Ray-Singer torsion and the Rumin Laplacian on lens spaces

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Abstract

The Rumin complex is the Bernstein-Gelfand-Gelfand complex (BGG complex) of the twisted de Rham complex of a flat vector bundle with respect to contact manifolds. As a typical theorem, the cohomology of the BGG complex coincides with the cohomology of the de Rham complex of a flat vector bundle. Moreover, the Rumin complex arises when we take the sub-Riemannian limit.

Let us consider what happens when we replace a concept defined using the de Rham complex with the Rumin complex. In this talk, we adapt this idea to analytic torsion. On flat vector bundles with a unitary holonomy over lens spaces, we express explicitly the analytic torsion functions associated with the Rumin complex in terms of the Hurwitz zeta function. In particular, we determine the analytic torsions, and it is written using the Betti numbers and the Ray-Singer torsion.

1 What is the Rumin complex?

1.1 Bernstein-Gelfand-Gelfand complex

The Bernstein-Gelfand-Gelfand sequence (BGG sequence) $(\mathcal{E}^\bullet(M,E), D)$ is defined for parabolic geometry on the twisted de Rham complex due to Čap-Slovák-Souček [5] and Calderbank-Dimer [4]. Rumin has also introduced a non G-invariant version in the context of sub-Riemannian geometry [21], which coincides with the Rumin complex [20] on contact manifolds (e.g. [23, §5.3], [7, §4]). Dave and Haller generalized the differential operator on filtered vector bundle with codifferentials of Kostant type [7]. If the BGG sequence is complex, we call it the BGG complex. As a typical theorem, on flat vector bundles, the cohomology of the BGG complex coincides with the cohomology of the de Rham complex of a flat vector bundle [5, Theorem 4.13], [4, Theorem 3.6], [7, Corollary 4.20]. This claim is a generalization of the result of the Rumin complex [20].

A filtered manifold is a smooth manifold M whose tangent bundle comes equipped with a filtration by smooth subbundles,

$$TM = T^{-r}M \supset \cdots \supset T^{-2}M \supset T^{-1}M \supset T^0M = \{0\},$$

which is compatible with the Lie bracket in the following sense: if $X \in \mathcal{C}^\infty(M, T^pM)$ and $Y \in \mathcal{C}^\infty(M, T^qM)$, then $[X,Y] \in \mathcal{C}^\infty(M, T^{p+q}M)$. We call a turple $(M, g, \{T^\bullet M\})$ a filtered Riemannian manifold if (M,g) is a Riemannian manifold which has a filtrattion

$\{T^\bullet M\}$, and there exist subbundles $\{\mathfrak{t}^p M\} \subset TM$ which satisfy

$$T^{-p}M = \bigoplus_{i=1}^{p} \mathfrak{t}^{-i}M \text{ and } g(\mathfrak{t}^i M, \mathfrak{t}^j M) = \{0\} \text{ for } i \neq j.$$

Since for all $X \in \mathcal{C}^\infty(M, T^pM)$ and $Y \in \mathcal{C}^\infty(M, T^qM)$ and $f \in \mathcal{C}^\infty(M)$,

$$[fX, Y] = f[X,Y] + (Yf)X = f[X,Y] \mod \mathcal{C}^\infty(M, T^{p+q}M),$$

the bracket on TM induces the smooth bracket $[\quad , \quad]_0$ on T_xM for $x \in M$ such that

$$[X,Y]_0 = [X,Y] \mod \mathcal{C}^\infty(M, T^{p+q}M).$$

A filtered vector bundle over a filtered manifold M is a vector bundle E over M which comes equipped with a filtration by smooth subbundles,

$$E = E^{-r_1} \supset E^{-r_1+1} \supset \cdots \supset E^{r_1} = \{0\}.$$

We call $\nabla\colon \mathcal{C}^\infty(M,E) \to \Omega^1(M,E)$ a filtration preserving connection on a filtered vector bundle E over a filtered manifold M if $\nabla_X \phi \in \mathcal{C}^\infty(M, E^{p+q})$ for all $X \in \mathcal{C}^\infty(M, T^pM)$ and $\phi \in \mathcal{C}^\infty(M, E^q)$. Let g_E be a metric of E. We set

$$\begin{aligned}
\mathrm{gr}_p(E) &:= E^p \cap (E^{p+1})^\perp, \\
\mathrm{gr}_p\left(\textstyle\bigwedge^k T^\vee M \otimes E\right) &:= \bigoplus_{p_1+\cdots+p_k-p_{k+1}=-p} \mathfrak{t}^{p_1\vee}M \wedge \cdots \wedge \mathfrak{t}^{p_k\vee}M \otimes \mathrm{gr}_{p_{k+1}} E, \\
\mathrm{gr}^p\left(\textstyle\bigwedge^k T^\vee M \otimes E\right) &:= \bigoplus_{q \geq p} \mathrm{gr}_q\left(\textstyle\bigwedge^k T^\vee M \otimes E\right).
\end{aligned}$$

