SUBRIEMANNIAN GEOMETRIES ON $\mathbb{S}^7$ AND SPECTRAL ANALYSIS

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Abstract. This paper provides a short survey on the geometry and spectral analysis of four different subriemannian (SR) structures on $\mathbb{S}^7$. In two of those cases with defining distributions of rank four we discuss the nilpotentization, Popp's measure as well as the SR isometry group and we determine the intrinsic sublaplacians. As a result it is observed that these SR manifolds are neither locally isometric around any point nor isospectral in the sense of SR geometry. However, by changing the measure on $\mathbb{S}^7$ a spectral inclusion can be proven.

1. Introduction

This paper surveys results in [10, 11, 12] on the geometry and analysis of a family of subriemannian structures (SRS) defined on the 7-dimensional Euclidean sphere $\mathbb{S}^7$. In particular, we consider SR geometries induced by the Hopf fibration and quaternionic Hopf fibration on $\mathbb{S}^7$, respectively. Both constructions are rather standard in the literature and have been frequently studied under different aspects, see [6, 7, 21, 24, 25, 26]. We compare these examples with so-called *trivializable* SR geometries introduced and studied in [10]. Analogous constructions exit on the 3-sphere $\mathbb{S}^3$ but essentially all of them coincide in this low dimensional setting, [24, 25]. However, on $\mathbb{S}^7$ most of the above mentioned geometries are non-isometric in the sense of subriemannian geometry and provide an interesting class of examples.

To each of these SRS we can assign an intrinsic sublaplacian Δ_{sub}, which occasionally is also referred to as *hypoelliptic Laplacian*. In fact, this operator is subelliptic (therefore hypoelliptic) based on the bracket generating condition (a.k.a. *Hörmander condition*) of the distribution defining the SR geometry (see [3] or Section 2 for further details). In particular, it can be shown that Δ_{sub} in each of the previously mentioned cases has discrete spectrum consisting of eigenvalues with finite multiplicities. Hence we can define the notion of *isospectral* regular SR manifolds with respect to their intrinsic sublaplacians in analogy to the case of a Riemannian manifold.

Among our examples there are two SR structures with defining rank four bracket generating distributions $\mathcal{D}_{\text{QH}}$ and $\mathcal{D}^{(4)}$, respectively. More precisely, $\mathcal{D}_{\text{QH}}$ is defined as the horizontal space in the quaternionic Hopf fibration and $\mathcal{D}^{(4)}$ is a trivial bundle spanned by four canonical vector fields on $\mathbb{S}^7$ (see [1, 10]). By restricting the standard Riemannian metric on the 7-sphere to $\mathcal{D}_{\text{QH}}$ and $\mathcal{D}^{(4)}$ we obtain two SR geometries on $\mathbb{S}^7$ which we denote by $\mathbb{S}^7_{\text{QH}}$ and $\mathbb{S}^7_{\text{T}}$, respectively.

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The aim of the present paper is to compare $\mathbb{S}^7_{\rm QH}$ and $\mathbb{S}^7_{\rm T}$ under geometric and analytic aspects. We calculate the corresponding Popp measures and study properties of the nilpotent approximations at each point as well as the size of the SR isometry groups [26] in both cases. As a result it is observed that $\mathbb{S}^7_{\rm QH}$ and $\mathbb{S}^7_{\rm T}$ are locally non-isometric at any point in the sense of SR geometry [2, 5]. In the second part we discuss the spectra and heat invariants of the intrinsic sublaplacians $\mathbb{S}^7_{\rm QH}$ and $\mathbb{S}^7_{\rm T}$. Although some relations between these objects exist we mention that both manifolds are not isospectral in a SR sense [12]. In conclusion, different from what was observed on $\mathbb{S}^3$, most of the above SR structures on $\mathbb{S}^7$ are rather different from a geometric and analytic point of view.

The structure of the paper is as follows. In Section 2 we recall the construction of four SR structures on $\mathbb{S}^7$ in [6, 7, 10, 26]. Some of these geometries are rather standard in the literature but the bracket generating property of the defining distributions requires a proof (here we do not present the detailed arguments). Section 3 recalls the notion of *nilpotentization* and *SR isometry group*. We mention Popp's measure and the construction of the intrinsic sublaplacian on a regular SR manifold in [3]. In Section 4 we explicitly express these objects in case of $\mathbb{S}^7_{\rm QH}$ and $\mathbb{S}^7_{\rm T}$. In particular, we observe that these SR structures are not locally isometric around any point. Finally, in Section 5 we compare the spectra and first heat invariants of $\mathbb{S}^7_{\rm QH}$ and $\mathbb{S}^7_{\rm T}$. As a result we conclude that both SR manifolds are not isospectral. However, by changing to the standard Riemannian measure on $\mathbb{S}^7$ in case of $\mathcal{D}^{(4)}$ we obtain a spectral inclusion. The paper ends with some remarks on what is known about the heat kernel and spectra of the remaining two SR structures on $\mathbb{S}^7$ of rank 5 and 6 that were introduced in Section 2.

