Generalized Schubert Eisenstein series

YoungJu Choie
Dept of Mathematics, POSTECH, Korea

Talk at RIMS Workshop 2022
Zeta functions and Their representations
March 6. 2022-March 8, 2022

1 Introduction

Schubert Eisenstein series has been defined as sums like usual Eisenstein series but with the summation restricted to elements coming from a particular Schubert cell [2]. This is no longer an automorphic form, but we may ask whether it has analytic continuation and at least some functional equations. Bump and the autor looked closely Schubert Eisenstein series at the special case, where $G = \mathrm{GL}(3)$ and suggested general lines of research for the general case in [2]:

(a) Does the Schubert Eisenstein series admit a meromorphic continuation ?

(b) Do a subset of the functional equations for the full Eisenstein series continue to hold for the Schubert Eisenstein series?

(c) Is it possible to find a linear combination of Schubert Eisenstein series which is entire?

Recently Getz and the author in [3] give a partial answer of the the above questions. In fact, in [3] Getz and author relate the above questions to the program of Braverman, Kazhdan, Lafforgue, Ngô, and Sakellaridis ([1, 4, 5, 6]), in which they prove the Poisson summation formula for certain schemes closely related to Schubert varieties and use it to refine and establish the above conjecture in many cases.

In this talk we try to explain one of the main result by Getz and author in [3], in which Theorem states a partial answer of (a) raised by Bump and author in [2].

This note is based on the paper by Getz and author in [3].

2 Preliminaries

Let $\mathbb{A}$ be the Adele ring of a global field F. Take G to be a split semisimple algebraic group over F, equipped with a Borel subgroup $B = TU$, where T represents its maximal split torus, and U is the unipotent radical. The Weyl group of T in G is denoted as $W(G,T) = N_G(T)/T$, where $N_G(T)$ is the normalizer of T.

For a place v of F, denote the group by $G_v = G(F_v)$ and similarly for algebraic subgroups of G. Let K_v be a maximal compact subgroup of $G_v = G(F_v)$.

2.1 Schubert Eisenstein series

Let χ be a quasi character of $T(\mathbb{A})/T(F)$. Let $(\pi_v(\chi_v), V_v(\chi_v))$ be the corresponding principal series representation. Thus $V_v(\chi_v)$ is the space of functions $f_v : G_v \longrightarrow \mathbb{C}$ that satisfy

$$f_v(bg) = \delta^{1/2}\chi_v(b)\, f_v(g)$$

for $b \in B_v = B(F_v)$, and which are K_v-finite. Here δ is the modular quasi character. If v is non-archimedean the group G_v acts by right-translation:

$$\pi_v(g_v)f_v(x) = f_v(xg_v).$$

If v is archimedean, this definition is wrong since $\pi_v(g_v)f_v$ may not be K_v-finite, but the K_v-finite vectors are invariant under the corresponding representation of the Lie algebra $\mathfrak{g}_v$ and so at an archimedean place v, $V_v(\chi_v)$ is a $(\mathfrak{g}_v, K_v)$-module.

Assume that the space of K_v-fixed vectors is nonzero. The vector space $V_v(\chi_v)$ has a K_v-fixed vector $f_v^\circ = f_{\chi_v}^\circ$ that is unique up toscalar multiple. We will normalize it so that $f_v^\circ(1) = 1$. Let $V(\chi)$ be the space of finite linear combinations of functions of theform $\prod_v f_v(g_v)$ where $f_v \in V_v(\chi_v)$ and $f_v = f_v^\circ$ for all but finitely many v. If the function f is of this form then we will write $f = \otimes_v f_v$. So the space $V(\chi)$ is the restricted tensor product ofthe local modules $V_v(\chi_v)$.

