Spectral projective schemes and their applications

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An $\mathbb{E}_\infty$-ring is a spectrum equipped with commutative multiplicative law up to coherent homotopy.

Lurie refined the notation of $\mathbb{E}_\infty$-ring by using ∞-operad $N_\Delta(\mathcal{F}\mathrm{in}_*)$. In 2001, Mandell, May, Schwede and Shipley considered diagram spectra, especially $\mathbb{Z}$-indexed ring spectra. Lurie consider $\mathbb{Z}$-grading on $\mathbb{E}_\infty$-ring by using $\mathbb{Z}$ with order preserving morphisms. I think some professionals already noticed that the symmetric monoidal structure on discrete graded rings and modules are obtained by the Day convolution.

We have symmetric monoidal ∞-categories $N_\Delta(\mathcal{O}_{\mathbb{N}}^{\otimes}) \to N_\Delta(\mathcal{F}\mathrm{in}_*)$ and $N_\Delta(\mathcal{O}_{\mathbb{Z}}^{\otimes}) \to N_\Delta(\mathcal{F}\mathrm{in}_*)$. By using these symmetric monoidal ∞-categories, we concretely define graded $\mathbb{E}_\infty$-rings and graded modules over them, and study their properties. We construct projective schemes associated to connective $\mathbb{N}$-graded $\mathbb{E}_\infty$-rings in spectral algebraic geometry. Actually, Torii and I gave these construction in more general situation, please see [8]. In this paper, I use an expedient ; I state their properties by using ∞-operads.

Under some finiteness conditions, we show that the ∞-category of almost perfect quasi-coherent sheaves over a spectral projective scheme $\mathrm{Proj}\,(A)$ associated to a connective $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring A can be described in terms of $\mathbb{Z}$-graded A-modules.

1. Graded $\mathbb{E}_\infty$-rings and graded modules

For a symmetric monoidal category consisting of one object $\{0\}$ with the unique isomorphism, we have a category $\mathcal{F}\mathrm{in}_*$. The map $\{0\} \to \mathbb{N}$ and $\{0\} \to \mathbb{Z}$ are the symmetric monoidal functors.

Definition 1.1. Let $\mathcal{C}^{\otimes} \to N_\Delta(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. We obtain a symmetric monoidal ∞-category $\mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}), \mathcal{C})^{\hat{\otimes}} \to N_\Delta(\mathcal{F}\mathrm{in}_*)$ by the Day convolution which is denoted by $\hat{\otimes}$.

We use the notation $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{M})$ for the fibration $\mathcal{M} \to \mathcal{O}$ of ∞-operads and given ∞-operads $\mathcal{O}' \to \mathcal{O}$.

For $X \in \mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}),\, \mathcal{C})^{\hat{\otimes}}$, X_i is the value at $i \in \mathbb{Z}$ of the underlying functor of X.

We take the symmetric monoidal ∞-categories $\mathrm{Sp}^{\otimes}$ and $\mathrm{Mod}_R^{\otimes}$, where R is an $\mathbb{E}_\infty$-ring.

Key diagram. Consider the following diagram

$$\begin{array}{ccccc} N_\Delta(\mathcal{F}\mathrm{in}_*) & \xleftarrow{\ i\ } & N_\Delta(\mathcal{O}^{\otimes}_{\mathbb{Z}}) & \xleftarrow{\ 0\ } & N_\Delta(\mathcal{F}\mathrm{in}_*) \\ & {\scriptstyle A(\infty)} \searrow & \downarrow {\scriptstyle A} & \swarrow {\scriptstyle A_0} & \\ & & \mathrm{Sp}^{\otimes} & & \end{array},$$

where the map 0 is induced from $\{0\} \to \mathbb{Z}$, the map i is the structure map of symmetric monoidal structure and $A(\infty)$ is the operadic left Kan extensions of A along i. Then,

(i) A_0 is an $\mathbb{E}_\infty$-ring, i.e., A_0 is an object in $\mathrm{Alg}_{N_\Delta(\mathcal{F}\mathrm{in}_*)/N_\Delta(\mathcal{F}\mathrm{in}_*)}(\mathrm{Sp}^{\otimes})$,

(ii) $A(\infty)$ is an $\mathbb{E}_\infty$-ring, i.e., an object in $\mathrm{Alg}_{N_\Delta(\mathcal{F}\mathrm{in}_*)/N_\Delta(\mathcal{F}\mathrm{in}_*)}(\mathrm{Sp}^{\otimes})$.

