Problems in microswimmer hydrodynamics

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Abstract

In this article, we present an informal introduction to the hydrodynamics of microscopic (self-)propelling particles, termed microswimmers. Due to the negligible inertia of the system, the dynamics are strongly restricted by the shape of the swimmer. Focusing on the mathematical structure of the system, we discuss some possible mathematical problems based on the author's interests.

1 Introduction

Even a single water droplet in a pond contains thousands of microorganisms. Many of these submillimeter living things are self-propelled often by using a slender appendage called cilia and flagella [1]. Since these organisms live in a fluid environment, the motion of the fluid is a strong mechanical constraint on their life. Studies on such biological locomotion under a microscope, termed *microswimming*, has rapidly expanded in these two decades, motivating the synthesis of artificial microswimmers such as magnetically- and electrically-activated microrobots, self-propelled colloids with chemical reaction, and combinations of engineered biological materials [2, 3].

This article is a (very) informal introduction to the hydrodynamics of microswimmer to showcase possible mathematical problems that might interest researchers working on rigorous mathematical analysis, though these problems are purely based on the author's interests and are far from a comprehensive survey or review of the field. For readers in a comprehensive review of the field, refer to some recent monographs on this topic [4, 5].

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In Sec. 2, we start with a brief introduction of the governing equations in the microswimmer hydrodynamics with a given shape gait, often referred to as a kinematic problem. We then proceed to the dynamics in a fluid flow and introduce a concept of hydrodynamic shape in Sec. 3. In Sec. 4, we extend our formulation to deal with a fluid-structure coupling, where the shape gait is determined by elasticity equations of a swimmer and fluid equation for the medium surrounding it. Concluding remarks are made in Sec. 5.

2 Kinematic microswimmer hydrodynamics

We first set up the governing equations of microswimmer hydrodynamics. Let $\mathcal{V} \subset \mathbb{R}^3$ be a region occupied by a fluid. We assume that the fluid satisfies the incompressible Navier-Stokes equations. Since the swimmer immersed in the fluid is microscopic, the Reynolds number for the fluid motion is typically very small. For a microorganism in water, the Reynolds number is at the order of 10^{-5} for a swimming bacterium, 10^{-3} for a mammalian sperm cell, and 10^{-1} for a larger ciliate such as Paramecium. Thus, as the governing equation of the system, we neglect all the inertia effects in the Navier-Stokes equations, yielding the so-called (steady) Stokes equations. For a velocity field u_i and a stress field σ_{ij} $(i, j \in \{1, 2, 3\})$, the Stokes equation reads,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \text{ and } \frac{\partial u_i}{\partial x_i} = 0, \tag{2.1}$$

where we use the Einstein summation convention for repeated indices. The stress tensor σ_{ij} is given by the Newton's constitutive relation, $\sigma_{ij} = -p\delta_{ij} + 2\mu E_{ij}$, where p is the pressure field, E_{ij} is the rate of strain tensor, given by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.2}$$

δ_{ij} is the Kronecker delta and μ is the fluid visocity and assumed to be constant.

We impose the no-slip boundary condition for the boundary of the fluid region and the fluid flow is then completely determined by the shape of the boundary and the values on it. Since the swimmer can deform by itself, the fluid region can move in time. The motion of the swimmer is represented by a summation of translation, rotation, and deformation. In the kinematic problem, we assume that the shape of the swimmer is a given function, and solve the translational and rotational velocities from the equations. Note that the shape gait is defined in the body-fixed coordinates and that the choice of the body-fixed coordinate possesses gauge freedom [6].

To close the system, we need the equations of motion for the center of mass of the swimmer. For microswimmers, the inertial effects are negligible and the momentum and angular momentum conservations are reduced into relations of force balance and torque balance. Let $\mathcal{S}$ be the surface of the swimmer and n_i be the outward (from the fluid region) normal to the surface, the balance relations are then written as

$$\int_{\mathcal{S}} \sigma_{ij} n_j \, dS = \int_{\mathcal{S}} \epsilon_{ijk}(x_j - X_j)\sigma_{k\ell} n_\ell \, dS = 0. \tag{2.3}$$

Here, ϵ_{ijk} is the Levi-Civita symbol, the position vector X_j denotes a point fixed in the space, and the surface integral is performed over the variable x_j. Then the kinematic swimmer problem is closed by these equations. Indeed, existence and uniqueness have been analyzed in a formal manner [7].

One of the fundamental properties of kinematic microswimmer hydrodynamics is kinematic reversibility. This is well-known as the scallop theorem [8], which states that after one period of deformation, a microswimmer goes back to its original position, including its orientation if its deformation is reciprocal in time. This theorem, therefore, is a strong mechanical constraint on the locomotion of a microscopic object. A formal proof of the theorem may be found in Ishimoto & Yamada [9].