Let $\mathrm{gr}_p\colon \bigwedge^k T^\vee M \otimes E \to \mathrm{gr}_p\left(\bigwedge^k T^\vee M \otimes E\right)$ be the fiberwise orthogonal projection with respect to the metrics g and g_E. For all linear operator $A\colon \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$, we set $\mathrm{gr}_p(A) := \sum_q \mathrm{gr}_{p+q} \circ A \circ \mathrm{gr}_q$, and A is called filtration preserving if $A\, \mathrm{gr}^q\, \Omega^\bullet(M,E) \subset \mathrm{gr}^q\, \Omega^\bullet(M,E)$. We set

$$d_0^\nabla := \mathrm{gr}_0(d^\nabla).$$

By Leibniz' rule, for $f \in \mathcal{C}^\infty(M)$, $u \in \mathrm{gr}_p(\Omega^k(M,E))$,

$$d_0^\nabla f\phi = \mathrm{gr}_p(df \wedge \phi + f d^\nabla \phi) = f d_0^\nabla \phi.$$

It means that d_0^∇ is a smooth bundle map. Henceforth, we assume that d_0^∇ is locally constant rank.

We set

$$\Box_0 := d_0^\nabla d_0^{\nabla *} + d_0^{\nabla *} d_0^\nabla. \quad \Box := d^\nabla d_0^{\nabla *} + d_0^{\nabla *} d^\nabla,$$

where $*$ is adjoint with respect to the metric g and g_E. Since d_0^∇ is locally constant rank, we define a subbundle of $\bigwedge^k T^\vee M \otimes E$ by for $x \in M$,

$$H^k(\mathfrak{t}M \otimes E)_x := \mathrm{Ker}\left(\Box_0\colon \textstyle\bigwedge^k T_x^\vee M \otimes E_x \to \bigwedge^k T_x^\vee M \otimes E_x\right),$$

and the fiberwise smooth projection $\Pi_{\mathcal{E}}\colon \bigwedge^k T^\vee M \otimes E \to H^k(\mathfrak{t}M \otimes E)$ is defined. We set

$$\mathcal{E}^k(M,E) := \mathcal{C}^\infty(M, H^k(\mathfrak{t}M \otimes E)).$$

Proposition 1.1. ([7, Lemma 4.4, Proposition 4.5])

Let ∇ be a flat filtration preserving connection on a filtrated vector bundle E with a metric g_E over a filtered Riemannian manifold $(M, g, \{T^\bullet M\})$. Assume that d_0^∇ is locally constant rank.

Then, there exists a unique filtration preserving differential operator $P\colon \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$ such that $P^2 = P$, $P\Box = \Box P$, $\mathrm{gr}_0(P) = \Pi_{\mathcal{E}}$.

In [11, page 29], Haller pointed out P coincides with Π_E in [21, Theorem 1, Lemma 1]. We set

$$d_0^\dagger := \begin{cases} d_0^* \Box_0^{-1}, & \text{on } \operatorname{Im} d_0, \\ 0, & \text{otherwise,} \end{cases}$$

and the nilpotent operator $N := d_0^{\nabla\dagger}(d^\nabla - d_0^\nabla)$. The operator P is given by

$$P = \mathrm{Id} - (\mathrm{Id} + N)^{-1} d_0^{\nabla\dagger} d^\nabla - d^\nabla (\mathrm{Id} + N)^{-1} d_0^\dagger. \tag{1.1}$$

We define the operator L by

$$L := P\Pi_{\mathcal{E}} + (1-P)(1-\Pi_{\mathcal{E}}).$$

We define the Bernstein-Gelfand-Gelfand operator (BGG operator) D by

$$D := \Pi_{\mathcal{E}} L^{-1} d^\nabla L \Pi_{\mathcal{E}}.$$

Proposition 1.2. ([7, Proposition 4.5, Corollary 4.20])

Let ∇ be a flat filtration preserving connection on a filtrated vector bundle E with a metric g_E over a filtered Riemannian manifold $(M, g, \{T^\bullet M\})$. Assume that d_0^∇ is locally constant rank.

Then $D^2 = 0$ and $L\colon \mathcal{E}^\bullet(M,E) \to \Omega^\bullet(M,E)$ provides a chain map, $d^\nabla L = LD$, inducing an isomorphism between the cohomologies of $(\Omega^\bullet(M,E), d)$ and $(\mathcal{E}^\bullet(M,E), D)$.

The complex $(\mathcal{E}^\bullet(M,E), D)$ is called the Bernstein-Gelfand-Gelfand complex (BGG complex) of $(\Omega^\bullet(M,E), d^\nabla)$.