Especially, the functor $(-)_0$ commutes with the graded tensor. We call $A(\infty)$ the underlying $\mathbb{E}_\infty$-ring of A. Roughly speaking, $A(\infty)$ is such as a form of direct sum with respect to grading.

Definition 1.2.

- We define the ∞-category of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings by
$$\mathrm{Alg}_{N_\Delta(\mathcal{F}\mathrm{in}_*)/N_\Delta(\mathcal{F}\mathrm{in}_*)}(\mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}), \mathrm{Sp})^{\hat{\otimes}}),$$
and call its object a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings.
- We say that a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-ring A is connective if each A_i for $i \in \mathbb{Z}$ is a connective spectrum.
- Let R be an $\mathbb{E}_\infty$-ring. We define the ∞-category of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings over R by
$$\mathrm{Alg}_{N_\Delta(\mathcal{F}\mathrm{in}_*)/N_\Delta(\mathcal{F}\mathrm{in}_*)}(\mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}), \mathrm{Mod}_R)^{\hat{\otimes}}),$$
and call its object a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings over R.

We denote by $\mathrm{CAlg}_R(\mathbb{Z})$ and $\mathrm{CAlg}_R(\mathbb{N})$ the ∞-category of $\mathbb{Z}$-graded and $\mathbb{N}$-graded $\mathbb{E}_\infty$-rings over R, respectively. We identify objects of $\mathrm{CAlg}(\mathbb{N})$ with that of $\mathrm{CAlg}(\mathbb{Z})$.

1.1. **Modules over graded $\mathbb{E}_\infty$-rings.**

Definition 1.3. For a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-ring A and an $\mathbb{E}_\infty$-ring R, the ∞-category of $\mathbb{Z}$-graded A-modules is
$$\mathrm{Mod}_A(\mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}),\, \mathrm{Sp})^{\hat{\otimes}}),$$
where the notation $\mathrm{Mod}_A(-)$ is in the sense of Lurie. Let us denote the ∞-category of $\mathbb{Z}$-graded A-modules by $\mathrm{Mod}_A(\mathbb{Z})$.

We call a morphism in $\mathrm{CAlg}(\mathbb{Z})$ and $\mathrm{Mod}_A(\mathbb{Z})$ a morphism of degree 0 or a morphism of graded $\mathbb{E}_\infty$-rings and of graded A-modules.

Remark 1.4. The ∞-category of $\mathbb{Z}$-graded A-modules over R can be defined as

$$\mathrm{Mod}_A(\mathrm{Fun}_{N_\Delta(\mathcal{F}\mathrm{in}_*)}(N_\Delta(\mathbb{Z}), \mathrm{Mod}_R)^{\hat{\otimes}}).$$

1.2. **Localizations of graded $\mathbb{E}_\infty$-rings.** For X in $\mathrm{Sp}(\mathbb{Z})$ and $g \in \mathbb{Z}$, we define a twisting $X(g)$ in $\mathrm{Sp}(\mathbb{Z})$ by $X(g)_{g'} \simeq X_{g+g'}$ for $g' \in \mathbb{Z}$.

Let A be a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-ring and let $a \in \pi_0(A)$ be homogeneous of degree $g \in \mathbb{Z}$. We regard a as a morphism $a : A \to A(g)$ of $\mathbb{Z}$-graded A-modules. Since $\mathrm{Mod}_A(\mathbb{Z})$ is a presentable ∞-category, there exists a localization functor

$$L : \mathrm{Mod}_A(\mathbb{Z}) \longrightarrow \mathrm{Mod}_A(\mathbb{Z})$$

with respect to the map $a : A \to A(g)$. As in the nongraded case, $L(M)$ is equivalent to $M[a^{-1}]$, where $M[a^{-1}]$ is a colimit of the sequence

$$M \xrightarrow{a} M(g) \xrightarrow{a} M(2g) \xrightarrow{a} \cdots$$

in $\mathrm{Mod}_A(\mathbb{Z})$. The localization L is smashing given by $L(-) \simeq A[a^{-1}] \otimes_A (-)$ and is compatible with the symmetric monoidal structure on $\mathrm{Mod}_A(\mathbb{Z})$. We can regard $l : A \to A[a^{-1}]$ a morphism in $\mathrm{CAlg}(\mathbb{Z})$. We obtain an adjunction

$$l_! : \mathrm{Mod}_A(\mathbb{Z}) \rightleftarrows \mathrm{Mod}_{A[a^{-1}]}(\mathbb{Z}) : l^*,$$

where the left adjoint $l_!$ is a symmetric monoidal functor given by $M \mapsto A[a^{-1}] \otimes_A M$, and the right adjoint l^* is a fully faithful lax symmetric monoidal functor.

