A Note on the Asymptotic Behavior in Time of the Kinetic Energy in a Liquid-Solid Interaction Problem

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1 Introduction

Consider a sufficiently smooth, rigid body $\mathscr{B}$ (the closure of a simply connected bounded domain of $\mathbb{R}^3$) completely immersed in a viscous liquid $\mathscr{L}$ that fills the entire space outside $\mathscr{B}$. We assume that the center of mass G of $\mathscr{B}$ is held fixed in a given position, while $\mathscr{B}$ is allowed to rotate around G. The motion of the coupled system $\mathscr{S} := \mathscr{B} \cup \mathscr{L}$ is driven by a time dependent torque with respect to G, $\mathrm{M}=\mathrm{M}(t)$, acting on $\mathscr{B}$.

Recently, the question of the large time behavior of $\mathscr{S}$ has attracted the attention of several authors, also in the more general case when G is free to move. More specifically, in [2] for $\mathscr{B}$ a sphere, and in [5, 7] in the general case, under diverse assumptions on the initial data and driving mechanism, the same conclusion is drawn, namely, that as time grows indefinitely large, the velocity u of $\mathscr{L}$ as well as translational (ξ) and angular (ω) velocities of $\mathscr{B}$ will tend to 0 in certain norms. Actually, in [2, 7] it is also shown that G (in absence of external forces and torques) will cover a finite distance.

However, in the more general and interesting case of a body of *arbitrary* shape, the asymptotic decay in time of the velocity field of $\mathscr{L}$ is established in norms that do not ensure that the total kinetic energy of the coupled system, defined as

$$E(t) := \tfrac{1}{2}\left\{\rho\int_{\mathcal{D}}|u(t)|^2 + m|\xi(t)|^2 + \omega(t)\cdot \mathsf{I}\cdot\omega(t)\right\}$$

with m and I mass and inertia tensor of $\mathscr{B}$, ultimately vanishes. Precisely, in [5, 7] it is proved that, as $t \to \infty$, while $\xi(t)$ and $\omega(t)$ tend pointwise to 0, the velocity field u tends to 0 in the L^q-norm if $q > 2$, thus excluding the case $q = 2$, representative of the kinetic energy of $\mathscr{L}$.[(1)]

Objective of this note is to show that the kinetic energy, E, of any solution belonging to a suitable function class, $\mathcal{C}$, will eventually tend to 0. As shown in [5], the class $\mathcal{C}$ is certainly not empty, provided the initial data are prescribed in appropriate function spaces with their magnitude is opportunely restricted, and $\mathrm{M}(t)$ vanishes as $t \to \infty$ in the L^2-sense. The method we use relies heavily upon establishing a space-weighted estimate on the solutions in combination with a uniform bound on the pressure field. Unfortunately, this approach does not seem to work if G is free to move, and therefore we defer to a future work the study of the more general case.

2 Equations of Motions and Preliminary Results

We shall describe the motion of the coupled system $\mathcal{S}$ with respect to a frame, $\mathscr{S}$, attached to $\mathcal{B}$ and with its origin at an interior point of $\mathscr{B}$. In this way, in particular, the domain occupied by $\mathcal{L}$ becomes time-independent, and we will denote it by $\mathcal{D}$ $(:= \mathbb{R}^3\backslash\mathcal{B})$ and by Σ its boundary. We suppose $\mathcal{D}$ of class

[(1)] As a matter of fact, in [5] it is only proved $\lim_{t\to\infty}\|u(t)\|_6 = 0$. However, by elementary interpolation, for any $q \in (2,6)$, we have $\|u(t)\|_q \le \|u(t)\|_2\|u(t)\|_6$, which, since $\|u(t)\|_2$ is uniformly bounded [5, Theorem 2.1], shows the claimed result.

C^2. Thus, with the notation introduced in the previous section, the governing equations of the motion of $\mathscr{S}$ are given by (see [4])

$$\left.\begin{aligned} \varrho\partial_t u &= \operatorname{div}\mathbb{T}(u,p) - \varrho[(u-V)\cdot\nabla u + \omega\times u] \\ \operatorname{div} u &= 0 \end{aligned}\right\}\text{ in } \mathcal{D}\times(0,\infty)$$

$$u = V \quad \text{at } \Sigma\times(0,\infty) \tag{2.1}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t\in(0,\infty)$$

$$\mathsf{I}\cdot\dot{\omega} + \omega\times(\mathsf{I}\cdot\omega) + \int_\Sigma x\times\mathbb{T}(u,p)\cdot n = \mathsf{M} \tag{2.2}$$