Remark 1.3. *In* [21], *Rumin constructed the BGG complex on Carnot-Caratheodry manifolds. However, to adapt this construction in* [21] *to filtered manifolds, we can extend the BGG complex on filtered manifolds.*

Proposition 1.4. *Let ∇ be a flat filtration preserving connection on a filtrated vector bundle E with a metric g_E over a filtered Riemannian manifold $(M, g, \{T^\bullet M\})$. Assume that d_0^∇ is locally constant rank.*

Then, the BGG sequence $(\mathcal{E}^\bullet(M,E), D)$ of the sequence $(\Omega^\bullet(M,E), d^\nabla)$ is given by for $1 \le p \le r$,

$$D_p := \mathrm{gr}_p(D) = \Pi_{\mathcal{E}} \sum_{p_1 + \cdots + p_\ell = p} (-1)^\ell \, \mathrm{gr}_{p_1}(d^\nabla) d_0^{\nabla\dagger} \mathrm{gr}_{p_2}(d^\nabla) \cdots d_0^{\nabla\dagger} \mathrm{gr}_{p_\ell}(d^\nabla) \Pi_{\mathcal{E}}.$$

Proof. We set

$$L' := L\Pi_{\mathcal{E}}.$$

The operator L' has

$$d_0^{\nabla *} L' = 0, \quad d_0^{\nabla *} d L' = 0, \quad \text{and } \Pi_{\mathcal{E}} L' = \mathrm{Id} \text{ on } \mathcal{E}^\bullet(M,E), \tag{1.2}$$

where $\Pi_{\operatorname{Ker} d_0^{\nabla *}}$ is the projection to $\operatorname{Ker} d_0^{\nabla *}$, see [7, Proposition 4.10]. The third equation of (1.2) follows that

$$\Pi_{\mathcal{E}} L = \mathrm{Id} \text{ on } \mathcal{E}^\bullet(M, E),$$

that is,

$$\Pi_{\mathcal{E}} = L^{-1} \text{ on } L\mathcal{E}^\bullet(M, E). \tag{1.3}$$

From (1.1), (1.3), and Proposition 1.2, the operator D is given by

$$D = \Pi_{\mathcal{E}} d^\nabla (\mathrm{Id} + d_0^{\nabla \dagger}(d^\nabla - d_0^\nabla))^{-1} \Pi_{\mathcal{E}}.$$

Since $d_0^{\nabla \dagger}(d^\nabla - d_0^\nabla)$ is nilpotent,

$$D = \sum_{p=0}^{\infty} \Pi_{\mathcal{E}} d^\nabla (-1)^p (d_0^{\nabla \dagger}(d^\nabla - d_0^\nabla))^p \Pi_{\mathcal{E}}.$$

Let $\mathrm{gr}_\bullet$ act on both sides, we obtain Proposition 1.4.

□

1.2 Rumin complex

Let (M, H) be a compact contact manifold of dimension $2n+1$ and E be the flat vector bundle with a unitary holonomy on M. Rumin [20] introduced a complex $(\mathcal{E}^\bullet(M, E), D^\bullet)$, which is a subquotient of the de Rham complex of E. This complex is the BGG complex with respect to contact manifolds. The operator d_0, $\mathrm{gr}_1(d)$, $\mathrm{gr}_2(d)$ are given by

$$d_0 = \mathrm{Int}_T \, d\theta \wedge, \quad \mathrm{gr}_2(d) = \theta \wedge \mathcal{L}_T,$$

and $\mathrm{gr}_1(d)\theta = 0$ and for $\phi \in C^\infty(M, \bigwedge^\bullet H^\vee)$

$$\mathrm{gr}_1(d)\phi = d\phi - \theta \wedge (\mathrm{Int}_T \, d\phi).$$

Here Int_T is the interior product and $\mathcal{L}_T$ is the Reeb vector field with respect to T. From definition of the BGG complex

$$\mathcal{E}^k(M) := \begin{cases} \left\{ \phi \in C^\infty\left(M, \bigwedge^k H^\vee\right) \,\middle|\, (d\theta\wedge)^* \phi = 0 \right\}, & k \le n, \\ \left\{ \phi \in \theta \wedge C^\infty\left(M, \bigwedge^{k-1} H^\vee\right) \,\middle|\, d\theta \wedge \phi = 0 \right\}, & k \ge n+1, \end{cases}$$

and Proposition 1.4, the Rumin complex is given by

$$\begin{aligned} D_1 &= \Pi_{\mathcal{E}} \, \mathrm{gr}_1(d) \Pi_{\mathcal{E}}, \\ D_2 &= \Pi_{\mathcal{E}} \left(\mathrm{gr}_2(d) - \mathrm{gr}_1(d) d_0^\dagger \, \mathrm{gr}_1(d) \right) \Pi_{\mathcal{E}} = \Pi_{\mathcal{E}} \theta \wedge \left(\mathcal{L}_T - \mathrm{gr}_1(d)(d\theta\wedge)^{-1} \mathrm{gr}_1(d) \right) \Pi_{\mathcal{E}}. \end{aligned}$$

A specific feature of the complex is that the operator $D^n = D_2 \colon \mathcal{E}^n(M, E) \to \mathcal{E}^{n+1}(M, E)$ in 'middle degree' is a second-order, while $D^k = D_1 \colon \mathcal{E}^k(M, E) \to \mathcal{E}^{k+1}(M, E)$ for $k \neq n$ are first order which are induced by the exterior derivatives.

