# SURJECTIVITY OF CONVOLUTION OPERATORS

TOMOYUKI KAKEHI

Abstract. In this article, we will explain the surjectivity of convolution operators on Euclidean spaces and on noncompact symmetric spaces. We will also give an application of our main result to PDE theory. Basically, this article is a brief summary of the joint work with Jens Christensen, Fulton Gonzalez, and Jue Wang. (See [CGK17] and [GWK21].)

## 1. Introduction

Let us start with a convolution operator on the Euclidean space. Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution on $\mathbb{R}^n$. Then a convolution operator $C_\mu : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is defined by

$$C_\mu f(x) := f * \mu(x) = \int_{\mathbb{R}^n} f(x-y)d\mu(y), \qquad \text{for } f \in C^\infty(\mathbb{R}^n). \tag{1.1}$$

We see easily that $C_\mu$ is well-defined as a continuous linear operator from $C^\infty(\mathbb{R}^n)$ to itself. The following are important examples of convolution operators.

**Example 1.1** (Mean value operators $M^r$ on $\mathbb{R}^n$)**.** For $r > 0$, we define the mean value operator $M^r$ on $C^\infty(\mathbb{R}^n)$ by

$$M^r f(x) := \frac{1}{\mathrm{Vol}(S_r)} \int_{x \in S_r} f(x+y)\, dS_r(y), \qquad \text{for } f \in C^\infty(\mathbb{R}^n), \tag{1.2}$$

where $S_r = \{y \in \mathbb{R}^n \,|\, |y| = r\}$ and $dS_r(y)$ denotes the canonical measure on $S_r$. If we take $\mu = \frac{1}{\mathrm{Vol}(S_r)}\delta_{S_r}$, then we see easily that $M^r f$ is rewritten as $M^r f = f * \mu = C_\mu f$. ($\delta_{S_r}$ denotes the delta density whose support is $S_r$.)

**Example 1.2** (Solution operator of the wave equation on $\mathbb{R}^n$)**.** Let $E$ be the solution to the wave equation

$$\partial_t^2 u - \triangle u = 0, \qquad u(0,x) = 0,\ u_t(0,x) = \delta(x), \tag{1.3}$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\delta$ denotes the delta density. Then the solution to the wave equation

$$\partial_t^2 u - \triangle u = 0, \qquad u(0,x) = f(x),\ u_t(0,x) = g(x), \tag{1.4}$$

is given by

$$u(t,x) = \partial_t E(t,\cdot) * f(x) + E(t,\cdot) * g(x). \tag{1.5}$$

---

*Date*: March 31, 2023.

2000 *Mathematics Subject Classification.* Primary: 33C67, 43A90, Secondary: 43A85.

*Key words and phrases.* convolution operator, surjectivity, noncompact symmetric spaces, slowly decreasing.

Here we note that

$$\mathrm{Supp}E(t,\cdot) \subset \{ x \in \mathbb{R}^n \mid |x| \leq |t| \}, \tag{1.6}$$

for each $t \in \mathbb{R}$. Therefore, $E(t,\cdot) \in \mathcal{E}'(\mathbb{R}^n)$. In particular, if $n \geq 3$ is odd and $f = 0$, then the solution operator is given by

$$u(t,x) = E(t,\cdot) * g(x) = \frac{1}{(n-2)!} \left( \frac{1}{t} \partial_t \right)^{\frac{n-3}{2}} \{ t^{n-2} M^{|t|} g(x) \}. \tag{1.7}$$

We also note that the mean value operator defined by (1.2) appears in the R.H.S. of (1.7).

Let us now consider the following.

<u>Problem</u>. When is the convolution operator $C_\mu$ surjective?

The difficulty lies in the fact that we can no longer consider $C_\mu$ to be a Fourier multiplier. In other words, the equality $\widehat{f * \mu} = \widehat{f}\,\widehat{\mu}$ does no longer hold for a general $f \in C^\infty(\mathbb{R}^n)$.

Ehrenpreis solved the above problem in [Ehr60]. From now on, we introduce a certain condition on entire functions on $\mathbb{C}^n$, and state the main theorem of [Ehr60].