By [3, Remark 7.3.2.13], this adjunction induces an adjunction

$$l_! : \mathrm{CAlg}_A \rightleftarrows \mathrm{CAlg}_{A[a^{-1}]} : l^*,$$

where the right adjoint l^* is fully faithful. Hence $l_! : \mathrm{CAlg}_A(\mathbb{Z}) \to \mathrm{CAlg}_{A[a^{-1}]}(\mathbb{Z})$ is a localization functor.

If a is invertible in $\pi_0(B)$, we have an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_A(\mathbb{Z})}(A[a^{-1}], B) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{CAlg}_A(\mathbb{Z})}(A, B)$$

since there is an equivalence $l_!A \simeq A[a^{-1}]$ in $\mathrm{CAlg}_A(\mathbb{Z})$.

We define an ∞-category

$$\mathrm{CAlg}_A^{\mathrm{Zar}}(\mathbb{Z})$$

to be the full subcategory of $\mathrm{CAlg}_A(\mathbb{Z})$ spanned by those objects of the form $A[a^{-1}]$ for some homogeneous element $a \in \pi_0(A)$. We define $\mathrm{CAlg}_{\pi_0(A)}^{\heartsuit\mathrm{Zar}}(\mathbb{Z})$ to be the full subcategory of $\mathrm{CRing}_{\pi_0(A)}(\mathbb{Z})$ spanned by those objects of the form $\pi_0(A)[a^{-1}]$ for some $a \in \pi_0(A)$.

Definition 1.5. A spectral scheme X is a projective spectral scheme if there exists a collection $\{U_a\}$ such that U_a covers X and there exists $A \in \mathrm{CAlg}(\mathbb{Z})$ such that $(U_a, \mathcal{O}_X) \simeq (\mathrm{Spec}\,(A[\alpha_a^{-1}]_0), \mathcal{O}_{\mathrm{Spec}\,(A[\alpha^{-1}]_0)})$ for each U_a and for degree (more that) 1 elements $\alpha_a \in \pi_0(A(\infty))$.

2. Quasi-coherent sheaves on projective spectral schemes

Definition 2.1. Let A be a connective $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring. We say that A is Noetherian if $\pi_0(A)$ is a Noetherian $\mathbb{N}$-graded commutative ring and $\pi_n(A)$ is a finitely generated $\mathbb{N}$-graded $\pi_0(A)$-module for any $n \in \mathbb{Z}$.

Let A be a connective Noetherian $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring. In this section we assume that A satisfies the following condition:

There are finitely many elements of $\pi_0(A_1)$ which generate $\pi_0(A)$ as an $\mathbb{N}$-graded commutative ring over $\pi_0(A_0)$.

Let A be a connective Noetherian $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring. Set $X = \mathrm{Proj}\,(A)$. We take a set $\{a_i\}_{i=1}^r \subset \pi_0(A_1)$ of generators of $\pi_0(A)$ as an $\mathbb{N}$-graded commutative ring over $\pi_0(A_0)$. We define a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-ring B by $B = A[a_1^{-1}] \times \cdots \times A[a_r^{-1}]$. Let $g : A \to B$ be the canonical morphism of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings. We take a Čech nerve

$$C(g)_+^\bullet$$

of g in the opposite ∞-category of $\mathrm{CAlg}(\mathbb{Z})$. Then $C(g)_+^\bullet$ is an augmented cosimplicial object of $\mathrm{CAlg}(\mathbb{Z})$ such that $C(g)_+^{-1} \simeq A$ and $C(g)_+^n \simeq B^n$ for $n \geq 0$, where B^n is given by

$$B^n = \overbrace{B \otimes_A \cdots \otimes_A B}^{n+1}.$$

By using the functor $(-)_0 : \mathrm{CAlg}(\mathbb{Z}) \to \mathrm{CAlg}$, we obtain $C(g)_0^\bullet$ as the composite of $(-)_0$ with the restriction $C(g)^\bullet = C(g)_+^\bullet|_\Delta$.