2. Subriemannian structures on $\mathbb{S}^7$

A subriemannian (SR) manifold is a triple $(M, \mathcal{D}, g)$, where M is a smooth, orientable manifold without boundary and $\dim M \geq 3$. Here $\mathcal{D}$ denotes a vector distribution inside the tangent bundle TM which is *bracket generating*. By this we mean that the Lie hull of all vector fields X taking values inside $\mathcal{D}$ evaluated at any point $p \in M$ coincides with the full tangent space T_pM at p. We call X a *horizontal* vector field. Moreover, $g = g_p$ denotes a family of inner products (called *subriemannian metric*) on $\mathcal{D}_p$ smoothly varying with the base point $p \in M$.

In this paper we fix M to be the Euclidean unit sphere $M = \mathbb{S}^7$ embedded into $\mathbb{R}^8$ in the usual way. The metric g is defined as the restriction to $\mathcal{D}$ of the standard Riemannian metric on the sphere. Hence different choices of a bracket generating distribution $\mathcal{D}$ will lead to a class of subriemannian structures on $\mathbb{S}^7$.

(Quaternionic) Hopf fibration: By $\mathbb{H} = \{q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \, : \, x_j \in \mathbb{R}\}$ we denote the quaterionic numbers. We may choose $\mathcal{D}$ as the horizontal space in the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^7 \to \mathbb{CP}^3$ or the quaterionic Hopf fibration $\mathrm{SU}(2) \to \mathbb{S}^7 \to \mathbb{HP}^1$, respectively. In the latter case we realize SU(2) as the group of unit quaternions

$$\mathrm{SU}(2) \cong \Big\{ q \in \mathbb{H} \, : \, \|q\|^2 := \sum_{j=0}^{3} x_j^2 = 1 \Big\} \cong \mathbb{S}^3,$$

which acts diagonally by left-multiplication on $\mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8 \supset \mathbb{S}^7$ leaving $\mathbb{S}^7$ invariant. More precisely, the horizontal distribution in each of the cases is the family of vector subspaces orthogonal to the fibers. In the following we denote these distributions as $\mathcal{D}_{\mathrm{H}}$ and $\mathcal{D}_{\mathrm{QH}}$, respectively.

Lemma 2.1 below is well-known. A short proof in case of the quaternionic Hopf fibration can be found in [8], see also [24, 25, 26].

Lemma 2.1. *The distributions $\mathcal{D}_{\mathrm{H}}$ and $\mathcal{D}_{\mathrm{QH}}$ are bracket generating and of step-two, i.e. horizontal vector fields X, Y together with their Lie brackets $[X, Y]$ span the tangent space at any point $q \in \mathbb{S}^7$.*

Trivializable subriemannian structures: We define a second pair of SR structures on $\mathbb{S}^7$ which have been introduced in [10]. They were called *trivializable* since the defining distributions are trivial as vector bundles. Let $\mathbb{S}^n$ denote the Euclidean sphere of dimension n and recall the following result, [1]:

Theorem 2.2 (J.F. Adams)**.** *The maximal dimension $\gamma(n)$ of a trivial subbundle of the tangent bundle $T\mathbb{S}^n$ is given by:*

$$\gamma(n) = 2^a + 8b - 1,$$

where $0 \leq a < 4$ and $0 \leq b$ are determined through the relation $n+1 = 2^{a+4b} \times [odd]$.

A set $\{X_1, \dots, X_{\gamma(n)}\}$ of linear vector fields X_j on $\mathbb{R}^{n+1} \supset \mathbb{S}^n$ that restrict to vector fields on $\mathbb{S}^n$ orthonormal at each point with respect to the induced standard Riemannian metric can be obtained as follows: consider a set of skew-symmetric matrices $A_\alpha \in \mathbb{R}^{(n+1)\times(n+1)}$, $\alpha = 1, \dots, \gamma(n)$ that fulfill the Clifford relations:

$$A_\alpha A_\beta + A_\beta A_\alpha = -2\delta_{\alpha\beta} I, \qquad \alpha, \beta = 1, \dots, \gamma(n). \tag{2.1}$$

With the standard coordinates $(x_1, \dots x_{n+1})$ of $\mathbb{R}^{n+1}$ we put

$$X_\alpha := \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^\alpha x_j \frac{\partial}{\partial x_j}, \qquad \textit{where} \qquad A_\alpha := (a_{ij}^\alpha) \in \mathbb{R}^{(n+1)\times(n+1)} \tag{2.2}$$

and we call X_α a *canonical vector field*. A direct calculation based on the Clifford relations (2.1) shows that iterated Lie brackets of canonical vector fields can be expressed by the canonical vector fields and their brackets of length two. More precisely,

$$\big[X_\alpha\big[X_\beta\big[X_\gamma \cdots\big]\big]\big] \in \operatorname{span}\Big\{X_i, \big[X_j, X_k\big] \;:\; i, j, k = 1, \cdots, \gamma(n)\Big\} = \mathcal{X}. \tag{2.3}$$

