Sum formulas for Schur multiple zeta values*

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1 Introduction

For an index $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{\geq 1}^d$ with $k_d \geq 2$ (such an index is called admissible), the multiple zeta values (MZVs) and multiple zeta-star values (MZSVs) are defined by

$$\zeta(\boldsymbol{k}) := \sum_{0<m_1<\cdots<m_d} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}, \qquad \zeta^\star(\boldsymbol{k}) := \sum_{0<m_1\leq\cdots\leq m_d} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}.$$

It is known that there are various relations among MZVs and MZSVs. Among them, we in this note focus on the following relations called the sum formulas which was proved in [3] and [4]; For integers $d \geq 1$ and $w \geq d+1$

$$\sum_{\substack{\boldsymbol{k}\,:\,\text{admissible} \\ \text{wt}(\boldsymbol{k})=w \\ \text{dep}(\boldsymbol{k})=d}} \zeta(\boldsymbol{k}) = \zeta(w), \qquad \sum_{\substack{\boldsymbol{k}\,:\,\text{admissible} \\ \text{wt}(\boldsymbol{k})=w \\ \text{dep}(\boldsymbol{k})=d}} \zeta^\star(\boldsymbol{k}) = \binom{w-1}{d-1}\zeta(w). \tag{1.1}$$

Here, $\zeta(w)$ is the Riemann zeta values and, for an index $\boldsymbol{k} = (k_1, \ldots, k_d)$, $\text{wt}(\boldsymbol{k}) = \sum_{i=1}^d k_i$ and $\text{dep}(\boldsymbol{k}) = d$ are the weight and the depth of $\boldsymbol{k}$, respectively. These relations are fundamental in the sense that many other relations include them as special cases.

The aim of this note is to generalize the above sum formulas for Schur multiple zeta values (Schur MZVs) introduced in [7], which are combinatoric and simultaneous generalizations of MZVs and MZSVs. In particular, we will establish what we call bounded-type sum formulas for Schur MZVs when their shapes are given by the following two specific cases: ribbons and those having only one corner.

2 Notations and definitions

A partition of $n \in \mathbb{Z}_{\geq 1}$ is a tuple $\lambda = (\lambda_1, \ldots, \lambda_h)$ of positive integers satisfying $\lambda_1 \geq \cdots \geq \lambda_h \geq 1$ and $n = |\lambda| = \lambda_1 + \cdots + \lambda_h$. A skew partition λ/μ is a pair of partitions $\lambda = (\lambda_1, \ldots, \lambda_h)$

*This note is based on the author's talk at RIMS conference "Zeta functions and their representations" and joint paper [2] with Henrik Bachmann (Nagoya University), Shin-ya Kadota (Niihama College), Yuta Suzuki (Rikkyo University) and Shuji Yamamoto (Keio University). For precise proofs of the results appeared in this note and detailed discussions, see [2].

and $\mu = (\mu_1, \ldots, \mu_r)$ satisfying $\mu \subset \lambda$, that is, $r \le h$ and $\mu_i \le \lambda_i$ for $i = 1, \ldots, h$ where we understand $\mu_i = 0$ for $i > r$. In the case where $\mu = \varnothing$ is the empty partition, we just write $\lambda/\mu = \lambda$. The weight of λ/μ is defined by $|\lambda/\mu| := |\lambda| - |\mu|$. With λ/μ, we associate the skew Young diagram $D(\lambda/\mu)$ defined by

$$D(\lambda/\mu) := \left\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le h\,, \mu_i < j \le \lambda_i\right\}\,.$$

An entry $(i,j) \in D(\lambda/\mu)$ is called an (outer) corner of λ/μ if $(i, j+1) \notin D(\lambda/\mu)$ and $(i+1, j) \notin D(\lambda/\mu)$. We denote the set of all corners of λ/μ by $C(\lambda/\mu)$. A Young tableau $\boldsymbol{k} = (k_{i,j})_{(i,j)\in D(\lambda/\mu)}$ of shape λ/μ is a filling of $D(\lambda/\mu)$ obtained by putting $k_{i,j} \in \mathbb{Z}_{\ge 1}$ into the (i,j)-entry of $D(\lambda/\mu)$. For shorter notation, we will also just write $(k_{i,j})$ in the following if the shape λ/μ is clear from the context. A Young tableau $(m_{i,j})$ is called semi-standard if $m_{i,j} < m_{i+1,j}$ and $m_{i,j} \le m_{i,j+1}$ for all possible i and j. The set of all Young tableaux and all semi-standard Young tableaux of shape λ/μ are denoted by $\mathrm{YT}(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$, respectively. For $\boldsymbol{k} = (k_{i,j}) \in \mathrm{YT}(\lambda/\mu)$, we define its weight by $\mathrm{wt}(\boldsymbol{k}) = \sum_{(i,j)\in D(\lambda/\mu)} k_{i,j}$ and call it admissible if $k_{i,j} \ge 2$ for all $(i,j) \in C(\lambda/\mu)$.

For an admissible $\boldsymbol{k} = (k_{i,j}) \in \mathrm{YT}(\lambda/\mu)$, the Schur multiple zeta value is defined by

$$\zeta(\boldsymbol{k}) := \sum_{(m_{i,j})\in\mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} \frac{1}{m_{i,j}^{k_{i,j}}}\,. \tag{2.1}$$

Note that the admissibility of $\boldsymbol{k}$ ensures the convergence of (2.1) (see [7, Lemma 2.1]). For the empty tableau $\boldsymbol{k} = \varnothing$, we have $\zeta(\varnothing) = 1$.

