\}_{t\ge0}$, called a Stokes semigroup, associated with problem (105). Especially, for any initial data $\mathbf{a} \in J_q(\mathbb{R}^N_+)$, $\mathbf{u} = T(t)\mathbf{a}$ is a unique solution of equations (105) with $\mathfrak{q} = K(T(t)\mathbf{a})$. Moreover, from (109) we have $\|T(t)\mathbf{a}\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}$ as well as

$$\|\mathcal{A}T(t)\mathbf{a}\|_{L_q(\mathbb{R}^N_+)} \leq Ct^{-1}\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}, \quad \|\mathcal{A}T(t)\mathbf{a}\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathcal{A}\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}.$$

Since we have the estimate:

$$\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathcal{A}\mathbf{v}\|_{L_q(\mathbb{R}^N_+)}$$

for any $\mathbf{v} \in \mathcal{D}(\mathcal{A})$, we have

$$\|\nabla^2 T(t)\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} \leq Ct^{-1}\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)}, \quad \|\nabla^2 T(t)\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathbf{f}\|_{\mathcal{D}(\mathcal{A})}.$$

Thus, by real interpolation method and the fact that $\partial_t T(t)\mathbf{f} = -\mathcal{A}T(t)\mathbf{f}$, we have

$$\int_0^\infty \|(\partial_t, \nabla^2)T(t)\mathbf{f}\|^p_{L_q(\mathbb{R}^N_+)}\,\mathrm{d}t \leq C\|\mathbf{f}\|_{(L_q(\mathbb{R}^N_+),\mathcal{D}(\mathcal{A}))_{1-1/p,p}}.$$

Let $\mathcal{D}_{p,q}(\mathbb{R}^N_+) = (L_q(\mathbb{R}^N_+), \mathcal{D}(\mathcal{A}))_{1-1/p,p}$. Note that $\mathcal{D}_{p,q}(\mathbb{R}^N_+) \subset J_q(\mathbb{R}^N_+) \cap B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)$. Moreover, if $\mathbf{v} \in J_q(\mathbb{R}^N_+) \cap B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)$ and the trace $\mathbf{D}(\mathbf{v})\mathbf{n}|_{\partial\mathbb{R}^N_+}$ exists, then $\mathbf{v} \in \mathcal{D}_{p,q}(\mathbb{R}^N_+)$. If $\mathbf{v} \in J_q(\mathbb{R}^N_+) \cap B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)$ but the trace $\mathbf{D}(\mathbf{v})\mathbf{n}|_{\partial\mathbb{R}^N_+}$ does not exist, then $\mathbf{v} \in \mathcal{D}_{p,q}(\mathbb{R}^N_+)$.

Summing up, we have obtained

Theorem 30. *Let $1 < p, q < \infty$. Let $\mathbf{F}$, G_{div}, $\mathbf{G}$ and $\mathbf{H}$ satisfy the conditions:*

$$\begin{aligned}
&\mathbf{F} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad G_{\mathrm{div}} \in L_p(\mathbb{R}, \dot{W}^1_q(\mathbb{R}^N_+)) \cap \dot{W}^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)),\\
&\mathbf{G} \in \dot{W}^1_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad \mathbf{H} \in L_p(\mathbb{R}, \dot{W}^1_q(\mathbb{R}^N_+)^N) \cap \dot{W}^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N).
\end{aligned}$$

Moreover, initial data $\mathbf{a} \in B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)$ satisfies the compatibility conditions:

$$\mathbf{a} - \mathbf{G}|_{t=0} \in J_q(\mathbb{R}^N_+), \quad (\mathbf{D}(\mathbf{a})\mathbf{n} - (\mathbf{D}(\mathbf{a})\mathbf{n}, \mathbf{n})\mathbf{n} - (\mathbf{H}|_{t=0} - (\mathbf{H}|_{t=0}, \mathbf{n})\mathbf{n}))|_{\partial\mathbb{R}^N_+} = 0.$$

Here, the second condition should not be satisfied if the trace does not exist. Then, problem (71) *admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ such that*

$$\partial_t\mathbf{u}, \ \partial_j\partial_k\mathbf{u} \in L_p((0,\infty), L_q(\mathbb{R}^N_+)^N), \quad \nabla\mathfrak{q} \in L_p((0,\infty), L_q(\mathbb{R}^N_+)^N)$$

for $j, k = 1, \ldots, N$ as well as

$$\begin{aligned}
&\|(\partial_t\mathbf{u}, \nabla^2\mathbf{u}, \nabla\mathfrak{q})\|_{L_p((0,\infty),L_q(\mathbb{R}^N_+))} + \sup_{t\in(0,\infty)}\|\mathbf{u}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)}\\
&\leq C(\|\mathbf{a}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)} + \|(\mathbf{F}, \partial_t\mathbf{G}, \nabla G_{\mathrm{div}}, \nabla\mathbf{H})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} + \|(G_{\mathrm{div}}, \mathbf{H})\|_{\dot{W}^{1/2}_p(\mathbb{R},L_q(\mathbb{R}^N_+))}).
\end{aligned}$$

Theorem 31. *Let $1 < p, q < \infty$ and $\gamma > 0$. Let $\mathbf{F}$, G_{div}, $\mathbf{G}$ and $\mathbf{H}$ satisfy the conditions:*

$$\begin{aligned}
&e^{-\gamma t}\mathbf{F} \in L_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}G_{\mathrm{div}} \in L_p(\mathbb{R}, \dot{W}^1_q(\mathbb{R}^N_+)) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)),\\
&e^{-\gamma t}\mathbf{G} \in W^1_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N), \quad e^{\gamma t}\mathbf{H} \in L_p(\mathbb{R}, \dot{W}^1_q(\mathbb{R}^N_+)^N) \cap W^{1/2}_p(\mathbb{R}, L_q(\mathbb{R}^N_+)^N).
\end{aligned}$$

Moreover, initial data $\mathbf{a} \in B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)$ satisfies the compatibility conditions:

$$\mathbf{a} - \mathbf{G}|_{t=0} \in J_q(\mathbb{R}^N_+), \quad (\mathbf{D}(\mathbf{a})\mathbf{n} - (\mathbf{D}(\mathbf{a})\mathbf{n}, \mathbf{n})\mathbf{n} - (\mathbf{H}|_{t=0} - (\mathbf{H}|_{t=0}, \mathbf{n})\mathbf{n}))|_{\partial\mathbb{R}^N_+} = 0.$$

Here, the second condition should not be satisfied if the trace does not exist. Then, problem (71) *admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ such that*

$$e^{-\gamma t}\mathbf{u} \in L_p((0,\infty), W^2_q(\mathbb{R}^N_+)^N) \cap W^1_p((0,\infty), L_q(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}\nabla\mathfrak{q} \in L_p((0,\infty), L_q(\mathbb{R}^N_+)^N),$$

as well as

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t \mathbf{u}, \nabla \mathfrak{q})\|_{L_p((0,\infty),L_q(\mathbb{R}^N_+))} + \|e^{-\gamma t}\mathbf{u}\|_{L_p((0,\infty),W^2_q(\mathbb{R}^N_+))} + \sup_{t\in(0,\infty)} e^{-\gamma t}\|\mathbf{u}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)}\\
&\quad \leq C(\|\mathbf{a}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_+)} + \|e^{-\gamma t}(\mathbf{F}, \partial_t\mathbf{G}, \nabla G_{\rm div}, \nabla \mathbf{H})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))}\\
&\qquad + \|e^{-\gamma t}(G_{\rm div}, \mathbf{H})\|_{W^{1/2}_p(\mathbb{R},L_q(\mathbb{R}^N_+))}).
\end{aligned}$$

3.6. **Global well-posedness of equations** (52)**.** In this subsection, we shall prove the global well-posedness of equations (52) with small initial data. Since Lagrange transformation (53) gives a C^1 diffeomorphism under the assumption (54) as a solution $\mathbf{u}$ exists, instead of equations (52) we shall prove the global well-posedness of equations (58) for small initial data. In fact, we have the following theorem

Theorem 32. *Let $N \geq 3$. Let q_0, q_1 and q_2 be exponents such that*

$$q_0 = \frac{N}{2(1+\theta)} > 1, \quad q_1 = 2q_0, \quad \frac{1}{q_2} + \frac{1+\theta}{N} = \frac{1}{q_1},$$

where θ is a small positive number. Then, there exist an exponent $p > 2$ and a small constant $\sigma > 0$ such that if initial data $\mathbf{a} \in \cap_{i=0}^2 B^{2(1-1/p)}_{q7,p}(\mathbb{R}^N_+)$ satisfying the compatibility conditions:

$$\mathrm{div}\,\mathbf{a} = 0 \ \textit{in}\ \mathbb{R}^N_+, \quad \mu\mathbf{D}(\mathbf{a})\mathbf{n}_0 - (\mu\mathbf{D}(\mathbf{a})\mathbf{n}_0, \mathbf{n}_0)\mathbf{n}_0 = 0 \ \textit{on}\ \mathbb{R}^N_+,$$

then problem (58) *admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ such that*

$$\begin{aligned}
\mathbf{u} &\in \bigcap_{i=0}^2 (L_p((0,\infty), \dot W^2_{q_i}(\mathbb{R}^N_+)^N) \cap \dot W^1_p((0,\infty), L_{q_i}(\mathbb{R}^N_+)^N)),\\
\nabla\mathfrak{q} &\in \bigcap_{i=0}^2 L_p((0,\infty), L_{q_i}(\mathbb{R}^N_+)^N),
\end{aligned}$$

as well as

$$\begin{aligned}
&\|\partial_t\mathbf{u}\|_{L_p((0,\infty),L_{q_0}(\mathbb{R}^N_+))} + \|\nabla^2\mathbf{u}\|_{L_p((0,\infty),L_{q_0}(\mathbb{R}^N_+))}\\
&\quad + \sum_{i=1}^2 \|(1+t)\partial_t\mathbf{u}\|_{L_p((0,\infty),L_{q_i}(\mathbb{R}^N_+))} + \|(1+t)\nabla^2\mathbf{u}\|_{L_p((0,\infty),L_{q_i}(\mathbb{R}^N_+))} \leq C\sigma.
\end{aligned}$$

for some constant C independent of $\sigma > 0$.

Remark 33. In the exterior domain case, the global well-posdeness was proved in Shiata's lecture note [30].

The free boundary problem in the half-space was proved by Oishi and Shibata [24] when $N \geq 3$. The proof of theorem presented here is slightly modified thanks to discussion with Piotr Mucha and Tomasz Piasecki, Warsaw University [26]. The essential point is to use the homogeneous spaces unlike Oishi and Shibata [24].

