RECENT PROGRESS IN THE STABILITY THEORY FOR THE SYMMETRIC HYPERBOLIC SYSTEM WITH GENERAL RELAXATION

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1. Introduction

In this note, we summarize the stability theory for the symmetric hyperbolic system with relaxation. The symmetric hyperbolic system with relaxation is described as follows.

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0. \tag{1.1}$$

Here, $u = u(t,x) \in \mathbb{R}^m$ over $t > 0$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $n \geq 1$ and $m \geq 2$ is an unknown vector function, and A^0, A^j and L are real valued $m \times m$ constant matrices for $1 \leq j \leq n$. We assume the following condition for the coefficient matrices of (1.1).

Condition (A): A^0 is real symmetric and positive definite, A^j $(1 \leq j \leq n)$ are real symmetric, while L is not necessarily real symmetric but are non-negative definite with the non-trivial kernel.

Namely, Condition (A) means that the constant matrices satisfy the followings.

$$\begin{aligned} &(A^j)^T = A^j \quad (0 \leq j \leq n), \\ &A^0 > 0, \quad L^\sharp \geq 0 \quad \text{on} \quad \mathbb{R}^m, \\ &\mathrm{Ker}(L) \neq \{0\}. \end{aligned}$$

Here and in the sequel, the superscript ^{T} stands for the transposition, and * stands for the Hermitian transposition. $X^\sharp$ and $X^\flat$ are denoted the Hermitian and skew-Hermitian part of the complex matrix X, respectively. Namely, $X^\sharp := (X + X^*)/2$ and $X^\flat := (X - X^*)/2$. Furthermore, a $m \times m$ complex matrix X is called positive definite (resp. semi-positive definite) on $\mathbb{C}^m$ if $\langle X^\sharp \varphi, \varphi \rangle > 0$ (resp. $\langle X^\sharp \varphi, \varphi \rangle \geq 0$) for any $\varphi \in \mathbb{C}^m \backslash \{0\}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^m$. For a complex matrix X, $\| \cdot \|$ denotes a spectral norm, that is,

$$\|X\| := \max_{\varphi \neq 0} \frac{|X\varphi|}{|\varphi|} = \max_{|\sigma|=1} |X\sigma|.$$

To analyze the dissipative structure of (1.1), we apply the Fourier transform to (1.1). This yields

$$A^0 \hat{u}_t + i|\xi| A(\omega) \hat{u} + L\hat{u} = 0. \tag{1.2}$$

Here, we used a notation that

$$A(\omega) := \sum_{j=1}^{n} A^j \omega_j,$$

where $\omega = (\omega_1, \cdots, \omega_n)$ is a unit vector in $\mathbb{R}^n$, which means $\omega \in S^{n-1}$. Then the corresponding eigenvalue problem is

$$\lambda A^0 \varphi + (irA(\omega) + L)\varphi = 0 \tag{1.3}$$

for $r \geq 0$ and $\omega \in S^{n-1}$. Here, $(\lambda, \varphi) \in \mathbb{C} \times \mathbb{C}^m \backslash \{0\}$ is a pair of the eigenvalue and the eigenvector of (1.3), To discuss the dissipative structure, we define the strict and uniform dissipativity for the system (1.1).

Definition 1.1. (Strict and uniform dissipativity ([17])) (i) *The system* (1.1) *is called strictly dissipative if real parts of all the eigenvalues of* (1.3) *are negative for each* $r > 0$ *and* $\omega \in S^{n-1}$. (ii) *The system* (1.1) *is called uniformly dissipative of the type* (α, β) *if all the eigenvalues* $\lambda(r, \omega)$ *of* (1.3) *satisfy*

$$\mathrm{Re}\lambda(r, \omega) \leq -c\frac{r^{2\alpha}}{(1 + r^2)^\beta}$$

for each $r \geq 0$ *and* $\omega \in S^{n-1}$*, where* c *is a certain positive constant and* (α, β) *is a pair of non-negative integers.*

Remark 1. *The uniform dissipativity of the type* (α, β) *with* $\alpha = \beta$ *or* $\alpha < \beta$ *is called the standard type or the regularity-loss type, respectively.*

If the relaxation matrix L of the system (1.1) has the symmetric property, Shizuta-Kawashima [13] and Umeda-Kawashima-Shizuta [23] introduced the useful stability condition called Kawashima-Shizuta condition or Stability Condition. Precisely, they introduced the following conditions.