Example 2.1. For integers $a, b, c, e, f \ge 1$ and $d, g \ge 2$,

$$\zeta\left(\begin{array}{cccc} & & & \boxed{a} \\ & \boxed{b} & \boxed{c} & \boxed{d} \\ \boxed{e} & \boxed{f} & \boxed{g} & \end{array}\right) = \sum_{\substack{ m_1 \\ \wedge \\ m_2 \le m_3 \le m_4 \\ \wedge \quad \wedge \\ m_5 \le m_6 \le m_7}} \frac{1}{m_1^a \cdot m_2^b m_3^c m_4^d \cdot m_5^e m_6^f m_7^g}\,.$$

The Schur MZVs are zeta-function analogues of the Schur functions and, moreover, generalizations of both MZVs and MZSVs in the sense that we recover them as special cases:

$$\zeta(k_1, \ldots, k_d) = \zeta\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_d \\ \hline\end{array}\right), \qquad \zeta^\star(k_1, \ldots, k_d) = \zeta\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_d \\ \hline\end{array}\right)\,.$$

Our main research object is the sum

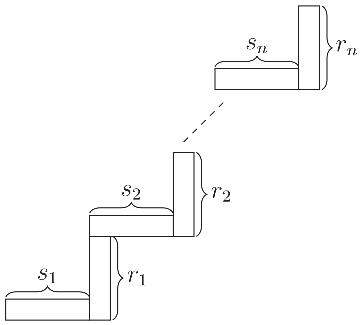
$$S_w(\lambda/\mu) := \sum_{\substack{\boldsymbol{k}\in\mathrm{YT}(\lambda/\mu) \\ \boldsymbol{k}\,:\,\text{admissible} \\ \mathrm{wt}(\boldsymbol{k})=w}} \zeta(\boldsymbol{k})\,.$$

Notice that when $\lambda = (1^d)$ or $\lambda = (d)$, this coincides with the left-hand side of the classical sum formulas (1.1). In this note, we say that we get a sum formula for Schur MZVs if we can establish a "good" expression of $S_w(\lambda/\mu)$. Among various expressions of $S_w(\lambda/\mu)$, we are interested in a "bounded-type" expression in the sense that it can be written as a $\mathbb{Q}$-linear combination of MZVs where the number of terms does not depend on w, but just on λ/μ.

3 Ribbons

A skew Young diagram is called a ribbon if it is connected and contains no 2×2 block of boxes. Using integers $s_1 \geq 0$ and $s_2, \ldots, s_n, r_1, \ldots, r_n > 0$, such a ribbon can be explicitly drawn as

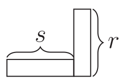
$$\mathrm{rib}\begin{pmatrix} s_1, \ldots, s_n \\ r_1, \ldots, r_n \end{pmatrix} :=$$



,

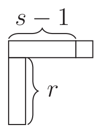
where n is the number of corners in the ribbon. This includes as special cases

$$\text{anti-hooks : } \mathrm{rib}\begin{pmatrix} s \\ r \end{pmatrix} =$$



$$= D(((s+1)^r)/(s^{r-1})) \quad (s \geq 0, r \geq 1)\,,$$

$$\text{hooks : } \mathrm{rib}\begin{pmatrix} 0, s-1 \\ r, \quad 1 \end{pmatrix} =$$



$$= D((s, 1^r)) \quad (s \geq 2, r \geq 1)\,.$$

In this section, we study the sum $S_w(\lambda/\mu)$ when the shape λ/μ is given by a ribbon.

For integers $w, s_1, \ldots, s_n \geq 0$ and $r_1, \ldots, r_n > 0$, define

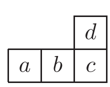
$$S_w\begin{pmatrix} s_1, \ldots, s_n \\ r_1, \ldots, r_n \end{pmatrix} := \sum_{\substack{\boldsymbol{l}_1, \ldots, \boldsymbol{l}_n \\ \mathrm{dep}(\boldsymbol{l}_i) = s_i \\ \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n \,:\, \text{admissible} \\ \mathrm{dep}(\boldsymbol{k}_i) = r_i \\ \sum_i \mathrm{wt}(\boldsymbol{k}_i) + \sum_i \mathrm{wt}(\boldsymbol{l}_i) = w}} \zeta\begin{pmatrix} \boldsymbol{l}_1, \ldots, \boldsymbol{l}_n \\ \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n \end{pmatrix},$$

where, for indices $\boldsymbol{l}_i = (l_{i,1}, \ldots, l_{i,s_i})$ of depth s_i and $\boldsymbol{k}_i = (k_{i,1}, \ldots, k_{i,r_i})$ of depth r_i,

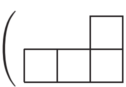
$$\zeta\begin{pmatrix} \boldsymbol{l}_1, \ldots, \boldsymbol{l}_n \\ \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n \end{pmatrix} := \sum_{\substack{0 < b_{i,1} \leq \cdots \leq b_{i,s_i+1} \\ 0 < a_{i,1} < \cdots < a_{i,r_i} \\ b_{i,s_i+1} = a_{i,r_i} \ (i=1,\ldots,n) \\ b_{i+1,1} < a_{i,1} \ (i=1,\ldots,n-1)}} \prod_{i=1}^{n} \frac{1}{a_{i,1}^{k_{i,1}} \cdots a_{i,r_i}^{k_{i,r_i}} b_{i,1}^{l_{i,1}} \cdots b_{i,s_i}^{l_{i,s_i}}}\,. \tag{3.1}$$