Let $2 \leq m \leq \gamma(n)$ and consider the trivial bundle

$$\mathcal{D}^{(m)} := \operatorname{span}\big\{X_1, \dots, X_m\big\} \subset T\mathbb{S}^7.$$

According to (2.3) above a necessary condition for $\mathcal{D}^{(m)}$ being bracket generating is

$$\rho(m) := |\mathcal{X}| = m + \binom{m}{2} \geq n.$$

In the case $m = \gamma(n)$ a complete list of dimensions $n \in \mathbb{N}$ such that $\rho := \rho(m) \geq n$ is given by the following table:

n	1	3	7	15	23	31	63
$\gamma(n)$	1	3	7	8	7	9	11
ρ	1	6	28	36	28	45	66

If $\mathcal{D}^{(m)}$ is bracket generating, then we call the induced SR structure on $\mathbb{S}^n$ trivializable. According to the above observation such structures can only exit in dimensions $n = 3, 7, 15, 23, 31, 63$. Moreover, in some cases the condition $\rho(m) \geq n$ could be fulfilled even if $m < \gamma(n)$. A classification of all spheres carrying a trivializable subriemannian structure via the above construction can be found in [10]. We recall the result:

Theorem 2.3 (W. Bauer, K. Furutani, C. Iwasaki)**.** *Trivializable SR structures on $\mathbb{S}^n$ via a Clifford module structure on $\mathbb{R}^{n+1}$ only exist in dimensions $n = 3, 7, 15$. On $\mathbb{S}^7$ the distributions $\mathcal{D}^{(m)}$ are bracket generating if and only if $m = 4, 5, 6, 7$.*

Throughout the paper we will focus on the SR manifold $\mathbb{S}^7_{\mathcal{D}} = (\mathbb{S}^7, \mathcal{D}, g)$ where the bracket generating distribution $\mathcal{D}$ is chosen from

$$\{\mathcal{D}_{\mathrm{H}}, \mathcal{D}_{\mathrm{QH}}, \mathcal{D}^{(4)}, \mathcal{D}^{(5)}\}. \tag{2.4}$$

In fact, it can be seen that $\mathcal{D}_{\mathrm{H}}$ and $\mathcal{D}^{(6)}$ essentially induce the same SR geometry on $\mathbb{S}^7$. Moreover, $\mathcal{D}^{(7)} = T\mathbb{S}^7$ yields the standard Riemannian structure on $\mathbb{S}^7$. Hence we have excluded the cases $m = 6, 7$ from the above list (2.4). We aim to compare two among the SR manifolds $\mathbb{S}^7_{\mathcal{D}}$ under geometric and spectral theoretical aspects. The detailed proofs and calculations will be omitted and can be found in [11, 12, 14].

3. Nilpotentization and intrinsic sublaplacian

To a (regular) SR manifold $(M, \mathcal{D}, h)$ we can associate a sheaf of graded nilpotent Lie algebras $\mathrm{Gr}(\mathcal{D})_q$ where $q \in M$ with Lie brackets induced by the brackets of vector fields. The Lie group $N_q = \exp(\mathrm{Gr}(\mathcal{D})_q)$ corresponding to $\mathrm{Gr}(\mathcal{D})_q$ is called the *nilpotentization* at $q \in M$ and it carries a naturally induced SR structure. In an appropriate sense N_q may be interpreted as a local model to the SR manifold M, see [26] for more details. Important information on the SR geometric structure of M can be obtained by determining this family of groups $(N_q)_{q \in M}$.

We shortly recall the construction of N_q in [26]. We may think of $\mathcal{D}$ as a sheaf of smooth horizontal (= tangent to $\mathcal{D}$) vector fields. Let $r \in \mathbb{N}$ and inductively define

$$\mathcal{D}^1 := \mathcal{D} \quad \textit{and} \quad \mathcal{D}^{r+1} := \mathcal{D}^r + \mathrm{span}\Big\{ \big[X, Y\big] \ : \ X \in \mathcal{D}, \ Y \in \mathcal{D}^r \Big\}.$$

One obtains a flag of sheafs of vector fields

$$\mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \ldots \subset \mathcal{D}^r \subset \mathcal{D}^{r+1} \subset \ldots \subset TM.$$

We write $\mathcal{D}^i_q$ for the evaluation of $\mathcal{D}^i$ at the point $q \in M$. The *bracket generating condition* on $\mathcal{D}$ means that for each $q \in M$ there is $r = r_q \in \mathbb{N}$, called the *step of nonholonomy*, such that $\mathcal{D}^{r_q}_q = T_q M$. If all the dimensions $\dim \mathcal{D}^i_q$ for each $i \in \mathbb{N}$ are constant under variation of the base point $q \in M$ we call the SR manifold *regular*. In particular, in case of a regular SR manifold M, the step of holonomy r_q does not depend on $q \in M$. Consider the vector space:

$$\mathrm{Gr}(\mathcal{D})_q = \mathcal{D}_q \oplus \mathcal{D}^2_q / \mathcal{D}_q \oplus \ldots \oplus \mathcal{D}^r_q / \mathcal{D}^{r-1}_q. \tag{3.1}$$