As is well-known, one of important points to prove the global well-posedness for small initial data is to show some suitable decay estimate for the linearized equations. Namely, for some large exponent r for the time variable, we can show that

$$\begin{aligned}
&\|(1+t)\partial_t\mathbf{u}\|_{L_r((1,\infty),L_{q_2}(\mathbb{R}^N_+))} + \|(1+t)\nabla^2\mathbf{u}\|_{L_r((1,\infty),L_{q_2}(\mathbb{R}^N_+))}\\
&\quad \leq C(\|((1+t)(\mathbf{F}, \mathbf{G}, \nabla G_{\rm div}, \nabla\mathbf{H})\|_{L_r(\mathbb{R},L_{q_1}\cap L_{q_2}(\mathbb{R}^N_+))} + \|(1+t)\mathbf{G}\|_{\dot W^1_r(\mathbb{R},L_{q_1}\cap L_{q_2}(\mathbb{R}^N_+))})\\
&\qquad + \|(1+t)(G_{\rm div}, \mathbf{H})\|_{\dot W^{1/2}_r(\mathbb{R},L_{q_1}\cap L_{q_2}(\mathbb{R}^N_+))}),
\end{aligned}$$

$$
\begin{aligned}
&\|(1+t)\partial_t\mathbf{u}\|_{L_r((1,\infty),L_{q_1}(\mathbb{R}^N_+))} + \|(1+t)\nabla^2\mathbf{u}\|_{L_r((1,\infty),L_{q_1}(\mathbb{R}^N_+))}\\
&\quad \le C(\|((1+t)(\mathbf{F},\mathbf{G},\nabla G_{\mathrm{div}}\,,\nabla\mathbf{H})\|_{L_r(\mathbb{R},L_{q_0}\cap L_{q_1}(\mathbb{R}^N_+))} + \|(1+t)\mathbf{G}\|_{\dot W^1_r(\mathbb{R},L_{q_0}\cap L_{q_1}(\mathbb{R}^N_+))})\\
&\qquad + \|(1+t)(G_{\mathrm{div}}\,,\mathbf{H})\|_{\dot W^{1/2}_r(\mathbb{R},L_{q_0}\cap L_{q_1}(\mathbb{R}^N_+))}).
\end{aligned}
$$

To obtain these estimates, we consider the equations satisfying $t\mathbf{u}$ which reads as

$$
\begin{cases}
\partial_t(t\mathbf{u}) - \mathrm{Div}\,(\mu\mathbf{D}(t\mathbf{u}) - (t\mathfrak{q})\mathbf{I}) = \mathbf{u} + t\mathbf{F} & \text{in } \mathbb{R}^N_+\times\mathbb{R},\\
\mathrm{div}\,(t\mathbf{u}) = tG_{\mathrm{div}} = \mathrm{div}\,(t\mathbf{G}) & \text{in } \mathbb{R}^N_+\times\mathbb{R},\\
(\mu\mathbf{D}(t\mathbf{u}) - (t\mathfrak{q})\mathbf{I})\mathbf{n} = t\mathbf{H}\mathbf{n}_0 & \text{on } \partial\mathbb{R}^N_+\times\mathbb{R}
\end{cases}
\tag{110}
$$

By using Theorem 18 and Weis's operator valued Fourier multiplier theorem, we have

$$
\begin{aligned}
&\|t\partial_t\mathbf{u}\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))} + \|t\nabla^2\mathbf{u}\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))}\\
&\quad \le Cr_b(\|\mathbf{u}\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))} + \|t(\mathbf{F},\nabla G_{\mathrm{div}}\,,\nabla\mathbf{H})\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))}\\
&\qquad + \|\partial_t(t\mathbf{G})\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))} + \|t(G_{\mathrm{div}}\,,\mathbf{H})\|_{\dot W^{1/2}_p(\mathbb{R},L_q(\mathbb{R}^N_+))}.
\end{aligned}
$$

Thus, the point is to estimate $\|\mathbf{u}\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+))}$.

A known idea to estimate this term is to use the homogeneous parabolic type embeddings. In the inhomogeneous case, such embeddings have been used in many cases, for example Solonnikov [33]. But, here we give a different method based on our spectral analysis given in Theorem 18.

To this end, we use the following Sobolev's imbedding theorem.

Lemma 34. *Let $1 < p < q < \infty$ and $s = N(1/p - 1/q) \le 1$. Then,*

$$
\|f\|_{L_q(\mathbb{R}^N_+)} \le C\|f\|^{1-s}_{L_p(\mathbb{R}^N_+)}\|\nabla f\|^s_{L_p(\mathbb{R}^N_+)}.
$$

Proof. In the $\mathbb{R}^N$ case, this lemma is known as Gagliardo-Nirenberg's inequality, cf. [34, Theorem 3.3]. For the proof in the $s = 1$, refer to [34, Lemma 3.7]. When $0 < s < 1$, we shall give a proof based on the L_p-L_q estimates of heat kernel.

Let $H(t)f = \int_{\mathbb{R}^N} E(t, x-y)f(y)\,\mathrm{d}y$, where $E(t) = (4\pi t)^{N/2}e^{-|x|^2/(4t)}$. This gives a solution of the heat equation:

$$
(\partial_t - \Delta)u = 0 \quad \text{in } \mathbb{R}^N, \quad u|_{t=0} = f
\tag{111}
$$

for $u = H(t)f$. As we know well, there hold

$$
\begin{aligned}
\|H(t)f\|_{L_q(\mathbb{R}^N)} &\le Ct^{-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|f\|_{L_p(\mathbb{R}^N)},\\
\|\nabla H(t)f\|_{L_q(\mathbb{R}^N)} &\le Ct^{-\frac12-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|f\|_{L_p(\mathbb{R}^N)}.
\end{aligned}
\tag{112}
$$

We write

$$
f = H(t)f - \int_0^t \partial_\tau H(\tau)f\,\mathrm{d}\tau
$$

Since $\partial_\tau H(\tau)f = \Delta H(\tau)f = \nabla H(\tau)(\nabla f)$, by (112) and (**??**) we have

$$
\begin{aligned}
\|f\|_{L_q(\mathbb{R}^N)} &\le C\|H(t)f\|_{L_q(\mathbb{R}^N)} + \int_0^t \|\nabla H(\tau)(\nabla f)\|_{L_q(\mathbb{R}^N)}\,\mathrm{d}\tau\\
&\le C(t^{-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|f\|_{L_p(\mathbb{R}^N)} + \int_0^t \tau^{-\frac12-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|\nabla f\|_{L_p(\mathbb{R}^N)}\,\mathrm{d}\tau)\\
&\le C(t^{-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|f\|_{L_p(\mathbb{R}^N)} + t^{\frac12}t^{-\frac{N}{2}\left(\frac1p-\frac1q\right)}\|\nabla f\|_{L_p(\mathbb{R}^N)}).
\end{aligned}
\tag{113}
$$

We choose $t > 0$ in such a way that $t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p(\mathbb{R}^N)} = t^{\frac{1}{2}}t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\nabla f\|_{L_p(\mathbb{R}^N)}$, and so $t = (\|f\|_{L_p(\mathbb{R}^B)}/\|\nabla f\|_{L_p(\mathbb{R}^N)})^2$. Inserting this relation into (113) gives

$$\begin{aligned}\|f\|_{L_q(\mathbb{R}^N)} &\leq C(\|f\|_{L_p(\mathbb{R}^N)}/\|\nabla f\|_{L_p(\mathbb{R}^N)})^{2(-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p(\mathbb{R}^N)}\\ &= \|f\|_{L_q(\mathbb{R}^N)}^{1-N\left(\frac{1}{p}-\frac{1}{q}\right)}\|\nabla f\|_{L_p(\mathbb{R}^N)}^{N\left(\frac{1}{p}-\frac{1}{q}\right)}.\end{aligned}$$

When f is define on $\mathbb{R}^N_+$, then applying Lemma 34 to the even extension of f implies the required estimate. This completes the proof of Lemma 34. □

Let $\mathcal{S}(\lambda)$ and $\mathcal{P}(\lambda)$ be solution operators given in Theorem 18, and set

$$\mathbf{u} = \mathcal{L}^{-1}[\mathcal{S}(\lambda)\mathcal{L}[\mathbf{F}]], \quad \mathfrak{q} = \mathcal{L}^{-1}[\mathcal{P}(\lambda)\mathcal{L}[\mathbf{F}]]$$

with $\mathcal{F} = (\mathbf{F}, \Lambda^{1/2}G_{\rm div}, \nabla G_{\rm div}, \partial_t\mathbf{G}, \Lambda^{1/2}\mathbf{H}, \nabla\mathbf{H})$, where $\Lambda^{1/2}f = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f]]$. Then, $\mathbf{u}$ and $\mathfrak{p}$ are unique solutions of equations (71). Let $F = \mathcal{L}[\mathcal{F}]$. Let Γ be a contour in $\mathbb{C}$ defined by $\Gamma = \Gamma_+ \cup \Gamma_-$ and $\Gamma_\pm = \{\lambda = re^{\pm i(\pi-\epsilon)} \mid 0 < r < \infty\}$ $(\lambda \in \Sigma_\epsilon)$. First, we consider the case where F is independent of λ, and set $U(t)F = \mathcal{L}[\mathcal{S}(\lambda)F](t)$. Since $\|\mathcal{S}(\lambda)F\|_{L_q(\mathbb{R}^N_+)} \leq r_b|\lambda|^{-1}\|F\|_{L_q(\mathbb{R}^N_+)}$ for any $\lambda \in \Sigma_\epsilon$ as follows from Theorem 18, we have

$$U(t)F = \frac{1}{2\pi}\int_{\Gamma+\gamma} e^{\lambda t}\mathcal{S}(\lambda)F\,\mathrm{d}\lambda \quad \text{for } t > 0, \quad U(t)F = 0 \quad (t < 0).$$

Write $\lambda = re^{\pm i(\pi-\epsilon)}$ and then $\mathrm{Re}\,\lambda = \gamma - r\cos\epsilon$ and $|\lambda| \geq \sin(\epsilon/2)(\gamma + r)$. By Lemma 34 and Theorem 18,

$$\|\mathcal{S}(\lambda)F\|_{L_q(\mathbb{R}^N_+)} \leq C\|\mathcal{S}(\lambda)F\|^{1-s}_{L_p(\mathbb{R}^N_+)}\|\nabla\mathcal{S}(\lambda)F\|^s_{L_p(\mathbb{R}^N_+)} \leq Cr_b|\lambda|^{-\frac{s}{2}}\|F\|_{L_p(\mathbb{R}^N_+)}$$

for $s = N(1/p - 1/q) \leq 1$ with $1 < p < q < \infty$. Thus,

$$\begin{aligned}\|U(\cdot,t)F\|_{L_q(\mathbb{R}^N_+)} &\leq \frac{1}{\pi}Cr_b\int_0^\infty e^{\gamma t}e^{-\cos\epsilon rt}(\sin(\epsilon/2)(\gamma+r))^{-1/2}\,\mathrm{d}r\|F\|_{L_p(\mathbb{R}^N_+)}\\ &\leq Cr_b\frac{e^{\gamma t}}{\pi}t^{-1+\frac{s}{2}}\int_0^\infty e^{-\cos\epsilon\ell}\ell^{-s/2}\,\mathrm{d}\ell\|F\|_{L_p(\mathbb{R}^N_+)}\end{aligned}$$

for $t \geq 1$. Since this inequality holds for any $\gamma > 0$, we have

$$\|U(\cdot,t)F\|_{L_q(\mathbb{R}^N_+)} \leq Cr_bt^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|F\|_{L_p(\mathbb{R}^N_+)}.$$

When $0 < t < 1$, using the estimate: $\|\mathcal{S}(\lambda)F\|_{L_q(\mathbb{R}^N_+)} \leq r_b|\lambda|^{-1}\|F\|_{L_q(\mathbb{R}^N_+)}$ and the well-known argument in the theory of analytic semigroup theory, we have

$$\|U(\cdot,t)F\|_{L_q(\mathbb{R}^N_+)} \leq C\|F\|_{L_q(\mathbb{R}^N_+)}.$$

Therefore, for $t > 0$ we have

$$\|U(t)F\|_{L_q(\mathbb{R}^N_+)} \leq (1+t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}(\|F\|_{L_p(\mathbb{R}^N_+)} + \|F\|_{L_q(\mathbb{R}^N_+)}). \tag{114}$$

Since $U(t)F = 0$ for $t < 0$, we have (114) for all $t \in \mathbb{R}\setminus\{0\}$.