**Definition 1.3.** An entire function $F$ on $\mathbb{C}^n$ is said to be *slowly decreasing* provided that there exist constants $A, B, C, D > 0$ such that

$$\sup\{ |F(\zeta)| \,;\, |\zeta - \xi| < A \log(2 + |\xi|) \} \geq B(C + |\xi|)^{-D} \tag{1.8}$$

for any point $\xi \in \mathbb{R}^n$.

Then Ehrenpreis proved the following.

**Theorem 1.4.** *(*[Ehr60]*; see also* [Hör05]*, Chapter XVI.) Let $\mu \in \mathscr{E}'(\mathbb{R}^n)$. Then the following two conditions on $\mu$ are equivalent:*

*(a) The convolution operator $C_\mu \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is surjective.*

*(b) The Fourier-Laplace transform $\widehat{\mu}$ is slowly decreasing.*

In the above theorem, the Fourier-Laplace transform $\widehat{\mu}$ is given by

$$\widehat{\mu}(\zeta) = \int_{\mathbb{R}^n} e^{-i\zeta\cdot x}\, d\mu(x), \qquad \text{for } \zeta \in \mathbb{C}^n. \tag{1.9}$$

As an application of Theorem 1.4, K. Lim showed the surjectivity of the mean value operator $M^r$ of Example 1.1 in his Ph. D. thesis [Lim12]. Following Lim's idea, we will prove that $M^r$ is surjective.

Let us put $\mu = \frac{1}{\mathrm{Vol}(S_r)} \delta_{S_r}$. Then the Fourier-Laplace transform $\widehat{\mu}$ is given by

$$\begin{aligned} \widehat{\mu}(\zeta) &= \frac{1}{\mathrm{Vol}(S_r)} \int_{x \in S_r} e^{-i\zeta\cdot x}\, dS_r(x) \\ &= \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\omega \in S^{n-1}} e^{-ir\zeta\cdot\omega}\, dS(\omega) = j_{\frac{1}{2}n-1}(r\sqrt{\zeta\cdot\zeta}), \end{aligned} \tag{1.10}$$

where $j_\nu(z)$ denotes the normalized Bessel function defined by

$$j_\nu(z) = \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left( \frac{z}{2} \right)^{2k}. \tag{1.11}$$

Here we note that in the R.H.S. of (1.10) $j_{\frac{1}{2}n-1}(r\sqrt{\zeta\cdot\zeta})$ is a holomorphic function of $\zeta\cdot\zeta = \sum_{j=1}^n \zeta_j^2$. Making use of the equality $\left(\frac{z}{2}\right)^\nu j_\nu(z) = \Gamma(\nu+1)J_\nu(z)$ and the asymptotic expansion formula for Bessel function $J_\nu(z)$, we have

$$j_\nu(z) \sim \sqrt{\frac{2}{\pi}}2^\nu\Gamma(\nu+1)\,|z|^{-\frac{\nu+1}{2}}\cos(\,z - \frac{\nu\pi}{2} - \frac{\pi}{4}) \qquad \text{as } |z|\to\infty, \tag{1.12}$$

which means that $\widehat{\mu}(\zeta)$ has a polynomial decay at infinity. In particular, $\widehat{\mu}(\zeta)$ is slowly decreasing. Hence, by Theorem 1.4, we have

**Theorem 1.5** ([Lim12]). *The mean value operator $M^r : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is surjective.*

*Remark* 1.6. K. Lim also obtains the surjectivity of the mean value operator on the 3-dimensional hyperbolic space $\mathbb{H}^3$ in his thesis [Lim12].

We will explain one more application of Theorem 1.4. Let $P(\zeta)$ be a nonzero polynomial of $\zeta\in\mathbb{C}^n$ and let $P(D)$ be the corresponding constant coefficient differential operator on $\mathbb{R}^n$. Here $D = \frac{1}{i}(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n})$. Then it was a famous but very difficult problem to show the surjectivity of $P(D) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, and finally Ehrenpreis and Malgrange solved the problem at the same time. (See [Ehr54], [Ehr56] and [Mal54].) If we use Theorem 1.4, then this problem is solved in the following way. Let $\mu = P(D)\delta \in \mathcal{E}'(\mathbb{R}^n)$, then we see that $P(D)$ is written as the convolution operator $C_\mu$. In addition, the Fourier-Laplace transform $\widehat{\mu}(\zeta)$ of $\mu$ is given by the polynomial $P(\zeta)$, which is obviously slowly decreasing. Thus by Theorem 1.4, $P(D) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is surjective. In fact, after Ehrenpreis proved the surjectivity of $P(D)$, he continued his research on this subject, and finally he obtained Theorem 1.4.