This is nonzero only when $w \geq s_1 + \cdots + s_n + r_1 + \cdots + r_n + n$. Note that $S_w\begin{pmatrix} s_1, \ldots, s_n \\ r_1, \ldots, r_n \end{pmatrix} = S_w\left(\mathrm{rib}\begin{pmatrix} s_1, \ldots, s_n \\ r_1, \ldots, r_n \end{pmatrix}\right)$ only when $s_2, \ldots, s_n > 0$. For example,

$$S_w\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \sum_{\substack{a,b,d \geq 1,\, c \geq 2 \\ a+b+c+d = w}} \zeta\left(\right.$$



$$\left.\right) = S_w\left(\right.$$



$$\left.\right)$$

but

$$S_w\begin{pmatrix}2,0\\1,1\end{pmatrix} = \sum_{\substack{a,b\geq 1,\, c,d\geq 2\\ a+b+c+d=w}} \zeta\left(\begin{array}{ccc} & & \boxed{d} \\ \boxed{a} & \boxed{b} & \boxed{c}\end{array}\right) \neq S_w\left(\begin{array}{ccc} & & \square \\ \square & \square & \square\end{array}\right).$$

3.1 A strategy

Our basic idea to compute the sum $S_w\begin{pmatrix}s_1,\ldots,s_n\\r_1,\ldots,r_n\end{pmatrix}$ is to reduce the number of corners n by using the following recursion formula, which can be shown by switching the inequality $b_{i+1,1} < a_{i,1}$ in (3.1) (for the given i) to the opposite $a_{i,1} \leq b_{i+1,1}$ and adding the corresponding Schur MZVs.

Proposition 3.1. *Let $s_1,\ldots,s_n \geq 0$ and $r_1,\ldots,r_n > 0$ be integers. For $1 \leq i \leq n-1$ with $r_i \geq 2$, we have*

$$\begin{aligned} &S_w\begin{pmatrix}s_1,\ldots,s_i,s_{i+1},\ldots,s_n\\r_1,\ldots,r_i,r_{i+1},\ldots,r_n\end{pmatrix} + S_w\begin{pmatrix}s_1,\ldots, & s_i, & s_{i+1}+1,\ldots,s_n\\r_1,\ldots,r_i-1, & r_{i+1}, & \ldots,r_n\end{pmatrix}\\ &= \sum_{\substack{w_1+w_2=w\\ w_1\geq s_1+\cdots+s_i+r_1+\cdots+r_i+i\\ w_2\geq s_{i+1}+\cdots+s_n+r_{i+1}+\cdots+r_n+n-i}} S_{w_1}\begin{pmatrix}s_1,\ldots,s_i\\r_1,\ldots,r_i\end{pmatrix}\cdot S_{w_2}\begin{pmatrix}s_{i+1},\ldots,s_n\\r_{i+1},\ldots,r_n\end{pmatrix}. \end{aligned}$$

Using this repeatedly, one can express $S_w\begin{pmatrix}s_1,\ldots,s_n\\r_1,\ldots,r_n\end{pmatrix}$ in terms of the values of the type $S_w\begin{pmatrix}s, & 0, & \ldots, & 0\\r_1, & r_2, & \ldots, & r_n\end{pmatrix}$. Moreover, using the Ohno relation for MZVs together with some combinatoric arguments, one can get the following expression for the latter sum.

Theorem 3.2. *For $w \geq 0$, $s \geq 0$ and $r_1,\ldots,r_n > 0$, we have*

$$S_w\begin{pmatrix}s, & 0, & \ldots, & 0\\r_1, & r_2, & \ldots, & r_n\end{pmatrix} = \sum_{\substack{t_1,\ldots,t_n\geq 0\\ t_1+\cdots+t_n=s}} \sum_{\substack{w_i\geq r_i+t_i+1\\ w_1+\cdots+w_n=w}} \prod_{i=1}^{n}\binom{w_i-1}{t_i}\cdot\zeta(w_1,\ldots,w_n).$$

Combining above two results, we have

Corollary 3.3. *For $n \geq 1$, $s_1,\ldots,s_n \geq 0$ and $r_1,\ldots,r_n > 0$, the sum $S_w\begin{pmatrix}s_1,\ldots,s_n\\r_1,\ldots,r_n\end{pmatrix}$ can be written as a $\mathbb{Q}$-linear combination of MZVs of weight w and depth $\leq n$.*

For example, when $n = 2$, we have

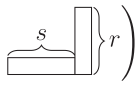
$$\begin{aligned} S_w\begin{pmatrix}s_1,s_2\\r_1,r_2\end{pmatrix} &= \sum_{i=0}^{s_2-1}(-1)^{s_2-i-1} \sum_{\substack{w_1\geq s_1+s_2+r_1-i+1\\ w_2\geq r_2+i+1\\ w_1+w_2=w}} \binom{w_1-1}{s_1}\binom{w_2-1}{i}\zeta(w_1)\zeta(w_2)\\ &\quad + (-1)^{s_2}\sum_{\substack{t_1,t_2\geq 0\\ t_1+t_2=s_1}} \sum_{\substack{w_1\geq s_2+r_1+t_1+1\\ w_2\geq r_2+t_2+1\\ w_1+w_2=w}} \binom{w_1-1}{t_1}\binom{w_2-1}{t_2}\zeta(w_1,w_2)\,. \end{aligned} \tag{3.2}$$

Notice that this is not a bounded-type expression in general.

3.2 Results

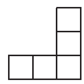
Using results in the previous subsection, we can actually evaluate the sum $S_w\binom{s_1,\dots,s_n}{r_1,\dots,r_n}$ explicitly in some special cases. First, we consider the cases of $n=1$, that is, the case of anti-hook shapes. The following theorem is an immediate consequence of Theorem 3.2.

Theorem 3.4. *For any integers $r \geq 1$, $s \geq 0$ and $w \geq s+r+1$, we have*

$$S_w\binom{s}{r} = S_w\Bigg($$  $$\Bigg) = \binom{w-1}{s}\zeta(w)\,.$$

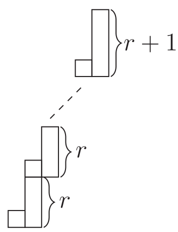
Notice that this is a simultaneous generalization of the classical sum formulas (1.1) for MZVs (the case of $s=0$) and MZSVs (the case of $r=1$).