Let $a_k = 1/\sqrt{|n-k|}$ for $k \neq n$ and $a_n = 1$. Then, $(\mathcal{E}^\bullet(M, E), d_{\mathcal{E}}^\bullet)$, where $d_{\mathcal{E}}^k = a_k D^k$, is also a complex. We call $(\mathcal{E}^\bullet(M, E), d_{\mathcal{E}}^\bullet)$ the *Rumin complex*. In virtue of the rescaling, $d_{\mathcal{E}}^\bullet$ satisfies Kähler-type identities on Sasakian manifolds [22, (34)], which include the case of lens spaces.

2 Harmonic forms and the Rumin complex on Sasakian manifolds

Let θ be a contact form of H and J be an almost complex structure on H. Then we may define a Riemannian metric $g_{\theta,J}$ on TM by

$$g_{\theta,J}(X,Y) := d\theta(X,JY) + \theta(X)\theta(Y) \text{ for } X,Y \in TM$$

Following [20], we define the Rumin Laplacians $\Delta_{\mathcal{E}}$ associated with $(\mathcal{E}^{\bullet}(M,E), d_{\mathcal{E}}^{\bullet})$ and the metric $g_{\theta,J}$ by

$$\Delta_{\mathcal{E}}^k := \begin{cases} (d_{\mathcal{E}} d_{\mathcal{E}}{}^*)^2 + (d_{\mathcal{E}}{}^* d_{\mathcal{E}})^2, & k \neq n,\, n+1, \\ (d_{\mathcal{E}} d_{\mathcal{E}}{}^*)^2 + D_2^* D_2, & k = n, \\ D_2 D_2^* + (d_{\mathcal{E}}{}^* d_{\mathcal{E}})^2, & k = n+1. \end{cases}$$

Rumin showed that $\Delta_{\mathcal{E}}$ has discrete eigenvalues with finite multiplicities.

Rumin [20] showed that $\mathrm{Ker}(\Delta_{\mathcal{E}})$ is isomorphic to $H^k(M)$. As a natural question, what is the difference between $\mathrm{Ker}(\Delta_{\mathrm{dR}})$ and $\mathrm{Ker}(\Delta_{\mathcal{E}})$ in set? The following theorem answers this question.

Theorem 2.1. ([16, Theorem 1.1]) *Let (M,H,θ,J) be a compact Sasakian manifold of dimension $2n+1$. Then, the kernel of the Rumin Laplacian agrees with that of the Hodge-de Rham Laplacian, that is,*

$$\mathrm{Ker}(\Delta_{\mathrm{dR}} \colon \Omega^k(M) \to \Omega^k(M)) = \mathrm{Ker}(\Delta_{\mathcal{E}} \colon \mathcal{E}^k(M) \to \mathcal{E}^k(M)).$$

Recently, Case showed that by [6, Proposition 12.10], for a compact Sasakian manifold M,

$$\begin{aligned} &\mathrm{Ker}(\Delta_{\mathcal{E}} \colon \mathcal{E}^k(M) \to \mathcal{E}^k(M)) \\ &= \bigoplus_{i+j=k} \mathrm{Ker}(\Delta_{\mathcal{E}} \colon \mathcal{E}^k(M) \to \mathcal{E}^k(M)) \cap C^\infty\left(M, \bigwedge^{i,j} H^\vee\right), \end{aligned} \tag{2.1}$$

where

$$\bigwedge^{i,j} H^\vee := \bigwedge^{i} \left\{\phi \in H^\vee \,\middle|\, J\phi = \sqrt{-1}\phi\right\} \otimes \bigwedge^{j} \left\{\phi \in H^\vee \,\middle|\, J\phi = -\sqrt{-1}\phi\right\}.$$

Using (2.1), he [6] recovered a topological obstruction [3, 10] to the existence of Sasakian structure on a given manifold in terms of its Betti numbers.