A Lie bracket on $\mathrm{Gr}(\mathcal{D})_q$ for each $q \in M$ is defined via brackets of vector fields on M and induces a graded Lie algebra structure. Identifying the first layer $\mathcal{D}_q$ of $\mathrm{Gr}(\mathcal{D})_q$ with a space of left-invariant vector fields the corresponding Lie group $N_q = \exp(\mathrm{Gr}(\mathcal{D})_q)$ is equipped with a left-invariant bracket generating distribution and a metric $\tilde{h} = h$. Hence to each point $q \in M$ we have assigned a graded nilpotent Lie group equipped with an induced SR structure, namely $(\mathrm{Gr}(\mathcal{D})_q, \mathcal{D}_q, \tilde{h}_q)$.

Definition 3.1 (see [2, 5])**.** Let $(M, \mathcal{D}, h)$ be a regular SR manifold. A diffeomorphism $\varphi : M \to M$ is called a *SR isometry* if its differential φ_* preserves the SR structure, i.e. for all $q \in M$:

(a) $\varphi_*(\mathcal{D}_q) = \mathcal{D}_{\varphi(q)}$

(b) $h_{\varphi(q)}(\varphi_* X, \varphi_* Y)) = h_q(X, Y)$ for all $X, Y \in \mathcal{D}_q$.

The collection $\mathrm{Iso}(M)$ of all SR isometries forms a subgroup of the diffeomorphism group which we call *SR isometry group*. More generally, we may define SR isometries between different SR manifolds and in an obvious way one defines the notion of *isometric* or *locally isometric* SR manifolds.

Proposition 3.2 ([5])**.** *If* $\mathrm{Iso}(M)$ *acts transitively on* M*, then there is a unique (up to multiplication by scalars) smooth measure* μ *on* M *such that*

$$\varphi^* \mu = \mu \quad \textit{for all} \quad \varphi \in \mathrm{Iso}(M). \tag{3.2}$$

In other words: each $\varphi \in \mathrm{Iso}(M)$ *is volume preserving.*

The existence of a smooth measure μ with (3.2) does not require the group $\mathrm{Iso}(M)$ to act transitively on M. A construction of such μ called *Popp's measure* is based on the bracket generating property of $\mathcal{D}$ as well as the regularity of the SR structure. Explicit formulas of *Popp's measure* in some cases can be found in [2, 5, 9, 26]. We recall that all SR structures on $M = \mathbb{S}^7$ considered in this paper are in fact regular so that the Popp measure exists (although the SR isometry group $\mathrm{Iso}(\mathbb{S}^7)$ does not act transitively in all cases).

Generalizing the Laplacian in Riemannian geometry we aim to assign a geometric operator Δ_{sub} to a SR manifold in an intrinsic way, i.e. only depending on the distribution $\mathcal{D}$ and the chosen SR metric h. The operator Δ_{sub} frequently is referred to as *sublaplacian* or *hypoelliptic Laplacian*. Here we recall the construction based on Popp's measure, see [3, 5] for further details.

Let ω and X be a smooth volume form and a vector field on M, respectively. Consider the *ω-divergence* div_ω and the *horizontal gradient* $\mathrm{grad}_{\mathcal{D}}$ which are defined through the following relations:

$$\mathcal{L}_X \omega = \mathrm{div}_\omega(X)\, \omega,$$

$$h_q\big(\mathrm{grad}_{\mathcal{D}}(\varphi), v\big) = d\varphi(v), \quad \textit{for all} \quad v \in \mathcal{D}_q$$

together with the horizontality condition $\mathrm{grad}_{\mathcal{D}}(\varphi) \in \mathcal{D}_q$. If M is regular with Popp measure $\omega = \mu$, then the intrinsic sublaplacian on M is defined by

$$\Delta_{\mathrm{sub}} = -\mathrm{div}_\mu \circ \mathrm{grad}_{\mathcal{D}}. \tag{3.3}$$

Let $[X_1, \ldots, X_m]$ with $m = \mathrm{rank}(\mathcal{D})$ be a local orthonormal frame of $\mathcal{D}$. Locally we can express Δ_{sub} as the sum of a *sum-of-squares* of vector fields and a first order operator, which depends on the measure μ:

$$\Delta_{\mathrm{sub}} = -\sum_{i=1}^{m} \big(X_i^2 + \mathrm{div}_\mu(X_i) X_i\big). \tag{3.4}$$