Now, we consider $F = \mathcal{L}[\mathcal{F}]$, and then

$$\mathbf{u}(\cdot,t) = \int_{\mathbb{R}} U(t-\ell)\mathcal{F}(\ell)\,\mathrm{d}\ell = \int_{-\infty}^t U(t-\ell)\mathcal{F}(\ell)\,d\ell.$$

Thus, choosing r in such a way that $r\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)>1$, by Minkowski's integral inequality

$$
\begin{aligned}
&\|\mathbf{u}\|_{L_r(0,\infty),L_q(\mathbb{R}^N_+))}\\
&\leq \Big\{\int_0^\infty\Big\{\int_{-\infty}^t \|U(t-\ell)\mathcal{F}(\ell)\|_{L_q(\mathbb{R}^N_+)}\,\mathrm{d}\ell\Big\}^r\,\mathrm{d}t\Big\}^{1/r}\\
&\leq \int_{-\infty}^\infty\Big\{\int_\ell^\infty \|U(t-\ell)\mathcal{F}(\ell)\|^r_{L_q(\mathbb{R}^N_+)}\,\mathrm{d}t\Big\}^{1/r}\,\mathrm{d}\ell\\
&\leq C\int_{-\infty}^\infty\Big\{\int_0^\infty (1+t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)r}(\|\mathcal{F}(\cdot,\ell)\|_{L_p(\mathbb{R}^N_+)}+\|\mathcal{F}(\cdot,\ell)\|_{L_q(\mathbb{R}^N_+)})^r\,\mathrm{d}t\Big\}^{1/r}\,\mathrm{d}\ell\\
&\leq C\int_{\mathbb{R}}(\|\mathcal{F}(\cdot,\ell)\|_{L_p(\mathbb{R}^N_+)}+\|\mathcal{F}(\cdot,\ell)\|_{L_q(\mathbb{R}^N_+)})\,\mathrm{d}\ell\\
&\leq C\Big(\int_{\mathbb{R}}(1+\ell)^{-r'}\,\mathrm{d}\ell\Big)^{1/r'}(\|(1+t)\mathcal{F}\|_{L_r(\mathbb{R},L_q(\mathbb{R}^N_+)}+\|(1+t)\mathcal{F}\|_{L_r(\mathbb{R},L_p(\mathbb{R}^N_+))},
\end{aligned}
$$

with $\mathcal{F}=(\mathbf{F},\Lambda^{1/2}G_{\rm div}\,,\nabla G_{\rm div}\,,\partial_t\mathbf{G},\Lambda^{1/2}\mathbf{H},\nabla\mathbf{H})$. In this way, we can show that

$$
\begin{aligned}
&\|(1+t)(\partial_t,\nabla^2)\mathbf{u}\|_{L_r((0,\infty),L_q(\mathbb{R}^N_+))}\\
&\quad\leq C(\|\mathbf{a}\|_{B^{2(1-1/r)}_{q,r}(\mathbb{R}^N_+)}+\|(1+t)(\mathbf{F},\nabla G_{\rm div}\,,\nabla\mathbf{H})\|_{L_r(\mathbb{R},L_p(\mathbb{R}^N_+)\cap L_q(\mathbb{R}^N_+))}\\
&\qquad+\|(1+t)\mathbf{G}\|_{\dot W^1_r(\mathbb{R},L_p(\mathbb{R}^N_+)\cap L_q(\mathbb{R}^N_+))}+\|(1+t)(G_{\rm div}\,,\mathbf{H})\|_{\dot W^{1/2}_r(\mathbb{R},L_p(\mathbb{R}^N_+)\cap L_q(\mathbb{R}^N_+))}
\end{aligned}
$$

with large r with $r(N/2)(1/p-1/q)>1$ for $1<p<q<\infty$.

4. Free boundary problem for the Navier-Stokes equations in the L_1 - Besov spaces maximal regularity framework

4.1. L_1 - Besov spaces maximal regularity for the Stokes equations with free boundary conditions. In this subsection, we discuss free boundary problem (58) in the L_1 in time and $\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$ in space framework. Here and in the sequel, $\mathcal{B}^s_{q,r}$ stands for the inhomogeneous Besov space $B^s_{q,r}$ or the homogeneous Besov space $\dot B^s_{q,r}$. To obtain the maximal L_1 in space and Besov in space regularity, there is no difference in technical issues between the homogeneous Besov space case and the inhomogeneous Besov space case, and so we write $\mathcal{B}^s_{q,r}$ to denote both of $B^s_{q,r}$ and $\dot B^s_{q,r}$ at the same time. The discussion in this section deeply depends on my joint work with Keiichi Watanabe, [32]. Let

$$
\begin{aligned}
&\mathcal{B}^{s+1}_{q,r}(\mathbb{R}^N_+)=\{f\in\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)\mid\nabla f\in\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^N\},\\
&\mathcal{B}^{s+2}_{q,r}(\mathbb{R}^N_+)=\{f\in\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)\mid\nabla f\in\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^N,\quad \nabla^2 f\in\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)^{N^2}\},\\
&\widehat{\mathcal{B}}^{s+1}_{q,r,0}(\mathbb{R}^N_+)=\{f\mid\exists\, g\in\mathcal{B}^{s+1}_{q,r,{\rm loc}}(\mathbb{R}^N)\ \text{such that}\ \nabla g\in\mathcal{B}^s_{q,1}(\mathbb{R}^N),\ \operatorname{supp} g\subset\overline{\mathbb{R}^N_+},\quad g|_{\mathbb{R}^N_+}=f\}.
\end{aligned}
$$

Remark 35. When $-1+1/q<s<1/q$, then $s+1>1/q$. Thus, for $f\in\widehat{B}^{s+1}_{q,r,0}(\mathbb{R}^N_+)$, $f|_{\partial\mathbb{R}^N_+}=0$.

To prove the L_1 - $\mathcal{B}^s_{q,1}$ maximal regularity of linear problem (71), in view of Theorem 16, it is sufficient to prove some estimates, given in Theorem 36 below, for the corresponding generalized resolvent problem (72). Namely, the main point of our proof of L_1-$\mathcal{B}^s_{q,1}$ maximal regularity is to prove the following theorem.

Theorem 36. *Let $1<q<\infty$, $-1+1/q<s<1/q$ and $\gamma>0$. Let $\gamma_b=0$ when $\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)=\dot B^s_{q,1}(\mathbb{R}^N_+)$ and let $\gamma_b=\gamma$ when $\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)=B^s_{q,1}(\mathbb{R}^N_+)$. Then, there exist operator families $\mathcal{S}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$
\begin{aligned}
&\mathcal{S}(\lambda)\in{\rm Hol}\,(\Sigma_\epsilon+\gamma_b,\mathcal{L}(\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^{M_N},\mathcal{B}^{s+2}_{q,1}(\mathbb{R}^N_+)^N)),\\
&\mathcal{P}(\lambda)\in{\rm Hol}\,(\Sigma_\epsilon+\gamma_b,\mathcal{L}(\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^{M_N},\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^N))
\end{aligned}
\tag{115}
$$

such that for any $\lambda \in \Sigma_\epsilon + \gamma_b$ *and* $\mathbf{f} \in B^s_q(\mathbb{R}^N_+)^N$, $g_{\rm div} \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$, $\mathbf{g} \in \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^N$, *and* $\mathbf{h} \in \mathcal{B}^{s+1}_{q,1}(\mathbb{R}^N_+)^N$, *problem* (72) *admits unique solutions* $\mathbf{u} \in \mathcal{B}^{s+2}_{q,1}(\mathbb{R}^N_+)^N$ *and* $\mathfrak{p} \in \mathcal{B}^{s+1}_{q,1}(\mathbb{R}^N_+) + \hat{\mathcal{B}}^{s+1}_{q,1,0}(\mathbb{R}^N_+)$ *with* $\mathbf{u} = \mathcal{S}(\lambda)\mathbf{F}$ *and* $\nabla\mathfrak{p} = \mathcal{P}(\lambda)\mathbf{F}$, *where* $\mathbf{F} = (\mathbf{f}, \lambda^{1/2}g_{\rm div}, \nabla g_{\rm div}, \lambda\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$, *as well as*

$$\begin{aligned} \|(\lambda, \lambda^{1/2}\nabla, \nabla^2_b)\mathcal{S}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b\|F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)}, \\ \|\nabla\mathcal{P}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b\|F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} \end{aligned} \tag{116}$$

for any $\lambda \in \Sigma_\epsilon + \gamma_b$. *Here and in the sequel, we write* $\nabla^2_b = \nabla^2$ *when* $\mathcal{B}^s_{q,1} = \dot{B}^s_{q,1}$ *and* $\nabla^2_b = \bar{\nabla}^2$ *when* $\mathcal{B}^s_{q,1} = B^s_{q,1}$, *and* C_b *denotes general constants which is independent of* γ *when* $\mathcal{B}^s_{q,1} = \dot{B}^s_{q,1}$ *and depends on* $\gamma > 0$ *when* $\mathcal{B}^s_{q,1}(\mathbb{R}^N_+) = B^s_{q,1}(\mathbb{R}^N_+)$.

Moreover, let $\sigma > 0$ *be a small positive number such that* $-1+1/q < s-\sigma < s < s+\sigma < 1/q$. *Then, for any* $\lambda \in \Sigma_\epsilon + \gamma_b$, *there hold*

$$\begin{aligned} \|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b|\lambda|^{-\frac{\sigma}{2}}\|F\|_{\mathcal{B}^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}, \\ \|\nabla\mathcal{P}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b|\lambda|^{-\frac{\sigma}{2}}\|F\|_{\mathcal{B}^{s+\sigma}_{q,1}(\mathbb{R}^N_+)} \end{aligned} \tag{117}$$

provided $F \in \mathcal{B}^{s+\sigma}_{q,1}(\mathbb{R}^N_+) \cap \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$, *as well as*

$$\begin{aligned} \|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b|\lambda|^{-(1-\frac{\sigma}{2})}\|F\|_{\mathcal{B}^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}, \\ \|\nabla\partial_\lambda\mathcal{P}(\lambda)F\|_{\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)} &\leq C_b|\lambda|^{-(1-\frac{\sigma}{2})}\|F\|_{\mathcal{B}^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \end{aligned} \tag{118}$$

provided $F \in \mathcal{B}^{s-\sigma}_{q,1}(\mathbb{R}^N_+) \cap \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$.

Following the argument in Subsection 2.3 and using Theorem 36, we have the following L_1-$\mathcal{B}^s_{q,1}$ maximal regularity theorem for equations (71).

Theorem 37. *Let* $1 < q < \infty$ *and* $-1+1/q < s < 1/q$. *Then, we have the following two maximal regularity theorem as follows:*
(1) (*Inhomogeneous Besov space case*) *Let* $\gamma > 0$. *Let* $\mathbf{F}$, $G_{\rm div}$, $\mathbf{G}$, $\mathbf{H}$ *and* $\mathbf{a}$ *be data for equations* (71) *such that*

$$\begin{aligned} &e^{-\gamma t}\mathbf{F} \in L_1(\mathbb{R}, B^s_{q,1}(\mathbb{R}^N_+)^N), \quad e^{-\gamma t}G_{\rm div} \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\mathbb{R}^N_+)) \cap W^{1/2}_{q,1}(\mathbb{R}, B^s_{q,1}(\mathbb{R}^N_+)), \\ &e^{-\gamma t}\mathbf{G} \in W^1_1(\mathbb{R}, B^s_{q,1}(\mathbb{R}^N_+)^N), \ e^{-\gamma t}\mathbf{H} \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\mathbb{R}^N_+)^N) \cap W^{1/2}_{q,1}(\mathbb{R}, B^s_{q,1}(\mathbb{R}^N_+)^N), \end{aligned}$$

as well as $\mathbf{a} \in B^s_{q,1}(\mathbb{R}^N_+)^N$ *satisfies the compatibility conditions:* $\operatorname{div}(\mathbf{a} - \mathbf{G}|_{t=0}) = 0$ *in* $\mathbb{R}^N_+$. *Then, problem* (71) *admits unique solutions* $\mathbf{u}$ *and* $\mathfrak{q}$ *such that*

$$\begin{aligned} &e^{-\gamma t}\mathbf{u} \in L_1((0,\infty), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,\infty), B^s_{q,1}(\mathbb{R}^N_+)^N), \\ &e^{-\gamma t}\nabla\mathfrak{q} \in L_1((0,\infty), B^s_{q,1}(\mathbb{R}^N_+)^N) \end{aligned}$$

as well as

$$\begin{aligned} &\|e^{-\gamma t}\mathbf{u}\|_{L_1((0,\infty),B^{s+2}_{q,1}(\mathbb{R}^N_+))} + \|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_1((0,\infty),B^s_{q,1}(\mathbb{R}^N_+))} + \|e^{-\gamma t}\nabla\mathfrak{q}\|_{L_1((0,\infty),B^s_{q,1}(\mathbb{R}^N_+)^N))} \\ &\quad \leq C(\|\mathbf{a}\|_{B^s_{q,1}(\mathbb{R}^N_+)} + \|e^{-\gamma t}(\mathbf{F}, \partial_t\mathbf{G}, \nabla G_{\rm div}, \nabla\mathbf{H})\|_{L_1(\mathbb{R},B^s_{q,1}(\mathbb{R}^N_+))} \\ &\qquad + \|e^{-\gamma t}(G_{\rm div}, \mathbf{H})\|_{W^{1/2}_1(\mathbb{R},B^s_{q,1}(\mathbb{R}^N_+))}). \end{aligned}$$