The scallop theorem only holds when all the inertia effects are neglected in Newtonian fluid dynamics. Then many studies have explored its extension and exact limitation of the condition that prohibits reciprocal swimming [10]. Indeed, some theoretical analyses with a simple mathematical model such as a deforming sphere, known as the squirmer model, demonstrate that the non-zero inertia generates net locomotion even with a reciprocal deformation [11, 12, 13, 14]. These perturbation analyses suggest a continuous breakdown of the scallop theorem due to finite, non-zero Reynolds number. On the other hand, numerical investigations of an oscillatory flapping [15, 16] suggest an existence of a non-zero critical Reynolds number above which the fluid motion becomes unstable and thus generates locomotion for a reciprocal deformation. The numerical study by Ota et al. [16] implies the symmetry breakdown is related to the symmetry of the shape gait. This comes to the first problem in which the author has long been interested.

Probem 1: Determine how the scallop theorem is broken by non-zero Reynolds numbers.

The next question is whether we can lift the assumption of the Newtonian fluid for the scallop theorem. Laboratory experiments with a viscoelastic, non-Newtonian fluid reveal that non-reciprocal microswimming is possible [17]. Results of theoretical analyses for a general deforming self-propelled object suggest that non-linear viscoelasticity is required for reciprocal locomotion [18, 19]. However, another linear viscoelastic model can generate net locomotion even with a reciprocal swimmer [20]. Linear viscoelastic models should be interpreted as an effective description of the full non-linear model since the linear representation violates the material objectivity in the constitutive relation of the material [21]. The second problem which the author is interested in is:

Problem 2: Determine how the scallop theorem is broken by non-Newtonian effects.

3 Microswimmers in fluid flows

We then proceed to the dynamics of passive and active particles in a background fluid flow. Let us assume the background fluid flow u_i^∞ is locally approximated as a linear flow as

$$u_i^\infty = U_i^\infty + \epsilon_{ijk}\Omega_k^\infty(x_k - X_k) + E_{ij}^\infty(x_k - X_k), \tag{3.1}$$

where U_i^∞, Ω^∞ and E_{ij}^∞ are the local linear velocity, local background rotational velocity, and the local rate-of-strain tensor of the background flow. With the linearity of the Stokes equation, one may solve the dynamics of the swimmer formally. Furthermore, the hydrodynamic forces cay be decomposed into propulsion and hydrodynamic force from the surrounding flow, the latter of which is given by

$$\mathcal{F}_a = \mathcal{K}_{ab}\mathcal{U}_b + \mathcal{G}_{ajk}E_{jk}^\infty. \tag{3.2}$$

Here, we introduce a six-dimensional velocity $\mathcal{U}_a$ and a six-dimensional force $\mathcal{F}_a$. The 6×6 tensor $\mathcal{K}_{ab}$ and $6\times 3\times 3$ tensor $\mathcal{G}_{ajk}$ are, respectively, called the grand resistance tensor and grand shear-resistance tensor. For precise definitions, refer to the recent review paper [22]. The indices a and b move from 1 to 6, whereas the indices j and k move from 1 to 3.

The components of resistance tensors depend on the body-fixed frame, but the tensors themselves are only determined by the (instantaneous) shape of the swimmer. We then introduce a symmetry of the resistance tensors, not from its actual shape. Let us consider a transformation of the body-fixed coordinates via $A_{ij} \in \mathrm{O}(3)$. If the representation of the resistance tensors $\mathcal{K}_{ab}$ and $\mathcal{G}_{ajk}$ are invariant under this transformation, then we can define the hydrodynamic symmetry associated with A_{ij}. For example, when we take A_{ij} as a rotational matrix around an axis by an arbitrary angle. The geometrical symmetry associated with this transformation is a body of revolution and is usually called an axisymmetric object. However, the hydrodynamic symmetry associated with this transformation indeed corresponds to a wider class of objects. Such an object is called a helicoidal object and may be interpreted as a hydrodynamically-axisymmetric object [23, 24].

The dynamics of a helicoidal object are exactly solvable. Let p_i be a unit vector to express its direction of the axis of helicoidal symmetry. The dynamics of this orientation vector are derived as [23]

$$\dot{p}_i = \epsilon_{ijk}\Omega_j^\infty p_k + B(\delta_{ij} - p_i p_j)E_{jk}^\infty p_k + +C\epsilon_{ijm}(\delta_{jk} - p_j p_k)E_{k\ell}^\infty p_\ell p_m + \dot{p}_i^{\mathrm{prop}}, \tag{3.3}$$

where B and C are parameters only determined by the instantaneous shape and the last term is a contribution from self-propulsive rotation. If the object is geometrically axisymmetric ($C = 0$), the orientational dynamics are known as Jeffery's equation named after G. B. Jeffery, who first derived this equation for a spheroidal object in 1922 [25]. Then the third term vanishes and the shape is only represented by a single parameter B, which is known as the Bretheron parameter and interpreted as an effective aspect ratio [26]. The new constant in Eq. (3.3) is a contribution from shape chirality. Indeed, the locomotion of a bacterial cell with a helical flagellum is well explained by the equation.

The shape parameters such as B and C integrate all the detailed shape geometry and thus the hydrodynamic shape is much simpler than the actual geometrical shape. It is therefore natural to ask the following question:

Problem 3: Determine all the possible hydrodynamic symmetries and shape parameters.