Note that there is a faithfully flat affine morphism $f : U \longrightarrow X$, where $U = \mathrm{Spec}(B_0)$, and we have an equivalence $U_\bullet \simeq \mathrm{Spec}(C(g)_0^\bullet)$ of simplicial objects of affine spectral schemes. Since $f : U \to X$ is an effective epimorphism, we have an equivalence

$$|U_\bullet| \xrightarrow{\simeq} X$$

in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, where the left hand side is the geometric realization of the simplicial object $U_\bullet$.

We have an ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X, which is stable presentable symmetric monoidal with unit $\mathcal{O}_X$ by [5, Proposition 2.2.4.2].

There is an equivalence

$$\mathrm{QCoh}(X) \simeq \lim_{\Delta} \mathrm{Mod}_{B_0^\bullet}$$

of symmetric monoidal stable ∞-categories.

Recall that $\mathrm{Sp}(\mathbb{Z})$ is a symmetric monoidal stable presentable ∞-category, so that it is a commutative algebra object of $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$, where $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ is the ∞-category of stable presentable ∞-categories and colimit-preserving functors. We denote by $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}})$ the ∞-category of commutative algebra objects of $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$. By [3, Theorem 4.8.5.16], we have a functor

$$\mathrm{Mod}_{(-)}(\mathbb{Z}) : \mathrm{CAlg}(\mathbb{Z}) \longrightarrow \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}),$$

which assigns to $D \in \mathrm{CAlg}(\mathbb{Z})$ the symmetric monoidal ∞-category $\mathrm{Mod}_D(\mathbb{Z})$ of $\mathbb{Z}$-graded D-modules.

By applying the functor $\mathrm{Mod}_{(-)}(\mathbb{Z})$ to $C(g)^{\bullet}_{+}$ and using the equivalences $\mathrm{Mod}_{B^n}(\mathbb{Z}) \simeq \mathrm{Mod}_{B^n_0}$ for $n \geq 0$, we obtain a symmetric monoidal functor

$$\mathrm{Mod}_A(\mathbb{Z}) \longrightarrow \lim_{\Delta} \mathrm{Mod}_{B^{\bullet}_0}.$$

Definition 2.2. We define a functor

$$\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \longrightarrow \mathrm{QCoh}(X)$$

to be the composite of the functor $\mathrm{Mod}_A(\mathbb{Z}) \to \lim_{\Delta} \mathrm{Mod}_{B^{\bullet}_0}$ with the equivalence between $\lim_{\Delta} \mathrm{Mod}_{B^{\bullet}_0}$ and $\mathrm{QCoh}(X)$. We call $\widetilde{M}$ the quasi-coherent sheaf on X associated to a $\mathbb{Z}$-graded A-modules M.

By the construction, the functor $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{QCoh}(X)$ is symmetric monoidal.

Recall that we have defined the shifting functor $(q) : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{Mod}_A(\mathbb{Z})$ given by $M(q)_n \simeq M_{q+n}$ for $M \in \mathrm{Mod}_A(Z)$ and $q, n \in \mathbb{Z}$. For $q \in \mathbb{Z}$, we define $\mathcal{O}_X(q)$ to be the quasi-coherent sheaf $\widetilde{A(q)}$ on X.

Proposition 2.3. *The quasi-coherent sheaf $\mathcal{O}_X(q)$ is locally free of rank 1 for any $q \in \mathbb{Z}$.*

Proof. It suffices to show that the restriction $\widetilde{A(q)}|_V$ is equivalent to $\widetilde{A}|_V$ for any affine open set $V = \mathrm{Spec}(\pi_0(A)[f^{-1}]_0)$ of the underlying projective scheme $\mathrm{Proj}\,(\pi_0(A))$, where f is an element of $\pi_0(A)$ of degree 1. The restriction $\widetilde{A(q)}|_V$ corresponds to an $A[f^{-1}]_0$-module $A[f^{-1}]_q$. The multiplication by f^q induces an equivalence of $A[f^{-1}]_0$-modules between $A[f^{-1}]_q$ and $A[f^{-1}]_0$. Thus, there is an equivalence of quasi-coherent sheaves between $\widetilde{A(q)}|_V$ and $\widetilde{A}|_V$. This completes the proof. □

For a quasi-coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on X and $q \in \mathbb{Z}$, we define $\mathscr{F}(q) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(q)$.

If $\mathscr{F}$ is the quasi-coherent sheaf associated to a $\mathbb{Z}$-graded A-module M, then we have an equivalence $\mathscr{F}(q) \simeq \widetilde{M(q)}$ of quasi-coherent sheaves.