## 2. Convolution operators on the Poincaré disk

In the introduction, we explained the problem of surjectivity for convolution operators on $\mathbb{R}^n$. We consider the same problem for convolution operators on general noncompact symmetric spaces. However, for the sake of simplicity, we first deal with the case of the Poincaré disk.

Let $\mathcal{D}$ be the Poincaré disk, namely, the unit disk $\{\,z\in\mathbb{C}\,|\,|z|<1\,\}$ with the Poincaré metric $ds^2 = (1-x^2-y^2)^{-2}(dx^2+dy^2)$, where $z = x+iy \in \mathcal{D}$. Let $r = \mathrm{dist}(z_1,z_2)$ for $z_1,z_2\in\mathcal{D}$. Then $r$ is given explicitly by the formula

$$\frac{|z_1-z_2|}{|1-\bar{z}_1 z_2|} = \tanh r. \tag{2.1}$$

Let

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b\in\mathbb{C},\ |a|^2-|b|^2 = 1 \right\}, \tag{2.2}$$

$$K = S(U(1)\times U(1)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \,\middle|\, |a| = 1 \right\}. \tag{2.3}$$

As is easily seen, $G$ acts transitively on $\mathcal{D}$ by

$$g\cdot z = \frac{az+b}{\bar{a}z+\bar{b}}, \qquad \text{for } g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G, \quad z\in\mathcal{D}. \tag{2.4}$$

and $K$ is the stabilizer of the origin $o$. Therefore, we can identify $\mathcal{D}$ with the quotient space $G/K$, which turns out to be a noncompact symmetric space. Here we note that in (2.4) the map $\mathcal{D} \ni z \mapsto g \cdot z \in \mathcal{D}$ is an isometry. Now we define a convolution on $\mathcal{D}$. Using the action of $G$ on $\mathcal{D}$, a function on $\mathcal{D}$ can be viewed as a funtion on $G$. More precisely, we correspond a function $f$ on $\mathcal{D}$ to a function $g \mapsto f(g \cdot o)$ on $G$. (In other words, the function $g \mapsto f(g \cdot o)$ is the pull back of $f$ by the canonical projection $G \to G/K = \mathcal{D}$.) Using the above identification and the convolution on $G$, we define the convolution $f_1 * f_2$ of $f_1,\ f_2 \in C_0^\infty(\mathcal{D})$ by

$$f_1 * f_2(z) = \int_{h \in G} f_1(h \cdot o) f_2(h^{-1} g \cdot o)\, dh, \qquad \text{for } \ z = g \cdot o. \tag{2.5}$$

Here we note that the equality $f_1 * f_2 = f_2 * f_1$ no longer holds for general $f_1,\ f_2 \in C_0^\infty(\mathcal{D})$ due to the fact that $G$ is non-commutative.

In a similar way to (2.5), for $f \in C^\infty(\mathcal{D})$ and a compactly supported distribution $\mu \in \mathcal{E}'(\mathcal{D})$, we define the convolution $f * \mu$. Finally, we define the convolution operator $C_\mu : C^\infty(\mathcal{D}) \to C^\infty(\mathcal{D})$ by $C_\mu f = f * \mu$.

As an important example of the convolution operator on $\mathcal{D}$, we consider the mean value operator. For a fixed $r > 0$, let

$$M^r f(z) = \frac{1}{\mathrm{Vol}(S_r(z))} \int_{w \in S_r(z)} f(w)\, dS_r(w), \qquad \text{for } \ z \in \mathcal{D},\ f \in C^\infty(\mathcal{D}), \tag{2.6}$$

where $S_r(z) = \{w \in \mathcal{D} \,|\, \mathrm{dist}(w, z) = r\}$ and $dS_r(w)$ denotes the canonical measure on $S_r(z)$. Then $M^r$ is defined as a continuous linear operator on $C^\infty(\mathcal{D})$. Here we take $\mu = \mathrm{Vol}(S_r(o))^{-1} \delta_{S_r(o)} \in \mathcal{E}'(\mathcal{D})$, where $\delta_{S_r(o)}$ denotes the delta density whose support is $S_r(o)$. Then we see easily that $M^r f = C_\mu f$. Moreover, we have the following.