In [24], the author specified all the objects with n-fold rotational symmetry ($n \geq 3$). In particular, when $n \geq 4$, the dynamics are found to be expressed by Eq. (3.3). The parameter B typically ranges from -1 (the limit of a disk) to 1 (the limit of a rod), however, there exists an object beyond this range [27]. The shape parameter C has also been estimated for several microswimmers [28, 29]. Then one may ask how we can manufacture an object with a very large C, for instance. The next problem the author would like to propose is:

Problem 4: Determine an actual shape of a micro object with given shape parameters.

The form of Eq. (3.3) is even applicable for a rapidly deforming swimmer as emergent time-average slow dynamics [30, 31]. In these studies, they demonstrate the asymptotic form of the dynamics using the classical multi-scale analysis. Interestingly, the effective shape parameter in the slow dynamics depends on the shape gait. Also, this generalized Jeffery equation covers a rapidly spinning swimmer such as wobbling bacterium [32, 33]. The emergent slow dynamics are derived by the multi-scale analysis and the effective shape parameters are affected by individual shape gait. For interested readers, refer to [22].

4 Microswimmer elastohydrodynamics

In the kinematic problem discussed in the previous section, the shape gait is a given function of time. However, many biological swimmers required internal actuation that deforms the shape of the swimmer with interacting surrounding viscous fluid. In this case, the shape is no more a given function, but an unknown function to be determined by equations of motion. The swimmer is often modeled as an elastic material and thus these swimming problems containing fluid-structure interactions are called elastohydrodynamics.

As we drive the deformation by internal actuation, the dynamics system becomes non-autonomoum and non-equilibrium. In the swimmer dynamics, the energy is typically injected from molecular scales, converted to deformation, and dissipated by the surrounding viscous medium. Such a non-equilibrium state of matter has recently gathered a special attention as a matter violating Newton's third law [34].

Recently, the description of an elastic material is extended to capture the energy injection from a microscopic scale, by non-symmetric components of the elastic matrix, termed odd elasticity [35, 36]. With this concept, we can write down the elastohydrodynamic system via an autonomous equation. We first label the shape of a swimmer by N dimensional vector, σ_α, where α moves from 1 to N. For a general swimmer in three-dimensional space, the state of the swimmer is designated by $N+6$-dimensional vector, $\boldsymbol{z} = (x_1, x_2, x_3, \theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2, \ldots, \sigma_\alpha, \ldots, \sigma_N)^{\mathrm{T}}$. The first six components indicate the position and the angle variables. We then write the odd-elastic fluid-structure interactions at low Reynolds number in the form [37, 38],

$$-M_{ab}\dot{z}_b = L_{ab}z_b, \tag{4.1}$$

with the indeces, a and b, moving from 1 to $N+6$. The matrix M_{ab} is a generalized resistance matrix that encodes the hydrodynamic interaction. The matrix on the right-hand side represents the elastic interaction, which also includes odd elasticity. Let us designate the right bottom components as $L_{\alpha+6,\beta+6} = K_{\alpha\beta}$, and other components of this matrix are set to be zero. The $N \times N$ matrix $K_{\alpha\beta}$ is a generalization of the elastic interactions between the units of the active material. In an ordinary elastic material, we usually assume that the elastic force is conservative and thus represented by a gradient of potential. The matrix $K_{\alpha\beta}$ then must be symmetric, as known as Maxwell-Betti reciprocity. An odd elastic matrix allows its non-symmetric components, and we consider such situations where $K_{\alpha\beta} \neq K_{\beta\alpha}$.

The resistance matrix on the right-hand side is determined only by the instantaneous shape, and we can also consider $K_{\alpha\beta}$ as a function of the shape. The simplest case, however, is a constant $K_{\alpha\beta}$ matrix and this is a generalization of the linear elasticity [37]. With the linear odd elasticity, the swimmer can generate a self-sustain wave pattern through geometrical non-linearity without any controlled, tuned actuation, although only a pusher-type swimmer, swimming with an appendage at the rear of the body, is possible. To express a general swimmer performing a periodic deformation, we can further extend the odd elasticity to a non-linear regime [38], which enables a unified description of active elastic material in a viscous fluid. A real biological cell can perceive external chemical and mechanical stimuli and respond to them, by changing its gait pattern. This fact further motivates the description of active soft material including smart behaviors of cells. This brings our final problem:

Problem 5: Determine and classify possible constitutive relations of active elastic materials

5 Concluding remarks

In this article, an informal introduction to microswimmer hydrodynamics is presented based on the author's interests. Microswimmers are still very active research fields where many research disciplines including physics, biology, medicine, engineering, data science, and mathematics, meet and are integrated.

Acknowledgements

The author acknowledges the Japan Society for the Promotion of Science (JSPS) KAKENHI for Transformative Research Areas A (Grant No. 21H05309), the Japan Science and Technology Agency (JST), PRESTO (Grant No. JPMJPR1921), and JST, FOREST (Grant No. JPMJFR212N). This study is supported in part by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located at Kyoto University.

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