Example 3.5. For the anti-hook shape $\lambda/\mu = (3^3)/(2^2)$, we have for $w \geq 6$

$$S_w\binom{2}{3} = S_w\Bigg($$  $$\Bigg) = \binom{w-1}{2}\zeta(w)\,.$$

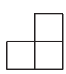
The next sum formula is for the "stair of tread one" shape. One can prove this by induction on n with Proposition 3.1 and Theorem 3.2.

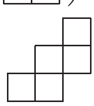
Theorem 3.6. *For any integers $r \geq 1$, $n \geq 1$ and $w \geq (r+2)n+1$, we have*

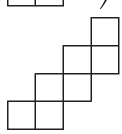
$$S_w\binom{\{1\}^{n-1},\ 1}{\{r\}^{n-1}, r+1} = S_w\Bigg($$  $$\Bigg) = c_{w,r}(n)\zeta(w)\,,$$

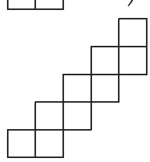
where $c_{w,r}(n) := \dfrac{w-1}{n}\dbinom{w-(r+1)n-2}{n-1} \in \mathbb{Z}_{>0}$.

Example 3.7 (The case of $r=1$)**.**

$$S_w\binom{1}{2} = S_w\Bigg($$  $$\Bigg) = \frac{w-1}{1}\binom{w-4}{0}\zeta(w) \quad (w \geq 4)\,,$$

$$S_w\binom{1,1}{1,2} = S_w\Bigg($$  $$\Bigg) = \frac{w-1}{2}\binom{w-6}{1}\zeta(w) \quad (w \geq 7)\,,$$

$$S_w\binom{1,1,1}{1,1,2} = S_w\Bigg($$  $$\Bigg) = \frac{w-1}{3}\binom{w-8}{2}\zeta(w) \quad (w \geq 10)\,,$$

$$S_w\binom{1,1,1,1}{1,1,1,2} = S_w\Bigg($$  $$\Bigg) = \frac{w-1}{4}\binom{w-10}{3}\zeta(w) \quad (w \geq 13)\,.$$

Let us say that the sum $S_w(\lambda/\mu)$ has a single-type expression if it can be written as a rational multiple of $\zeta(w)$. In general, it seems to be difficult to find a shape λ/μ (or, more strongly, a family of shapes $\{\lambda_n/\mu_n\}_n$) for which $S_w(\lambda/\mu)$ (or $S_w(\lambda_n/\mu_n)$ for all n) has a single-type expression.

We finally give a sum formula for general ribbons with two corners.

Theorem 3.8. *For $s_1, s_2 \geq 0$, $r_1, r_2 > 0$ and $w \geq s_1+s_2+r_1+r_2+2$, we have*

$$S_w\begin{pmatrix} s_1, s_2 \\ r_1, r_2 \end{pmatrix} = \binom{w-2}{s_1+s_2}\zeta(w) + \sum_{\substack{w_1,w_2\geq 2\\ w_1+w_2=w}} A^{s_1,s_2,r_1,r_2}_{w_1,w_2}\zeta(w_1)\zeta(w_2) + \sum_{\substack{w_1\geq 1, w_2\geq 2\\ w_1+w_2=w}} B^{s_1,s_2,r_1,r_2}_{w_1,w_2}\zeta(w_1,w_2), \tag{3.3}$$

where the integers $A^{s_1,s_2,r_1,r_2}_{w_1,w_2}$ and $B^{s_1,s_2,r_1,r_2}_{w_1,w_2}$ are explicitly given by

$$\begin{aligned} A^{s_1,s_2,r_1,r_2}_{w_1,w_2} := {} & (-1)^{w_1} C^{s_1,s_2}_{w_1,w_2} \\ & - \mathbf{1}_{w_1\leq s_1+r_1 \text{ or } w_2\leq s_2+r_2-1}\binom{w_1-1}{s_1}\binom{w_2-2}{s_2-1} \\ & + \mathbf{1}_{w_1>s_1+r_1}(-1)^{s_1+r_1+w_1}\binom{w_1-1}{s_1}\binom{w_2-2}{s_1+s_2+r_1-w_1} \\ & + \mathbf{1}_{r_2<w_2\leq s_2+r_2-1}(-1)^{s_2+r_2+w_2}\binom{w_1-1}{s_1}\binom{w_2-2}{r_2-1}, \\ B^{s_1,s_2,r_1,r_2}_{w_1,w_2} := {} & C^{s_1,s_2}_{w_1,w_2} - (-1)^{s_2}\sum_{\substack{t_1,t_2\geq 0\\ t_1\geq w_1-(s_2+r_1) \text{ or } t_2\geq w_2-r_2\\ t_1+t_2=s_1}}\binom{w_1-1}{t_1}\binom{w_2-1}{t_2} \end{aligned}$$

with

$$C^{s_1,s_2}_{w_1,w_2} := (-1)^{s_2}\sum_{\substack{0\leq i\leq s_1\\ 1\leq j\leq s_2\\ i+j=w_1}}\binom{w_1-1}{i}\binom{w_2-1}{s_1-i} - (-1)^{s_1}\binom{w_1-1}{s_1}\binom{w_2-2}{s_1+s_2-w_1}.$$

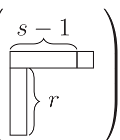
Here, $\mathbf{1}_P$ is the indicator function for the condition P.