Note that Δ_{sub} is a positive, second order differential operator on $L^2(M, \mu)$. The bracket generating condition on $\mathcal{D}$ (also called *Hörmander condition*) combined with the representation (3.4) and a classical result by L. Hörmander in [22] implies *hypoellipticity* of the sublaplacian Δ_{sub}. Consider the heat operator P in SR geometry:

$$P := \frac{\partial}{\partial t} + \Delta_{\text{sub}} \quad \textit{on} \quad \mathbb{R}_+ \times M,$$

where $t \in \mathbb{R}_+$. Another application of [22] proves hypoellipticity of P on $\mathbb{R}_+ \times M$ and the existence of a smooth SR heat kernel K (= *fundamental solution* of P) is guaranteed. More precisely:

$$K(t,x,y) : \mathbb{R}_+ \times M \times M \to \mathbb{R} \tag{3.5}$$

is a smooth function with:

$$\begin{cases} PK(t,\cdot,y) = 0, & \textit{for all } t > 0,\ y \in M \\ \lim_{t\downarrow 0} K(t,x,\cdot) = \delta_x & \textit{in the sense of distributions.} \end{cases}$$

The SR heat kernel is of interest to us, since an explicit formula or its asymptotic properties as $t \downarrow 0$ provides important information on the spectrum of Δ_{sub}. It provides a link between geometric and analytic objects on M, see [4, 13, 15, 16, 23].

4. Rank-4 SR geometries on $\mathbb{S}^7$

In this section we compare the rank-4 SR structures on $\mathbb{S}^7$ induced by $\mathcal{D}_{\text{QH}}$ and $\mathcal{D}^{(4)}$, respectively. Recall that $\mathcal{D}_{\text{QH}}$ is the horizontal distribution in the quaternionic Hopf fibration whereas $\mathcal{D}^{(4)}$ is trivial as a vector bundle and spanned by globally define canonical vector fields.

Theorem 4.1 (Corollary 3.4 in [11]). *The distribution $\mathcal{D}_{\text{QH}}$ on $\mathbb{S}^7$ does not admit a nowhere vanishing and globally defined vector field (section of the bundle). In particular, $\mathcal{D}_{\text{QH}}$ is not trivial as a vector bundle.*

The SR manifolds $\mathbb{S}^7_{\text{QH}} := (\mathbb{S}^7, \mathcal{D}_{\text{QH}}, g)$ and $\mathbb{S}^7_T := (\mathbb{S}^7, \mathcal{D}^{(4)}, g)$ are not isometric in the sense of Definition 3.1. Below we will state an even stronger result. In [11] Popp's measures μ_{QH} and μ_{T} have been derived explicitly in case of $\mathbb{S}^7_{\text{QH}}$ and $\mathbb{S}^7_T$, respectively. The calculation is based on a formula in [5] and in both cases Popp's measure $\mu_\bullet$ has the form

$$\mu_\bullet(q) = \frac{1}{\sqrt{\det B_\bullet(q)}} d\sigma(q), \qquad q \in \mathbb{S}^7, \tag{4.1}$$

where $d\sigma$ denotes the standard Riemannian volume form on $\mathbb{S}^7$ and $B_\bullet$ is a certain matrix-valued function on $\mathbb{S}^7$ encoding the SR structure of $\mathbb{S}^7_{\text{QH}}$ and $\mathbb{S}^7_T$, respectively. We state the result:

Lemma 4.2 (Lemma 5.2 in [11]). *With respect to the standard inclusion $\mathbb{S}^7 \subset \mathbb{R}^8$ we split the coordinates $q = (x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \supset \mathbb{S}^7$. Then*

$$\mu_{\text{QH}}(q) = \frac{1}{16} d\sigma(q),$$
$$\mu_{\text{T}}(q) = \Big(16(1 - 2\|x\|^2\|y\|^2)\Big)^{-\frac{3}{2}} d\sigma(q),$$

where $\|x\|^2 = \sum_{i=1}^4 x_i^2$ denotes the square of the Euclidean norm of $x \in \mathbb{R}^4$.

It can be seen that the SR isometry group $\mathrm{Iso}(\mathbb{S}^7_{\mathrm{QH}})$ acts transitively on $\mathbb{S}^7$ and by a standard argument (s. [11, Lemma 5.1]) it follows that μ_{T} and $d\sigma$ are equal up to a constant factor $\lambda > 0$. The proof of Lemma 4.2 shows that this factor is $\lambda = \frac{1}{16}$. On the other hand, since $\mu_{\mathrm{QH}}(q) = \rho(q)d\sigma(q)$, where $\rho(q)$ is the non-constant function specified in Lemma 4.2, it follows:

Corollary 4.3. *The SR isometry group* $\mathrm{Iso}(\mathbb{S}^7_{\mathrm{T}})$ *does not act transitively on* $\mathbb{S}^7$.