**Theorem 2.1** ([CGK17]). *$M^r : C^\infty(\mathcal{D}) \to C^\infty(\mathcal{D})$ is surjective.*

The above theorem is a special case of the main result of our paper [CGK17]. See Theorem 3.1 below.

## 3. Fourier-Laplace transform on the Poincaré disk

In this section, we introduce the Fourier-Laplace transform on the Poincaré disk $\mathcal{D}$ in order to state the surjectivity theorem for convolution operators on $\mathcal{D}$.

For $z \in \mathcal{D}$ and $b \in \partial\mathcal{D} \cong S^1$, we define $\langle z, b\rangle$ by

$$e^{2\langle z,b\rangle} = \frac{1 - |z|^2}{|z - b|^2}. \tag{3.1}$$

We note that $\langle z, b\rangle$ is constant on a given horocycle tangent to $b \in \partial\mathcal{D}$. For this reason, the function $\mathcal{D} \ni z \mapsto e^{(i\lambda+1)\langle z,b\rangle}$ can be regarded as a *plane wave* on $\mathcal{D}$ for $\lambda \in \mathbb{R}$. As is well known, a plane wave $\mathbb{R}^n \ni x \mapsto e^{ir\omega\cdot x}$ $(r \in \mathbb{R}, \omega \in \mathbf{S}^{n-1})$ on $\mathbb{R}^n$ satisfies $\triangle_{\mathbb{R}^n} e^{ir\omega\cdot x} = -r^2 e^{ir\omega\cdot x}$. Similarly a plane wave on $\mathcal{D}$ satisfies

$$\triangle_{\mathcal{D}}\, e^{(i\lambda+1)\langle z,b\rangle} = -(\lambda^2 + 1)\, e^{(i\lambda+1)\langle z,b\rangle}, \tag{3.2}$$

where $\triangle_{\mathcal{D}}$ denotes the Laplace-Beltrami operator on $\mathcal{D}$ corresponding to the Poincaré metric. Using the above Poincaré disk analogue of plane waves, for $f \in C_0^\infty(\mathcal{D})$ , we define a function $\mathcal{F}f(\lambda, b)$ of $(\lambda, b) \in \mathbb{R} \times \partial\mathcal{D}$ by

$$\mathcal{F}f(\lambda, b) = \int_{\mathcal{D}} e^{(-i\lambda+1)\langle z,b\rangle}\, f(z)\, dm(z), \tag{3.3}$$

where $dm(z) = (1-|z|^2)^{-2}dxdy$ is the canonical measure on $\mathcal{D}$. We call the above defined function $\mathcal{F}f$ on $\mathbb{R} \times \partial\mathcal{D}$ the Fourier transform of $f$.

For $f \in C^\infty(\mathcal{D})$ and $g \in G$, we put $\rho_g f(z) := f(g^{-1} \cdot z)$. We say $f$ is $K$-invariant if $\rho_k f = f$ for any $k \in K$. If $f \in C_0^\infty(\mathcal{D})$ is $K$-invariant, we see that $\mathcal{F}f$ is also $K$-invariant in the sense that $\mathcal{F}f(\lambda, k \cdot b) = \mathcal{F}f(\lambda, b)$ for any $k \in K$ and for any $(\lambda, b) \in \mathbb{R} \times \partial\mathcal{D}$. In other words, if $f \in C_0^\infty(\mathcal{D})$ is $K$-invariant, $\mathcal{F}f(\lambda, b)$ is independent of $b$. So for simplicity, for a $K$-invariant function $f \in C_0^\infty(\mathcal{D})$, we write the Fourier transform of $f$ as $\mathcal{F}f(\lambda)$. Thus for a $K$-invariant function $f \in C_0^\infty(\mathcal{D})$, $\mathcal{F}f(\lambda)$ is extended to an entire function of $\lambda \in \mathbb{C}$, which we call the Fourier-Laplace transform of $f$. In the same way, a $K$-invariant distribution on $\mathcal{D}$ is defined. In addition, if $\mu \in \mathcal{E}'(\mathcal{D})$ is $K$-invariant, then the Fourier-Laplace transform $\mathcal{F}\mu(\lambda)$ is similarly defined as an entire function of $\lambda \in \mathbb{C}$.