Stability Condition (SC): For each $(\mu, \omega) \in \mathbb{R} \times S^{n-1}$,

$$\mathrm{Ker}(\mu I + (A^0)^{-1}A(\omega)) \cap \mathrm{Ker}(L) = \{0\}.$$

Craftsmanship Condition (K): There is a real compensating matrix $K(\omega) \in C(S^{n-1})$ with the following properties: $K(-\omega) = -K(\omega)$, $(K(\omega)A^0)^T = -K(\omega)A^0$ and

$$L + (K(\omega)A(\omega))^\sharp > 0 \quad \text{on} \quad \mathbb{R}^m$$

for each $\omega \in S^{n-1}$.

On the other hand, and Beaucherd-Zuazua [1] discussed the different condition called Kalman Rank Condition for the system (1.1), that is stated as follows.

Kalman Rank Condition (R): For each $\omega \in S^{n-1}$, the $m^2 \times m$ Kalman matrix has rank m, namely,

$$\mathrm{rank}\begin{pmatrix} L \\ L(A^0)^{-1}A(\omega) \\ L((A^0)^{-1}A(\omega))^2 \\ \vdots \\ L((A^0)^{-1}A(\omega))^{m-1} \end{pmatrix} = m.$$

Then they derived the following relation to the above conditions and dissipative structure for (1.1).

Theorem 1.2. ([1, 13, 23]) *Suppose the system* (1.1) *satisfies Condition* (A) *with*

$$\mathrm{Ker}(L) = \mathrm{Ker}(L^\sharp). \tag{1.4}$$

Then, for the system (1.1)*, the following conditions are equivalent.*
(i) *Stability Condition* (SC) *holds.*
(ii) *Kalman Rank Condition* (R) *holds.*
(iii) *Craftsmanship Condition* (K) *holds.*
(iv) *System* (1.1) *is strictly dissipative.*
(v) *System* (1.1) *is uniformly dissipative of the type* $(1,1)$.

The typical property of the type $(1,1)$ is that the high-frequency part decays exponentially while the low-frequency part decays polynomially with the rate of the heat kernel. A lot of physical models satisfies these conditions and can be applied Theorem 1.2. For example, the model system of the compressible fluid gas and the discrete Boltzmann equation are studied by Kawashima [7] and Shizuta-Kawashima [13], respectively. On the other hand, some complicated physical models which possess the weak dissipative structure called the regularity-loss structure was studied in recent 10 years. For example, the dissipative Timoshenko system was discussed in [5, 6, 12], the Euler-Maxwell system was studied in [21, 22]. We would like to emphasize that these physical models do not satisfy (1.4) but Condition (A). Namely, we can apply Theorem 1.2 to these models no longer. Considering this situation, Ueda-Duan-Kawashima [17] introduced the new condition called Condition (S) for the system (1.1) as follows.

Condition (S): There is a real compensating matrix S with the following properties: $(SA^0)^T = SA^0$ and

$$(SL)^\sharp + L^\sharp \geq 0 \quad \text{on} \quad \mathbb{R}^m, \qquad \mathrm{Ker}((SL)^\sharp + L^\sharp) \subseteq \mathrm{Ker}(L),$$
$$i(SA(\omega))^\flat = 0 \quad \text{on} \quad \mathrm{Ker}(L^\sharp)$$

for each $\omega \in S^{n-1}$.

It is important to research for the dissipative structure of linear differential equations. From the detailed analysis of the dissipative structure, we can lead to the decay estimate for the solutions. The analysis of the dissipative structure for a problem is based on the analysis of the corresponding eigenvalue problem. The profile of the eigenvalues gives a useful information concerned with the dissipative structure.