Next, we consider the nilpotentizations of $\mathbb{S}^7_{\mathrm{QH}}$ and $\mathbb{S}^7_{\mathrm{T}}$. In both cases $\mathrm{Gr}(\mathcal{D})_q$ with $\mathcal{D} \in \{\mathcal{D}_{\mathrm{QH}}, \mathcal{D}^{(4)}\}$ is a graded nilpotent Lie algebra of step two, i.e.

$$\mathrm{Gr}(\mathcal{D})_q = \mathcal{D}_q \oplus \mathcal{D}^2_q/\mathcal{D}_q \tag{4.2}$$

has two levels for each $q \in \mathbb{S}^7$, (see (2.3) in case of the trivializable SR structure $\mathbb{S}^7_{\mathrm{T}}$.) In case of $\mathbb{S}^7_{\mathrm{QH}}$ the nilpotentization (4.2) at each point can be identified with the *quaterionic Heisenberg Lie algebra* of dimension seven, see [23, 20].

To analyze the nilpotentization $\mathrm{Gr}(\mathcal{D}^{(4)})_q$ we now introduce the notion of a *singular Carnot algebra*. Consider a Carnot algebra $(\mathfrak{g}, [\cdot,\cdot])$ of step 2. This means that $\mathfrak{g}$ can be expressed as a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \quad \textit{and} \quad [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}.$$

Let $\langle \cdot,\cdot \rangle$ denote an inner product on the first level $\mathfrak{g}_1$. Every element Z in the dual space $\mathfrak{g}_2^*$ induces a representation map

$$J_Z : \mathfrak{g}_1 \to \mathfrak{g}_1 \quad \textit{defined by} \quad \big\langle J_Z X, Y \big\rangle := Z\big([X,Y]\big), \quad X, Y \in \mathfrak{g}_1.$$

Definition 4.4. The Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is called *non-singular*, if for all $Z \in \mathfrak{g}_2^* \setminus \{0\}$ the induced map J_Z is invertible. Otherwise $\mathfrak{g}$ is called *singular*.

It has been shown in [11]:

Proposition 4.5 (Lemma 6.3 in [11])**.** *Let* $q = (x, y) \in \mathbb{S}^7 \subset \mathbb{R}^8 \cong \mathbb{R}^4 \times \mathbb{R}^4$. *Then the nilpotentization* $\mathrm{Gr}(\mathcal{D}^{(4)})_q$ *at* q *is non-singular if and only if* $\|x\| \neq \|y\|$.

Since the property of being non-singular is preserved under isometric Lie algebra isomorphisms and since the quaterionic Heisenberg Lie algebra is non-singular it follows that $\mathbb{S}^7_{\mathrm{QH}}$ and $\mathbb{S}^7_{\mathrm{T}}$ are not locally isometric as SR manifolds at $q = (x, y) \in \mathbb{S}^7$ with $\|x\| = \|y\|$. By refined arguments we can prove a stronger result:

Theorem 4.6 (Theorem 6.4 in [11])**.** *The SR manifolds* $\mathbb{S}^7_{\mathrm{QH}}$ *and* $\mathbb{S}^7_{\mathrm{T}}$ *are not locally isometric around any point* $q \in \mathbb{S}^7$.

5. Spectrum and first heat invariants

Now we determine the intrinsic sublaplacians $\Delta^{\mathrm{QH}}_{\mathrm{sub}}$ and $\Delta^{\mathrm{T}}_{\mathrm{sub}}$ on $\mathbb{S}^7_{\mathrm{QH}}$ and $\mathbb{S}^7_{\mathrm{T}}$ and we compare their spectra $\sigma(\Delta^{\mathrm{QH}}_{\mathrm{sub}})$ and $\sigma(\Delta^{\mathrm{T}}_{\mathrm{sub}})$, respectively. Since both manifolds are compact and based on subelliptic estimates it can be shown that both spectra consist of eigenvalues of finite multiplicities. Moreover, $\Delta^{\mathrm{QH}}_{\mathrm{sub}}$ commutes with the Laplacian $\Delta_{\mathbb{S}^7}$ and therefore leaves $\Delta_{\mathbb{S}^7}$-eigenspaces invariant.

Definition 5.1. We call two regular and compact SR manifolds *isospectral* if they have the same sets of eigenvalues of their intrinsic sublaplacian, when those are counted with multiplicities.

As was already mentioned a useful tool in the spectral analysis on a SR manifold is the small time asymptotic behaviour of the corresponding SR heat kernels.