endowed with initial conditions

$$u(x,0) = u_0(x),\ x\in\mathcal{D}\,, \quad \omega(0) = \omega_0\,. \tag{2.3}$$

In the above equations, p is the pressure field of $\mathcal{L}$, ϱ its (constant) density, and $V(x,t) := \omega(t)\times x$. Also, $\mathbb{T}$ is the Cauchy stress tensor given by

$$\mathbb{T}(u,p) = 2\mu\,\mathbb{D}(u) - p\,\mathbb{I}, \quad 2\mathbb{D}(u) := \nabla u + (\nabla u)^\top\,,$$

with μ shear-viscosity coefficient and $\mathbb{I}$ identity. Moreover, m is the mass of $\mathcal{B}$ and I its inertia tensor relative to G. Furthermore,

$$\mathsf{M}(t) = \mathbb{Q}^\top(t)\cdot\mathrm{M}(t)\,, \tag{2.4}$$

with the tensor $\mathbb{Q}$ satisfying the following equation

$$\begin{cases} \dot{\mathbb{Q}} = -\mathbb{Q}\cdot\mathbb{O}(\omega) \\ \mathbb{Q}(0) = \mathbb{I} \end{cases} \qquad \mathbb{O}(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{2.5}$$

In particular, $\mathbb{Q}$ is proper orthogonal, that is,

$$\mathbb{Q}^\top(t)\cdot\mathbb{Q}(t) = \mathbb{Q}(t)\cdot\mathbb{Q}^\top(t) = \mathbb{I}, \quad \det\mathbb{Q}(t) = 1, \quad \text{for all}\, t\in\mathbb{R}\,.$$

We wish to introduce a suitable class of functions satisfying (2.1). To this end, let [(2)]

$$\mathcal{R} := \{\overline{u}\in C^\infty(\mathbb{R}^3) : \overline{u}(x) = \overline{u}\times x,\ \ \overline{u}\in\mathbb{R}^3\}\,,$$

and define

$$\mathcal{V}(\mathcal{D}) = \{u\in W^{1,2}(\mathcal{D}) : \operatorname{div} u = 0 \text{ in } \mathcal{D},\ \ u|_\Sigma = \overline{u},\ \text{for some } \overline{u}\in\mathcal{R}\}.$$

We also set

$$B_R := \{x\in\mathbb{R}^3 :\ |x| < R\}\,; \quad R_* := 2\inf\{R\in(0,\infty) : \mathscr{B}\cap B_R \supset \mathscr{B}\}\,;$$

$$\mathcal{D}_R := \mathcal{D}\cap B_R\,, \quad \mathcal{D}^R = \mathcal{D}\backslash\overline{\mathcal{D}_R}\,, \quad R > R_*\,.$$

Definition 2.1 *A triple (u,p,ω) is in the class $\mathcal{C}$, if*

$$u\in L^\infty(0,\infty;\mathcal{V}(\mathcal{D})), \quad \nabla u\in L^2(0,\infty;W^{1,2}(\mathcal{D}))$$

$$\omega\in W^{1,2}(0,\infty), \quad \nabla p\in L^2(0,\infty;L^2(\mathcal{D}))\,,$$

$$u\in C([0,\tau];W^{1,2}(\mathcal{D}_R))\,, \quad \partial_t u,\ p\in L^2(0,\infty;L^2(\mathcal{D}_R))\,, \quad \textit{for all}\, R\geq R_*\,,$$

and, in addition, (u,p,ω) satisfies (2.1).

(2) We shall use standard notation for function spaces, see [1]. So, for instance, $L^q(\mathcal{A})$, $W^{m,q}(\mathcal{A})$, $W_0^{m,q}(\mathcal{A})$, etc., will denote the usual Lebesgue and Sobolev spaces on the domain $\mathcal{A}$, with norms $\|\cdot\|_{q,\mathcal{A}}$ and $\|\cdot\|_{m,q,\mathcal{A}}$, respectively. Whenever confusion will not arise, we shall omit the subscript $\mathcal{A}$. Occasionally, for X a Banach space, we denote by $\|\cdot\|_X$ its associated norm. Moreover $L^q(I;X)$, $C(I;X)$ I real interval, denote classical Bochner spaces.

The class $\mathcal{C}$ is not empty, as secured by the following result, which is a particular case of [5, Theorem 2.1].

Theorem 2.1 *Let* $\mathrm{M} \in L^2(0,\infty;\mathbb{R}^3)$ *and* $u_0 \in \mathcal{V}(\mathcal{D})$ *with* $u_0|_\Sigma = \omega_0 \times x$. *Then, there is* $\delta > 0$ *such that if*

$$\|u_0\|_{1,2} + |\omega_0| + \|\mathrm{M}\|_{L^2(0,\infty)} \leq \delta\,, \tag{2.6}$$

there exists at least one solution $(\boldsymbol{u}, p, \omega, \mathbb{Q})$ *to* (2.1) − (2.5) *with* (u, p, ω) *in the class* $\mathcal{C}$.