(2) (*Homogeneous Besov space case*) *Let* $\mathbf{F}$, $G_{\rm div}$, $\mathbf{G}$, $\mathbf{H}$ *and* $\mathbf{a}$ *be data for equations* (71) *such that*

$$\begin{aligned} &\mathbf{F} \in L_1(\mathbb{R}, \dot{B}^s_{q,1}(\mathbb{R}^N_+)^N), \quad G_{\rm div} \in L_1(\mathbb{R}, \dot{B}^{s+1}_{q,1}(\mathbb{R}^N_+)) \cap \dot{W}^{1/2}_{q,1}(\mathbb{R}, \dot{B}^s_{q,1}(\mathbb{R}^N_+)), \\ &\mathbf{G} \in \dot{W}^1_1(\mathbb{R}, \dot{B}^s_{q,1}(\mathbb{R}^N_+)^N), \ \mathbf{H} \in L_1(\mathbb{R}, \dot{B}^{s+1}_{q,1}(\mathbb{R}^N_+)^N) \cap \dot{W}^{1/2}_{q,1}(\mathbb{R}, \dot{B}^s_{q,1}(\mathbb{R}^N_+)^N), \end{aligned}$$

as well as $\mathbf{a} \in \dot B^s_{q,1}(\mathbb{R}^N_+)^N$ *satisfies the compatibility conditions:* $\mathrm{div}\,(\mathbf{a} - \mathbf{G}|_{t=0}) = 0$ *in* $\mathbb{R}^N_+$. *Then, problem* (71) *admits unique solutions* $\mathbf{u}$ *and* $\mathfrak{q}$ *such that*

$$\mathbf{u}, \partial_j\partial_k\mathbf{u}, \nabla\mathfrak{q} \in L_1((0,\infty), B^s_{q,1}(\mathbb{R}^N_+)^N), \quad \mathbf{u} \in BC([0,\infty), B^s_{q,1}(\mathbb{R}^N_+)^N),$$
$$t^{1/2}\nabla\mathbf{u} \in L_\infty((0,\infty), B^s_{q,1}(\mathbb{R}^N_+)^N)$$

for $j, k =, \ldots, N$*, as well as there holds*

$$\|(\partial_t\mathbf{u}, \nabla^2\mathbf{u}, \nabla\mathfrak{q})\|_{L_1((0,\infty),\dot B^s_{q,1}(\mathbb{R}^N_+))} + \sup_{t\in(0,\infty)} \|\mathbf{u}(\cdot,t)\|_{\dot B^s_{q,1}(\mathbb{R}^N_+))} + \sup_{t\in(0,\infty)} t^{1/2}\|\nabla\mathbf{u}(\cdot,t)\|_{\dot B^s_{q,1}(\mathbb{R}^N_+)}$$
$$\leq C(\|\mathbf{a}\|_{\dot B^s_{q,1}(\mathbb{R}^N_+)} + \|(\mathbf{F}, \partial_t\mathbf{G}, \nabla G_{\rm div}, \nabla\mathbf{H})\|_{L_1(\mathbb{R},\dot B^s_{q,1}(\mathbb{R}^N_+))} + \|(G_{\rm div}, \mathbf{H})\|_{\dot W^{1/2}_1(\mathbb{R},\dot B^s_{q,1}(\mathbb{R}^N_+))}).$$

Idea of my proof of Theorem 36. Since my proof is based on interpolation theorems and since my method seems to be applicable to prove the L_1 - $\mathcal{B}^s_{q,1}$ maximal regularity in many initial boudary value problems for the system of parabolic or hyperbolic-parabolic equations appearing in mathematical physics, I will focus on how to use the interpolation results.

We assume that $1 < q < \infty$ and $-1 + 1/q < s < 1/q$. $\mathcal{B}^s_{q,1}$ is taken as a basic space, and the reason is only that $C^\infty_0(\Omega)$ is dense in $\mathcal{B}^s_{q,1}(\Omega)$ for $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. In the sequel, $T(\lambda)$ denotes one of $\lambda\mathcal{S}(\lambda)$, $\lambda^{1/2}\nabla\mathcal{S}(\lambda)$. Analytic evaluation of operators is only initial evaluation in $\mathcal{H}^1_q(\mathbb{R}^N_+)$. In the sequel, we write

$$H^\alpha_q(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) \mid \|f\|_{H^\alpha_q(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\alpha/2}\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} < \infty\},$$
$$\dot H^\alpha_q(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}(\mathbb{R}^N) \mid \|f\|_{H^\alpha_q(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[|\xi|^\alpha\mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} < \infty\}.$$

Here, $\mathcal{P}(\mathbb{R}^N)$ denotes the set of all polynomials on $\mathbb{R}^N$. Note that $H^1_q(\mathbb{R}^N) = W^1_q(\mathbb{R}^N)$ and $\dot H^1_q(\mathbb{R}^N) = \{f \in L_{q,\rm loc}(\mathbb{R}^N) \mid \nabla f \in L_q(\mathbb{R}^N)^N\}/\{constants\}$. Here, $\{\cdot\}/\{constants\}$ means that if $\nabla f = 0$, then $f = 0$ as a member of $\{\cdot\}$. Note that $\mathcal{H}^\alpha_q(\mathbb{R}^N) = (L_q(\mathbb{R}^N), \mathcal{H}^1_q(\mathbb{R}))_{[\alpha]}$ for $\alpha \in (0,1)$, where $(\cdot,\cdot)_{[\alpha]}$ stands for complex interpolation functors.

Let

$$\mathcal{H}^\alpha_q(\mathbb{R}^N_+) = \{f \mid \exists\, g \in \mathcal{H}^\alpha_q(\mathbb{R}^N) \text{ such that } g|_{\mathbb{R}^N_+} = f\},$$
$$\|f\|_{\mathcal{H}^\alpha_q(\mathbb{R}^N_+)} = \inf\{\|g\|_{\mathcal{H}^\alpha_q(\mathbb{R}^N)} \mid g \in \mathcal{H}^\alpha_q(\mathbb{R}^N) \text{ such that } g|_{\mathbb{R}^N_+} = f\}.$$

We see that $\mathcal{H}^\alpha_q(\mathbb{R}^N_+) = (L_q(\mathbb{R}^N_+), \mathcal{H}^1_q(\mathbb{R}^N_+))_{[\alpha]}$ for $\alpha \in (0,1)$. Moreover, $H^1_q(\mathbb{R}^N_+) = W^1_q(\mathbb{R}^N_+)$ and $\dot H^1_q(\mathbb{R}^N_+) = \{f \in L_{q,\rm loc}(\mathbb{R}^N_+) \mid \nabla f \in L_q(\mathbb{R}^N_+)^N\}/\{constants\}$.

We denote that $\mathcal{H}^\alpha_q(\Omega) = H^\alpha_q(\Omega)$ when $\mathcal{B}^s_{q,1}(\Omega) = B^s_{q,1}(\Omega)$ and $\mathcal{H}^\alpha_q(\Omega) = \dot H^\alpha_q(\Omega)$ when $\mathcal{B}^s_{q,1}(\Omega) = \dot B^s_{q,1}(\Omega)$, where $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. Let $\mathcal{H}^\alpha_{q,0}(\mathbb{R}^N_+)$ denotes the closure of $C^\infty_0(\mathbb{R}^N_+)$ in $\mathcal{H}^\alpha_q(\mathbb{R}^N_+)$.

We use the following results concerning the real and complex interpolations.

Proposition 38. *Let* $1 < q < \infty$ *and* $q' = q/(q-1)$. *Let* $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. *Then, the following assertions are valid.*

1. *For* $1 \leq r \leq \infty$ *and* $-\infty < s < \infty$*, it follows that* $(\mathcal{H}^s_q(\mathbb{R}^N))' = \mathcal{H}^{-s}_{q'}(\mathbb{R}^N)$, $(H^s_{q,0}(\mathbb{R}^N_+))' = H^{-s}_{q'}(\mathbb{R}^N_+)$.
2. *For* $-d/q' < s < d/q$*, it follows that* $(\dot H^s_q(\mathbb{R}^N_+))' = \dot H^{-s}_{q',0}(\mathbb{R}^N_+)$ *and* $(\dot H^s_{q,0}(\mathbb{R}^N_+))' = \dot H^{-s}_{q'}(\mathbb{R}^N_+)$.
3. *Let* $1 \leq q_0, q_1, r_0, r_1, r \leq \infty$, $-\infty < s_0, s_1 < \infty$, $s_0 \neq s_1$*, and* $0 < \theta < 1$. *Let* s *and* q *be defined by* $s = (1-\theta)s_0 + \theta s_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then, there hold*

$$(H^{s_0}_q(\Omega), H^{s_1}_q(\Omega))_{\theta,r} = B^s_{q,r}(\Omega), \tag{119}$$
$$(B^{s_0}_{q,r_0}(\Omega), B^{s_1}_{q,r_1}(\Omega))_{\theta,r} = B^s_{q,r}(\Omega), \tag{120}$$
$$[H^{s_0}_{q_0}(\Omega), H^{s_1}_{q_1}(\Omega)]_\theta = H^s_q(\Omega) \tag{121}$$

If s_0, s_1, and s satisfy additionally $s_j > -1+1/q_j$, $j \in \{0,1\}$, and $s < d/q$ (or $s \le d/q$ if $r=1$), then there hold

$$(\dot H^{s_0}_q(\Omega), \dot H^{s_1}_q(\Omega))_{\theta,r} = \dot B^s_{q,r}(\Omega), \tag{122}$$

$$(\dot B^{s_0}_{q,r_0}(\Omega), \dot B^{s_1}_{q,r_1}(\Omega))_{\theta,r} = \dot B^s_{q,r}(\Omega), \tag{123}$$

$$[\dot H^{s_0}_{q_0}(\Omega), \dot H^{s_1}_{q_1}(\Omega)]_\theta = \dot H^s_q(\Omega). \tag{124}$$

with $s := (1-\theta)s_0 + \theta s_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$.

(4) *For $1 \le q_1 \le q_0 \le \infty$ and $1 \le r_1 \le r_0 \le \infty$, and $s \in \mathbb{R}$, it follows that $\mathcal{B}^s_{q_0,r_0}(\Omega) \hookrightarrow \mathcal{B}^{s+d(\frac{1}{q_1}-\frac{1}{q_0})}_{q_1,r_1}(\Omega)$.*

Roughly speaking, the idea of my proof of Theorem 36 is the following: Below, we assume that $1<q<\infty$ and $-1+1/q<s<1/q$.
(1) When $0<s<1/q$, the starting evaluation is done in $\mathcal{H}^1_q$. Then, using the complex interpolation to obtain the estimates in $\mathcal{H}^\mu_q$ $(0<\mu<1/q)$. Finally, by real interpolation, we arrive at the estimates in $\mathcal{B}^s_{q,1}$.
(2) When $-1+1/q<s<0$. First, we consider the dual operator and we evaluate it in $\mathcal{H}^1_{q'}$. Secondly, we use the complex interpolation to obtain the estimates of dual operators in $\mathcal{H}^\mu_{q'}$ $(0<\mu<1/q'=1-1/q)$. Thirdly, by the duality argument, we obtain the estimates in $\mathcal{H}^{-\mu}_q$. Finally, by real interpolation, we arrive at the estimates in $\mathcal{B}^s_{q,1}$.
(3) The esitmates in $\mathcal{B}^0_{q,1}$ follows from the real interpolations between $\mathcal{B}^s_{q,1}$ and $\mathcal{B}^{-s}_{q,1}$.