From Theorem 2.1 and (2.1), we give another proof of the following corollary:

Corollary 2.2. ([26, Theorems 7.1, 8.1], [10, Corollary 4.2]) *In the setting of Theorem 2.1, for $\phi \in \mathrm{Ker}(\Delta_{\mathrm{dR}} \colon \Omega^k(M) \to \Omega^k(M))$,*

(1) if $k \leq n$, we have $\mathrm{Int}_T\, \phi = 0$, $(d\theta\wedge)^*\phi = 0$,

(2) if $k \geq n+1$, we have $\theta \wedge \phi = 0$, $d\theta \wedge \phi = 0$,

(3) we have $J\phi \in \mathrm{Ker}(\Delta_{\mathrm{dR}})$, that is, $J\phi$ is also a harmonic form,

where Int_T *is the interior product with respect to* T*.*

We recall Proposition 1.2. The cohomology of the BGG complex coincides with that of the de Rham complex. To the author's knowledge, the Sasakian manifolds are the only cases when the kernel of $D + D^*$ agrees with the harmonic space. It is an interesting question: whether on the filtered manifolds the kernel of $D + D^*$ coincides with the harmonic space or not.

Next, one can view the Rumin complex as arising naturally the sub-Riemannian limit of Δ_{dR} induced by the filtration $H \subset TM$ [22]. An analytic approach to sub-Riemannian limit, for fiber bundles, was developed by Mazzeo and Melrose [17], and, for Riemann foliations, was by Forman [8]. On contact manifolds, Albin-Quan solved the asymptotical equation of Δ_{dR}, which was introduced by Forman [8], and its asymptotic behavior can be explicitly written by the Rumin complex [1].

Let $t \in [0, \infty)$. We set

$$d_t := d_0 + t\,\mathrm{gr}_1(d) + t^2\,\mathrm{gr}_2(d).$$

Let $X := M \times [0, \infty)$ and $\Delta_t := d_t d_t^* + d_t^* d_t \colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ for $t \in [0, \infty)$, where d_t^* is the formal adjoint of d_t for the L^2-inner product on $\Omega^\bullet(M)$. We define the space of the sub-Riemannian limit differential forms by

$$\begin{aligned} &{}^{\mathrm{sR}}\Omega^k(X) \\ &:= \left\{ u_0 + tu_1 + \cdots + t^q u_q \in \Omega^k(X) \,\middle|\, u_j \in \Omega^k(M),\ q \in \mathbb{Z}_{\geq 0},\ t \geq 0 \right\}, \end{aligned}$$

and set

$$\mathscr{F}_p^k(\Delta_t) := \left\{ u \in \Omega^k(M) \,\middle|\, \exists \widetilde{u} \in {}^{\mathrm{sR}}\Omega^k(X) \text{ s.t. } \widetilde{u}\big|_{t=0} = u,\ \Delta_t \widetilde{u} = O(t^p) \right\},$$

for $p \geq 0$. In [1, p. 18], Albin-Quan showed that

$$\begin{aligned} \mathscr{F}_3^k(\Delta_t) &= \mathrm{Ker}(\Delta_{\mathcal{E}} \colon \mathcal{E}^k(M) \to \mathcal{E}^k(M)) && \text{for } k \neq n, n+1, \\ \mathscr{F}_5^k(\Delta_t) &= \mathrm{Ker}(\Delta_{\mathcal{E}} \colon \mathcal{E}^k(M) \to \mathcal{E}^k(M)) && \text{for } k = n, n+1. \end{aligned}$$

By Corollary 2.2, we obtain the following:

Corollary 2.3. ([16, Corollary 1.3]) *In the setting of Theorem 2.1,*

$$\mathrm{Ker}(\Delta_{\mathrm{dR}}) = \bigcap_{t>0} \mathrm{Ker}(\Delta_t).$$

By Theorem 2.1 and [1, p. 18], we have

Corollary 2.4. ([16, Corollary 1.4]) *In the setting of Theorem 2.1,*

$$\begin{aligned} \mathscr{F}_3^k(\Delta_t) &= \bigcap_{t>0} \mathrm{Ker}(\Delta_t \colon \Omega^k(M) \to \Omega^k(M)) && \text{for } k \neq n, n+1, \\ \mathscr{F}_5^k(\Delta_t) &= \bigcap_{t>0} \mathrm{Ker}(\Delta_t \colon \Omega^k(M) \to \Omega^k(M)) && \text{for } k = n, n+1. \end{aligned}$$

It means that for "$k \neq n, n+1$ and $u \in \mathscr{F}_3^k(\Delta_t)$" or "$k = n, n+1$ and $u \in \mathscr{F}_5^k(\Delta_t)$", taking $\widetilde{u} = u$, we see

$$\Delta_t \widetilde{u} = 0 \text{ for } t > 0.$$

In [1], on compact contact manifolds, for $u \in \mathrm{Ker}(\Delta_{\mathcal{E}})$ Albin-Quan constructed $\widetilde{u}$ such that "for $k \neq n, n+1$, $\Delta_t \widetilde{u} = O(s^3)$" and "for $k = n, n+1$, $\Delta_t \widetilde{u} = O(s^5)$" by using d_0, d_b, d_T. On compact Sasakian manifolds, we give a simple construction of $\widetilde{u}$.