The functor $\widetilde{(-)}$ is obtained from the augmented cosimplicial diagram $\mathrm{Mod}_{(-)}(\mathbb{Z}) \circ C(g)^{\bullet}_{+} : \Delta_+ \to \mathrm{CAlg}(\mathbb{Z}) \to \widehat{\mathrm{Cat}}_{\infty}$. Since the functor $\mathrm{Mod}_{(-)}(\mathbb{Z}) : \mathrm{CAlg}(\mathbb{Z}) \to \widehat{\mathrm{Cat}}_{\infty}$ factors through the ∞-category $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ of stable presentable ∞-categories and colimit-preserving functors, we see that the functor $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{QCoh}(X)$ is a morphism in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$.

Thus, there exists a right adjoint $\Gamma_*(X, -) : \mathrm{QCoh}(X) \longrightarrow \mathrm{Mod}_A(\mathbb{Z})$ to $\widetilde{(-)}$.

The equivalence $\mathrm{QCoh}(X) \xrightarrow{\simeq} \lim_\Delta \mathrm{Mod}_{B^\bullet}(\mathbb{Z})$ of ∞-categories implies an equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(X)}(\widetilde{M}, \mathscr{F}) \simeq \lim_{\Delta} \mathrm{Map}_{\mathrm{Mod}_{B^\bullet}(\mathbb{Z})}(B^\bullet \otimes_A M, \Gamma(U_\bullet, \mathscr{F}(*)))$$

of mapping spaces. Since we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{B^\bullet}(\mathbb{Z})}(B^\bullet \otimes_A M, \Gamma(U_\bullet, \mathscr{F}(*))) \simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathbb{Z})}(M, \Gamma(U_\bullet, \mathscr{F}(*)))$$

of cosimplicial spaces, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(X)}(\widetilde{M}, \mathscr{F}) \simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathbb{Z})}(M, \lim_{\Delta} \Gamma(U_\bullet, \mathscr{F}(*))).$$

Hence we obtain $\Gamma_*(X, \mathscr{F}) \simeq \lim_\Delta \Gamma(U_\bullet, \mathscr{F}(*))$.

We show that the functor $\Gamma_*(X, -)$ is fully faithful. Recall that $B = A[a_1^{-1}] \times \cdots \times A[a_r^{-1}]$, where $\{a_i\}_{i=1}^r \subset \pi_0(A_1)$ is the set of generators of $\pi_0(A)$ as an $\mathbb{N}$-graded commutative ring over $\pi_0(A_0)$. We have the faithfully flat affine morphism $f : U \to X$, where $U = \mathrm{Spec}(B_0)$. Note that there is an equivalence $\Gamma(U, f^*\mathcal{O}_X(*)) \simeq B$ of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings and hence that $\Gamma(U, f^*\mathscr{F}(*))$ is a $\mathbb{Z}$-graded B-module for a quasi-coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on X. We have the restriction map $\Gamma_*(X, \mathscr{F}) \to \Gamma(U, f^*\mathscr{F}(*))$, which induces a map

$$B \otimes_A \Gamma_*(X, \mathscr{F}) \to \Gamma(U, f^*\mathscr{F}(*))$$

of $\mathbb{Z}$-graded B-modules. We shall compare $\Gamma(U, f^*\mathscr{F}(*))$ with the scalar extension $B \otimes_A \Gamma_*(X, \mathscr{F})$.

Lemma 2.4 (cf. [8])**.** *Let $\mathscr{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-module on X. There is a natural equivalence*

$$B \otimes_A \Gamma_*(X, \mathscr{F}) \xrightarrow{\simeq} \Gamma(U, f^*\mathscr{F}(*))$$

of $\mathbb{Z}$-graded B-modules.

Proof. We have $U = V_1 \times \cdots \times V_r$, where $V_i = \mathrm{Spec}(A[a_i^{-1}]_0)$ for $1 \leq i \leq r$. This implies a decomposition

$$\Gamma(U, f^*\mathscr{F}(*)) \simeq \Gamma(V_1, \mathscr{F}(*)) \times \cdots \times \Gamma(V_r, \mathscr{F}(*)),$$

where $\Gamma(V_i, \mathscr{F}(*))$ is a $\mathbb{Z}$-graded $A[a_i^{-1}]$-module for $1 \leq i \leq r$. Since $\Gamma(V_i, \mathscr{F}(*))$ is a $\mathbb{Z}$-graded $A[a_i^{-1}]$-module, the restriction map $\Gamma_*(X, \mathscr{F}) \to \Gamma(U, f^*\mathscr{F}(*))$ induces a map $\Gamma_*(X, \mathscr{F})[a_i^{-1}] \to \Gamma_*(V_i, \mathscr{F})$ of $\mathbb{Z}$-graded $A[a_i^{-1}]$-modules. It suffices to show that this map is an equivalence for any i with $1 \leq i \leq r$.