First, we consider the case $0<s<1/q$. We assume that

Assumption 4.1. Let $1<q<\infty$ and $\gamma>0$. We assume that the starting evaluations hold as follows:

For any $f \in C^\infty_0(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_\epsilon + \gamma_b$, the following estimates hold:

$$\|T(\lambda)f\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)} \le C_b\|f\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)}, \tag{125}$$

$$\|T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C_b\|f\|_{L_q(\mathbb{R}^N_+)}; \tag{126}$$

$$\|T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C_b|\lambda|^{-1/2}\|f\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)} \tag{127}$$

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)} \le C_b|\lambda|^{-1}\|f\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)}, \tag{128}$$

$$\|\partial_\lambda T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C_b|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)}; \tag{129}$$

$$\|\partial_\lambda T(\lambda)F\|_{\mathcal{H}^1_q(\mathbb{R}^N_+)} \le C_b|\lambda|^{-1/2}\|f\|_{L_q(\mathbb{R}^N_+)}. \tag{130}$$

Here and in the sequel, $\gamma=0$ when $\mathcal{H}^\alpha_q = \dot H^\alpha_q$ and $\gamma_b=\gamma>0$ when $\mathcal{H}^\alpha_q = H^\alpha_q$, and C_b is a condtant independent of γ when $\mathcal{H}^\alpha_q = \dot H^\alpha_q$ and depending on γ when $\mathcal{H}^\alpha_q = H^\alpha_q$. Then, we have

Proposition 39. *We assume that Assumption 4.1 above holds. Let q and γ be the same as in Assumption 1. Let $1 \le r \le \infty$. Let $0<s<1/q$ and let $\sigma>0$ be numbers such that $0<s-\sigma<s<s+\sigma<1/q$. Then, for any $\lambda \in \Sigma_\epsilon+\gamma_b$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \le C_b\|f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)}, \tag{131}$$

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \le C_b|\lambda|^{-\frac{\sigma}{2}}\|f\|_{\mathcal{B}^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}, \tag{132}$$

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \le C_b|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{\mathcal{B}^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{133}$$

Here and in the sequel, $\gamma_b=0$ when $\mathcal{B}^\mu_{q,1} = \dot B^\mu_{q,1}$ and $\gamma_b=\gamma>0$ when $\mathcal{B}^\mu_{q,1} = B^\mu_{q,1}$ and C_b is a condtant independent of γ when $\mathcal{B}^\mu_{q,1} = \dot B^s_{q,1}$ and depending on γ when $\mathcal{B}^\mu_{q,1} = B^\mu_{q,1}$.

Proof. Below, we always assume that $f \in C^\infty_0(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_\epsilon+\gamma_b$. Choose μ and μ' in such a way that $0<s<s+\sigma<\mu'<\mu<1/q$. Estimates (125), (126), ande (127) are interpolated with complex interpolation method to obtain

$$\|T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_b\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{134}$$

$$\|T(\lambda)f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)} \leq C_b\|f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)}, \tag{135}$$

$$\|T(\lambda)f\|_{\mathcal{H}^{\mu'}_q(\mathbb{R}^N_+)} \leq C_b\|f\|_{\mathcal{H}^{\mu'}_q(\mathbb{R}^N_+)}, \tag{136}$$

$$\|T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-\mu/2}\|f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)} \tag{137}$$

By interpolating (134) and (135) with real interpolation method,

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b\|f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)}. \tag{138}$$

Choosing $\theta = s/\mu'$ and setting $A = \mu(1 - s/\mu')$, by (136) and (137) wtih real interpolation method,

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-\frac{A}{2}}\|f\|_{\mathcal{B}^{s+A}_{q,r}(\mathbb{R}^N_+)} \tag{139}$$

Now, we choose μ and μ' in such a way that $s < s + \sigma < s + A$, that is, we choose μ and μ' in such a way that $\sigma/\mu + s/\mu' < 1$ and $s + \sigma < \mu' < \mu < 1/q$. Thus, choosing $\theta \in (0, 1)$ in such a way that $s + \sigma = (1 - \theta)s + \theta(s + A)$, that is, $\theta = \sigma/A$, by (138) and (139) we have

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-\frac{\sigma}{2}}\|f\|_{\mathcal{B}^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{140}$$

Therefore, we have (131) and (132)

Now, we shall prove (133). Let μ be a number such that $0 < s < s+\sigma < \mu < 1/q$. Combining (128) and (129), and (128) and (130) with complex interpolation method, implies that

$$\|\partial_\lambda T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{141}$$

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)}, \tag{142}$$

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{\mu}{2})}\|f\|_{L_q(\mathbb{R}^N_+)}. \tag{143}$$

Combining (141) and (142) with real interpolation method yields

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)}. \tag{144}$$

Now, choosing μ' and θ in such a way that $0 < \mu' < \mu$ and $\theta = \mu'/\mu \in (0, 1)$ and combining (142) and (143) with complex interpolation, we have

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^\mu_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-(1/2)(\mu-\mu'))}\|f\|_{\mathcal{H}^{\mu'}_q(\mathbb{R}^N_+)}. \tag{145}$$

as follows from $\theta + (1 - \mu/2)(1 - \theta) = 1 - (\mu/2)(1 - \theta) = 1 - \frac{\mu}{2}(1 - \frac{\mu'}{\mu}) = 1 - (1/2)(\mu - \mu')$.

Next, we will combine (141) and (145) with real interpolation method for $s = \theta\mu$, Namely, we choose $\theta = s/\mu \in (0, 1)$ and so $\theta\mu' = (\mu'/\mu)s$,

$$(1 - (1/2)(\mu - \mu'))\theta + (1 - \theta) = 1 - \frac{\theta}{2}(\mu - \mu') = (1 - \frac{s}{2\mu}(\mu - \mu')).$$

Thus, we have

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{s}{2\mu}(\mu-\mu'))}\|f\|_{\mathcal{B}^{\frac{\mu'}{\mu}s}_{q,r}(\mathbb{R}^N_+)} \tag{146}$$

Finally, we will combine (144) and (146) with real interpolation method. We choose $0 < \mu' < \mu$ in such a way that $(\mu'/\mu)s < s - \sigma < s$, that is $0 < \mu' < (1 - \dfrac{\sigma}{s})\mu$. And, we choose $\theta \in (0, 1)$ in such a way that $s - \sigma = (1 - \theta)s + \theta(\mu'/\mu)s$, that is $\theta = \sigma/A$ with $A = s(1 - \mu'/\mu)$. In this case, we have

$$(1 - \theta) + \theta(1 - \frac{s}{2\mu}(\mu - \mu')) = 1 - \frac{s}{2}(1 - \frac{\mu'}{\mu})\theta = 1 - \frac{s}{2}\frac{A}{s}\frac{\sigma}{A} = 1 - \frac{\sigma}{2}.$$

Thus, by (144) and (146), we have

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{\mathcal{B}^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{147}$$

Therefore, we have proved Proposition 39. □

Next, we consider the case where $-1+1/q < s < 0$, that is $0 < |s| < 1-1/q = 1/q'$. We assume the existence of dual operators $T(\lambda)^*$ and $\partial_\lambda T(\lambda)^*$ such that
Assumption 4.2. Let $1 < q < \infty$, $q' = q/(q-1)$ and $\gamma > 0$. For any $\varphi \in C^\infty_0(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_\epsilon + \gamma_b$, there hold

$$\|T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \leq C_b\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)}, \tag{148}$$

$$\|T(\lambda)^*\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)} \leq C_b\|\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)}, \tag{149}$$

$$\|T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1/2}\|\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)}, \tag{150}$$

$$\|\partial_\lambda T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)}, \tag{151}$$

$$\|\partial_\lambda T(\lambda)^*\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)}, \tag{152}$$

$$\|\partial_\lambda T(\lambda)^*\varphi\|_{\mathcal{H}^1_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1/2}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)}. \tag{153}$$

Here, the dual operators means that for any f, $\varphi \in C^\infty_0$ and $\lambda \in \Sigma_\epsilon + \gamma_b$ there hold

$$|(T(\lambda)f, \varphi| = |(f, T(\lambda)^*\varphi)|, \quad |(\partial_\lambda T(\lambda)f, \varphi| = |(f, \partial_\lambda T(\lambda)^*\varphi)|$$

where $(a,b) = \int_{\mathbb{R}^N_+} a(x)b(x)\,\mathrm{d}x$. Note that we do not take the complex conjugate to define the dual operator in order not to consider the operator for parameter $\overline{\lambda}$.

We shall prove the following proposition.

Proposition 40. *We assume that Assumption 4.2 above holds. Let q and γ be the same as in Assumption 2. Let $1 \leq r \leq \infty$. Let $-1+1/q < s < 0$ and let $\sigma > 0$ be a number such that $-1+1/q < s-\sigma < s < s+\sigma < 0$. Then, for any $\lambda \in \Sigma_\epsilon + \gamma_b$ and $f \in C^\infty_0(\mathbb{R}^N_+)$, there hold*

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b\|f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)}, \tag{154}$$

$$\|T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-\frac{\sigma}{2}}\|f\|_{\mathcal{B}^{s+\sigma}_{q,r}(\mathbb{R}^N_+)}, \tag{155}$$

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{\sigma}{2})}\|f\|_{\mathcal{B}^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}. \tag{156}$$

Proof. Since $-1+1/q < s < 0$, we have $0 < |s| < 1-1/q = 1/q'$. Let μ, μ' and σ be positive number such that

$$0 < \mu' < |s| - \sigma < |s| < \mu < 1/q'. \tag{157}$$

In the same manner as in the proof of Proposition 39, using the complex interpolation method, by (148), (149), and (150), we have

$$\begin{aligned}
\|T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} &\leq C_b\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|T(\lambda)^*\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)} &\leq C_b\|\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)},\\
\|T(\lambda)^*\varphi\|_{\mathcal{H}^{\mu'}_{q'}(\mathbb{R}^N_+)} &\leq C_b\|\varphi\|_{\mathcal{H}^{\mu'}_{q'}(\mathbb{R}^N_+)},\\
\|T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} &\leq C_b|\lambda|^{-\mu/2}\|\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)}.
\end{aligned}$$

By the duality argument, we have

$$\|T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C_b\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{158}$$

$$\|T(\lambda)f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)} \le C_b\|f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)}, \tag{159}$$

$$\|T(\lambda)f\|_{\mathcal{H}^{-\mu'}_q(\mathbb{R}^N_+)} \le C_b\|f\|_{\mathcal{H}^{-\mu'}_q(\mathbb{R}^N_+)}, \tag{160}$$

$$\|T(\lambda)f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)} \le C_b|\lambda|^{-\mu/2}\|f\|_{L_q(\mathbb{R}^N_+)} \tag{161}$$

In fact, note that $\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+) = (\mathcal{H}^{\mu}_{q',0}(\mathbb{R}^N_+))^*$. For any f and $\varphi \in C^\infty_0(\mathbb{R}^N_+)$, by the dual argument we have

$$\begin{aligned}|(T(\lambda)f,\varphi)| &= |(f, T(\lambda)^*\varphi)| \\ &\le \|f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)}\|T(\lambda)^*\varphi\|_{\mathcal{H}^{\mu}_{q'}(\mathbb{R}^N_+)} \\ &\le \|f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)}C_b\|\varphi\|_{\mathcal{H}^{\mu}_{q'}(\mathbb{R}^N_+)},\end{aligned}$$

which implies (159). Likewise, we have (160) and (158). And also,

$$\begin{aligned}|(T(\lambda)f,\varphi)| &= |(f, T(\lambda)^*\varphi)| \\ &\le \|f\|_{L_q(\mathbb{R}^N_+)}\|T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \\ &\le \|f\|_{L_q(\mathbb{R}^N_+)}C_b|\lambda|^{-\mu/2}\|\varphi\|_{\mathcal{H}^{\mu}_{q'}(\mathbb{R}^N_+)},\end{aligned}$$

which imples (161).

Now, we shall prove (154) and (155) in Proposition 40. Combining (158) and (159) with real interpolation method, we have

$$\|T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C_b\|f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)}, \tag{162}$$

which shows (154).

Next, recall that $0 < \mu' < |s| - \sigma < |s| < \mu < 1/q'$ as follows from (157). Choose $\theta \in (0,1)$ in such a way that $-|s| = -\mu(1-\theta) - \mu'\theta$, that is $\theta = \dfrac{\mu - |s|}{\mu - \mu'}$. Combining (160) and (161) with real interpolation method implies that

$$\|T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C_b|\lambda|^{-\frac{\mu}{2}(1-\theta)}\|f\|_{\mathcal{B}^{(-\mu')\theta}_{q,r}}.$$

Therefore, we have

$$\|T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,1}(\mathbb{R}^N_+)} \le C_b|\lambda|^{-\frac{\mu}{2}\frac{|s|-\mu'}{\mu-\mu'}}\|f\|_{\mathcal{B}^{-\frac{\mu'(\mu-|s|)}{\mu-\mu'}}_{q,r}}. \tag{163}$$

Since $0 < \mu' < |s| - \sigma$ and $0 < \mu - |s| < \mu - \mu'$, we have

$$-|s| < -|s| + \sigma < -\frac{\mu'(\mu - |s|)}{\mu - \mu'}.$$

Choose $\theta \in (0,1)$ in such a way that

$$-|s| + \sigma = (1-\theta)(-|s|) + \theta(-\frac{\mu'(\mu-|s|)}{\mu-\mu'})$$

Combining (162) and (163) with real interpolation method implies that

$$\|T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C_b|\lambda|^{-\frac{\mu}{2}\frac{|s|-\mu'}{\mu-\mu'}\theta}\|f\|_{\mathcal{B}^{-|s|+\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

Inserting $\theta = \dfrac{(\mu-\mu')\sigma}{\mu(|s|-\mu')}$, we have

$$\|T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C_b\||\lambda|^{-\frac{\sigma}{2}}\|f\|_{\mathcal{B}^{-|s|+\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

which shows(155).