The usual Eisenstein series are sums over the integer points in the flag variety $X = B(F)\backslash G(F)$. Furthermore, the Bruhat decomposition of G gives the decomposition of the flag variety into Schubert cells

$$X = \bigcup_{w \in W(G,T)} Y_w$$

where Y_w is the image of BwB in $B\backslash G$. The closure of Y_w is the closed Schubert variety

$$X_w = \bigcup_{u \leqslant w} Y_u$$

where $\leqslant$ is the Bruhat order. It seems a natural question to consider the *Schubert Eisenstein series*

$$E_w(g,\nu) = \sum_{\gamma \in X_w(F)} f_\nu(\gamma g). \tag{2.1.1}$$

This is no longer an automorphic form, but we may ask whether it has analytic continuation and at least some functional equations. Bump and the author in [2] looked closely Schubert Eisenstein series at the special case, where $G = \mathrm{GL}(3)$ and suggest general lines of research for the general case (see [2] for details). Recently Getz and author in [3] give a partial answer of the above first question (a).

2.2 Generalized Schubert Eisenstein series

This section is a part of the joint work with Getz [3].

Consider a parabolic subgroup P of G such that

$$T \leq B \leq P \leq G$$

and let M be the the Levi subgroup of P containing T. Take an isomorphism

$$\omega_P : M^{ab} \longrightarrow G_m^{k+1}$$

where $M^{ab} = M/M^{\mathrm{der}}$ is the abelianization of M. Denote M^{der} is the derived group of an algebraic group M.

Let $\chi : (A_{G_m} F^\times \backslash \mathbb{A}_F^\times)^{k+1} \to C^\times$ be a character, where $A_{G_m} < F_\infty^\times$ is a subgroup. For $s \in \mathbb{C}^{k+1}$ define

$$\chi_s(a_0, \ldots, a_k) := \chi(a_0, \ldots, a_k) \prod_{i=0}^{k} |a_i|^{s_i}$$

and form the induced representation

$$I_P(\chi_s) := \mathrm{Ind}_P^G(\chi_s \circ \omega_P),$$

normalized so that it is unitary when $s \in (i\mathbb{R})^{k+1}$.

The Bruhat decomposition of G implies the following decomposition of the generalized flag variety

$$P\backslash G = \coprod_{w \in W(M,T)\backslash W(G,T)} P\backslash PwB.$$

Let X_w be the (Zariski) closure of the Schubert cell $P\backslash PwB$ in $P\backslash G$. It is a Schubert variety. In [3] the definition of Schubert Eisenstein series was further generalized and with much greater generality questions (a), (b) and (c) in the introduction answered affirmatively when we regard $E_w(g,\nu)$ as a function of s_0 assuming the s_i with $i \neq 0$ are fixed with large real part. We refer these results with details to [3].

Now take an arbitrary algebraic subgroup H of G and consider

$$P'\gamma H$$

where $P \leq P' \leq G$ are a pair of parabolic subgroups, $\gamma \in G$. From the automorphic point of view this may be the most important situation. Schubert cells are often nonsmooth, whereas the image of any set of the form $P'\gamma H$ in $P\backslash G$ is a smooth subscheme (see [2] for details).

In order to treat Eisenstein series indexed by sets of the form Y and $P\bar{w}B$ simultaneously we work with an arbitrary (locally closed) subscheme $Y \subseteq G$ that is stable under left multiplication by P'. Let $X_P^\circ := P^{\mathrm{der}}\backslash G$ be the Braverman-Kazhdan space associated to P and G. Let

$$Y_P = \mathrm{Im}(Y \longrightarrow X_P^\circ). \tag{2.2.1}$$

To be more precise, the set theoretic image of $Y \to X_P^\circ$ is locally closed (see [3]). This set is the underlying topological space of a subscheme Y_P of X_P°. The subscheme $Y_P \subseteq X_P^\circ$ is quasi-affine. Let X_P be the affine closure of X_P° and let

$$Y_{P,P'} \subseteq X_P \tag{2.2.2}$$

be the partial closure of Y_P in X_P. Furthermore, if we assume

$$P \text{ is maximal in } P'. \tag{2.2.3}$$

there is a unique parabolic subgroup $P^* < P'$ with Levi subgroup M that is not equal to P.

2.3 Schwartz space

when F is nonarchimedean. Let $K \leq M^{\mathrm{ab}} \times G$ be a compact open subgroup. Let $\mathcal{C}_{\beta_0}(X_P)$ be the space of K-finite $f \in C^\infty(X_P^\circ)$ such that for $\mathrm{Re}(s_{\beta_0})$ sufficiently large the integral defining the Mellin transform f_{χ_s} converges absolutely and defines a good section.