From Definition 2.1 and Sobolev inequality, we infer that

$$p \in L^2(0,\infty; L^6(\mathcal{D}))\,, \tag{2.7}$$

while we only have $p \in L^2(0,\infty; L^2(\mathcal{D}_R)$ for all $R > R_*$. Our first objective is to prove that the latter property holds, in fact, in the whole of $\mathcal{D}$. Precisely, we have the following.

Proposition 2.1 *Let* $(u, p, \omega) \in \mathcal{C}$. *Then,*

$$p \in L^2(0,\infty; L^2(\mathcal{D}))\,.$$

In order to prove the proposition, we need the next two results, whose proofs are given in [5, Lemma 3.2] and [3, Lemma 3.1], respectively.

Lemma 2.1 *Let* (u, p, ξ, ω, Q) *be in the class* $\mathcal{C}$. *Then for a.a.* $t \in (0,\infty)$

$$\nabla p \in L^{q_1}(\mathcal{D}^{2R_*})\,, \quad p \in L^{q_2}(\mathcal{D}^{2R_*})\,, \quad \textit{for all } q_1 \in (1,6]\,,\, q_2 \in (\tfrac{3}{2},\infty]\,.$$

Lemma 2.2 *Let* $g \in C_0^\infty(\mathcal{D})$. *Then the Neumann problem*

$$\begin{aligned} &\Delta\varphi = g \quad \text{in } \mathcal{D} \\ &\frac{\partial\varphi}{\partial n} = 0 \quad \text{at } \Sigma \end{aligned} \tag{2.8}$$

with the side condition

$$\lim_{|x|\to\infty} \nabla\varphi(x) = 0. \tag{2.9}$$

has one and only one solution such that for all $s \in (1,3)$

$$D^2\varphi \in L^s(\mathcal{D}), \quad \nabla\varphi \in L^{3s/(3-s)}(\mathcal{D})$$

and

$$\int_{\mathcal{D}_K} \varphi = 0, \tag{2.10}$$

for a fixed $K > R_*$. *Moreover,* $\varphi \in L^\infty(\mathcal{D})$ *and*

$$\|\varphi\|_{s,\mathcal{D}_K} + \|\nabla\varphi\|_{3s/(3-s)} + \|D^2\varphi\|_s \leq c\|g\|_s\,, \tag{2.11}$$

where the constant c *depends only on* s, K *and* $\mathcal{D}$.