In the first step we calculate the divergence $\mathrm{div}_{\mu_\bullet}$ for the Popp measures μ_{QH} and μ_{T} on $\mathbb{S}^7_{\mathrm{QH}}$ and $\mathbb{S}^7_{\mathrm{T}}$, respectively. In case of a volume form $\tilde{\mu}(q) = \tilde{\rho}(q)d\sigma(q)$ and $\tilde{\rho} \in C^\infty(\mathbb{S}^7)$ with $\tilde{\rho} > 0$ we have:

$$\mathrm{div}_{\tilde{\mu}}(X) = \mathrm{div}_\sigma(X) + X(\log \tilde{\rho}).$$

First we consider μ_{T} in Lemma 4.2. Since the canonical vector fields X_α in (2.2) are Killing vector fields and hence σ-divergence free we conclude that

$$\mathrm{div}_{\mu_T}(X_\alpha) = X_\alpha(\varphi) \quad \textit{with} \quad \varphi(q) := -\frac{3}{2}\log\big(1 - 2\|x\|^2\|y\|^2\big).$$

According to (3.4) it follows that the intrinsic sublaplacian on $\mathbb{S}^7_{\mathrm{T}}$ has a non-trivial first order term and it is given by:

$$\Delta_{\mathrm{sub}} = \Delta^{\mathrm{T}}_{\mathrm{sub}} = -\sum_{i=1}^4 \big(X_i^2 + X_i(\varphi)X_i\big). \tag{5.1}$$

Remark: By X_i, $i = 1, \dots, 4$ we denote canonical vector fields that are induced via (2.2) from a set of skew symmetric matrices $A_1, \dots, A_4$. Moreover A_j fulfill the Clifford relations (2.1) and $\mathcal{D}^{(4)} = \mathrm{span}\{X_1, \dots, X_4\}$. Although the operator $\Delta^{\mathrm{T}}_{\mathrm{sub}}$ in general depends on the choice of vector fields, its spectrum can be shown to be independent of the choice of Clifford generators $A_1, \dots, A_4$.

The intrinsic sublaplacian of $\mathbb{S}^7_{\mathrm{QH}}$ - roughly speaking - has the form (see [11, 6])

$$\Delta_{\mathrm{sub}} = \Delta^{\mathrm{QH}}_{\mathrm{sub}} = \Delta_{\mathbb{S}^7} + X^2 + Y^2 + W^2,$$

where X, Y, W are globally defined vector fields on $\mathbb{S}^7$ pairwise orthonormal at each point and tangent to the fibers of the quaternionic Hopf fibration. Here $\Delta_{\mathbb{S}^7} > 0$ denotes the standard Laplace-Beltrami operator on $\mathbb{S}^7$.

Let K_{T} and K_{QH} denote the SR heat kernels (3.5) on $\mathbb{S}^7_{\mathrm{T}}$ and $\mathbb{S}^7_{\mathrm{QH}}$, respectively. According to [15, 19] the existence of the small time asymptotic expansions (5.2) below is guaranteed at each point $q \in \mathbb{S}^7$:

$$\begin{aligned} K_{\mathrm{T}}(t,q,q) &= \frac{1}{t^5}\Big(c_0^{\mathrm{T}}(q) + c_1^{\mathrm{T}}(q)t + O(t^2)\Big), \qquad \textit{as} \quad t \downarrow 0, \\ K_{\mathrm{QH}}(t,q,q) &= \frac{1}{t^5}\Big(c_0^{\mathrm{QH}}(q) + c_1^{\mathrm{QH}}(q)t + O(t^2)\Big). \end{aligned} \tag{5.2}$$

We remark that the kernel K_{QH} has been calculated in [6] whereas no explicit expression of K_{T} seems to be known. In the following the coefficients $c_j^{\mathrm{T}}(q)$ and $c_j^{\mathrm{QH}}(q)$ in the asymptotic expansion will be called $(j+1)$-th *SR heat invariants.* The first heat invariants $c_0^{\mathrm{T}}(q)$ and $c_0^{\mathrm{QH}}(q)$ can be obtained from the nilpotentization of $\mathbb{S}^7_{\mathrm{T}}$ and $\mathbb{S}^7_{\mathrm{QH}}$ at q, respectively, together with a well-known integral representation of the heat kernel for the sublaplacian on step-2 nilpotent Lie groups (see [16]). The explicit formulas below have been obtained in Theorem 8.6 and Corollary 8.9 of [11].

Let $q = (x, y) \in \mathbb{S}^7 \subset \mathbb{R}^4 \times \mathbb{R}^4$, then:

$$c_0^{\mathrm{T}}(q) = \frac{1}{(2\pi)^5\rho(q)}\int_{\mathbb{R}^3} \frac{\|\tau\|}{\sinh\|\tau\|}\,\frac{(\|x\|^2 - \|y\|^2)\|\tau\|}{\sinh(\|x\|^2 - \|y\|^2)\|\tau\|}d\tau,$$

where

$$\rho(q) = \Big(16(1-2\|x\|^2\|y\|^2)\Big)^{-\frac{3}{2}}$$

denotes the function in Lemma 4.2. We observe that $c_0^{\mathrm{T}}(q)$ may vary with the point $q \in \mathbb{S}^7$. In case of $\mathbb{S}^7_Q$ the situation is different. Since the nilpotentization of $\mathbb{S}^7_Q$ is the same at each point we conclude that $c_0^{\mathrm{QH}}(q) \equiv c_0^{\mathrm{QH}}$ is *constant* on $\mathbb{S}^7$ with value

$$c_0^{\mathrm{QH}} = \frac{16^{\frac{3}{2}}}{(2\pi)^5} \int_{\mathbb{R}^3} \left(\frac{\|\tau\|}{\sinh \|\tau\|} \right)^2 d\tau.$$