Let P be the partially ordered set of all nonempty finite subsets of $\{1, \ldots, r\}$. We set $V_I = \cap_{i \in I} V_i$ for $I \in P$. By [5, Proposition 1.1.4.4], we have an equivalence

$$\Gamma_*(X, \mathscr{F}) \simeq \lim_{I \in P} \Gamma(V_I, \mathscr{F}(*))$$

of $\mathbb{Z}$-graded A-modules. Note that the right hand side is a finite limit indexed by P. Since filtered colimits commute with finite limits, we obtain an equivalence

$$\Gamma_*(X, \mathscr{F})[a_i^{-1}] \simeq \lim_{I \in P} (\Gamma(V_I, \mathscr{F}(*))[a_i^{-1}])$$

of $\mathbb{Z}$-graded $A[a_i^{-1}]$-modules. By definition, we have an equivalence

$$\Gamma(V_I, \mathscr{F}(*))[a_i^{-1}] \simeq \Gamma(V_{I \cup \{i\}}, \mathscr{F}(*))$$

for any $I \in P$. We consider a functor $\theta : P \to \mathrm{Mod}_{A[a_i^{-1}]}(\mathbb{Z})$ which assigns to $I \in P$ the $\mathbb{Z}$-graded $A[a_i^{-1}]$-module $\Gamma(V_{I \cup \{i\}}, \mathscr{F}(*))$. Let P_i be the subset of P consisting of finite subsets of $\{1, \ldots, r\}$ which contain i. Since the functor θ is a right Kan extension of the restriction to P_i, we have an equivalence

$$\lim_{I \in P} \Gamma(V_{I \cup \{i\}}, \mathscr{F}(*)) \simeq \lim_{J \in P_i} \Gamma(V_J, \mathscr{F}(*)).$$

By [5, Proposition 1.1.4.4], we have an equivalence

$$\Gamma(V_i, \mathscr{F}(*)) \simeq \lim_{J \in P_i} \Gamma(V_J, \mathscr{F}(*)).$$

Thus, we have an equivalence $\Gamma(V_i, \mathscr{F}(*)) \simeq \Gamma_*(X, \mathscr{F})[a_i^{-1}]$ of $\mathbb{Z}$-graded $A[a_i^{-1}]$-modules. □

Especially, we can see that he functor $\Gamma_*(X, -) : \mathrm{QCoh}(X) \to \mathrm{Mod}_A(\mathbb{Z})$ is fully faithful.

We have the adjunction $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \rightleftarrows \mathrm{QCoh}(X) : \Gamma_*(X, -)$ of ∞-categories. Since the left adjoint $\widetilde{(-)}$ is a symmetric monoidal functor, the right adjoint $\Gamma_*(X, -)$ is a lax symmetric monoidal functor. In particular, $\Gamma_*(X, \mathcal{O}_X)$ is a $\mathbb{Z}$-graded $\mathbb{E}_\infty$-ring and there is a map

$$A \longrightarrow \Gamma_*(X, \mathcal{O}_X)$$

of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings. For a quasi-coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on X, $\Gamma_*(X, \mathscr{F})$ is a $\mathbb{Z}$-graded $\Gamma_*(X, \mathcal{O}_X)$-module. We note that the $\mathbb{Z}$-graded A-module structure on $\Gamma_*(X, \mathscr{F})$ is obtained from the $\mathbb{Z}$-graded $\Gamma_*(X, \mathcal{O}_X)$-module structure through the map $A \to \Gamma_*(X, \mathcal{O}_X)$ of $\mathbb{Z}$-graded $\mathbb{E}_\infty$-rings.

3. The properties of Quasi-coherent sheaves on spectral projective schemes

The definition of projective schemes by using quasi-coherent sheaves may be valuable in the non-commutative geometry. For example, for a field k and finitely generated commutative graded k-algebra which is generated by degree 1 elements, Artin and Zhan shows that there is an categorical equivalence between the category of certain coherent sheaves on projective scheme of A and the category of finite graded right A-modules. Verevkin also studied injective objects and Ext-groups in the category of finite graded A-modules.