Proof of Proposition 2.1. By formally applying the div operator on both sides of $(2.1)_1$ and observing that
$$\operatorname{div}[\rho(\partial_t u + \omega\times x\cdot\nabla u - \omega\times u) - \Delta u] = 0\,,$$
one easily deduces that p satisfies for a.a. $t\in(0,\infty)$ the following Neumann problem in the distributional sense
$$\begin{aligned}&\Delta p = \operatorname{div}(u\cdot\nabla u) \quad \text{in } \mathcal{D}\,,\\ &\frac{\partial p}{\partial n} = -[\rho\dot\omega\times x + \operatorname{curl} a]\cdot n \quad \text{at } \mathcal{D}\,, \quad a := \operatorname{curl} u\,.\end{aligned} \tag{2.12}$$
Let g and φ be as in Lemma 2.2. Multiplying both sides of $(2.12)_1$ by g, integrating by parts over $\mathcal{D}_r$, and taking into account (2.8) and $(2.12)_2$, we show
$$\int_{\mathcal{D}_r} p\,g = -\int_{\Sigma\cup\partial B_r}\varphi\frac{\partial p}{\partial n} + \int_{\mathcal{D}_r}\varphi\operatorname{div}(u\cdot\nabla u) = \int_\Sigma \varphi\,[\rho\,\dot\omega\times x + \operatorname{curl} a]\cdot n + \rho\int_{\mathcal{D}_r}\varphi\,\partial_i\partial_j(u_iu_j) + \sigma_1(r)\,, \tag{2.13}$$
where we used the identity $\operatorname{div}(u\cdot\nabla u) = \partial_i\partial_j(u_iu_j)$, and set
$$\sigma_1(r) := -\int_{\partial B_r}(\varphi\frac{\partial p}{\partial n} - p\frac{\partial\varphi}{\partial n})\,.$$
By a double integration by parts, we infer
$$\begin{aligned}\int_{\mathcal{D}_r}\varphi\,\partial_i\partial_j(u_iu_j) &= \int_{\mathcal{D}_r} u_iu_j\partial_i\partial_j\varphi + \int_{\Sigma\cup\partial B_r}[\varphi\,u\cdot\nabla u\cdot n - u\cdot\nabla\varphi\,u\cdot n]\\ &= \int_{\mathcal{D}_r} u_iu_j\partial_i\partial_j\varphi + \int_\Sigma[\varphi\,\omega\times x\cdot\nabla u\cdot n - (\omega\times x\cdot\nabla\varphi)\,\omega\times x\cdot n] + \sigma_2(r)\,,\end{aligned} \tag{2.14}$$
where
$$\sigma_2(r) := \int_{\partial B_r}[\varphi\,u\cdot\nabla u\cdot n - u\cdot\nabla\varphi\,u\cdot n]\,.$$
By employing Hölder inequality, we deduce
$$\int_{R_*}^\infty |\sigma_2(r)|\mathrm{d}r \le \|\varphi\|_\infty\|u\|_2\|\nabla u\|_2 + \|\nabla\varphi\|_2\|u\|_4^2\,, \tag{2.15}$$
and, likewise, we infer
$$\int_{2R_*}^\infty r^{-\frac12}|\sigma_1(r)|\mathrm{d}r \le \|\varphi\|_\infty\|r^{-\frac12}\|_{3,(2R_*,\infty)}\|\nabla p\|_{\frac32,\mathcal{D}^{2R_*}} + (2R_*)^{-\frac12}\|\nabla\varphi\|_2\|p\|_2\,. \tag{2.16}$$
In view of Lemma 2.1, and the fact that $u\in\mathcal{C}$, we find that the right-hand side in both equations (2.15) and (2.16) is finite. Therefore, there exists an unbounded sequence $\{r_n\}$ such that
$$\lim_{r_n\to\infty}[\sigma_1(r_n) + \sigma_2(r_n)] = 0\,.$$
Employing this information in (2.13), (2.14) we conclude
$$\begin{aligned}\int_{\mathcal{D}} p\,g &= \int_\Sigma\{\varphi\,[\rho\,\dot\omega\times x + \operatorname{curl} a]\cdot n + \rho\,[\varphi\,\omega\times x\cdot\nabla u\cdot n - (\omega\times x\cdot\nabla\varphi)\,\omega\times x\cdot n]\} + \rho\int_{\mathcal{D}} u_iu_j\partial_i\partial_j\varphi\\ &:= I_{\Sigma1} + I_{\Sigma2} + I_{\Sigma3} + I_{\Sigma4} + I_{\mathcal{D}}.\end{aligned} \tag{2.17}$$
In the following estimates, we shall use several times the classical trace inequality
$$\|w\|_{1,\Sigma} \le c\|w\|_{1,1,\mathcal{D}_K}\,. \tag{2.18}$$

We thus have

$$|I_{\Sigma 1}| + |I_{\Sigma 4}| \le c\left(|\dot{\omega}| + |\omega|^2\right) \|\varphi\|_{2,2,\mathcal{D}_K} \,. \tag{2.19}$$

Moreover, employing (2.18) along with Schwarz inequality, we get

$$|I_{\Sigma 3}| \le c\, \||\varphi||\nabla u|\|_{1,1,\mathcal{D}_K} \le c\left(\|\varphi\|_{2,\mathcal{D}_K}\|\nabla u\|_{2,\mathcal{D}_K} + \|\varphi\|_{1,2,\mathcal{D}_K}\|\nabla u\|_{1,2,\mathcal{D}_K}\right)\,,$$

which, in turn, gives

$$|I_{\Sigma 3}| \le c\, \|\nabla u\|_{1,2,\mathcal{D}_K}\|\varphi\|_{2,2,\mathcal{D}_K} \,. \tag{2.20}$$

Also, by Schwarz inequality,

$$|I_{\mathcal{D}}| \le \|u\|_4^2 \|D^2\varphi\|_2 \,. \tag{2.21}$$

The estimate for $I_{\Sigma 2}$ requires a little care. Let ζ be a function which is one in a neighborhood of Σ and zero at large distances. We have

$$I_{\Sigma 2} = -\int_{\mathcal{D}} \mathrm{div}\,(\varphi \mathrm{curl}\,(\zeta a)) = -\int_{\mathcal{D}} \nabla\varphi\cdot \mathrm{curl}\,(\zeta a). \tag{2.22}$$