Now, we shall show (156). Combining (151), (152) and (153) with complex interpolation method for $|s| < \mu, \mu' < 1 - 1/q = 1/q'$, we have

(164) $$\|\partial_\lambda T(\lambda)^*\varphi\|_{L_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)},$$

(165) $$\|\partial_\lambda T(\lambda)^*\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)},$$

(166) $$\|\partial_\lambda T(\lambda)^*\varphi\|_{\mathcal{H}^{\mu'}_{q'}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|\varphi\|_{\mathcal{H}^{\mu'}_{q'}(\mathbb{R}^N_+)},$$

(167) $$\|\partial_\lambda T(\lambda)^*\varphi\|_{\mathcal{H}^\mu_{q'}(\mathbb{R}^N_+)} \leq C_b\||\lambda|^{-(1-\frac{\mu}{2})}\|\varphi\|_{L_{q'}(\mathbb{R}^N_+)}.$$

Thus, by the dual argument we have

(168) $$\|\partial_\lambda T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{L_q(\mathbb{R}^N_+)},$$

(169) $$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)},$$

(170) $$\|\partial_\lambda T(\lambda)f\|_{\mathcal{H}^{-\mu'}_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{\mathcal{H}^{-\mu'}_q(\mathbb{R}^N_+)},$$

(171) $$\|\partial_\lambda T(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{\mu}{2})}\|f\|_{\mathcal{H}^{-\mu}_q(\mathbb{R}^N_+)}.$$

Noting that $-1 + 1/q < -\mu < -|s| < 0$ and combining (168) and (169) with real interpolation method implies

(172) $$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,1}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-1}\|f\|_{\mathcal{B}^{-|s|}_{q,1}(\mathbb{R}^N_+)}.$$

Choosing $\theta \in (0, 1)$ in such a way that $|s| = \mu'\theta$ and combining (170) and (171) with real interpolation method, we have

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-a}\|f\|_{\mathcal{B}^c_{q,r}(\mathbb{R}^N_+)}.$$

Here,

$$a = -\theta - (1-\theta)(1-\frac{\mu}{2}) = -1 + \frac{\mu}{2}(1-\frac{|s|}{\mu'})$$
$$c = -\mu'\theta - \mu(1-\theta) = -\mu'\frac{|s|}{\mu'} - \mu(1-\frac{|s|}{\mu'}) = -|s| - \mu(1-\frac{|s|}{\mu'}) = -(|s| + \mu(1-\frac{|s|}{\mu'})).$$

Thus, we have obtained

(173) $$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C_b|\lambda|^{-(1-\frac{\mu}{2}(1-\frac{|s|}{\mu'}))}\|f\|_{\mathcal{B}_{q,r}^{-(|s|+\mu(1-\frac{|s|}{\mu'}))}(\mathbb{R}^N_+)}.$$

Now, we choose $\mu' \in (0, 1)$ in such a way that

$$-|s| > -|s| - \sigma > -|s| - \mu(1-\frac{|s|}{\mu'}),$$

that is

(174) $$\frac{\mu|s|}{\mu-\sigma} < \mu' < 1 - 1/q.$$

Since $\sigma > 0$ may be chosen so small that $\mu/(\mu - \sigma)$ is very close to 1, we can choose μ' in such a way that $|s| < \mu'$ and (174) holds.

We choose $\theta \in (0, 1)$ in such a way that

$$-|s| - \sigma = -|s|\theta - (|s| + \mu(1-\frac{s|}{\mu'}))(1-\theta).$$

Combining (172) and (173) with real interpolation method implies that

$$\|\partial_\lambda T(\lambda)f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \leq C|\lambda|^{-d}\|f\|_{\mathcal{B}^{-|s|-\sigma}_{q,r}(\mathbb{R}^N_+)},$$

where

$$d = \theta + (1-\theta)(1 - \frac{\mu}{2}(1 - \frac{|s|}{\mu'})) = 1 - \frac{\sigma}{2}.$$

Thus, we have

$$\|\partial_\lambda T(\lambda) f\|_{\mathcal{B}^{-|s|}_{q,r}(\mathbb{R}^N_+)} \le C_b |\lambda|^{-(1-\frac{\sigma}{2})} \|f\|_{\mathcal{B}^{-|s|-\sigma}_{q,r}(\mathbb{R}^N_+)}.$$

Namely, we have (156), which completes the proof of Proposition 40. □

When $s = 0$, we have the results for $s = \pm\omega$ with very small $\omega > 0$. Thus, by real interpolation method, we have

$$\|T(\lambda) f\|_{\mathcal{B}^0_{q,r}(\mathbb{R}^N_+)} \le C_b \|f\|_{\mathcal{B}^0_{q,r}(\mathbb{R}^N_+)},$$
$$\|T(\lambda) f\|_{\mathcal{B}^0_{q,r}(\mathbb{R}^N_+)} \le C_b |\lambda|^{-\frac{\sigma}{2}} \|f\|_{\mathcal{B}^\sigma_{q,r}(\mathbb{R}^N_+)},$$
$$\|\partial_\lambda T(\lambda) f\|_{\mathcal{B}^0_{q,r}(\mathbb{R}^N_+)} \le C_b |\lambda|^{-(1-\frac{\sigma}{2})} \|f\|_{\mathcal{B}^{-\sigma}_{q,r}(\mathbb{R}^N_+)}$$

provided that Assumption 1 and Assumption 2 hold.

Summing up, we have obtained the following theorem.

Theorem 41. *Let $1 < q < \infty$, $1 \le r \le \infty$, $-1 + 1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Let $\sigma > 0$ be a small number such that $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$. Assume that Assumption 4.1 and Assumption 4.2 hold. Let $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. Then, we have the following two assertions:*
(1) For any $f \in C^\infty_0(\Omega)$ and $\lambda \in \Sigma_\epsilon$, there hold

$$\|T(\lambda) f\|_{\dot B^s_{q,1}(\Omega)} \le C \|f\|_{\dot B^s_{q,1}(\Omega)},$$
$$\|T(\lambda) f\|_{\dot B^s_{q,1}(\Omega)} \le C |\lambda|^{-\frac{\sigma}{2}} \|f\|_{\dot B^{s+\sigma}_{q,1}(\Omega)},$$
$$\|\partial_\lambda T(\lambda) f\|_{\dot B^s_{q,1}(\Omega)} \le C |\lambda|^{-1-\frac{\sigma}{2}} \|f\|_{\dot B^{s-\sigma}_{q,1}(\Omega)}$$

for some constant C.
(2) Let $\gamma > 0$. For any $f \in C^\infty_0(\Omega)$ and $\lambda \in \Sigma_\epsilon + \gamma$, there hold

$$\|T(\lambda) f\|_{B^s_{q,1}(\Omega)} \le C_\gamma \|f\|_{B^s_{q,1}(\Omega)},$$
$$\|T(\lambda) f\|_{B^s_{q,1}(\Omega)} \le C_\gamma |\lambda|^{-\frac{\sigma}{2}} \|f\|_{B^{s+\sigma}_{q,1}(\Omega)},$$
$$\|\partial_\lambda T(\lambda) f\|_{B^s_{q,1}(\Omega)} \le C_\gamma |\lambda|^{-1-\frac{\sigma}{2}} \|f\|_{B^{s-\sigma}_{q,1}(\Omega)}$$

for some constant C_γ depending on γ.

Applying Theorem 41 to $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}(\lambda)$ and $\nabla\mathcal{P}(\lambda)$, we have Theorem 36 when $\mathcal{B}^s_{q,1}(\Omega) = \dot B^s_{q,1}(\Omega)$ and $\gamma_b = 0$. And, applying Theorem 41 to $(\lambda, \lambda^{1/2}\nabla, \bar\nabla^2)\mathcal{S}(\lambda)$ and $\nabla\mathcal{P}(\lambda)$, we have Theorem 36 when $\mathcal{B}^s_{q,1}(\Omega) = B^s_{q,1}(\Omega)$ and $\gamma_b = \gamma > 0$

4.2. **Free boundary problems in the L_1-$\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$ maximal regularity framework.** In this subsection, first I consider equations (58), and I will state the global well-posedness for small initial data and the local well-posedness for large initial data. In the small data case, the proof relies on the linear theory, namely the unique existence theory follows from the Banach fixed point theore in the framework of the L_1-$\mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$ maximal regularity theory for the Stokes equations with free boundary conditions (71). But, for large initial data, even for the local well-posedness we need some idea to treat the nonlinear terms $\mathbf{H}(\mathbf{u})$ because we have to use the non-local norm $\|\mathbf{H}(\mathbf{u})\|_{W^{1/2}_1(\mathbb{R}, B^s_{q,1}(\mathbb{R}^N_+))}$.

First, I would like to mention our theorem obtained in [32].

Theorem 42 (Local well-posedness)**.** *Let $N - 1 < q \le N$ and $-1 + N/q < s < 1/q$. Let $\mathbf{a} \in B^s_{q,1}(\mathbb{R}^N_+)$ be initial data which satisfies the compatibility condition:* $\operatorname{div} \mathbf{a} = 0$ *in $\mathbb{R}^N_+$. Then,* *there exists time $T > 0$ depending on* $\mathbf{a}$ *such that problem fbp.2 admits unique solutions* $\mathbf{u}$ *and* $\mathfrak{q}$ *with*

$$\mathbf{u} \in L_1((0,T), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)^N)$$

satisfying the estimate:

$$\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1}(\mathbb{R}^N_+))} + \|\partial_t \mathbf{u}\|_{L_1((0,T),B^s_{q,1}(\mathbb{R}^N_+))} \leq C\|\mathbf{a}\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Theorem 43 (Global well-posedness)**.** *Let $N-1 < q < 2N$ and $s = -1 + N/q$. Then, there exists a small constant $c_0 > 0$ such that for any initial data $\mathbf{a} \in \dot{B}^s_{q,1}(\mathbb{R}^N_+)$ satisfying the compatibility condition* $\operatorname{div}\mathbf{a} = 0$ *in $\mathbb{R}^N_+$ and the smallness condition: $\|\mathbf{a}\|_{\dot{B}^s_{q,1}(\mathbb{R}^N_+)} \leq c_0$, then problem* (58) *admits unique solutions* $\mathbf{u}$ *and* $\mathfrak{q}$ *with*

$$\partial_t \mathbf{u},\ \ \partial_j\partial_k \mathbf{u},\ \ \nabla\mathfrak{q} \in L_1((0,\infty), \dot{B}^s_{q,1}(\mathbb{R}^N_+))^N)$$

for $j, k = 1, \ldots, N$ satisfying the estimate:

$$\|(\partial_t \mathbf{u}, \nabla^2 \mathbf{u}, \nabla\mathfrak{q})\|_{L_1((0,\infty),\dot{B}^s_{q,1}(\mathbb{R}^N_+))} + \sup_{t\in(0,\infty)} \|\mathbf{u}(\cdot,t)\|_{\dot{B}^s_{q,1}(\mathbb{R}^N_+)} \leq Cc_0$$

where C is some constant independent of c_0.