Generally it is difficult to get a property of the eigenvalues from the coefficient matrices of the system. Fortunately, if the coefficient matrices of our system have a good property, it might be easy to analyze the eigenvalue problem. Indeed, for the symmetric hyperbolic system with symmetric relaxation, Shizuta-Kawashima [13] and Umeda-Kawashima-Shizuta [23] introduced the stability conditions for the coefficient matrices and disclosed the relation between these conditions and the eigenvalues. In recent 10 years, a lot of physical models which do not satisfy the condition discussed in [13] appeared, namely, the dissipative Timoshenko system, Euler-Maxwell system, etc. To analyze these physical models, Ueda-Duan-Kawashima [17, 20] tried to extend the stability condition. However, these results are not enough to treat the general symmetric hyperbolic system with relaxation. Inspired by these results, Ueda [15] introduced the new stability condition which is the extension of the classical stability condition derived in [13]. This new stability condition can be cover the general symmetric hyperbolic system with relaxation. In this situation, our purpose of this article is to study the detail of the dissipative structure induced by the new stability condition obtained in [15].

Then Ueda-Duan-Kawashima [17] derived the sufficient condition which is a combination of Condition (K) and Condition (S) to get the uniform dissipativity of the type $(1,2)$, which is the regularity-loss type. Noting that the dissipative structure of the regularity-loss type is frailer than the one of the standard type. Precisely, $\mathrm{Re}\lambda(r,\omega)$ may tend to zero as $r\to\infty$. This structure causes more regularity for the initial data when we derive decay estimates of solutions. This is the reason why this structure is called the regularity-loss type. Indeed, the dissipative Timoshenko system, the Euler-Maxwell system and the thermoelastic plate equation with Cattaneo's law have the dissipative structure of type $(1,2)$. For the detail, we refer to readers [5, 6, 11, 21, 22]. Unfortunately, the stability condition constructed in [17] is not completely enough to understand the regularity-loss structure. In fact, some physical models which possess the regularity-loss structure do not satisfy the stability condition in [17] (e.g. [3, 11, 18]). Moreover, we can construct artificial models which have several kinds of the regularity-loss structures (in detail, see [19]). Furthermore, in recent, Ueda-Duan-Kawashima in [20] succeeded to extend Condition (K) and Condition (S), and analyzed more complicated dissipative structures.

This situation tells us that it is difficult to characterize the dissipative structure for the regularity-loss type. In this situation, Ueda [15] succeeded to extend Classical Stability Condition (SC) and derive the sufficient and necessary conditions to get the strict dissipativity for (1.1). To mention this result, we use notations that $\mathbb{R}_+ := (0,\infty)$ and

$$\mathcal{A}(\nu,\omega) := (A^0)^{-1}(\nu A(\omega) - iL^\flat).$$

Then Ueda [15] introduced the following conditions.

General Stability Condition (GSC): For each $(\mu,\nu,\omega)\in\mathbb{R}\times\mathbb{R}_+\times S^{n-1}$,

$$\mathrm{Ker}(\mu I + \mathcal{A}(\nu,\omega))\cap \mathrm{Ker}(L^\sharp) = \{0\}.$$

General Kalman Rank Condition (GR): For each $(\nu,\omega)\in\mathbb{R}_+\times S^{n-1}$, the $m^2\times m$ Kalman matrix has rank m, that is,

$$\operatorname{rank}\begin{pmatrix} L^\sharp \\ L^\sharp\mathcal{A}(\nu,\omega) \\ L^\sharp\mathcal{A}(\nu,\omega)^2 \\ \vdots \\ L^\sharp\mathcal{A}(\nu,\omega)^{m-1}\end{pmatrix} = m.$$

Based on these new conditions, Ueda [15] got the relation for the stability condition and the eigenvalue of (1.3).

Theorem 1.3. ([15]) *Suppose the system* (1.1) *satisfies Condition* (A)*. Then, for the system* (1.1)*, the following conditions are equivalent.*
(i) *General Stability Condition* (GSC) *holds.*
(ii) *General Kalman Rank Condition* (GR) *holds.*
(iii) *System* (1.1) *is strictly dissipative.*

To make sure the relation between Theorem 1.2 and 1.3, we introduce Condition (GK) which is the extension for Condition (K).