We remark that, though the expression is complicated, this is actually of bounded-type because $A^{s_1,s_2,r_1,r_2}_{w_1,w_2} = B^{s_1,s_2,r_1,r_2}_{w_1,w_2} = 0$ unless $w_1 \leq s_1+s_2+r_1$ or $w_2 \leq \max\{s_2+r_2-1, s_1+r_2\}$.

As a special case of Theorem 3.8, we obtain the following sum formula for hook shapes.

Corollary 3.9. *For $s, r \geq 1$ and $w \geq s+r+2$, we have*

$$S_w\begin{pmatrix} 0, s-1 \\ r, \ \ 1 \end{pmatrix} \left(= S_w\left(\text{[diagram]} \right) \text{ when } s \geq 2 \right)$$

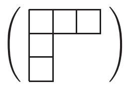


$$= \binom{w-2}{s-1}\zeta(w) - \sum_{k=1}^{s-1}\binom{w-k-2}{s-k-1}\zeta(k,w-k) + (-1)^s\sum_{k=s}^{s+r-1}\zeta(k,w-k)$$

$$-\sum_{k=2}^{s-1}(-1)^k\binom{w-k-2}{s-k-1}\zeta(k)\zeta(w-k)-\sum_{k=2}^{r}\binom{w-k-2}{s-2}\zeta(k)\zeta(w-k)$$
$$+(-1)^r\sum_{k=r+1}^{s+r-1}(-1)^k\binom{w-k-2}{r+s-1-k}\zeta(k)\zeta(w-k)\,.$$

Example 3.10. For the hook shape $\lambda=(3,1^2)$, we have for $w\geq 7$

$$S_w\begin{pmatrix}0,2\\2,1\end{pmatrix}=S_w\left(\right)$$



$$=\binom{w-2}{2}\zeta(w)-(w-3)\zeta(2)\zeta(w-2)-(w-5)\zeta(3)\zeta(w-3)+\zeta(4)\zeta(w-4)$$
$$-(w-3)\zeta(1,w-1)-\zeta(2,w-2)-\zeta(3,w-3)-\zeta(4,w-4)\,.$$

We here give an outline of the proof of Theorem 3.8. To get (3.3), we start from (3.2), which we again notice that is not a bounded-type expression. Applying the harmonic product formula $\zeta(w_1)\zeta(w_2)=\zeta(w_1,w_2)+\zeta(w_2,w_1)+\zeta(w_1+w_2)$ to the first sum in the right-hand side of (3.2), one sees that it can be written as weighted sums of MZVs with the weight given by a product of binomial coefficients, that is,

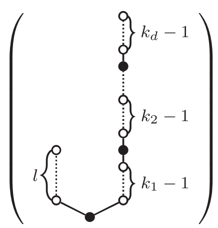
$$P_l(k_1,k_2):=\sum_{\substack{w_1\geq 1,w_2\geq 2\\ w_1+w_2=k_1+k_2+l}}\binom{w_1-1}{k_1-1}\binom{w_2-1}{k_2-1}\cdot\zeta(w_1,w_2)\,. \tag{3.4}$$

More generally, for indices $\boldsymbol{n}=(n_1,\dots,n_d)$, $\boldsymbol{k}=(k_1,\dots,k_d)$ and an integer $l\geq 0$, define

$$P_l(\boldsymbol{n};\boldsymbol{k}):=\sum_{\substack{\boldsymbol{w}=(w_1,\dots,w_d):\,\text{admissible}\\ w_i\geq n_i\ (i=1,\dots,d)\\ \mathrm{wt}(\boldsymbol{w})=\mathrm{wt}(\boldsymbol{k})+l}}\prod_{i=1}^{d}\binom{w_i-n_i}{k_i-1}\cdot\zeta(w_1,\dots,w_d)\,. \tag{3.5}$$

Put $P_l(\boldsymbol{k})=P_l((1,\dots,1);\boldsymbol{k})$, which coincides with (3.4) when $d=2$. We see that if $\boldsymbol{k}$ is admissible, then $P_l(\boldsymbol{k})$ has the so-called Yamamoto integral representation defined in [8] as

$$P_l(\boldsymbol{k})=$$



Based on this observation, we can obtain the following bounded-type expression of $P_l(\boldsymbol{k})$ (in the sense that the number of terms appearing in the expression does not depend on l but only on $\boldsymbol{k}$) for not only admissible $\boldsymbol{k}$ but also those are not admissible. This is a recursive formulas with respect to l, which can be described by using the Schur MZVs of anti-hook shape

$$\zeta\begin{pmatrix}l_1,\dots,l_s\\k_1,\dots,k_r\end{pmatrix}:=\zeta\left(\begin{array}{|c|c|c|c|}\hline \multicolumn{3}{c|}{} & k_1\\ \cline{4-4}\multicolumn{3}{c|}{} & \vdots\\ \hline l_1 & \cdots & l_s & k_r\\ \hline\end{array}\right)=\sum_{0<a_1<\cdots<a_r\geq b_s\geq\cdots\geq b_1>0}\frac{1}{a_1^{k_1}\cdots a_r^{k_r}b_1^{l_1}\cdots b_s^{l_s}}\,. \tag{3.6}$$

Proposition 3.11. *Let $d \geq 2$ and $l > 0$.*

(i) *For an admissible index $\boldsymbol{k} = (k_1, \ldots, k_d)$, we have*

$$P_l(\boldsymbol{k}) = \sum_{i=1}^{d} \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\cdots+k_{i-1}+a_i} P_{k_i-1-a_i}(k_{i-1}, \ldots, k_1, l+1)\, P_{a_i}(k_{i+1}, \ldots, k_d)\,.$$