Using the identity

$$-\mathrm{curl}\, A\cdot B + \mathrm{curl}\, B\cdot A = \mathrm{div}\,(A\times B)$$

with $A = \zeta a$ and $B = \nabla\varphi$, from (2.22), (2.18) and Schwarz inequality we show

$$|I_{\Sigma 2}| = \left|\int_\Sigma a\times\nabla\varphi\cdot n\right| \le c\||\nabla u||\nabla\varphi|\|_{1,1,\mathcal{D}_K} \le c\left(\|\nabla u\|_{2,\mathcal{D}_K}\|\nabla\varphi\|_{1,2,\mathcal{D}_K} + \|D^2 u\|_{2,\mathcal{D}_K}\|\nabla\varphi\|_{2,\mathcal{D}_K}\right)\,,$$

which leads to

$$|I_{\Sigma 2}| \le c\, \|\nabla u\|_{1,2,\mathcal{D}_K}\|\varphi\|_{2,2,\mathcal{D}_K} \,. \tag{2.23}$$

If we employ (2.19)–(2.23) in (2.17) and take into account (2.11), we arrive at

$$\left|\int_{\mathcal{D}} pg\right| \le c\,\left(|\dot{\omega}| + |\omega|^2 + \|u\|_4^2 + \|\nabla u\|_2 + \|D^2 u\|_2\right)\|g\|_2 \,. \tag{2.24}$$

Since g is arbitrary in $C_0^\infty(\mathcal{D})$, and, by Sobolev embedding theorem,

$$\|u\|_4^2 \le c\, \|u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{3}{2}}\,, \tag{2.25}$$

from (2.24) it follows that $p\in L^2(\mathcal{D})$ for a.a. $t\in(0,\infty)$ and

$$\|p\|_2 \le c\,\left(|\dot{\omega}| + |\omega|^2 + \|u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{3}{2}} + \|\nabla u\|_2 + \|D^2 u\|_2\right)\,. \tag{2.26}$$

The proposition is then a consequence of (2.26) and the fact that (u,ω) is in the class $\mathcal{C}$.

□

We conclude this section with another preparatory result concerning the asymptotic behavior of functions in the class $\mathcal{C}$.

Lemma 2.3 *Let (u,p,ω) be in the class $\mathcal{C}$. Then,*

$$\lim_{t\to\infty}\|u(t)\|_6 = 0\,. \tag{2.27}$$

Proof. For $R \geq 2R_*$, let $\psi_R = \psi_R(r)$ be a smooth, non-increasing "cut-off" function such that $\psi_R = 1$ for $r \leq R$, $\psi_R = 0$ for $r \geq 2R$, and $|\nabla \psi_R| \leq C\,R^{-1}$, for some constant C independent of R. We then test $(2.1)_1$ by $\psi_R \,\mathrm{div}\,\mathbb{T}(u,p)$ to get

$$\int_{\mathcal{D}} \psi_R \partial u_t \cdot \mathrm{div}\, T = \|\sqrt{\psi_R}\mathrm{div}\, T\|_2^2 - \varrho \int_{\mathcal{D}} \left(\psi_R u \cdot \nabla u \cdot \mathrm{div}\, T - \psi_R \Phi \cdot \mathrm{div}\, T\right) . \tag{2.28}$$

By integration by parts, we formally show the following identity that can be rigorously justified by a standard approximation procedure

$$\begin{aligned}\int_{\mathcal{D}} \psi_R \partial u_t \cdot \mathrm{div}\, \mathbb{T} \quad &= \int_{\mathcal{D}} [\mathrm{div}\,(\psi_R\, \partial_t u \cdot \mathbb{T}) - 2\mu\, \psi_R\, \mathbb{D}(\partial_t u) : \mathbb{D}(u)] \\ &= \int_{\Sigma} \dot{V} \cdot \mathbb{T} \cdot n - \mu \frac{d}{dt} \|\sqrt{\psi_R}\mathbb{D}(u)\|_2^2 - \int_{\mathcal{D}} \nabla \psi_R \cdot \mathbb{T} \cdot \partial_t u \,.\end{aligned}$$

Set $\Phi := \omega \times x \cdot \nabla u - \omega \times u$ and recall [6, Lemma 2.4(b)]

$$\Phi \cdot n = 0 \quad \text{at } \Sigma \tag{2.29}$$

Thus, integrating by parts and with the help of (2.29) we show

$$\int_{\mathcal{D}} \psi_R \Phi \cdot \mathrm{div}\, \mathbb{T} = 2\mu \int_{\Sigma} \Phi \cdot \mathbb{D}(u) \cdot n - 2\mu \int_{\mathcal{D}} \psi_R \partial_i \Phi_j (\mathbb{D}(u))_{ij} - \int_{\mathcal{D}} \nabla \psi_R \cdot \mathbb{T} \cdot \Phi \,. \tag{2.30}$$