General Craftsmanship Condition (GK): There is a $m\times m$ complex valued compensating matrix $\mathcal{K}(\nu,\omega)\in C(\mathbb{R}_+\times S^{n-1})$ with the following properties:

$$\bar{\mathcal{K}}(\nu,-\omega) = -\mathcal{K}(\nu,\omega), \qquad \mathcal{K}(\nu,\omega)^* = -\mathcal{K}(\nu,\omega)$$

for each $(\nu,\omega)\in\mathbb{R}_+\times S^{n-1}$, and there exists c_K and C_K such that

$$\langle (L^\sharp + (\mathcal{K}(\nu,\omega)\mathcal{A}(\nu,\omega))^\sharp)\sigma,\sigma\rangle > c_K\frac{\nu^2}{1+\nu^2}|\mathcal{K}(\nu,\omega)\sigma|^2, \qquad \|\mathcal{K}(\nu,\omega)\|\le C_K \tag{1.5}$$

for each $(\nu,\omega,\sigma)\in\mathbb{R}_+\times S^{n-1}\times\mathbb{S}^{m-1}$.

Here, we used the notation $\mathbb{S}^{m-1} := \{\sigma\in\mathbb{C}^m; |\sigma|=1\}$. In this situation, we extend Theorem 1.3 and get the following result.

Theorem 1.4. ([16]) *Suppose the system* (1.1) *satisfies Condition* (A)*. Then, for the system* (1.1)*, the following conditions are equivalent.*
(i) *Stability Condition* (GSC) *holds.*
(ii) *Kalman Rank Condition* (GR) *holds.*
(iii) *Craftsmanship Condition* (GK) *holds.*
(iv) *The system* (1.1) *is strictly dissipative.*

The property (iv) in Theorem 1.4 tells us that the solution to (1.1) should decay as a time tends to infinity. To conclude this conjecture rigorously, we have to get more information from Condition (SC), Condition (R) and Condition (K). Summarizing this situation, our main purpose of this article is to characterize the dissipative structure for (1.1) and extend Theorem 1.2.

Theorem 1.5. ([16]) *Let* $n=1$*. Suppose the system* (1.1) *satisfies Condition* (A)*. Then, for the system* (1.1)*, the conditions* (i)–(iv) *appeared in Theorem 1.4 are equivalent to the following condition.*

(v) *The system* (1.1) *is uniformly dissipative of type* $(m-1, 2(m-1))$.

We need some additional assumption for the coefficient matrices to get the same result in the case $n \geq 2$. To this end, we introduce notations. For matrices X and Y, we introduce the binomial expansion for matrices that

$$(X+Y)^k = \sum_{\ell=0}^{k} b_{\ell,k-\ell}(X,Y),$$

where $b_{\ell,k-\ell}(X,Y)$ denotes a polynomial of X and Y, which degrees of X and Y are ℓ and $k-\ell$, respectively. Namely,

$$b_{\ell,k-\ell}(X,Y) := X^\ell Y^{k-\ell} + X^{\ell-1}YXY^{k-\ell-1} + X^{\ell-2}YX^2Y^{k-\ell-1} + \cdots .$$

Remark that $b_{0,k}(X,Y) = Y^k$ and $b_{k,0}(X,Y) = X^k$. Then $\mathcal{A}(r,\omega)^k$ is represented by

$$\begin{aligned} \mathcal{A}(r,\omega)^k &= \big((A^0)^{-1}(rA(\omega) - iL^\flat)\big)^k \\ &= \sum_{\ell=0}^{k} r^\ell b_{\ell,k-\ell}\big((A^0)^{-1}A(\omega), -i(A^0)^{-1}L^\flat\big) \\ &= \sum_{\ell=0}^{k} r^{k-\ell} b_{\ell,k-\ell}\big(-i(A^0)^{-1}L^\flat, (A^0)^{-1}A(\omega)\big). \end{aligned} \tag{1.6}$$

Under the additional assumption of A^0, $A(\omega)$ and L, we derive the following theorem for $n \geq 2$.

Theorem 1.6. ([16]) *Let* $n \geq 2$. *Suppose the system* (1.1) *satisfies that Condition* (A) *and*

$$\mathrm{Ker}\big(L^\sharp b_{\ell,k-\ell}\big((A^0)^{-1}A(\omega), -i(A^0)^{-1}L^\flat\big)\big)$$

does not depend on $\omega \in S^{n-1}$ *for each* $0 \leq k \leq m-1$ *and* $0 \leq \ell \leq k$. *Then, for the system* (1.1), *the conditions* (i)–(iv) *appeared in Theorem 1.4 are equivalent to* (v) *appeared in Theorem 1.5.*

Theorem 1.5 and 1.6 tell us that at least the solution of (1.1) must have the quantitative decay estimate stated in Corollary 3.3 if the system satisfies Condition (GSC). On the other hand, if the system does not satisfy Condition (GSC), there is no chance to get the similar decay estimate. This situation means Condition (GSC) has the very important role as the index to analyze the dissipative structure for (1.1).