(ii) *For a non-admissible index $\boldsymbol{k} = (k_1, \ldots, k_d)$, we have*

$$\begin{aligned} P_l(\boldsymbol{k}) &= \sum_{i=1}^{d} \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\cdots+k_{i-1}+a_i} P_{k_i-1-a_i}(k_{i-1}, \ldots, k_1, l+1) P_{a_i}(k_{i+1}, \ldots, k_{d-1}, 1) \\ &\quad + \sum_{i=1}^{d-1} (-1)^{l+d+k_i} \sum_{\substack{(b_0, \ldots, b_{d-1}) \in \mathbb{Z}_{\geq 1}^{d} \\ b_i = 2 \\ b_0+\cdots+b_{d-1} = \mathrm{wt}(\boldsymbol{k})+l+1}} (-1)^{b_0+b_1+\cdots+b_{i-1}} \binom{b_0-1}{l} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{d-1} \binom{b_j-1}{k_j-1} \right\} \\ &\quad \times \sum_{j=i}^{d-1} \sum_{c_j=1}^{b_j-1} (-1)^{c_j+j+b_{j+1}+\cdots+b_{d-1}} \zeta\begin{pmatrix} c_j, b_{j+1}, \ldots, b_{d-1} \\ b_{i-1}, \ldots, b_1, b_0 \end{pmatrix} \zeta(b_{i+1}, \ldots, b_{j-1}, b_j - c_j + 1)\,. \end{aligned}$$

We note that to obtain the expression of the latter case, we have used the Yamamoto integral representation of Schur MZVs (3.6) of anti-hook shape obtained by Kaneko and Yamamoto [6]. When $d = 2$, the above expressions can be unified as follow, which we have actually used to obtain (3.3).

Corollary 3.12. *For $k_1, k_2 \geq 1$ and $l > 0$, we have*

$$\begin{aligned} P_l(k_1, k_2) &= (-1)^{k_2} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2 = k_1+k_2+l}} (-1)^{w_1} \binom{w_1-1}{k_2-1} \binom{w_2-1}{l} \zeta(w_1)\zeta(w_2) \\ &\quad + (-1)^{k_1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2 = k_1+k_2+l}} \binom{w_1-1}{k_1-1} \binom{w_2-1}{l} \zeta(w_1, w_2) + \mathbf{1}_{k_2=1} \binom{l+k_1-1}{k_1-1} \zeta\begin{pmatrix} 1 \\ l+k_1 \end{pmatrix}. \end{aligned}$$

4 Shapes having only one corner

In this section, we study the sum $S_w(\lambda/\mu)$ when the shape λ/μ has only one corner. To do that, we first introduce the notion of the semi-standard decompositions of a Young diagram; For a skew shape λ/μ, we call a tuple $(D_1, \ldots, D_r)$ of non-empty subsets of $D(\lambda/\mu)$ a semi-standard decomposition if it satisfies the following two conditions.

(i) $D(\lambda/\mu) = D_1 \sqcup \cdots \sqcup D_r$.

(ii) The Young tableau $(t_{i,j}) \in \mathrm{YT}(\lambda/\mu)$ defined by $t_{i,j} = a$ for $(i, j) \in D_a$ with $a = 1, \ldots, r$ is semi-standard.

We denote by $\mathrm{SSD}(\lambda/\mu)$ the set of all semi-standard decompositions of $D(\lambda/\mu)$. Notice that we always identify $(D_1,\dots,D_r)\in\mathrm{SSD}(\lambda/\mu)$ with $(t_{i,j})\in\mathrm{SSYT}(\lambda/\mu)$ defined in the condition (ii) above. For example, $(D_1,D_2)=(\{(1,1),(1,2)\},\{(2,1)\})\in\mathrm{SSD}\left(\ydiagram{2,1}\right)$ is identified with $\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\in\mathrm{SSYT}\left(\ydiagram{2,1}\right)$. When λ/μ has only one corner, we see that for any admissible Young tableau $\boldsymbol{k}=(k_{i,j})\in\mathrm{YT}(\lambda/\mu)$

$$\zeta(\boldsymbol{k})=\sum_{(D_1,\dots,D_r)\in\mathrm{SSD}(\lambda/\mu)}\zeta\left(\sum_{(i,j)\in D_1}k_{ij},\dots,\sum_{(i,j)\in D_r}k_{ij}\right). \tag{4.1}$$

For an index $\boldsymbol{k}=(k_1,\dots,k_d)$, let

$$\begin{aligned}Q_l(\boldsymbol{k}) &:= P_l((1,\dots,1,2);\boldsymbol{k})\\ &=\sum_{\substack{\boldsymbol{w}=(w_1,\dots,w_d)\,:\,\text{admissible}\\ \mathrm{wt}(\boldsymbol{w})=\mathrm{wt}(\boldsymbol{k})+l}}\binom{w_1-1}{k_1-1}\cdots\binom{w_{d-1}-1}{k_{d-1}-1}\binom{w_d-2}{k_d-1}\zeta(\boldsymbol{w}),\end{aligned}$$

where $P_l(\boldsymbol{n};\boldsymbol{k})$ is defined in (3.5). Then, using (4.1), one has the following expression for $S_w(\lambda/\mu)$, which is crucial for our evaluation of $S_w(\lambda/\mu)$.