Lemma 3.1. *For $M \in \mathrm{Mod}_A(\mathbb{Z})$, we have $\widetilde{M} \simeq 0$ if and only if $\pi_n(M)$ is Zariski locally bounded above for any $n \in \mathbb{Z}$.*

Proof. We take a set $\{a_i\}_{i=1}^r \subset \pi_0(A_1)$ of generators of $\pi_0(A)$ as an $\mathbb{N}$-graded commutative ring over $\pi_0(A_0)$. Then there is an affine open covering $\{V_i\}_{i=1}^r$ of X, where $V_i = \mathrm{Spec}\,(A[a_i^{-1}]_0)$. We have $\widetilde{M} \simeq 0$ if and only if $\widetilde{M}|_{V_i} \simeq 0$ for $i = 1, \ldots, r$. Under the equivalence $\mathrm{QCoh}(V_i) \simeq \mathrm{Mod}_{A[a_i^{-1}]_0}$, the restriction $\widetilde{M}|_{V_i}$ corresponds to an $A[a_i^{-1}]_0$-module $M[a_i^{-1}]_0$. Hence $\widetilde{M} \simeq 0$ if and only if $M[a_i^{-1}]_0 \simeq 0$ for $i = 1, \ldots, r$. This is equivalent to the condition that $\pi_n(M)[a_i^{-1}]_0 = 0$ for any $n \in \mathbb{Z}$ and $i = 1, \ldots, r$. □

Definition 3.2. We say that a $\mathbb{Z}$-graded A-module M is locally bounded above in homotopy groups if the $\mathbb{Z}$-graded $\pi_0(A)$-module $\pi_n(M)$ is locally bounded above for each $n \in \mathbb{Z}$. We define $\mathrm{Mod}_A^{\mathrm{lbah}}(\mathbb{Z})$ to be the full subcategory of $\mathrm{Mod}_A(\mathbb{Z})$ spanned by those objects that are locally bounded above in homotopy groups.

We have the adjunction $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \rightleftarrows \mathrm{QCoh}(X) : \Gamma_*(X, -)$ of stable presentable ∞-categories, where the left adjoint $\widetilde{(-)}$ is symmetric monoidal and the right adjoint $\Gamma_*(-)$ is lax symmetric monoidal and fully faithful. By Lemma 3.1, we have $\widetilde{M} \simeq 0$ if and only if $\pi_n(M)$ is locally bounded above for any $n \in \mathbb{Z}$.

Hence we obtain the following proposition.

Proposition 3.3. *The functor $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{QCoh}(X)$ induces an equivalence*

$$\mathrm{Mod}_A(\mathbb{Z})/\mathrm{Mod}_A^{\mathrm{lbah}}(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{QCoh}(X)$$

of stable symmetric monoidal presentable ∞-categories. Here, W be the class of all morphisms in $\mathrm{Mod}_A(\mathbb{Z})$ whose cofiber lies in $\mathrm{Mod}_A^{\mathrm{lbah}}(\mathbb{Z})$ and the left hand side is the localization with respect to the class W.

□

3.1. **The equivalence of Serre theorem.** In this subsection, we give a short survey of a generalization of Serre theorem as in [8]. It is given by restricting the equivalence in Proposition 3.3 to "finitely generated part". We prepare some notation for "finiteness".

Definition 3.4. Let A be a connective $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring.

(i) Let R be a connective Noetherian $\mathbb{E}_\infty$-ring. Recall that an R-module M is almost perfect if $\pi_n(M)$ is a finitely generated $\pi_0(R)$-module for any $n \in \mathbb{Z}$ and if $\pi_n(M) = 0$ for $n \ll 0$ [3, Proposition 7.2.4.17].

(ii) We let $\mathrm{Mod}_A^{\mathrm{perf}}(\mathbb{Z})$ be the smallest stable subcategory of $\mathrm{Mod}_A(\mathbb{Z})$ which contains $A(q)$ for all $q \in \mathbb{Z}$ and is closed under retracts. We say that a $\mathbb{Z}$-graded A-module M is perfect if it belongs to the full subcategory $\mathrm{Mod}_A^{\mathrm{perf}}(\mathbb{Z})$.

(iii) The quasi-coherent sheaf $\widetilde{M}$ is almost perfect if a $\mathbb{Z}$-graded A-module M is almost finitely generated.