The **Schwartz space** of $Y_{P,P'}$ is the space of restrictions to Y_P of functions in $\mathcal{C}_{\beta_0}(X_P)$:

$$\mathcal{S}(Y_{P,P'}) = \mathrm{Im}(\mathcal{C}_{\beta_0}(X_P) \longrightarrow C^0(Y_P). \tag{2.3.1}$$

For Archimedean F see [3] for definition of Schwartz space.

3 Main Theorem

Now consider Schwartz spaces

$$\mathcal{S}(Y_{Q,P'}(\mathbb{A}_F))$$

for $Q \in \{P, P^*\}$ together with a Fourier transform

$$\mathcal{F}_{P|P^*} : \mathcal{S}(Y_{P,P'}(\mathbb{A}_F)) \tilde{\longrightarrow} \mathcal{S}(Y_{P^*,P'}(\mathbb{A}_F)). \tag{3.0.1}$$

The Schwartz space $\mathcal{S}(Y_{Q,P'}(\mathbb{A}_F))$ is contained in the set of restrictions to $Y_P(\mathbb{A}_F)$ of functions in $C^\infty(X_P^\circ(\mathbb{A}_F))$. Let $H \leq G$ be a subgroup, and consider the action of H on G by right multiplication. Assume that Y is stable under the action of H. Then the Schwartz spaces $\mathcal{S}(Y_{P,P'}(\mathbb{A}_F))$ and $\mathcal{S}(Y_{P^*,P'}(\mathbb{A}_F))$ are preserved under the action of $M^{ab}(\mathbb{A}_F) \times H(\mathbb{A}_F)$ and the Fourier transform satisfies a twisted equivariance property (see Lemma 3.4 in [3]).

Let $I_{P^*}^*(\chi_s) := \mathrm{Ind}_{P^*}^G(\chi_s \circ \omega_P)$. The $*$ indicates that we are inducing $\chi_s \circ \omega_P$, not $\chi_s \circ \omega_{P^*}$. The group M^{ab} acts on Y_P and Y_{P^*} on the left, and hence we obtain Mellin transforms

$$\begin{aligned} \mathcal{S}(Y_{P,P'}(\mathbb{A}_F)) &\longrightarrow I_P(\chi_s)|_{Y_P(\mathbb{A}_F)} \\ f &\longmapsto f_{\chi_s}(\cdot) := f_{\chi_s,P}(\cdot) := \int_{M^{ab}(F)} \delta_P^{1/2}(m)\chi_s(\omega_P(m))f(m^{-1}\cdot)dm, \\ \mathcal{S}(Y_{P^*,P'}(\mathbb{A}_F)) &\longrightarrow I_{P^*}^*(\chi_s)|_{Y_{P^*}(\mathbb{A}_F)} \\ f &\longmapsto f_{\chi_s}^*(\cdot) := f_{\chi_s,P^*}^*(\cdot) := \int_{M^{ab}(F)} \delta_{P^*}^{1/2}(m)\chi_s(\omega_P(m))f(m^{-1}\cdot)dm. \end{aligned} \tag{3.0.2}$$

Here δ_Q is the modular quasi-character of an algebraic group Q. The fact that the Mellin transform f_{χ_s} (resp. $f^*_{\chi_s}$) is absolutely convergent for $\mathrm{Re}(s_0)$ large (resp. $\mathrm{Re}(s_0)$ small) is built into the definition of the Schwartz space.

For $f_1 \in \mathcal{S}(Y_{P,P'}(\mathbb{A}_F))$, $f_2 \in \mathcal{S}(Y_{P^*,P'}(\mathbb{A}_F))$ define **generalized Schubert Eisenstein series**

$$\begin{aligned} E_{Y_P}(f_{1\chi_s}) &:= \sum_{y \in M^{ab}(F)\backslash Y_P(F)} f_{1\chi_s}(y), \\ E^*_{Y_{P^*}}(f^*_{2\chi_s}) &:= \sum_{y^* \in M^{ab}(F)\backslash Y_{P^*}(F)} f^*_{2\chi_s}(y^*). \end{aligned} \tag{3.0.3}$$

These sums converge absolutely for $\mathrm{Re}(s_0)$ sufficiently large (resp. small).