We now consider problem (52). Let $T_b = T < \infty$ when $\mathcal{B}^s_{q,1} = B^s_{q,1}$ and $T_b = \infty$ when $\mathcal{B}^s_{q,1} = \dot{B}^s_{q,1}$. We know existence of solutions $\mathbf{u}$ and $\mathfrak{q}$ for equations (58) with Lagrange coordinates and $\mathbf{u} \in L_1((0,T_b), \mathcal{B}^{s+2}_{q,1}(\mathbb{R}^N_+)^N)$ or $\mathbf{u} \in BC((0,T_b), \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^N)$ and $\nabla\mathbf{u} \in L_1((0,T_b), \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)^{N^3})$, thus the Lagrange map:

$$x = X_{\mathbf{u}}(y,t) = y + \int_0^t \mathbf{u}(y,\ell)\,\mathrm{d}\ell$$

is C^1 diffeomorphism from $\mathbb{R}^N_+$ onto Ω_t, where $\Omega_t = \{x = X_{\mathbf{u}}(y,t) \mid y \in \mathbb{R}^N_+\}$. Moreover, we know the smallness condition (56) holds, and so there exists an inverse map: $y = X^{-1}_{\mathbf{u}}(x,t)$ for each $t \in (0,T_b)$. For any function $F \in \mathcal{B}^s_{q,1}(\mathbb{R}^N_+)$, $1 < q < \infty$, $s \in (-\min(N/q, N/q'), N/q)$, it follows from the chain rulde that

$$\|F \circ X^{-1}_{\mathbf{u}}\|_{\mathcal{B}^s_{q,1}(\Omega_t)} \leq C\|F\|_{B^s_{q,1}(\mathbb{R}^N_+)}$$

with some constant $C > 0$. Setting $\mathbf{v}(x,t) = \mathbf{u}(X_{\mathbf{u}}(x,t),t)$, we see that $\mathbf{v} \in BC$ in time with value in $B^s_{q,1}(\Omega_t)$ and $\partial_j\partial_k\mathbf{v} \in L_1((0,T_b), \mathcal{B}^s_{q,1}(\Omega_t)$. In fact, setting $\mathbb{A}_{\mathbf{u}} = (\nabla_y X_{\mathbf{u}})^{-1}$ and $\mathbb{A}^\top_{\mathbf{u}} = (A_{j,k})$, we have

$$\partial_{x_j}\partial_{x_k}\mathbf{v} = \sum_{\ell,\ell'=1}^N \Big(A_{j,\ell}\partial_{y_\ell}(A_{k,\ell'}\partial_{y_{\ell'}}\mathbf{u})\Big) \circ X^{-1}_{\mathbf{u}}, \quad j,k = 1,\ldots,N.$$

Moreover, for the time derivative of $\mathbf{v}$, we have

$$\partial_t \mathbf{v} = (\partial_t \mathbf{u}) \circ X^{-1}_{\mathbf{u}} - \Big((\mathbf{u} \cdot X^{-1}_{\mathbf{u}}) \cdot \nabla_x)\mathbf{v}.$$

From these observations and Theorems 42 and 43, we have the theorems for problem (52) as follows.

Theorem 44 (Local well-posedness)**.** *Let $N-1 < q \leq N$ and $-1 + N/q < s < 1/q$. Let $\mathbf{a} \in B^s_{q,1}(\mathbb{R}^N_+)$ be initial data which satisfies the compatibility condition:* $\operatorname{div}\mathbf{a} = 0$ *in $\mathbb{R}^N_+$. Then, there exists time $T > 0$ depending on* $\mathbf{a}$ *such that problem 52 admits unique solutions* $\mathbf{v}$ *and* $\mathfrak{p}$ *with*

$$\mathbf{v} \in B^{s+2}_{q,1}(\Omega_t)^N,\ \ \partial_t\mathbf{v} \in B^s_{q,1}(\Omega_t)^N,\ \ \nabla\mathfrak{p} \in B^s_{q,1}(\Omega_t)^N$$

for each $t \in (0,T)$ which satisfy the estimate:

$$\int_0^T \|(\bar{\nabla}^2\mathbf{v}, \partial_t\mathbf{v}, \nabla\mathfrak{p})(\cdot,t)\|_{B^s_{q,1}(\Omega_t)})\,\mathrm{d}t \leq C\|\mathbf{a}\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Theorem 45 (Global well-posedness)**.** *Let $N-1<q<2N$ and $s=-1+N/q$. Then, there exists a small constant $c_0>0$ such that for any initial data $\mathbf{a}\in\dot{B}^s_{q,1}(\mathbb{R}^N_+)$ satisfying the compatibility condition* $\operatorname{div}\mathbf{a}=0$ *in $\mathbb{R}^N_+$ and the smallness condition: $\|\mathbf{a}\|_{\dot{B}^s_{q,1}(\mathbb{R}^N_+)}\le c_0$, then problem* (58) *admits unique solutions $\mathbf{v}$ and $\mathfrak{p}$ with*

$$\partial_t\mathbf{v},\ \ \partial_j\partial_k\mathbf{v},\ \ \nabla\mathfrak{p}\in\dot{B}^s_{q,1}(\Omega_t))^N$$

for $j,k=1,\dots,N$ and $t\in(0,\infty)$ which satisfy the estimate:

$$\int_0^\infty\|(\partial_t\mathbf{v},\nabla^2\mathbf{v},\nabla\mathfrak{p})(\cdot,t)\|_{\dot{B}^s_{q,1}(\Omega_t)}\,\mathrm{d}t+\sup_{t\in(0,\infty)}\|\mathbf{v}(\cdot,t)\|_{\dot{B}^s_{q,1}(\Omega_t)}\le Cc_0$$

where C is some constant independent of c_0.

Concerning the proof, one of the points is the following propositions concerning the products estimate in the Besov spaces, which was proved in [1] and [18].

Proposition 46. *Let $1\le q\le q_1\le\infty$. If $s\in\mathbb{R}$ satisfies*

$$\begin{cases}-\dfrac{d}{q_1}<s<\dfrac{d}{q_1} & \text{if}\quad \dfrac{1}{q}+\dfrac{1}{q_1}\le 1,\\ -\dfrac{d}{q'}<s<\dfrac{d}{q_1} & \text{if}\quad \dfrac{1}{q}+\dfrac{1}{q_1}>1,\end{cases}\tag{175}$$

then for every $u\in B^s_{q,1}(\mathbb{R}^N_+)$ and $v\in B^{d/q_1}_{q_1,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)$, there holds

$$\|uv\|_{B^s_{q,1}(\mathbb{R}^N_+)}\le C\|u\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{d/q_1}_{q_1,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)}.\tag{176}$$

We introduce propositions to estimate our nonlinear terms expressed by (59).

Proposition 47. *Let $1<q<\infty$. If $s\in\mathbb{R}$ satisfies*

$$\begin{cases}-1+\dfrac{d}{q}\le s<\dfrac{d}{q} & \text{if}\quad 1<q<2d,\\ \quad -\dfrac{d}{q}<s<\dfrac{d}{q} & \text{if}\quad 2d\le q<\infty.\end{cases}\tag{177}$$

then for every $u\in B^s_{q,1}(\mathbb{R}^N_+)$ and $v\in B^{d/q}_{q,1}(\mathbb{R}^N_+)$, there holds

$$\|uv\|_{B^s_{q,1}(\mathbb{R}^N_+)}\le C\|u\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)}.\tag{178}$$

Proof. First, we consider the case that $q<2$. In this case, setting $q_1=q$ in the second case of Proposition 46, for $s\in(-d/q',d/q)$, we have

$$\|uv\|_{B^s_{q,1}(\mathbb{R}^N_+)}\le C\|u\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{d/q}_{q,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)}.\tag{179}$$

Here, notice that there holds $-d/q'<-1+d/q$. Since we have $B^{d/q}_{q,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)\hookleftarrow B^{d/q}_{q,1}(\mathbb{R}^N_+)$ as follows from Proposition 38, we obtain (178) for the case $q<2$. On the other hand, if $q\ge 2$, we choose $q_1=q$ in the first case of Proposition 46. Then for $s\in(-d/q,d/q)$ we see that

$$\|uv\|_{B^s_{q,1}(\mathbb{R}^N_+)}\le C\|u\|_{B^s_{q,1}}\|v\|_{B^{d/q}_{q,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)}.\tag{180}$$

When $q<2d$, it holds $-d/q<-1+d/q$. Thus, noting that $B^{d/q}_{q,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)\hookleftarrow B^{d/q}_{q,1}(\mathbb{R}^N_+)$, we have (178) provided that $2\le q<2d$. On the other hand, when $2d\le q<\infty$, it holds $-1+d/q\le -d/q$, and hence (178) holds for $-d/q<s<1/q$. The proof is complete. □

Proposition 48. *Let $d-1<q\le d$ and $-1+d/q<s<1/q$. For every $u\in B^{s-1}_{q,1}(\mathbb{R}^N_+)$ and $v\in B^{d/q}_{q,1}(\mathbb{R}^N_+)$ there holds*

$$\|uv\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}\le C\|u\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)}.\tag{181}$$

Proof. We first consider the case $q < 2$. From the proof of Proposition 47, we have (181) provided that $d-1 < q < 2$ and $-d/q' < s-1 < d/q$. In addition, we see that $1 - d/q' \leq -1 + d/q$ due to $d \geq 2$, and hence (181) holds provided that $-1 + d/q < s$ and $q < 2$. Concerning the remaining case $q \geq 2$, we infer from the proof of Proposition 47 that (181) is valid provided that $-d/q < s - 1 < d/q$. Since $q \leq d$ is equivalent to $1 - d/q \leq -1 + d/q$, we obtain the desired estimate assuming that $2 \leq q \leq d$ and $-1 + d/q < s$. □

Proposition 49. *Let $1 \leq q \leq \infty$. For every u, $v \in B^{d/q}_{q,1}(\mathbb{R}^N_+)$, there holds*

$$\|uv\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)} \leq C\|u\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)}. \tag{182}$$

Namely, $B^{d/q}_{q,1}(\mathbb{R}^N_+)$ is a Banach algebra.

Proof. According to [18, Prop. 2.3], there holds

$$\|uv\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)} \leq C(\|u\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)}\|v\|_{L_\infty(\mathbb{R}^N_+)} + \|u\|_{L_\infty(\mathbb{R}^N_+)}\|v\|_{B^{d/q}_{q,1}(\mathbb{R}^N_+)} \tag{183}$$

provided that $1 \leq q \leq \infty$. By $B^{d/q}_{q,1}(\mathbb{R}^N_+) \hookrightarrow L_\infty(\mathbb{R}^N_+)$, we have the desired estimate. □

The following result on composite functions is stated in [18, Prop. 2.4] (cf. [4, Thm. 2.87]).

Proposition 50. *Let $I \subset \mathbb{R}$ be open. Let $s > 0$ and σ be the smallest integer such that $\sigma \geq s$. Let $\mathsf{F}\colon I \to \mathbb{R}$ satisfy $\mathsf{F}(0) = 0$ and $\mathsf{F}' \in W^\sigma_\infty(I)$. Assume that $v \in B^s_{q,r}(\mathbb{R}^N_+)$ has values in $J \Subset I$. Then it holds $\mathsf{F}(v) \in B^s_{q,r}(\mathbb{R}^N_+)$ and there exists a constant C depending only on s, I, J, and d such that*

$$\|\mathsf{F}(v)\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C\Big(1 + \|v\|_{L_\infty(\mathbb{R}^N_+)}\Big)^\sigma \|\mathsf{F}'\|_{W^\sigma_\infty(I)}\|v\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{184}$$

To prove Theorems 42 and 43, we use the Banach fixed point argument. Namely, given $\mathbf{w}$, let $\mathbf{u}$ and $\mathfrak{q}$ be solutions to the linear system of equations:

$$\begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}) = \mathbf{F}(\mathbf{w}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \mathrm{div}\,\mathbf{u} = G_{\mathrm{div}}(\mathbf{u}) = \mathrm{div}\,\mathbf{G}(\mathbf{w}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q})\mathbf{n} = \mathbf{H}(\mathbf{w})\mathbf{n}_0 & \text{on } \partial\mathbb{R}^N_+ \times (0,T), \\ \mathbf{u}|_{t=0} = \mathbf{a} & \text{in } \mathbb{R}^N_+, \end{cases} \tag{185}$$

Noting that $\mathbf{F}(\mathbf{w})$, $\mathbf{G}(\mathbf{w})$ and $\mathbf{H}(\mathbf{w})$ vanish at $t = 0$, we extend them suitable to $\mathbb{R}$. When initial data are small, using Theorem 37, we prove the map $\mathbf{w} \to \mathbf{u}$ is a contractive on some underlying space $\mathcal{H}$ when the initial data are small enough. The proof is quite standard.