Next, since $\mathrm{div}\, u = \mathrm{div}\, V = 0$, we get

$$2\partial_i \Phi_j (\mathbb{D}(u))_{ij} = \mathrm{div}\,(\Phi \cdot \nabla u + \tfrac{1}{2} V\, |\nabla u|^2) + \omega \times \nabla u_i \cdot \nabla u_i - \nabla(\omega \times u) : \nabla u \,,$$

so that, substituting the latter in (2.30) and using Gauss theorem, we infer

$$\begin{aligned}\int_{\mathcal{D}} \psi_R \Phi \cdot \mathrm{div}\, \mathbb{T} = \; & -\mu \int_{\mathcal{D}} \psi_R \left(\omega \times \nabla u_i \cdot \nabla u_i - \nabla(\omega \times u) : \nabla u\right) + \mu \int_{\Sigma} \left(n \cdot \nabla u \cdot \Phi - \tfrac{1}{2} V \cdot n |\nabla u|^2\right) \\ & + \int_{\mathcal{D}} \nabla \psi_R \cdot \left[2\mu (\nabla u^\top \cdot \Phi + \tfrac{1}{2} V |\nabla u|^2) - \mathbb{T} \cdot \Phi\right] \,.\end{aligned} \tag{2.31}$$

Collecting (2.28), (2.30) and (2.31) we deduce

$$\begin{aligned}\mu \frac{d}{dt} \|\sqrt{\psi_R}\mathbb{D}(u)\|_2^2 + \|\sqrt{\psi_R}\mathrm{div}\, \mathbb{T}\|_2^2 = & \int_{\Sigma} \left[\dot{V} \cdot \mathbb{T} \cdot n - \varrho\mu \left(n \cdot \nabla u \cdot \Phi - \tfrac{1}{2} V \cdot n |\nabla u|^2\right)\right] \\ & + \varrho \int_{\mathcal{D}} \psi_R \left[u \cdot \nabla u \cdot \mathrm{div}\, \mathbb{T} + \mu \left(\omega \times \nabla u_i \cdot \nabla u_i - \nabla(\omega \times u) : \nabla u\right)\right] \\ & - \varrho \int_{\mathcal{D}} \nabla \psi_R \cdot \left[\mathbb{T} \cdot \partial_t u + 2\mu (\nabla u^\top \cdot \Phi + \tfrac{1}{2} V |\nabla u|^2) - \mathbb{T} \cdot \Phi\right] := I_\Sigma + I_{\mathcal{D}} + I_R\end{aligned} \tag{2.32}$$

Arguing exactly as in the proof of [5, Eq. (3.22)] we show

$$\lim_{R \to \infty} \int_0^t I_R = 0\,, \ \text{ for all } t > 0\,. \tag{2.33}$$

Furthermore, as shown in the proof of [5, Theorem 2.1], we have

$$|I_D| \leq c \left(\|\mathbb{D}(u)\|_2^3 + \|\mathbb{D}(u)\|_2^4 + \|\mathbb{D}(u)\|_2^6\right) + \tfrac{1}{4} \|\mathrm{div}\, \mathbb{T}\|_2^2 \tag{2.34}$$

Finally, using (2.18) multiple times along with the inequality [4, Lemma 4.9]

$$|\omega| \leq c\, \|\nabla u\|_2 \,, \tag{2.35}$$

one can prove, in a way entirely similar to [5, Eq. (4.11)]

$$|I_\Sigma| \le c\,\left[|\dot\omega|(\|\nabla u\|_{1,2}+\|p\|_{1,2})+\|\mathbb{D}(u)\|_2^3+\|\mathbb{D}(u)\|_2^4\right]+\tfrac14\|\mathrm{div}\,\mathbb{T}\|_2^2\,. \tag{2.36}$$

We now integrate both sides of (2.31) over $(0,t)$, $t\in(0,\infty)$, let $R\to\infty$ and employ (2.33), along with Lebesgue dominated convergence theorem. If we differentiate with respect to t the resulting equation, and take into account the estimates (2.34) and (2.36), we conclude, in particular,

$$\frac{d}{dt}\|\mathbb{D}(u)\|_2^2 \le c\,\left[|\dot\omega|(\|\nabla u\|_{1,2}+\|p\|_{1,2})+\|\mathbb{D}(u)\|_2^3+\|\mathbb{D}(u)\|_2^4+\|\mathbb{D}(u)\|_2^6\right]:=h(t) \tag{2.37}$$