Lemma 4.1. *When λ/μ has only one corner, we have for $w>|\lambda/\mu|$*

$$S_w(\lambda/\mu)=\sum_{(D_1,\dots,D_r)\in\mathrm{SSD}(\lambda/\mu)}Q_{w-|\lambda/\mu|}(|D_1|,\dots,|D_r|). \tag{4.2}$$

Now we rewrite (4.2) in terms of Hoffman algebra [5]. Denote by $\mathfrak{H}^1=\mathbb{Q}\langle z_k\mid k\geq 1\rangle$ the non-commutative polynomial ring in the variables z_k for $k\geq 1$. A monic monomial in $\mathfrak{H}^1$ is called a word and the empty word will be denoted by $\mathbf{1}$. We define the stuffle product $*$ and the index shuffle product $\widetilde{\sqcup\!\sqcup}$ on $\mathfrak{H}^1$ as the $\mathbb{Q}$-bilinear products, which satisfy $\mathbf{1}*w=w*\mathbf{1}=w$ and $\mathbf{1}\,\widetilde{\sqcup\!\sqcup}\,w=w\,\widetilde{\sqcup\!\sqcup}\,\mathbf{1}=w$ for any word $w\in\mathfrak{H}^1$, and for any $i,j\geq 1$ and words $w_1,w_2\in\mathfrak{H}^1$

$$\begin{aligned}z_iw_1*z_jw_2&=z_i(w_1*z_jw_2)+z_j(z_iw_1*w_2)+z_{i+j}(w_1*w_2),\\ z_iw_1\,\widetilde{\sqcup\!\sqcup}\,z_jw_2&=z_i(w_1\,\widetilde{\sqcup\!\sqcup}\,z_jw_2)+z_j(z_iw_1\,\widetilde{\sqcup\!\sqcup}\,w_2).\end{aligned}$$

By [5, Theorem 2.1], we obtain a commutative $\mathbb{Q}$-algebra $\mathfrak{H}^1_*$. For $k\geq 1$ and $n\in\mathbb{Z}$, define

$$z_k^n=\begin{cases}\underbrace{z_k\cdots z_k}_{n} & \text{if } n>0,\\ \mathbf{1} & \text{if } n=0,\\ 0 & \text{if } n<0.\end{cases}$$

Moreover, for each index $\boldsymbol{k}=(k_1,\dots,k_d)$, we put $z_{\boldsymbol{k}}:=z_{k_1}\cdots z_{k_d}$ so that we define $Q_l:\mathfrak{H}^1\to\mathbb{R}$ by setting $Q_l(z_{\boldsymbol{k}}):=Q_l(\boldsymbol{k})$ and extending it linearly. Define the element $\varphi(\lambda/\mu)$ of $\mathfrak{H}^1$ by

$$\varphi(\lambda/\mu):=\sum_{(D_1,\dots,D_r)\in\mathrm{SSD}(\lambda/\mu)}z_{|D_1|}\cdots z_{|D_r|}.$$

Then, (4.2) can be written as

$$S_w(\lambda/\mu) = Q_{w-|\lambda/\mu|}\left(\varphi(\lambda/\mu)\right) . \tag{4.3}$$

Moreover, using the Jacobi-Trudi type formula for Schur MZVs (cf. [7, Theorem 1.1] and [1, Theorem 4.7]), we show that $\varphi(\lambda/\mu)$ has the following determinant expression.

Proposition 4.2. *For any skew shape λ/μ, let $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ and $\mu' = (\mu'_1, \ldots, \mu'_s)$ be the conjugates of λ and μ, respectively. Then we have the identity*

$$\varphi(\lambda/\mu) = \det\nolimits_* \left[z_1^{\lambda'_i - \mu'_j - i + j} \right]_{1 \le i,j \le s} , \tag{4.4}$$

where $\det_*$ *denotes the determinant performed in the stuffle algebra* $\mathfrak{H}^1_*$.

Therefore, if we have a good evaluation formula for Q_l, combining (4.3) and (4.4), we get a nice expression of $S_w(\lambda/\mu)$. Actually, since

$$Q_l(k_1, \ldots, k_d) = \sum_{j=0}^{k_d - 1} (-1)^j P_{l+j}(k_1, \ldots, k_{d-1}, k_d - j) ,$$

employing Proposition 3.11, one can obtain a bounded-type expression of $S_w(\lambda/\mu)$ when λ/μ has only one corner.

Example 4.3. (i) When $\lambda/\mu = (2^2)$, we have for $w \ge 5$

$$\begin{aligned} S_w\left(\ydiagram{2,2} \right) &= Q_{w-4}\left(\det\nolimits_* \begin{bmatrix} z_1^2 & z_1^3 \\ z_1 & z_1^2 \end{bmatrix} \right) \\ &= 2Q_{w-4}(z_1^4) + Q_{w-4}(z_2 \mathbin{\widetilde{ш}} z_1^2) + Q_{w-4}(z_2^2) \\ &= -(w-2)\zeta(1, w-1) + (w-4)\zeta(2, w-2) + 2\zeta(3, w-3) \\ &\quad - 2\zeta(3)\zeta(w-3) + (w-2)\zeta(2)\zeta(w-2) . \end{aligned}$$

(ii) When $\lambda/\mu = (3^2)/(1)$, we have for $w \ge 6$

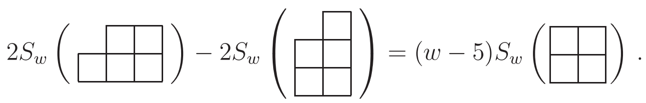
$$\begin{aligned} S_w\left(\ydiagram{1+2,3} \right) &= Q_{w-5}\left(\det\nolimits_* \begin{bmatrix} z_1 & z_1^3 & z_1^4 \\ 1 & z_1^2 & z_1^3 \\ 0 & z_1 & z_1^2 \end{bmatrix} \right) \\ &= 5Q_{w-5}(z_1^5) + 3Q_{w-5}(z_2 \mathbin{\widetilde{ш}} z_1^3) + 2Q_{w-5}(z_2^2 \mathbin{\widetilde{ш}} z_1) + Q_{w-5}(z_3 \mathbin{\widetilde{ш}} z_1^2) + Q_{w-5}(z_3 \mathbin{\widetilde{ш}} z_2) \\ &= \binom{w-2}{2}\zeta(2)\zeta(w-2) - \frac{5}{4}\zeta(4)\zeta(w-4) + \binom{w-3}{2}\zeta(2, w-2) - \binom{w-2}{2}\zeta(1, w-1) \\ &\quad + \zeta(2)\zeta(2, w-4) - \zeta(2)\zeta(1, w-3) + (w-3)\zeta(3, w-3) + (w-3)\zeta(1, 1, w-2) \\ &\quad - (w-5)\zeta(1, 2, w-3) - 2\zeta(1, 3, w-4) + \zeta(2, 1, w-3) - \zeta(2, 2, w-4) . \end{aligned}$$