3 The eigenvalue of the Rumin Laplacian on the standard CR sphere

In this section, we see the result [14] of the eigenvalues of $\Delta_{\mathcal{E}}$ on the trivial bundle $\underline{\mathbb{C}}$ over the standard CR spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$. Here the *standard CR sphere* is the triple (S^{2n+1}, θ, J), where θ is given the contact form by $\theta = \sqrt{-1}(\overline{\partial} - \partial)|z|^2$ and J is an almost complex structure J induced from the complex structure of $\mathbb{C}^{n+1}$. To state our result we need to introduce notation for highest weight representations of the unitary group $\mathrm{U}(n+1)$ which acts on S^{2n+1}. The irreducible representations of $\mathrm{U}(n+1)$ are classified by the highest weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$; the corresponding representation will be denoted by $V(\lambda)$. Julg and Kasparov [12] showed that the complexification of $\mathcal{E}^k(S^{2n+1})$, as a $\mathrm{U}(n+1)$-module, is decomposed into the irreducible of the form

$$\Psi_{(q,j,i,p)} := V(q, \underbrace{1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{i \text{ times}}, -p).$$

Since $\Delta_{\mathcal{E}}$ commutes with the $\mathrm{U}(n+1)$-action, it acts as a scalar on each $\Psi_{(q,j,i,p)}$.

Theorem 3.1. ([14, Theorem 0.1]) *Let S^{2n+1} be the standard CR sphere with the contact from $\theta = \sqrt{-1}(\overline{\partial} - \partial)|z|^2$. Then, on the subspaces of the complexification of $\mathcal{E}^\bullet(S^{2n+1})$ corresponding to the representations $\Psi_{(q,j,i,p)}$, the eigenvalue of $\Delta_{\mathcal{E}}$ is*

$$\frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}.$$

This theorem claims that the eigenvalues of $\Delta_{\mathcal{E}}$ are determined by the highest weight. This phenomenon also appears in the case of the Hodge-de Rham Laplacian Δ_{dR} on symmetric spaces G/K. Ikeda and Taniguchi [13] showed that on the subspaces of k-forms of G/K corresponding to $V(\lambda)$, the eigenvalue of Δ_{dR} is determined by λ. It is a natural question to ask whether the eigenvalues of $\Delta_{\mathcal{E}}$ on a contact homogeneous space G/K are determined by the highest weight of G.

Theorem 3.1 unifies the following results on the eigenvalues of Rumin Laplacians on the spheres. Julg and Kasparov [12] determined the eigenvalues of $D_2^* D_2$. Folland [9] calculated the eigenvalue of the sub-Laplacian Δ_b, which agrees with $\Delta_{\mathcal{E}}$ on $\mathcal{E}^0(S^{2n+1})$. Seshadri [25] determined the eigenvalues of $d_{\mathcal{E}} d_{\mathcal{E}}^*$ on $\mathcal{E}^1(S^{2n+1})$ in the case S^3. Ørsted and Zhang [18] determined eigenvalues of the Laplacian of the holomorphic and anti-holomorphic part of D except for the ones containing D_2.

Note that Ørsted and Zhang used D in place of $d_{\mathcal{E}}$. As a result, the eigenvalues of the Laplacian in their paper are not determined by the highest weights. This also explains the importance the scaling factor a_k.

4 Ray-Singer torsion and the Rumin Laplacian on lens spaces

We introduce the analytic torsion and metric of the Rumin complex $(\mathcal{E}^\bullet(M, E), d_{\mathcal{E}}^\bullet)$ by following [2, 14, 24]. We define the *contact analytic torsion function* associated with

$(\mathcal{E}^\bullet(M,E), d_\mathcal{E}^\bullet)$ by

$$\kappa_\mathcal{E}(M,E,g_{\theta,J})(s) := \sum_{k=0}^{n} (-1)^{k+1}(n+1-k)\zeta(\Delta_\mathcal{E}^k)(s), \tag{4.1}$$

where $\zeta(\Delta_\mathcal{E}^k)(s)$ is the spectral zeta function of $\Delta_\mathcal{E}^k$, and the *contact analytic torsion* $T_\mathcal{E}$ by

$$2\log T_\mathcal{E}(M,E,g_{\theta,J}) = \kappa_\mathcal{E}(M,E,g_{\theta,J})'(0).$$

Let $H^\bullet(\mathcal{E}^\bullet, d_\mathcal{E}^\bullet)$ be the cohomology of the Rumin complex. We define the contact metric on $\det H^\bullet(\mathcal{E}^\bullet, d_\mathcal{E}^\bullet)$ by

$$\| \quad \|_\mathcal{E}(M,E,g_{\theta,J}) = T_\mathcal{E}^{-1}(M,E,g_{\theta,J})| \quad |_{L^2(\mathcal{E}^\bullet)},$$

where the metric $| \quad |_{L^2(\mathcal{E}^\bullet)}$ is induced by L^2 metric on $\mathcal{E}^\bullet(M,E)$ via identification of the cohomology classes by the harmonic forms on $\mathcal{E}^\bullet(M,E)$.