The analysis of the nonlinear system is also important. In fact, Bianchini-Hanouzet-Natalini [2], Hanouzet-Natalini [4], and Kawashima-Yong [9, 10] studied the hyperbolic balance laws and analyzed the dissipative structure. The essential parts of the analysis for nonlinear problems are derivation of the entropy functional and analysis of the corresponding linearized system. In the above references, the authors assumed (1.4) and the analysis of the linearized system lies on a standard argument. On the other hand, if the linearized system does not satisfy (1.4), the situation becomes more complicated, and we need the detailed analysis insisted in the present article. Kawashima-Ueda [8] suggested the extended entropy condition for the hyperbolic balance laws, which is applicable to these complicated situations. Our future work is to study the hyperbolic balance laws which linearized system does not satisfy (1.4) and introduce the general condition which induces the nonlinear stability.

2. Lyapunov function

We show the key lemma to prove the main theorems in this section. Precisely, we introduce the Lyapunov function for (1.2). To this end, we give a preparation. Let κ be a small positive number. We introduce κ_k for $k = 0, 1, \cdots, m$ such that

$$
\begin{aligned}
&0 = \kappa_0 < \kappa_1 < \cdots < \kappa_m, \\
&\kappa_k - \frac{1}{2}(\kappa_{k-1} + \kappa_{k+1}) \geq \kappa > 0, \quad k = 1, \cdots, m-1.
\end{aligned} \tag{2.1}
$$

Then we construct the useful Lyapunov function as follows.

Lemma 2.1. *Define the Lyapnov function for the system* (1.2) *that*

$$
\mathcal{E}(|\xi|, \omega, \hat{u}) := \langle A^0 \hat{u}, \hat{u} \rangle + \delta h(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{\mathrm{Im}\langle L^\sharp \mathcal{A}(|\xi|, \omega)^{k-1} \hat{u}, L^\sharp \mathcal{A}(|\xi|, \omega)^k \hat{u} \rangle}{\|\mathcal{A}(|\xi|, \omega)\|^{2k}}
$$

for $\delta > 0$ and $\varepsilon > 0$, where κ_k is introduced in (2.1) *and*

$$
h(|\xi|, \omega) := \frac{\|\mathcal{A}(|\xi|, \omega)\|^2}{(\|\mathcal{A}(|\xi|, \omega)\| + \|(A^0)^{-1}\| \|L^\sharp\|)^2}. \tag{2.2}
$$

Then, under Condition (A)*, there exist positive constants δ_0 and ε_0 such that*

$$
\frac{d}{dt} \mathcal{E}(|\xi|, \omega, \hat{u}) + c_0 |L^\sharp \hat{u}|^2 + c_1 h(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{|L^\sharp \mathcal{A}(|\xi|, \omega)^k \hat{u}|^2}{\|\mathcal{A}(|\xi|, \omega)\|^{2k}} \leq 0 \tag{2.3}
$$

and

$$
c_* |\hat{u}|^2 \leq \mathcal{E}(|\xi|, \omega, \hat{u}) \leq C_* |\hat{u}|^2, \tag{2.4}
$$

provided by $\delta = \delta_0$ and $0 < \varepsilon \leq \varepsilon_0$, where c_0, c_1, c_* and C_* are certain positive constants.

The Lyapunov function has a different property which depends on $A(\omega)$ and $L^\flat$. We characterize the property of the Lyapunov function as follows.