The heat kernel and, in particular, the SR heat invariants provide important information on the spectrum of both sublaplacians. Since the manifold $\mathbb{S}^7$ is compact we can form the heat trace and consider its asymptotic expansion:

$$\sum_{j=1}^{\infty} m_j e^{-\lambda_j t} = \mathrm{Tr}\big(e^{-t\Delta_{\mathrm{sub}}}\big) \sim \frac{1}{t^5}\big(\alpha_0 + \alpha_1 t + O(t^2)\big) \qquad as \qquad t \downarrow 0.$$

Here $(\lambda_j)_{j=1}^{\infty}$ denotes the set of distinct eigenvalues of Δ_{sub} in increasing order with corresponding multiplicities $m_j \in \mathbb{N}$.

The coefficients α_j, $j = 0, 1, \ldots$ in case of the SR manifolds $\mathbb{S}^7_{\mathrm{T}}$ and $\mathbb{S}^7_{\mathrm{QH}}$ are obtained by integrating the SR heat invariants $c_j^{\mathrm{T}}(q)$ and $c_j^{\mathrm{QH}}(q)$ with respect to the corresponding Popp measures over the manifold $\mathbb{S}^7$. It is interesting to note [11, Remark 8.8] that c_0^{QH} solves a minimization problem, namely:

$$c_0^{\mathrm{QH}} = \inf \big\{ c_0^{\mathrm{T}}(q) \ : \ q \in \mathbb{S}^7 \big\}.$$

Moreover, based on the explicit integral expressions of the first heat invariants above one can check:

$$\kappa(q) := \rho(q) c_0^{\mathrm{T}}(q) - c_0^{\mathrm{QH}} \geq 0$$

and $\kappa > 0$ on an open subset of $\mathbb{S}^7$. In conclusion, the coefficients α_0 in the heat trace expansions of the intrinsic sublaplacians on $\mathbb{S}^7_{\mathrm{T}}$ and $\mathbb{S}^7_{\mathrm{QH}}$ do not coincide.

Corollary 5.2 (Corollary 8.9 of [11])**.** $\mathbb{S}^7_{\mathrm{QH}}$ *and* $\mathbb{S}^7_{\mathrm{T}}$ *are not isospectral.*

More can be said if we replace Popp's measure on $\mathbb{S}^7_{\mathrm{T}}$ by the standard Riemannian measure σ. With our previous notation the corresponding (non-intrinsic) sublaplacian is given by:

$$\tilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}} = -\mathrm{div}_\sigma \circ \mathrm{grad}_{\mathcal{D}^{(4)}} = -\sum_{j=1}^{4} X_i^2. \tag{5.3}$$

This operator and its spectrum have been studied earlier in [10, 11]. In particular, it was shown in [11, Theorem 9.3] that there is a (strict) spectral inclusion:

$$\sigma\big(\Delta^{\mathrm{QH}}_{\mathrm{sub}}\big) \subset \sigma\big(\tilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}}\big).$$

Based on an integral representation of the SR heat kernel K_{QH} in [6] one can calculate $\sigma(\Delta^{\mathrm{QH}}_{\mathrm{sub}})$ explicitly. However, the full spectrum of $\tilde{\Delta}^{\mathrm{T}}_{\mathrm{sub}}$ and properties of induced spectral functions (e.g. its spectral zeta function) seem to be unknown.

Final remarks: In this paper we have not discussed more in detail the SR manifolds $(\mathbb{S}^7, \mathcal{D}_{\mathrm{H}}, g)$ and $(\mathbb{S}^7, \mathcal{D}^{(5)}, g)$ corresponding to the rank 6 and rank 5 distributions $\mathcal{D}_{\mathrm{H}}$ and $\mathcal{D}^{(5)}$ in (2.4), respectively. We remark that in both cases the Popp measures are constant multiples of the standard volume form on $\mathbb{S}^7$ and the SR isometry groups act transitively. Explicit integral representations of the corresponding SR heat kernels K have been obtained, see [7, 12, 21]. One method of constructing K uses a suitable decomposition of the sublaplacian into commuting operators and their geometric interpretation together with a change from real to complex variables. These ideas were established in [6, 7] in the case of $\mathbb{S}^7_{\mathrm{H}}$ and $\mathbb{S}^7_{\mathrm{QH}}$ and generalize to spheres of dimensions $2n+1$ and $4n+3$, $n \in \mathbb{N}$, respectively. As an application, such heat kernel expressions can be used to obtain a full spectral decomposition of the intrinsic sublaplacians and an exact form of the fundamental solution of the so-called conformal sublaplacians. Details of the calculation in the case of the rank 5 trivializable case can be found in the forthcoming paper [12].

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