(iv) We say that a $\mathbb{Z}$-graded A-module M is almost finitely generated if the following conditions are satisfied: for each $n \in \mathbb{Z}$, the $\mathbb{Z}$-graded $\pi_0(A)$-module $\pi_n(M)$ is finitely generated, and, for $n \ll 0$, $\pi_n(M) = 0$. We define an ∞-category $\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})$ to be the full subcategory of $\mathrm{Mod}_A(\mathbb{Z})$ spanned by almost finitely generated $\mathbb{Z}$-graded A-modules.

(v) Let M be an almost finitely generated $\mathbb{Z}$-graded A-module. We say that M is almost torsion if the $\mathbb{Z}$-graded $\pi_0(A)$-module $\pi_n(M)$ is bounded above for each $n \in \mathbb{Z}$. We define an ∞-category $\mathrm{Mod}_A^{\mathrm{ator}}(\mathbb{Z})$ to be the full subcategory of $\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})$ spanned by almost torsion $\mathbb{Z}$-graded A-modules.

We give a characterization of M in $\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})$ satisfying $\widetilde{M} \simeq 0$ in terms of the $\mathbb{Z}$-graded $\pi_0(A)$-modules $\pi_n(M)$ for $n \in \mathbb{Z}$. By Lemma 3.1, we have $\widetilde{M} \simeq 0$ if and only if M is almost torsion.

If we restrict the symmetric monoidal functor $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{QCoh}(X)$ to the full ∞-subcategory $\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})$, it factors through $\mathrm{QCoh}(\mathrm{Proj}\,(A))^{\mathrm{aperf}}$. By the above argument, it also induces a symmetric monoidal exact functor

$$\widetilde{(-)} : \mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})/\mathrm{Mod}_A^{\mathrm{ator}}(\mathbb{Z}) \longrightarrow \mathrm{QCoh}(X)^{\mathrm{aperf}}.$$

The Serre theorem describes the ∞-category $\mathrm{QCoh}(X)^{\mathrm{aperf}}$ of almost perfect quasi-coherent sheaves on X in terms of $\mathbb{Z}$-graded A-modules. It requires that the above symmetric monoidal exact functor gives an equivalence of symmetric monoidal ∞-categories. To see this, especially, we need the essentially surjectivity of this functor.

The key proposition is the following, which is proved by calculating the spectral sequence.

Proposition 3.5 ([8]). *If $\mathscr{F} \in \mathrm{QCoh}(X)$ is almost perfect, then we have*

- *the* $\mathbb{Z}$*-graded* $\pi_0(A)$*-module* $\pi_n(\Gamma_*(X, \mathscr{F}))$ *is strongly quasi-finitely generated for each* $n \in \mathbb{Z}$*, and*
- $\pi_n(\Gamma_*(X, \mathscr{F})) = 0$ *for* $n \ll 0$*.*

□

The ∞-category $\mathrm{QCoh}(X)^{\mathrm{aperf}}$ of almost perfect quasi-coherent sheaves on X can be related with the ∞-category $\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})$ of almost finitely generated $\mathbb{Z}$-graded A-modules. Let $f \in \pi_0(A)$ be homogeneous of positive degree and let $U = \mathrm{Spec}\,(A[f^{-1}]_0)$ be an affine open subscheme of X. The restriction $\widetilde{M}|_U$ corresponds to the $A[f^{-1}]_0$-module $M[f^{-1}]_0$. We see that $M[f^{-1}]_0$ is almost perfect since M is almost finitely generated.

Conversly, by Proposition 3.5, we obtain

- If $\mathscr{F} \in \mathrm{QCoh}(X)$ is almost perfect, then $\pi_n\mathscr{F}$ is a coherent sheaf of $\mathcal{O}_{\mathsf{X}}$-modules on X for any $n \in \mathbb{Z}$.
- $\mathscr{F} \in \mathrm{QCoh}(X)$ is almost perfect, then $\pi_n\mathfrak{F} = 0$ for $n \ll 0$.

By proceeding local argument, we have the following.

Theorem 3.6 ([8])**.** *The functor* $\widetilde{(-)} : \mathrm{Mod}_A(\mathbb{Z}) \to \mathrm{QCoh}(X)$ *induces an equivalence*

$$\mathrm{Mod}_A^{\mathrm{afg}}(\mathbb{Z})/\mathrm{Mod}_A^{\mathrm{ator}}(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{QCoh}(X)^{\mathrm{aperf}}$$

of small stable symmetric monoidal ∞*-categories.*

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