Let us now state the following.

**Theorem 3.1.** *For a $K$-invariant distribution $\mu \in \mathcal{E}'(\mathcal{D})$, the convolution operator $C_\mu : C^\infty(\mathcal{D}) \to C^\infty(\mathcal{D})$ is surjective if and only if $\mathcal{F}\mu(\lambda)$ is slowly decreasing.*

The above theorem is a special case of the results of [CGK17] and [GWK21]. See Theorem 4.1 below. In particular, as a corollary of Theorem 3.1, we obtain Theorem 2.1. For the detail of the proof, see [CGK17].

## 4. Convolution operators on noncompact symmetric spaces

In this section, we will state our main theorem on the surjectivity of convolution operators on noncompact symmetric spaces.

Let $X = G/K$ be a noncompact Riemannian symmetric space, where $G$ is a noncompact real semisimple Lie group and $K$ is a maximal compact subgroup of $G$. Similarly as in the case of the Poincaré disk, a function $f$ on $X$ can be regarded as a function on $G$. Then, for $f_1, f_2 \in C_0^\infty(X)$, the convolution $f_1 * f_2$ of $f_1$ and $f_2$ is defined by

$$f_1 * f_2(x) = \int_{h\in G} f_1(hK) f_2(h^{-1}gK)\, dh, \qquad \text{for} \quad x = gK \in G/K. \tag{4.1}$$

In the same manner as in (4.1), for a smooth function $f$ on $X$ and a compactly supported distribution $\mu$ on $X$, the convolution $f * \mu$ is defined and becomes a smooth function on $X$. So we define the convolution operator $C_\mu : C^\infty(X) \to C^\infty(X)$ by $f \mapsto f * \mu$.

Next, we will briefly explain the Fourier-Laplace transform on $X$. Let $G = NAK$ be the corresponding Iwasawa decomposition, and let $A = \exp \mathfrak{a}$. For each $g \in G$, we accordingly write $g = n(g) \exp A(g) k(g)$, where $n(g) \in N, A(g) \in \mathfrak{a}, k(g) \in K$. Here we note that $\mathrm{rank} X = \ell$ means $\mathfrak{a} \cong \mathfrak{a}^* \cong \mathbb{R}^\ell$. So if we denote the complexification of $\mathfrak{a}^*$ by $\mathfrak{a}^*_{\mathbb{C}}$, then $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^\ell$. Next, let $B = \partial X$ be the boundary of $X$. For $x \in X$ and $b \in B$, we denote by $A(x, b) \in \mathfrak{a}$ the "directed distance" from the origin of $X$ to the horocycle which passes through $x$ and is normal to $b$. Then for $f \in C_0^\infty(X)$, the Fourier transform $\mathcal{F}f$ of $f$ is defined by

$$\mathcal{F}f(\lambda, b) = \int_X e^{(-i\lambda+\rho)A(x,b)} f(x)\, dx \qquad \text{for} \ (\lambda, b) \in \mathfrak{a}^* \times B, \tag{4.2}$$

where $\rho = \frac{1}{2} \sum_{\alpha:\text{positive root}} m_\alpha \alpha$. (We omit the definition of the root system of a symmtric space.) For the details on symmetric spaces, see Helgason's books, [Hel01], [Hel00], and [Hel08]. For a compactly supported distribution $\mu$, the Fourier transform $\mathcal{F}\mu$ is defined in the same manner. Let us denote the space of compactly supported and $K$-invariant distributions on $X$ by $\mathcal{E}_K{}'(X)$. If $\mu \in \mathcal{E}_K{}'(X)$, then $\mathcal{F}\mu$ is also $K$-invariant in the sense that $\mathcal{F}\mu(\lambda, k \cdot b) = \mathcal{F}\mu(\lambda, b)$ for any $k \in K$ and any $(\lambda, b) \in \mathfrak{a}^* \times B$. Thus, $\mathcal{F}\mu(\lambda, b)$ is independent of $b$ and is regarded as a function of $\lambda$. So let us write $\mathcal{F}\mu(\lambda)$ for $\mu \in \mathcal{E}_K{}'(X)$. We see easily that $\mathcal{F}\mu(\lambda)$ is extended to an entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, which we call the Fourier-Laplace transform of $\mu$.