Since $(u,p,\omega)\in\mathcal{C}$, and also in view of Proposition 2.1 we infer, on the one hand, $h\in L^1(0,\infty;\mathbb{R})$ and, on the other hand, the existence of an unbounded sequence of times $\{t_n\}$ such that

$$\lim_{t_n\to\infty}\|\mathbb{D}(u(t_n))\|_2\to 0\,.$$

Thus, integrating both sides of (2.37) over the interval (t_n,t), $t>t_n$, we get

$$\|\mathbb{D}(u(t))\|_2^2\le\|\mathbb{D}(u(t_n))\|_2^2+\int_{t_n}^\infty h(s)\,\mathrm{d}s$$

which implies

$$\lim_{t\to\infty}\|\mathbb{D}(u(t))\|_2\to 0\,.$$

The latter furnishes the desired result after we use the inequality [4, Eq. (4.75)]

$$\|u\|_6\le c\,\|\mathbb{D}(u)\|_2\,,\quad u\in\mathcal{V}.$$

□

3 Main Result

In this section we will give a proof of the following result, representing the major achievement of this note.

Theorem 3.1 *Let (u,p,ω) be in the class $\mathcal{C}$. Suppose that*

$$\sqrt{\ln r}\,u_0\in L^2(\mathcal{D})\,,\quad (r:=\sqrt{x_ix_i})\,.$$

Then,

$$\lim_{t\to\infty}E(t)\equiv\tfrac12\lim_{t\to\infty}\left(\|u(t)\|_2^2+\omega(t)\cdot\mathsf{I}\cdot\omega(t)\right)=0\,.$$

Proof. We begin to observe that by assumption $\omega\in W^{1,\infty}(0,\infty;\mathbb{R}^3)$ which delivers

$$\lim_{t\to\infty}|\omega(t)|=0\,.$$

Therefore, we only have to show

$$\lim_{t\to\infty}\|u(t)\|_2=0\,. \tag{3.1}$$

To this end, let $\psi_R=\psi_R(r)$ be the "cut-off" function introduced in Lemma 2.3. By dot-multiplying through both sides of (2.1) by $\psi_R\ln r\,u$, integrating by parts over $\mathcal{D}$, and using the fact that $\nabla\psi_R\cdot\omega\times x=$

0, we show

$$\tfrac12\frac{d}{dt}\|\psi_R\sqrt{\ln r}\,u(t)\|_2^2 = \tfrac12\int_{\mathcal D}\left(\ln r\,u^2u\cdot\nabla\psi_R+\psi_R u^2\,u\cdot\frac{x}{r^2}\right)-2\int_{\mathcal D}\ln r\,\mathbb D(u):\mathbb D(u)$$
$$-2\int_{\mathcal D}(\mathbb D(u))_{ij}[\ln r(u_j\partial_i\psi_R+u_i\partial_j\psi_R)+\psi_R(u_j\frac{x_i}{r^2}+u_i\frac{x_j}{r^2})] \tag{3.2}$$
$$+\int_{\mathcal D}p\,u\cdot\left(\nabla\psi_R\,\ln r+\psi_R\frac{x}{r^2}\right)+\int_\Sigma\psi_R\,\ln r\,\omega\times x\cdot\mathbb T(u,p)\cdot n\,.$$

Employing Hölder inequality multiple times, and recalling the properties of ψ_R, we get

$$\int_{\mathcal D}\{\ln r\,u^2u\cdot\nabla\psi_R-2(\mathbb D(u))_{ij}\;[\ln r(u_j\partial_i\psi_R+u_i\partial_j\psi_R)]+p\,u\cdot\nabla\psi_R\}$$
$$\le\|\ln r|\nabla\psi_R|\|_{4,\mathcal D^R}\big(\|u\|_4^3+2\|\mathbb D(u)\|_2\|u\|_4+\|p\|_2\|u\|_4\big)$$
$$\le c\,R^{-\frac12}\left(\|u\|_4^3+\|\mathbb D(u)\|_2^{\frac32}+\|p\|_2^{\frac32}\right)\,.$$

On account of $(u,p)\in\mathcal C$ and (2.26) we thus infer

$$\lim_{R\to\infty}\int_0^t\left\{\int_{\mathcal D}\{\ln r\,u^2u\cdot\nabla\psi_R-2(\mathbb D(u))_{ij}[\ln r(u_j\partial_i\psi_R+u_i\partial_j\psi_R)]+p\,u\cdot\nabla\psi_R\}\right\}\mathrm ds=0\,,\quad\text{for all } t>0\,. \tag{3.3}$$