Corollary 2.2. *The Lyapnov function $\mathcal{E}$ introduced in Lemma 2.1 satisfies*

$$
\frac{\partial}{\partial t} \mathcal{E}(|\xi|, \omega, \hat{u}) + c \mathcal{D}(|\xi|, \omega, \hat{u}) \leq 0, \tag{2.5}
$$

where c is a certain positive constants and $\mathcal{D}$ is defined by

$$
\mathcal{D}(|\xi|, \omega, \hat{u}) := \begin{cases}
\displaystyle\sum_{k=0}^{m-1} \varepsilon^{\kappa_k} \frac{1}{(1+|\xi|^2)^k} |L^\sharp \mathcal{A}(|\xi|, \omega)^k \hat{u}|^2 & \text{if } A(\omega) \not\equiv O, \ L^\flat \neq O, \\
\displaystyle\frac{|\xi|^2}{1+|\xi|^2} \sum_{k=0}^{m-1} \varepsilon^{\kappa_k} |L^\sharp A(\omega)^k \hat{u}|^2 & \text{if } A(\omega) \not\equiv O, \ L^\flat = O, \\
\displaystyle\sum_{k=0}^{m-1} \varepsilon^{\kappa_k} |L^\sharp (L^\flat)^k \hat{u}|^2 & \text{if } A(\omega) \equiv O, \ L^\flat \neq O.
\end{cases} \tag{2.6}
$$

3. Pointwise estimate

Using the Lyapnov function introduced in Lemma 2.1, we derive the pointwise estimates for the solution to our system in Fourier space. For the system (1.1), we introduce the initial data

$$u(0,x) = u_0(x) \tag{3.1}$$

for $x \in \mathbb{R}^n$. Then, applying the Fourier transform to (3.1), this yields

$$\hat{u}(0,\xi) = \hat{u}_0(\xi) \tag{3.2}$$

for $\xi \in \mathbb{R}^n$. In this situation, Corollary 2.2 gives the following pointwise estimates for solutions to (1.2), (3.2).

Theorem 3.1. *Let $n = 1$. Suppose the system* (1.1) *satisfies Condition* (A) *and Condition* (R). *Then the solutions to* (1.2), (3.2) *satisfy the following pointwise estimate in Fourier space.*

$$|\hat{u}(t,\xi)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)| \tag{3.3}$$

with

$$\rho(\xi) := \begin{cases} \dfrac{|\xi|^{2(m-1)}}{(1+|\xi|^2)^{2(m-1)}} & \text{if} \quad A(\omega) \not\equiv O, \quad L^\flat \ne O, \\[2mm] \dfrac{|\xi|^2}{1+|\xi|^2} & \text{if} \quad A(\omega) \not\equiv O, \quad L^\flat = O, \\[2mm] 1 & \text{if} \quad A(\omega) \equiv O, \quad L^\flat \ne O, \end{cases} \tag{3.4}$$

where c and C are certain positive constants.

Theorem 3.2. *Let $n \ge 2$. Suppose the system* (1.1) *satisfies the same assumptions as in Theorem 1.6 and Condition* (R). *Then the solutions to* (1.2), (3.2) *satisfy the pointwise estimate* (3.3) *with* (3.4) *in Fourier space.*

Corollary 3.3. *Suppose that the solutions to* (1.2), (3.2) *satisfy the pointwise estimate* (3.3) *with* (3.4) *in Fourier space. Then the corresponding solutions to* (1.1), (3.1) *satisfy the following decay estimates.*

$$\begin{aligned} \|\partial_x^k u(t)\|_{L^2} &\le C(1+t)^{-\frac{n}{4(m-1)}-\frac{k}{2(m-1)}}\|u_0\|_{L^1} \\ &\qquad + C(1+t)^{-\frac{\ell}{2(m-1)}}\|\partial_x^{k+\ell}u_0\|_{L^2} \quad \text{if } A(\omega) \not\equiv O,\ L^\flat \ne O, \\ \|\partial_x^k u(t)\|_{L^2} &\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \quad \text{if } A(\omega) \not\equiv O,\ L^\flat = O, \\ |u(t,x)| &\le Ce^{-ct}|u_0(x)| \quad \text{if } A(\omega) \equiv O,\ L^\flat \ne O \end{aligned}$$

for $k \ge 0$ and $\ell \ge 0$, where c and C are certain positive constants.

Theorem 3.1 and Theorem 3.2 tell us that the condition (ii) is the sufficient condition to get the condition (v) in Theorem 1.5 and Theorem 1.6. Corollary 3.3 is derived by Theorem 3.1 and Theorem 3.2. The proof is straightforward argument and omitted here. For the detail, we refer [14, 17, 20] to readers.

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