Let us now state our main theorem.

**Theorem 4.1.** [CGK17], [GWK21] *Let $\mu \in \mathcal{E}_K{}'(X)$. Then the following two conditions on $\mu$ are equivalent:*

*(a) The convolution operator $C_\mu \colon C^\infty(X) \to C^\infty(X)$ is surjective.*
*(b) The Fourier-Laplace transform $\mathcal{F}\mu$ is slowly decreasing.*

*Remark* 4.2. Theorem 4.1 is a symmetric space analogue of Theorem 1.4. In Theorem 4.1, the proof of $(b) \Longrightarrow (a)$ is given in [CGK17], and the proof of $(a) \Longrightarrow (b)$ is given in [GWK21].

As an application of Theorem 4.1, we have the following.

**Corollary 4.3.** [Hel73] *Let $P(\neq 0)$ be a $G$-invariant differntial operator on $X$. Then $P : C^\infty(X) \to C^\infty(X)$ is surjective.*

This is one of the main results by Helgason [Hel73]. Let $o$ denote the origin of $X$and $\delta_o$ denote the delta density with support $o$. Then $Pf = f * (P\delta_o)$. In addition, we see that $P\delta_o \in \mathcal{E}_K{}'(X)$. On the other hand, as is well known in harmonic analysis on symmetric spaces, there exists a polynomial $\Gamma_P$ on $\mathfrak{a}_{\mathbb{C}}^*$ such that $\mathcal{F}(P\delta_o)(\lambda) = \Gamma_P(\lambda), \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$. In particular, $\mathcal{F}(P\delta_o)$ is slowly decreasing, so Theorem 4.1 applies.

*Acknowledgments.* I would like to thank the organizers of the conference "Geometric Structures and Differential Equations" Professor Daisuke Tarama, Professor Kenro Furutani, and Professor Hiroaki Yoshimura for giving me an opportunity to talk about our results.

## References

[CGK17] Jens Christensen, Fulton Gonzalez, and Tomoyuki Kakehi. Surjectivity of mean value operators on noncompact symmetric spaces. *J. Funct. Anal.*, 272(9):3610–3646, 2017.

[Ehr54] Leon Ehrenpreis. Solution of some problems of division. I. Division by a polynomial of derivation. *Amer. J. Math.*, 76:883–903, 1954.

[Ehr56] Leon Ehrenpreis. Solutions of some problems of division. III. Division in the spaces, $\mathcal{D}', \mathcal{H}, \mathcal{Q}_A, \mathcal{O}$. *Amer. J. Math.*, 78:685–715, 1956.

[Ehr60] Leon Ehrenpreis. Solution of some problems of division. IV. Invertible and elliptic operators. *Amer. J. Math.*, 82:522–588, 1960.

[GWK21] Fulton Gonzalez, Jue Wang, and Tomoyuki Kakehi. Surjectivity of convolution operators on noncompact symmetric spaces. *J. Funct. Anal.*, 280(2):Paper No. 108805, 31, 2021.

[Hel73] Sigurdur Helgason. The surjectivity of invariant differential operators on symmetric spaces. I. *Ann. of Math. (2)*, 98:451–479, 1973.

[Hel00] Sigurdur Helgason. *Groups and geometric analysis*, volume 83 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[Hel08] Sigurdur Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2008.

[Hör05] Lars Hörmander. *The analysis of linear partial differential operators. II.* Classics in Mathematics. Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients, Reprint of the 1983 original.

[Lim12] Kyung-Taek Lim. *The Spherical Mean Value Operators on Euclidean and Hyperbolic Spaces.* PhD thesis, Tufts University, Medford, MA, 2012.

[Mal54] Bernard Malgrange. Equations aux dérivées partielles à coefficients constants. II. Equations avec second membre. *C. R. Acad. Sci. Paris*, 238:196–198, 1954.

University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571 Japan

*Email address*: kakehi@math.tsukuba.ac.jp