Rumin and Seshadri [24] defined another analytic torsion function κ_R from $(\mathcal{E}^\bullet(M,E), D^\bullet)$, which is different from $\kappa_\mathcal{E}$ except in dimension 3.

Proposition 4.1. ([24])

(1) In dimension 3, $\kappa_\mathrm{R}(M,E,g_{\theta,J})(0)$ is a contact invariant, that is, independent of the metric $g_{\theta,J}$.

(2) For flat bundles E with a unitary holonomy on 3-dimensional Sasakian manifolds with S^1 action,

$$\kappa_\mathrm{R}(M,E,g_{\theta,J})(0) = 0, \quad T_\mathcal{E}(M,E,g_{\theta,J}) = T_\mathrm{dR}(M,E,g_{\theta,J}).$$

To extend Proposition 4.1 (2) on the standard CR sphere S^{2n+1}, with $d_\mathcal{E}$ instead of D, the author [14] showed the following:

Proposition 4.2. ([14, Theorem 0.2, Corollary 0.1]) *On trivial line bundle $\underline{\mathbb{C}}$ over the standard CR spheres $S^{2n+1}(\subset \mathbb{C}^{n+1})$, we have*

$$\kappa_\mathrm{R}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J})(0) = 0, \quad T_\mathcal{E}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J}) = n! T_\mathrm{dR}(S^{2n+1}, \underline{\mathbb{C}}, g_{\theta,J}).$$

Moreover, Albin and Quan [1] showed the difference between the Ray-Singer torsion and the contact analytic torsion is given by some integrals of universal polynomials in the local invariants of the metric on contact manifolds:

Proposition 4.3. ([1, Corollary 3 and (4)]) *Let M be the compact contact manifold with dimension $2n+1$. for all unitary holonomy $\alpha\colon \pi_1(M) \to \mathrm{U}(r)$, we have*

$$\begin{aligned}&\log T_\mathcal{E}(K, E_\alpha, g_{\theta,J}) - \log T_\mathrm{dR}(K, E_\alpha, g_{\theta,J})\\ &= \int_M \exists \textit{universal polynomials in the local invariants of } g_{\theta,J}.\end{aligned}$$

In this section, we extend this coincidence on lens spaces and determine explicitly the analytic torsion functions associated with the Rumin complex in terms of the Hurwitz zeta function. Let g_std be the standard metric on S^{2n+1} and we note that $g_{\theta,J} = 4g_\mathrm{std}$.

Let $\mu, \nu_1, \ldots, \nu_{n+1}$ be integers such that the ν_j are coprime to μ. Let Γ be the subgroup of $(S^1)^{n+1}$ generated by

$$\gamma = (\gamma_1, \ldots, \gamma_{n+1}) := \big(\exp(2\pi\sqrt{-1}\nu_1/\mu), \cdots, \exp(2\pi\sqrt{-1}\nu_{n+1}/\mu)\big).$$

We denote the lens space by

$$K := S^{2n+1}/\Gamma.$$

Let $\underline{\mathbb{C}}$ be the trivial line bundle on K. Fix $u \in \mathbb{Z}$ and consider the unitary representation $\alpha_u\colon \pi_1(K) = \Gamma \to \mathrm{U}(1)$, defined by

$$\alpha_u\left(\gamma^\ell\right) := \exp\left(2\pi\sqrt{-1}u\ell/\mu\right) \text{ for } \ell \in \mathbb{Z}.$$

Let E_α be the flat vector bundle associated with the unitary representation $\alpha\colon \pi_1(K) = \Gamma \to \mathrm{U}(r)$, and $E_{\alpha_u} = E_u$, which can be considered as α_u-equivariant functions on S^{2n+1}.

Our main result is

Theorem 4.4. ([15, Theorem 1.1]) *Let K be the lens space with the contact form and the almost complex structure which are induced by the action Γ on the standard CR sphere S^{2n+1}.*

(1) *The contact analytic torsion function of $(K, \underline{\mathbb{C}})$ is given by*

$$\kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})(s) = -(n+1)\big(1 + 2^{2s+1}\mu^{-2s}\zeta(2s)\big), \tag{4.2}$$

where ζ is the Riemann zeta function. In particular, we have

$$\kappa_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J})(0) = 0, \tag{4.3}$$

$$T_{\mathcal{E}}(K, \underline{\mathbb{C}}, g_{\theta,J}) = \left(\frac{4\pi}{\mu}\right)^{n+1}. \tag{4.4}$$

(2) *The contact analytic torsion function of (K, E_u) for $u \in \{1, \ldots, \mu - 1\}$ is given by*