Similarly, using Hardy's inequality

$$\int_{\mathcal D}\frac{u^2}{r^2}\le4\|\nabla u\|_2^2$$

furnishes

$$\int_{\mathcal D}\frac1r\left|u^2u\cdot\frac xr+p\,u\cdot\frac xr-2(\mathbb D(u))_{ij}(u_j\frac{x_i}{r}+u_i\frac{x_j}{r})\right|\le c\|r^{-1}u\|_2\left(\|u\|_4^2+\|p\|_2+\|\mathbb D(u)\|_2\right)$$
$$\le c\,\left(\|u\|_4^4+\|p\|_2^2+\|\nabla u\|_2^2\right),$$

which, in turn, since $(u,p)\in\mathcal C$, with the help of Proposition 2.1 and (2.25) entails

$$\int_0^\infty\left\{\int_{\mathcal D}\frac1r\left|u^2u\cdot\frac xr+p\,u\cdot\frac xr-2(\mathbb D(u))_{ij}(u_j\frac{x_i}{r}+u_i\frac{x_j}{r})\right|\right\}\mathrm dt<\infty\,. \tag{3.4}$$

Finally, using (2.18) with $w=\mathbb D(u)$ and $w=p$, and recalling (2.35), we obtain

$$\left|\int_\Sigma\psi_R\,\ln r\,\omega\times x\cdot\mathbb T(u,p)\cdot n\right|\le c\,\|\nabla u\|_2\left(\|\nabla u\|_2+\|D^2u\|_2)+\|p\|_{1,2}\right)$$

which, because $(u,p)\in\mathcal C$, with the help of Proposition 2.1 provides

$$\int_0^\infty\left|\int_\Sigma\psi_R\,\ln r\,\omega\times x\cdot\mathbb T(u,p)\cdot n\right|\mathrm dt<\infty\,. \tag{3.5}$$

We now integrate both sides of (3.2) over $[0,t]$, arbitrary $t>0$, and then pass to the limit $R\to\infty$. Taking into account (3.3), that $\psi_R\le1$, and employing Fubini's theorem, we thus receive, in particular,

$$\|\sqrt{\ln r}\,u(t)\|_2^2-\|\sqrt{\ln r}\,u_0\|_2^2\le2\int_0^t\left\{\int_{\mathcal D}\frac1r\left|u^2u\cdot\frac xr+p\,u\cdot\frac xr-2(\mathbb D(u))_{ij}(u_j\frac{x_i}{r}+u_i\frac{x_j}{r})\right|\right\}\mathrm ds\,, \tag{3.6}$$

which, by (3.4), (3.5) and the assumption, furnishes

$$\sup_{t\ge0}\|\sqrt{\ln r}\,u(t)\|_2^2\le M<\infty\,. \tag{3.7}$$

For any fixed R, employing also Hölder inequality, we have

$$\|u(t)\|_2^2 = \|u(t)\|_{2,\mathcal{D}_R}^2 + \|u(t)\|_{2,\mathcal{D}^R}^2 \le c\,R^2\|u(t)\|_6^2 + \frac{1}{\ln R}\|\sqrt{\ln r}\,u(t)\|_{2,\mathcal{D}^R}^2\,,$$

which, by (3.7), entails

$$\|u(t)\|_2^2 \le c\,R^2\|u(t)\|_6^2 + \frac{M}{\ln R}\,.$$

Therefore, if we operate with $\limsup_{t\to\infty}$ on both sides of this relation, use (2.27) and then let $R\to\infty$, we arrive at (3.1), thus completing the proof of the theorem.

$\square$

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References

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, New York (1975)

[2] Ervedoza, S., Maity, D., and Tucsnak, M., Large time behaviour for the motion of a solid in a viscous incompressible fluid. *Math. Ann.* **385** (2023), no. 1-2, 631–691

[3] Galdi, G.P., Slow motion of a body in a viscous incompressible fluid with application to particle sedimentation. *Recent developments in partial differential equations*, 1–35, Quad. Mat., **2**, Dept. Math., Seconda Univ. Napoli, Caserta, 1998.

[4] Galdi, G.P., On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications, *Handbook of Mathematical Fluid Mechanics*, Elsevier Science, Vol. 1 653–791 (2002)

[5] Galdi, G.P., Large-time behavior of a rigid body of arbitrary shape in a viscous fluid under the action of prescribed forces and torques. *J. Math. Fluid Mech.* **25** (2023), no. 3, Paper No. 43, 11 pp.

[6] Galdi, G.P., Silvestre, A.L.: Strong Solutions to the Problem of Motion of a Rigid Body in a Navier–Stokes Liquid Under the Action of Prescribed Forces and Torques. Nonlinear Problems in Mathematical Physics and Related Topics, I, pp. 121–144. International Mathematical Series (N. Y.), vol. 1. Kluwer/Plenum, New York (2002)

[7] Maity, D., and Tucsnak, M., Motion of rigid bodies of arbitrary shape in a viscous incompressible fluid: Wellposedness and Large Time Behaviour. *J. Math. Fluid Mech.* **25** (2023), no. 3, 74

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