(iii) When $\lambda/\mu = (2^3)/(1)$, we have for $w \ge 6$

$$S_w\left(\ydiagram{1+1,2,2} \right) = Q_{w-5}\left(\det\nolimits_* \begin{bmatrix} z_1^2 & z_1^4 \\ z_1 & z_1^3 \end{bmatrix} \right)$$

$$
\begin{aligned}
&= 5Q_{w-5}(z_1^5) + 2Q_{w-5}(z_2 \,\widetilde{ш}\, z_1^3) + Q_{w-5}(z_2^2 \,\widetilde{ш}\, z_1) \\
&= (w-2)\zeta(2)\zeta(w-2) + (w-5)\zeta(3)\zeta(w-3) - \frac{5}{4}\zeta(4)\zeta(w-4) \\
&\quad - \zeta(2)\zeta(1,w-3) + \zeta(2)\zeta(2,w-4) + (2-w)\zeta(1,w-1) + (w-4)\zeta(2,w-2) \\
&\quad + 2\zeta(3,w-3) + (w-3)\zeta(1,1,w-2) - (w-5)\zeta(1,2,w-3) \\
&\quad - 2\zeta(1,3,w-4) + \zeta(2,1,w-3) - \zeta(2,2,w-4)\,.
\end{aligned}
$$

Comparing these results, one find that



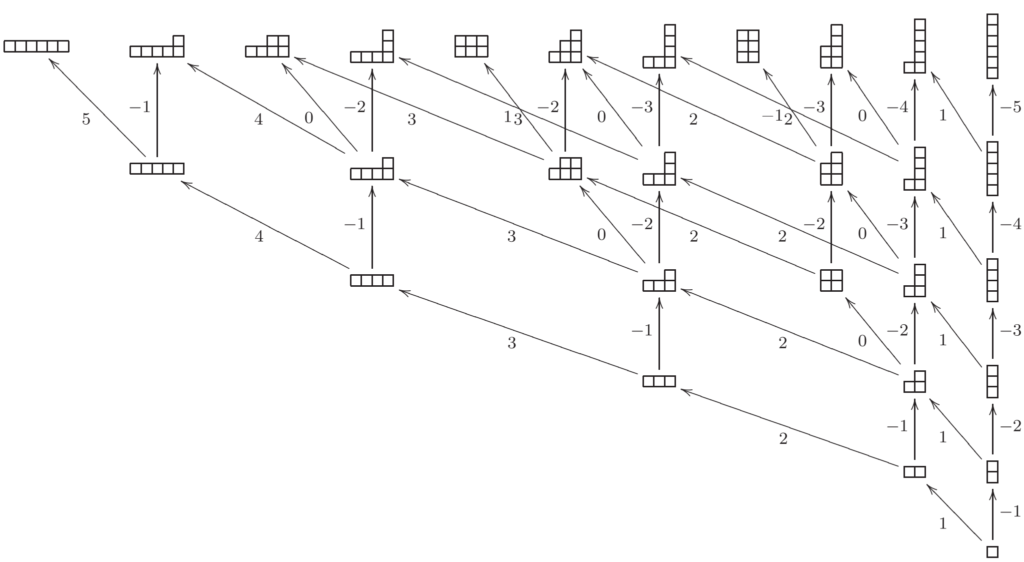
More generally, we can show the following relation[1] among $S_w(\lambda/\mu)$ for different shapes λ/μ. We remark that this can be proved without any relation among MZVs.

Theorem 4.4. *For a partition λ, let $\lambda^\dagger$ be the rotation of λ by π. Then, we have for $w \geq 1$*

$$
\sum_{\lambda^+=(\lambda_i^+)_i} (\lambda_{i_0}^+ - i_0) S_w\left((\lambda^+)^\dagger\right) = (w - |\lambda| - 1) S_w(\lambda^\dagger)\,, \tag{4.5}
$$

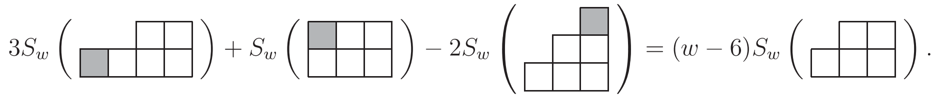
where the sum runs over all partitions $\lambda^+ = (\lambda_i^+)_i$ obtained by adding one box to i_0-th row of λ.

The following is the weighted directed graph whose weight of each arrow gives the coefficient of $S_w((\lambda^+)^\dagger)$, i.e., $\lambda_{i_0}^+ - i_0$, in the left-hand side of (4.5) for small λ.

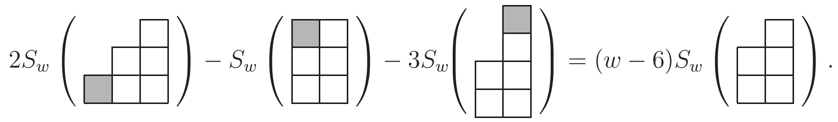


[1] This relation is another description of the one proved in [2, Theorem 4.8]; They are the same.

Example 4.5. (i) When $\lambda = (3, 2)$, we have



(ii) When $\lambda = (2^2, 1)$, we have



Here, for each λ^+, the gray box of λ^+ represents the box added to λ.

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