On the other hand, to prove the local well-posedness, we have to treat the largeness of the initial data, and so we need some idea. As far as I understand, to prove the local well-posedness for the large initial data gives us some difficulty usually. At this time, such difficulty of the proof is due to the fact that we have to estimate $\|\mathbf{H}(\mathbf{w})\|_{W^{1/2}_1(\mathbb{R}, B^s_{s,1}(\mathbb{R}^N_+))}$, which is non-local. Thus, instead of using the norm $\|\cdot\|_{W^{1/2}_1(\mathbb{R}, B^s_{s,1}(\mathbb{R}^N_+))}$, we use $\|\cdot\|_{W^1_1((0,T), B^{s-1}_{q,1}(\mathbb{R}^N_+))}$. Namely, we use the properties: $W^1_1((0,T), B^{s-1}_{q,1}(\mathbb{R}^N_+)) \cap L_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+)) \subset W^{1/2}_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)) = (L_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)), W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+))_{[1/2]}$. Then, we have to pay the price to product estimates (cf. Proposition 48). Namely, the range of s is only $-1 + d/q < s < 1/q$ and $d - 1 < q \leq d$. Especially, $0 < s < 1/q$. We can not consider problems in non-positive order spaces unlike the small data case.

The key argument is the following. $\mathbf{H}(\mathbf{w})$ has the following form:

$$\mathbf{H}(\mathbf{w}) = \nabla\mathbf{w}\,\mathsf{F}\Big(\int_0^t \nabla\mathbf{w}\,\mathrm{d}\tau\Big)$$

with some nonlinear function F with $\mathsf{F}(0)=0$. It follows from Propositions 48, 49, and 50 that

$$
\begin{aligned}
&\Big\|\partial_t\Big(\nabla\mathbf{w}\,\mathsf{F}\Big(\int_0^t\nabla\mathbf{w}\,\mathrm{d}\tau\Big)\Big)\Big\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}\\
&\le\Big\|(\partial_t\nabla\mathbf{w})\,\mathsf{F}\Big(\int_0^t\nabla\mathbf{w}\,\mathrm{d}\tau\Big)\Big\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}+\Big\|\nabla\mathbf{w}\nabla\mathbf{w}\,\mathsf{F}'\Big(\int_0^t\nabla\mathbf{w}\,\mathrm{d}\tau\Big)\Big\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}\\
&\le C\Big\{\|\partial_t\mathbf{w}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\Big\|\mathsf{F}\Big(\int_0^t\nabla\mathbf{w}\,\mathrm{d}\tau\Big)\Big\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)}\\
&\qquad+\|\nabla\mathbf{w}\|_{B^{s-1}_{q,1}(\mathbb{R}^N_+)}\Big\|\nabla\mathbf{w}\,\mathsf{F}'\Big(\int_0^t\nabla\mathbf{w}\,\mathrm{d}\tau\Big)\Big\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)}\Big\}\\
&\le C\Big(\|\partial_t\mathbf{w}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|\nabla\mathbf{w}\|_{L_1((0,T),B^{N/q}_{q,1}(\mathbb{R}^N_+))}+\|\mathbf{w}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|\mathbf{w}\|_{B^{N/q+1}_{q,1}(\mathbb{R}^N_+)}\Big).
\end{aligned}
$$

Essential assumption is that $-1+N/q<s$, that is $N/q+1<s+2$. Thus, we may choose $\theta\in(0,1)$ such that $N/q+1=s(1-\theta)+(s+2)(1-\theta)$, and hence we infer from the interpolation inequality that $\|\mathbf{w}\|_{B^{d/q+1}_{q,1}(\mathbb{R}^N_+)}\le C\|\mathbf{w}\|^{1-\theta}_{B^s_{q,1}(\mathbb{R}^N_+)}\|\mathbf{w}\|^{\theta}_{B^{s+2}_{q,1}(\mathbb{R}^N_+)}$. Thus, by the embedding $B^{d/q+1}_{q,1}(\mathbb{R}^N_+)\hookrightarrow B^{s+1}_{q,1}(\mathbb{R}^N_+)$ and the Young inequality, there holds

$$
\begin{aligned}
\|\mathbf{w}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|\mathbf{w}\|_{B^{d/q+1}_{q,1}(\mathbb{R}^N_+)}&\le C\|\mathbf{w}\|^{2-\theta}_{B^s_{q,1}(\mathbb{R}^N_+)}\|\mathbf{w}\|^{\theta}_{B^{s+2}_{q,1}(\mathbb{R}^N_+)}\\
&\le C_\theta\Big(\epsilon\|\mathbf{w}\|_{B^s_{q,1}(\mathbb{R}^N_+)}+\epsilon^{-\frac{\theta}{1-\theta}}\|\mathbf{w}\|^{\frac{2-\theta}{1-\theta}}_{B^s_{q,1}(\mathbb{R}^N_+)}\Big)
\end{aligned}
\tag{186}
$$

for every $\epsilon>0$. Thus, we have

$$
\begin{aligned}
\|\partial_t\mathbf{H}(\mathbf{w})\|_{L_1((0,T),B^{s-1}_{q,1}(\mathbb{R}^N_+)}&\le C\Big(\epsilon\|\mathbf{w}\|_{L_1((0,T),B^{s+2}_{q,1}(\mathbb{R}^N_+))}+\epsilon^{-\frac{\theta}{1-\theta}}\|\mathbf{w}\|^{\frac{2-\theta}{1-\theta}}_{L_\infty((0,T),B^s_{q,1}(\mathbb{R}^N_+))}T\\
&\qquad+\|\partial_t\mathbf{w}\|_{L_1((0,T),B^s_{q,1}(\mathbb{R}^N_+))}\|\mathbf{w}\|_{L_1((0,T),B^{s+2}_{q,1}(\mathbb{R}^N_+))}\Big).
\end{aligned}
\tag{187}
$$

Thus, the first term of the right hand side can be controlled by first choosing $\epsilon>0$ small as much as we want and second choosing $T>0$ small enough according to $\epsilon^{-\theta/(1-\theta)}$. The second term is a normal squre term. This is an idea to control the boundary term when the initial data are arbitrary large.

5. Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, real numbers and complex numbers, respectively. Set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $\partial_t=\partial/\partial t$ and $\partial^\alpha_x=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial^{\alpha_N}_{x_N}$ for any multi-index $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{N}_0^N$, where $|\alpha|=\alpha_1+\cdots+\alpha_N$. Let $\nabla f=\{\partial^\alpha_x f\mid|\alpha|=1\}$, $\bar\nabla f=\{\partial^\alpha_x f\mid|\alpha|\le1\}$, $\nabla^2 f=\{\partial^\alpha_x f\mid|\alpha|=2\}$, and $\bar\nabla^2 f=\{\partial^\alpha_x f\mid|\alpha|\le2\}$.

Let X be a Banach space with norm $\|\cdot\|_X$. Let $L_q(\Omega,X)$, $W^m_q(\Omega,X)$, $B^s_{q,r}(\Omega,X)$ and $\dot B^s_{q,r}(\Omega,X)$ denote the standard X-valued Lebesgue spaces, Sobolev spaces, inhomogeneus Besov spacse, and homogeneous Besov spaces while $\|\cdot\|_{L_q(\Omega,X)}$, $\|\cdot\|_{W^m_q(\Omega,X)}$, $\|\cdot\|_{B^s_{q,r}(\Omega,X)}$, and $\|\cdot\|_{\dot B^s_{q,r}(\Omega,X)}$ denote their norms. When $X=\mathbb{R}$ or $\mathbb{C}$, we omit X, namely for example, $L_q(\Omega)$ and $\|\cdot\|_{L_q(\Omega)}$.

$$
\widehat W^1_q(\Omega)=\{f\in L_{q,\mathrm{loc}}(\Omega)\mid\nabla f\in L_q(\Omega)\},\quad\widehat W^1_{q,0}(\mathbb{R}^N_+)=\{f\in\widehat W^1_q(\mathbb{R}^N_+)\mid f|_{\partial\mathbb{R}^N_+}=0\}.
$$

Let

$$
\begin{aligned}
\dot W^m_q(\mathbb{R}^N) &= \{f \in W^m_{q,\mathrm{loc}}(\mathbb{R}^N) \mid \partial^\alpha f \in L_q(\mathbb{R}^N) \ \ (|\alpha| = m)\},\\
\dot W^s_q(\mathbb{R}^N) &= (L_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N))_{[s]} = \{f \mid \|\mathcal{F}^{-1}[|\xi|^s \mathcal{F}[f](\xi)]\|_{L_q(\mathbb{R}^N)} < \infty\},\\
\dot W^m_q(\Omega) &= \{f \mid \exists g \in \dot W^m_q(\mathbb{R}^N) \text{ such that } g|_\Omega = f\},\\
\|f\|_{\dot W^m_q(\Omega)} &= \inf\{\|g\|_{\dot W^m_q(\mathbb{R}^N)} \mid \exists g \in \dot W^m_q(\mathbb{R}^N) \text{ such that } g|_\Omega = f\},\\
\dot W^s_q(\Omega) &= \{f \mid \exists g \in \dot W^s_q(\mathbb{R}^N) \text{ such that } g|_\Omega = f\},\\
\|f\|_{\dot W^m_q(\Omega)} &= \inf\{\|g\|_{\dot W^m_q(\mathbb{R}^N)} \mid \exists g \in \dot W^s_q(\mathbb{R}^N) \text{ such that } g|_\Omega = f\},\\
J_q(\mathbb{R}^N_+) &= \{\mathbf{f} \in L_q(\mathbb{R}^N_+)^N \mid (\mathbf{f}, \nabla\varphi) = 0 \quad \text{for every } \varphi \in \widehat{W}^1_{q',0}(\mathbb{R}^N_+)\},\\
W^1_p(\mathbb{R}, X) &= \{f \in L_p(\mathbb{R}, X) \mid \partial_t f \in L_p(\mathbb{R}, X)\},\\
\dot W^1_p(\mathbb{R}, X) &= \{f \in W^1_{p,\mathrm{loc}}(\mathbb{R}, X) \mid \partial_t f \in L_p(\mathbb{R}, X)\},\\
W^s_q(\mathbb{R}, X) &= (L_q(\mathbb{R}, X), W^1_q(\mathbb{R}, X))_{[s]}, \quad \dot W^s_q(\mathbb{R}, X) = (L_q(\mathbb{R}, X), \dot W^1_p(\mathbb{R}, X))_{[s]},
\end{aligned}
$$

where $m \geq 2$, $0 < s < 1$ and $1 < q < \infty$. $(\cdot,\cdot)_{[\theta]}$ denote complex interpolation functors and $(\cdot,\cdot)_{\theta,p}$ denote real interpolation functors for $\theta \in (0,1)$ and $1 \leq p \leq \infty$. For $\gamma > 0$ we write

$$
\|e^{-\gamma t} f\|_{L_q(I,X)} = \Big\{\int_I (e^{-\gamma t}\|f(t)\|_X)^q \, dt\Big\}^{1/q}.
$$

For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operator from X into Y, and we write $\mathcal{L}(X,X) = \mathcal{L}(X)$. Let U be a domain in $\mathbb{C}$ and let $\mathrm{Hol}\,(U,X)$ be the set of all X-valued holomorphic functions defined in U.

Let $\mathcal{F}$ and $\mathcal{F}^{-1}_\xi$ be respective the Fourier transform with respect to $x \in \mathbb{R}^N$ and its inversion formula defined by

$$
\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}^{-1}_\xi[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-ix\cdot\xi} g(\xi)\, d\xi.
$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}_\xi$ be respective the Laplace transform with respect to $t \in \mathbb{R}$ and its inversion formula defined by

$$
\mathcal{L}[f](\xi) = \int_{\mathbb{R}} e^{-\lambda t} f(t)\, dt = \mathcal{F}[e^{-\gamma t} f](\tau) \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{t\lambda} g(\tau)\, d\tau = e^{\gamma t}\mathcal{F}^{-1}[g](t)
$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$. Let

$$
\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \epsilon\}, \quad \Sigma_\epsilon + \gamma = \{\gamma + \lambda \mid \lambda \in \Sigma_\epsilon\}.
$$

For any two N vectors $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$, $(\mathbf{a}, \mathbf{b}) = \mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^N a_j b_j$. The character C denotes general constants and $C(a, b, \cdots) = C_{a,b,\cdots}$ denotes that the constant C depends on a, b, $\cdots$. C and $C(a, b, \cdots)$, $C_{a,b,\cdots}$ may change from line to line.

For $x = (x_1, \ldots, x_N)$, we write $x' = (x_1, \ldots, x_{N-1})$.

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