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(s) = -2^{2s}\mu^{-2s}\sum_{j=1}^{n+1}\Big(\zeta\big(2s, A_\mu(u\tau_j)/\mu\big) + \zeta\big(2s, A_\mu(-u\tau_j)/\mu\big)\Big), \tag{4.5}$$

where $\zeta(s,a) := \sum_{q=0}^{\infty}(q+a)^{-s}$ is the Hurwitz zeta function for $0 < a \leq 1$, $A_\mu(w)$ is the integer between 1 and μ such that $A_\mu(w) \equiv w \mod \mu$ and $\tau_j\nu_j \equiv 1 \mod \mu$. In particular, we have

$$\kappa_{\mathcal{E}}(K, E_u, g_{\theta,J})(0) = 0, \tag{4.6}$$

$$T_{\mathcal{E}}(K, E_u, g_{\theta,J}) = \prod_{j=1}^{n+1}\left|e^{2\pi\sqrt{-1}u\tau_j/\mu} - 1\right|. \tag{4.7}$$

The equations (4.2) and (4.5) extend the following results of $\kappa_{\mathcal{E}}$ the spheres to on lens spaces. Rumin and Seshadri [24, Theorem 5.4] showed (4.2) in the case of 3-dimensional lens spaces. The author [14] showed (4.2) in the case of $(S^{2n+1}, \underline{\mathbb{C}})$ for arbitrary n.

From (4.3) and (4.6), we see that the metric $\|\ \ \|_{\mathcal{E}}$ on $(K, E_u, g_{\theta,J})$ is invariant under the constant rescaling $\theta \mapsto C\theta$. The argument is exactly the same as the one in [24].

In the same way as [14], the fact that the representations determine the eigenvalues of $\Delta_{\mathcal{E}}$ cause several cancellations in the linear combination (4.1), which significantly simplifies the computation of $\kappa_{\mathcal{E}}(s)$. We cannot get such a simple formula for the contact analytic torsion function κ_{R} of $(\mathcal{E}^\bullet(M,E), D^\bullet)$ for dimensions higher than 3.

Let us compare the contact analytic torsion with the Ray-Singer torsion on lens spaces. Ray [19] showed the following:

Proposition 4.5. ([19]) *For* $u\,(=1,\ldots,\mu-1)$,

$$T_{\mathrm{dR}}(K,E_u,4g_{\mathrm{std}})=\prod_{j=1}^{n+1}\left|e^{2\pi\sqrt{-1}u\tau_j/\mu}-1\right|.$$

Weng and You [27] calculate the Ray-Singer torsion on spheres:

Proposition 4.6. ([27])

$$T_{\mathrm{dR}}(S,\underline{\mathbb{C}},g_{\mathrm{std}})=\frac{2\pi^{n+1}}{n!}.$$

The author extended their results for the trivial bundle on lens spaces:

Proposition 4.7. ([15, Proposition 1.2]) *In the setting of Theorem 4.4, we have*

$$T_{\mathrm{dR}}(K,\underline{\mathbb{C}},4g_{\mathrm{std}})=\frac{(4\pi)^{n+1}}{n!\mu^{n+1}}.$$

The metric $4g_{\mathrm{std}}$ agrees with the metric $g_{\theta,J}$ defined from the contact form $\theta=\sqrt{-1}(\overline{\partial}-\partial)|z|^2$. Since the cohomology of $(\mathcal{E}^\bullet(M,E),d_{\mathcal{E}}^\bullet)$ coincides with that of $(\Omega^\bullet(M,E),d)$ (e.g. [20, p 286]), there is a natural isomorphism
$\det H^\bullet(\mathcal{E}^\bullet(M,E),d_{\mathcal{E}}^\bullet)\cong\det H^\bullet(\Omega^\bullet(M,E),d)$, which turns out to be isometric for the L^2 metrics. Therefore (4.4) and (4.7) give

Corollary 4.8. ([15, Corollary 1.3]) *In the setting of Theorem 4.4, for all unitary holonomy* $\alpha\colon\pi_1(K)\to\mathrm{U}(r)$, *we have*

$$\begin{aligned}T_{\mathcal{E}}(K,E_\alpha,g_{\theta,J})&=n!^{\dim H^0(K,E_\alpha)}T_{\mathrm{dR}}(K,E_\alpha,g_{\theta,J}),\\ \|\quad\|_{\mathcal{E}}(K,E_\alpha,g_{\theta,J})&=n!^{-\dim H^0(K,E_\alpha)}\|\quad\|_{\mathrm{dR}}(K,E_\alpha,g_{\theta,J}),\end{aligned}$$

via the isomorphism $\det H^\bullet(\mathcal{E}^\bullet(M,E_\alpha),d_{\mathcal{E}}^\bullet)\cong\det H^\bullet(\Omega^\bullet(M,E_\alpha),d)$.

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