3.1 Main Theorem

Let M_{β_0} be the simple normal subgroup of the Levi subgroup M' of P'. For any topological abelian group A we denote by $\widehat{A}$ the set of quasi-characters of A, that is, continuous homomorphisms $A \to \mathbb{C}^\times$.

For any

$$(m, f, \chi, s) \in M_{\beta_0}(\mathbb{A}_F) \times \mathcal{S}(X_{P\cap M_{\beta_0}}(\mathbb{A}_F)) \times \widehat{A_{\mathbb{G}_m}F^\times\backslash\mathbb{A}_F^\times} \times \mathbb{C}$$

let $\chi_s := \chi|\cdot|^s$ and form the degenerate Eisenstein series

$$E(m, f_{\chi_s}) = \sum_{x \in (P\cap M_{\beta_0})\backslash M_{\beta_0}(F)} f_{\chi_s}(xm)$$

They converge for $\mathrm{Re}(s)$ large enough (resp. $\mathrm{Re}(s)$ small enough). Here f_{χ_s} is the Mellin transforms of (3.0.2) in the special case $P' = M_{\beta_0}$.

Let $K \leq M_{\beta_0}(\mathbb{A}_F)$ be a maximal compact subgroup. The following conjecture appeared in the statements of Theorem 1 :

Conjecture 1 *For each character $\chi \in \widehat{A_{\mathbb{G}_m}F^\times\backslash\mathbb{A}_F^\times}$ there is a finite set $\Upsilon(\chi) \subset \mathbb{C}$ such that if $E(m, f_{\chi_s})$ has a pole for any K-finite $f \in \mathcal{S}(X_{P\cap M_{\beta_0}}(\mathbb{A}_F))$ then $s \in \Upsilon(\chi)$.*

Remark 1 *Conjecture 1 is proved in [3] when M_{β_0} is SL_n.*

Theorem 1 *[3] Let* $f \in \mathcal{S}(Y_{P,P'}(\mathbb{A}_F))$*. Assume that* F *is a number field and Conjecture 1 is valid. Fix* $s_1, \ldots, s_k$ *with* $\mathrm{Re}(s_i)$ *sufficiently large. Then* $E_{Y_P}(f_{\chi_s})$ *and* $E_{Y_{P^*}}(\mathcal{F}_{P|P^*}(f)^*_{\chi_s})$ *are meromorphic in* s_0*. Moreover one has*

$$E_{Y_P}(f_{\chi_s}) = E^*_{Y_{P^*}}(\mathcal{F}_{P|P^*}(f)^*_{\chi_s}).$$

Proof We relate this problem to the program of Braverman, Kazhdan[1], Lafforgue[4], Ngô[5], and Sakellaridis [6] aimed at establishing generalizations of the Poisson summation formula. We prove the Poisson summation formula for certain schemes closely related to Schubert varieties and use it to refine and establish Theorem. See the details in [3]. □

Acknowledgement I would like to thank to organizers of the workshop for their kind invitation and hospitality while I was visiting RIMS, Japan.

References

[1] A. Braverman and D. Kazhdan , On the Schwartz space of the basic affine space, Selecta Mathematica. New Series, Vol5, (1999), No. 1, 1–28.

[2] D. Bump and Y. Choie, Schubert Eisenstein series, American Journal of Mathematics, Vol136, (2014), No.6, 1581–1608.

[3] Y. Choie and J. Getz, Schubert Eisenstein series and Poisson Summation for Schubert varieties, Preprint (2021).

[4] V. Lafforgue , Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale, Journal of the American Mathematical Society, Vol.31, (2018), No. 3, 719–891.

[5] B. C. Ngô, , Hankel transform, Langlands functoriality and functional equation of automorphic L-functions, Japanese Journal of Mathematics, Vol 15 (2020) No. 1, 121–167.

[6] Y. Sakellaridis, Spherical varieties and integral representations of L-functions , Algebra & Number Theory, Vol. 6, (2012), No. 4, 611–667.